O‘ZBEKISTON RESPUDLIKASI OLIY VA O‘RTA MAXSUSS TA’LIM VASIRLIGI

QARSHI DAVLAT UNIVERSITETI

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ODDIY DIFFERENSIAL TENGLAMALARDAN MISOLLAR, MASALALAR VA TOPSHIRIQLAR

5130100 – ”Matematika” ta’lim yo‘nalishi talabalari uchun o‘quv qo‘llanma

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Mazkur kitob 5130100 – ”matematika” ta’lim yo‘nalishi talabalari uchun oddiy differensial tenglamalar kursi bo‘yicha o‘quv qo‘llanma. Unda yordamchi nazariy ma’lumotlar, masalalar yechish metodlari, javobli masalalar, mustaqil ishlar topshiriqlari va ularni bajarish tartibi keltirilgan. Kitob amaldagi o‘quv dasturining barcha mavzularini qamrab olgan.

Kitobдан “mexanika”, ”amaliy matematika va informatika”, “fizika” yo`nalishi talabalari hamda oddiy differensial tenglamalarni mustaqil o‘rganmoqchi bo‘lgan barcha xohlovchilar samarali foydalanishlari mumkin.

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MUNDARIJA

SO‘Z BOSHI

Mazkur kitob oddiy differensial tenglamalarni yechish usullarini o‘rganish va mustaqil ishlarni bajarish uchun oquv qo‘llanma sifatida yozilgan. U “matematika” (5130100) yo‘nalishi bo‘yicha tahsil oluvchi bakalavriat talabalari uchun oddiy differensial tenglamalar kursi dasturining barcha mavzularini o‘z ichiga qamrab olgan. Bundan tashqari, differensial tenglamalarni kompyuter matematikasi “Maple” yordamida tekshirishni o‘rganishga oid mavzu ham bor. Kitobdan oddiy differensial tenglamalarni mustaqil o‘rganmoqchi bo‘lgan barcha xohlovchilar ham unumli foydalanishlari mumkin.

Tushunarliki, har qanday kursni yoki nazariyani o‘rganish, asosan, mos mustaqil ishlarni bajarish orqali amalga oshiriladi. Shu sababli mazkur kitobdagi masalalarni yechish va mustaqil ishlarni bajarish differensial tenglamalar kursini o‘zlashtirish uchun zarurdir. O‘qituvchi ba`zi mustaqil ishlarning ba’zi qismlarini bajartirmasligi yoki ba`zilarini birlashtirishi mumkin.

Kitobda 21 ta mustaqil ishni bajarish uchun zarur bo‘lgan nazariy ma’lumotlar, ularni tushuntiruvchi misollar, javobli masalalar, mustaqil ish topshiriqlari va ularni bajarish tartibi keltirilgan. Bundan tashqari, bilimlarni chuqurlashtirish va mustahkamlash uchun tavsiyalar ham berilgan.

Birinchi tartibli differensial tenglamalarga oid mustaqil ishlar kurs uchun poydevor bo‘lganligi sababli ularga oid ishlar soni (1-8 mustaqil ishlar) maydalanish evaziga nisbatan ko‘p.

Paragraflar yordamchi ma’lumotlar, masalalar va mustaqil ish topshiriqlari bo‘limlaridan tuzilgan. Yordamchi ma’lumotlar bo‘limida zarur nazariy ma’lumotlar, ularga oid yechib ko‘rsatilgan masalalar, masalalar bo‘limida mavzuni o‘zlashtirish va mustahkamlash maqsadida mustaqil yechish uchun masalalar tavsiya etilgan. Bu masalalarning javoblari yoki/va yechimlari (ko‘satmalar) ham keltirilgan.

Har bir mustaqil ish (MI) topshiriqlari (masalalari) o‘qituvchi ko‘rsatmasiga binoan olinadi (masalan, talaba familiyasining guruh jurnalidagi tartib raqamiga mos ravishda). Talaba mustaqil ishni bajarish jarayonida shu kitob va [1-11] adabiyotlardan foydalanib nazariy bilimlarini chuqurlashtiradi va masalalar yechish ko‘nikmalarini hosil qiladi. So‘ngra u MI nomi, maqsadi, yordamchi nazariy ma’lumotlar keltirilgan va masalalarning yechilishi tushuntirilgan mustaqil ish yozadi va uni (o‘qituvchiga topshiradi) o‘qituvchi oldida himoya qiladi.

Agar talaba mavzuni yetarli darajada o‘zlashtirmagan yoki hisobot tushunilmagan tarzda ko‘chirib kelingan bo‘lsa, mustaqil ish bajarilmagan deb hisoblanadi va talabaga mavzuni to‘la o‘zlashtirish topshirig‘i beriladi. Topshirish muddatlari oldindan e’lon qilinadi. Odatda MI bir haftada bajariladi. Har bir mustaqil ish odatda navbatdagi mustaqil ish muddatigacha topshirilishi kerak.

Talabalarning joriy va oraliq baholash ballari bosqich (qism, modul) larga bo‘lib amalga oshiriladi. Har bir bosqichda talaba uchun joriy va oraliq baholashlar balini qo‘yishda shu bosqichga taaluqli bo‘lgan mustaqil ishlarni talaba tomonidan bajarish muddati va sifati hisobga olinadi.

Mustaqil ishlarni o‘z vaqtida bajarib va topshirib borgan talabalar differensial tenglamalar kursi bo‘yicha o‘tish (minimal) balini to‘play olishlari tajribada sinalgan.

Qo‘llanmadagi misol va masalalarning ko‘pchiligi muallif tomonidan tuzilgan, qolganlari esa mavjud adabiyotlardan olingan.

Ushbu o`quv qo`llanma muallifning “Differensial tenglamalardan misollar, masalalar va mustaqil ishlar” nomli o`quv qo`llanmasining kamchiliklari tuzatilgan va to`ldirilgan variantidir. Bunda  kamchiliklar tuzatilishi bilan bir qatorda, ba’zi paragraflarga masalalar qo`shildi, § 2 amaliy masalalar bilan kenggaytirildi hamda “Aralash differensial tenglamalar” deb nomlangan yangi § 8 yozildi.

Muallif qo‘llanmadagi mustaqil ishlarni aprobatsiyadan o‘tkazishda yordam bergan barcha shogirdlari va talabalaridan hamda kasbdoshlaridan, bundan tashqari, taqrizchilardan ham, minnatdor ekanligini e’tirof etadi.

Kitob haqidagi fikr va mulohazalaringizni nosir_d@mail.ru elektron manzilga yozsangiz, muallif sizdan mamnun bo‘ladi.

ASOSIY BELGILASHLAR RO‘YXATI

$\forall$ — har qanday, ixtiyoriy, har bir (umumiylik kvantori).

$\exists$ — mavjud, kamida bitta mavjud (mavjudlik kvantori).

$\Rightarrow$ — kelib chiqadi (implikatsiya belgisi).

$\Leftrightarrow$ — teng kuchli (ekvivalent).

$\overset{def}{=}$ — ta’rifga ko‘ra teng.

$\{x\in E\,|\,P(x)\}$ — E to‘plamning $P(x)$ xossaga ega bo‘lgan barcha x elementlari to‘plami.

$\mathbb{N}$ — natural sonlar to‘plami. $n\in\mathbb{N}$.

$\mathbb{R}$ — haqiqiy sonlar to‘plami.

$\mathbb{C}$ — kompleks sonlar to‘plami.

$\mathbb{R}^n$ — n o‘lchamli haqiqiy Evklid fazosi.

$c, c_1, c_2, \ldots$ — ixtiyoriy o‘zgarmaslar (doimiylar).

const — o‘zgarmas (doimiy).

$(a,b)\overset{def}{=}\{x\in\mathbb{R}\,|\,a<x<b\}$ $(a<b)$ — interval.

$[a,b]\overset{def}{=}\{x\in\mathbb{R}\,|\,a\le x\le b\}$ $(a<b)$ — segment.

$(a,b]\overset{def}{=}\{x\in\mathbb{R}\,|\,a<x\le b\}$ $(a<b)$ — yarim segment.

$[a,b)\overset{def}{=}\{x\in\mathbb{R}\,|\,a\le x<b\}$ $(a<b)$ — yarim segment.

$\mathbb{R}_+\overset{def}{=}[0,+\infty)$.

I — sonli oraliq (ichi bo‘sh bo‘lmagan bog‘lanishli sonli to‘plam).

D — soha, ya’ni bo‘shmas, ochiq va bog‘lanishli to‘plam.

$\max E$ — E sonli to‘plamning maksimumi (eng katta elementi).

$\min E$ — E sonli to‘plamning minimumi (eng kichik elementi).

$\sup E$ — E sonli to‘plamning supremumi (yuqori chegaralarning eng kichigi).

$\inf E$ — E sonli to‘plamning infimumi (quyi chegaralarning eng kattasi).

$\|\ \|$ — norma (yoki matritsa) belgisi.

∂E — E to‘plamning chegarasi.

E^C — E to‘plamning (qaralayotgan fazogacha) to‘ldiruvchisi.

$B_\delta(a)$ — δ radiusli a markazli (ochiq) shar.

$B_\delta = B_\delta(o)$

$X\times Y$ — to‘plamlarning to‘g‘ri (Dekart) ko‘paytmasi.

$f: X\to Y$ — X to‘plamda aniqlangan, qiymatlari Y to‘plamda joylashgan f funksiya

$D(f)$ — f funksiyaning aniqlanish to‘plami (sohasi).

$f|_E$ — f funksiyaning E to‘plamga torayishi.

$f|_a = f(a)$

$g\circ f$ — f va g funksiyalar kompozitsiyasi (ketma-ket bajarilishi).

$f(x)=o(g(x)),\ x\to a,$ — asimptotik tenglik (kichik o); u $f(x)=\varepsilon(x)\cdot g(x)$, $\lim\limits_{x\to a}\varepsilon(x)=0$, ekanligini anglatadi.

$f(x)=O(g(x)),\ x\to a,$— (katta o); u $f(x)$ funksiya $g(x)$ ni a nuqtaning biror atrofida chegaralangan $h(x)$ funksiyaga ko‘paytirishdan hosil bo‘lishini ($f(x)= h(x)\cdot g(x)$) anglatadi.

$C(X,Y)$ — barcha uzluksiz $f: X\to Y$ funksiyalar sinfi (to‘plami).

$C(X)=C(X,\mathbb{R})$.

$C^k(X,Y))$ — k- tartibli barcha hosilalari (demak, undan past tartiblilari ham) uzluksiz bo‘lgan $f: X\to Y$ funksiyalar sinfi.

$C^k(X)=C^k(X,\mathbb{R})$

dist(X,Y) — to‘plamlar orasidagi masofa (distance – masofa).

dim$X$ — X fazoning o‘lchami (dimension – o‘lcham).

deg$P$ — P ko‘phadning darajasi (degree – daraja).

$\mathbb{M}_{n\times n}(\mathbb{R})$ — haqiqiy sonlardan tuzilgan $n\times n$ o‘lchamli matritsalar to‘plami.

$\mathbb{M}_{n\times n}(\mathbb{C})$ — kompleks sonlardan tuzilgan $n\times n$ o‘lchamli matritsalar to‘plami.

$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{c}, \boldsymbol{h}, \boldsymbol{f}, \boldsymbol{m}, \boldsymbol{n}, \boldsymbol{p}, \boldsymbol{q}, \ldots$ (qalin harflar) — vektorlar.

MYaT — mavjudlik va yagonalik teoremasi.

MI — mustaqil ish.

ODT — oddiy differensial tenglama.

DT = ODT.

🗝 — masala (misol) yechilishining boshlanish belgisi.

👍 — masala (misol) yechilishining tugallanganlik belgisi.

1. DIFFERENSIAL TENGLAMA VA UNING YECHIMI

Maqsad – differensial tenglama va uning yechimi tushunchalarini hamda egri chiziqlarning berilgan oilasi uchun bu oila qanoatlantiruvchi differensial tenglamani tuzishni o‘rganish.

Yordamchi ma’lumotlar:

Ushbu

$$F(x, y, y', y'', ..., y^{(n)}) = 0 \quad \text{yoki} \quad F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^ny}{dx^n}\right) = 0 \qquad (1)$$

ko‘rinishdagi tenglik $y = y(x)$ funksiyaga nisbatan ***n-tartibli oddiy differensial tenglama*** deyiladi, bunda $F(x, y, p_1, p_2, ..., p_n)$ funksiya $x, y, p_1, p_2, ..., p_n$ haqiqiy argumentlarning berilgan haqiqiy funksiyasi, $x-$ erkli haqiqiy o‘zgaruvchi, $y = y(x) - x$ o‘zgaruvchining noma’lum haqiqiy funksiyasi, $y', y'', ..., y^{(n)} -$ uning hosilalari; $y^{(n)} -$ haqiqatdan ham (1) tenglamada qatnashgan deb hisoblanadi (F funksiyaning qolgan argumentlari bu tenglamada ishtirok etmasligi mumkin).

Agar $y = y(x)$ funksiya I oraliqda aniqlangan $\left(I \subset D(y)\right)$ hamda

1⁰. $y(x) \in C^n(I)$ ($y = y(x)$ funksiyaning $n-$tartibli hosilasi I da uzluksiz, funksiya I da n marta uzluksiz differensiallanuvchi, ya’ni $y^{(n)}(x) \in C(I)$),

2⁰. ixtiyoriy $x \in I$ uchun $F(x, y(x), y'(x), y''(x), ..., y^{(n)}(x)) = 0$ $\big($ $y = y(x)$ funksiya (1) tenglamani I da qanoatlantiradi (ayniyatga aylantiradi)$\big)$

shartlar bajarilsa, u holda bu $y = y(x)$ funksiya (1) tenglamaning ***I oraliqda aniqlangan yechimi*** deyiladi.

Yechim oshkormas ko‘rinishda, ya’ni

$$\Phi(x, y) = 0 \qquad (2)$$

tenglama bilan ham berilishi mumkin. Bunda (2) munosabat biror I oraliqda (1) ning biror $y = y(x)$ yechimini aniqlaydi deb tushuniladi (kitob oxiridagi ilovaga qarang).

Ushbu

$$\Phi\left(x, y, c_1, c_2, ..., c_n\right) = 0 \tag{3}$$

tenglama bilan berilgan n parametrli silliq egri chiziqlar oilasini qaraylik; bunda $c_1, c_2, ..., c_n$ –biror sohada o‘zgaruvchi parametrlar. Bu chiziqlar oilasi qanoatlantiradigan differensial tenglamani tuzish uchun quyidagicha ish tutamiz. Dastlab (3) munosabatdan topiluvchi $y = y(x)$ funksiyani n marta uzluksiz differensiallanuvchi deb faraz qilgan holda (3) ni n marta x bo‘yicha differensiallaymiz:

$$\begin{cases} \dfrac{\partial \Phi}{\partial x} + \dfrac{\partial \Phi}{\partial y} \cdot y' = 0 \\ \dfrac{\partial^2 \Phi}{\partial x^2} + 2\dfrac{\partial^2 \Phi}{\partial x \partial y} y' + \dfrac{\partial^2 \Phi}{\partial y^2} y'^2 + \dfrac{\partial \Phi}{\partial y} y'' = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{\partial^n \Phi}{\partial x^n} + \ldots + \dfrac{\partial \Phi}{\partial y} y^{(n)} = 0 \end{cases} \tag{4}$$

Endi (3) va (4) munosabatlardan (ular $n+1$ ta) $c_1, c_2, ..., c_n$ parametrlarni yo‘qotamiz. Natijada izlangan differensial tenglama hosil bo‘ladi.

Misol 1. Ushbu

$$y = \varphi(x), \ \ \varphi(x) = x + \frac{1}{x},$$

funksiya

$$y' + y^2 - x^2 - 3 = 0$$

birinchi tartibli differensial tenglamaning $(0;+\infty)$ intervalda yechimi ekanligini ko‘rsating.

⊶Yechim ta’rifidagi shartlarning bajarilishini tekshiramiz.

1⁰. Berilgan funksiya $(0;+\infty)$ intervalda uzluksiz differensiallanuvchi, chunki uning hosilasi shu intervalda uzluksiz:

$$\varphi'(x) = \left(x + \frac{1}{x}\right)' = 1 - \frac{1}{x^2} \in C(0;+\infty).$$

2⁰. Endi $y = \varphi(x)$ bo‘lganda berilgan tenglama $(0;+\infty)$ intervalda ayniyatga aylanishini tekshiramiz:

$$y' + y^2 - x^2 - 3 = \varphi'(x) + (\varphi(x))^2 - x^2 - 3 = 1 - \frac{1}{x^2} + \left(x + \frac{1}{x}\right)^2 - x^2 - 3 =$$

$$= 1 - \frac{1}{x^2} + x^2 + 2 + \frac{1}{x^2} - x^2 - 3 = 0.$$

Demak, berilgan funksiya berilgan tenglamaning $(0;+\infty)$ intervalda yechimi. 👍

Izoh. Berilgan funksiya berilgan tenglamaning $(-\infty;0)$ intervalda ham yechimi ekanligi yuqoridagilardan ravshan.

Misol 2. Ushbu

$$x\frac{dy}{dx} - y - x^2 \cos x^2 = 0 \ \left(\text{yoki } x\frac{dy}{dx} - y = x^2 \cos x^2\right)$$

tenglama berilgan. Ushbu

$$\varphi(x) = x\int_0^x \cos t^2 dt, \ \ x \in (-\infty;+\infty),$$

funksiyaning bu tenglama yechimi ekanligini ko‘rsating.

🗝 Yechim ta’rifiga ko‘ra quyidagilarni bajaramiz:

1⁰. $\varphi(x)$ funksiya $(-\infty;+\infty)$ da uzluksiz differensiallanuvchi, chunki uning hosilasi

$$\frac{d\varphi(x)}{dx} = \frac{d}{dx}\left(x\int_0^x \cos t^2 dt\right) = \int_0^x \cos t^2 dt + x \cdot \cos x^2,$$

ravshanki, $(-\infty;+\infty)$ da uzluksiz.

2⁰. Endi $y = \varphi(x)$ bo‘lganda berilgan tenglamaning qanoatlanishiga ishonch hosil qilamiz:

$$x\frac{d\varphi(x)}{dx} - \varphi(x) = x \cdot \left(\int_0^x \cos t^2 dt + x\cos x^2\right) - x\int_0^x \cos t^2 dt = x^2 \cos x^2, x \in (-\infty;+\infty) .$$

Shunday qilib, biz berilgan $y=\varphi(x)$ funksiya berilgan tenglamaning $(-\infty;+\infty)$ da yechimi ekanligini ko‘rsatdik. ♘

Misol 3. Ushbu $y=|x-1|$ funksiya biror differensial tenglamaning $(0;2)$ oraliqda aniqlangan yechimi bo‘la oladimi?

⊶ Yo‘q, chunki berilgan $y=|x-1|$ funksiya $(0;2)$ intervalda uzluksiz differensiallanuvchi emas ($x=1\in(0;2)$ nuqtada hosila mavjud emas). ♘

Misol 4. $y=x^2-2$ funksiya $y''-xy=y^2$ tenglamaning biror I oraliqda yechimi bo‘la oladimi?

⊶ Tushunarliki, $y'=2x,\ y''=1\in C(\mathbb{R})\Rightarrow y=x^2-2\in C^2(\mathbb{R})$. Agar berilgan funksiya berilgan tenglamani biror I da qanoatlantirsa,

$$2-x(x^2-2)=(x^2-2)^2,\ x\in I,$$

yoki

$$2-x^3+2x=x^4-4x^2+4,\ x\in I,$$

ya’ni

$$x^4+x^3-4x^2-2x+2=0$$

tenglik barcha $x\in I$ lar uchun o‘rinli bo‘lishi kerak. Bunday bo‘la olmaydi, chunki 4- darajali ko‘phad (oliy algebraning asosiy teoremasiga ko‘ra) ko‘pi bilan to‘rtta nuqtada nolga aylanishi mumkin xolos; ixtiyoriy oraliqda esa cheksiz ko‘p nuqtalar bor.

Demak, berilgan $y=x^2-2$ funksiya berilgan $y''-xy=y^2$ differensial tenglamaning hech qanday oraliqda yechimi bo‘la olmaydi. ♘

Misol 5. $y=\ln\left(1+(x+c)^2\right)$ funksiya c ning ixtiyoriy qiymatida $(-\infty;+\infty)$ intervalda $y''+y'^2=2e^{-y}$ tenglamaning yechimi ekanligini isbotlang.

⊶ Dastlab berilgan funksiyaning birinchi va ikkinchi tartibli hosilalarini hisoblaymiz ($c=\text{const}$):

$$y'=\left(\ln(1+(x+c)^2\right)'=\frac{1}{1+(x+c)^2}\left((x+c)^2\right)'=\frac{2(x+c)}{1+(x+c)^2},$$

$$y''=\frac{2\left(1+(x+c)^2\right)-2(x+c)\cdot 2(x+c)}{\left(1+(x+c)^2\right)^2}=\frac{2\left(1-(x+c)^2\right)}{\left(1+(x+c)^2\right)^2};\ y''\in C(-\infty,+\infty).$$

Endi tenglamaning qanoatlanishini tekshiramiz. Berilgan tenglamaning chap tomoni

$$y''+y'^2=\frac{2\left(1-(x+c)^2\right)}{\left(1+(x+c)^2\right)^2}+\frac{4(x+c)^2}{\left(1+(x+c)^2\right)^2}=\frac{2+2(x+c)^2}{\left(1+(x+c)^2\right)^2}=\frac{2}{1+(x+c)^2}. \qquad (5)$$

Uning o‘ng tomoni

$$2e^{-y}=2e^{-\ln(1+(x+c)^2)}=\frac{2}{1+(x+c)^2}. \qquad (6)$$

Demak, (5) va (6) tengliklarga ko‘ra c ning ixtiyoriy tayinlangan qiymatida berilgan funksiya $(-\infty;+\infty)$ intervalda berilgan tenglamani qanoatlantiradi. ☝

Misol 6. c ning ixtiyoriy tayinlangan qiymatida

$$y=\text{arctg}(x+y)+c$$

tenglik aniqlovchi uzluksiz differensiallanuvchi $y=y(x)$ funksiya ushbu

$$(x+y)^2y'=1$$

tenglamaning yechimi bo‘lishini ko‘rsating.

⊶Berilgan tenglik tayinlangan c da biror $y=\varphi(x)\in C^1(I)$ funksiyani oshkormas ko‘rinishda aniqlasin ($I=(-\infty;+\infty)$, 1.1- rasm). U holda

$$\varphi(x)=\text{arctg}(x+\varphi(x))+c, x\in I,$$

bo‘ladi. Oxirgi tenglikni ($x\in I$ ga nisbatan ayniyatni) x bo‘yicha differensiallaymiz:

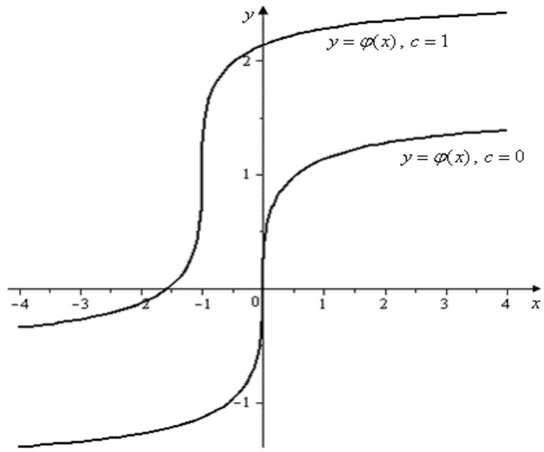
$$\varphi'(x)=\frac{1}{1+\left(x+\varphi(x)\right)^2}\cdot\left(1+\varphi'(x)\right), x\in I.$$

Bundan $\varphi'(x)$ ni topamiz:

$$\varphi'(x)\cdot\left(1+\left(x+\varphi(x)\right)^2\right)=1+\varphi'(x),\ \ \varphi'(x)=1/\left(x+\varphi(x)\right)^2,\ x\in I\ .$$

Demak,

$$\left(x+\varphi(x)\right)^2\cdot\varphi'(x)=1,\ \ x\in I.$$



1.1-rasm. $y=\mathrm{arctg}\left(x+y\right)+c$ tenglama grafigi.

Shunday qilib, $I=(-\infty,+\infty)$ oraliqda $y=\mathrm{arctg}(x+y)+c$ tenglama bilan oshkormas ko‘rinishda berilgan $y=\varphi(x)$ funksiya shu I oraliqda $(x+y)^2y'=1$ tenglamaning yechimi. ☝

Misol 7. Ushbu

$$x+c\sqrt{1+x^2}-y=0 \qquad (c-\text{parametr}) \tag{7}$$

chiziqlar oilasi qanoatlantiruvchi differensial tenglamani tuzing.

⊶ Berilgan tenglikni x bo‘yicha differensiallaymiz ($y=y(x)$):

$$1+\frac{cx}{\sqrt{1+x^2}}-y'=0. \tag{8}$$

Endi (7) va (8) tengliklardan c ni yo‘qotamiz. (7) dan $c=\dfrac{y-x}{\sqrt{1+x^2}}$ ni topib, uni (8) ga qo‘yamiz:

$$1+\frac{y-x}{\sqrt{1+x^2}}\cdot\frac{x}{\sqrt{1+x^2}}-y'=0.$$

Bundan izlangan differensial tenglamani topamiz

$$(1+x^2)y'=xy+1.$$

Izoh. Bu tenglamani (7) dan topilgan $c=\dfrac{y-x}{\sqrt{1+x^2}}$ tenglikni differensiallash natijasida ham hosil qilish mumkin. ☝

Misol 8. Shunday differensial tenglamani tuzingki, uni ushbu

$$y-x\sin(\ln cx)=0 \tag{9}$$

chiziqlar oilasi qanoatlantirsin.

🗝Berilgan tenglikni x bo‘yicha differensiallab topamiz($y=y(x)$):

$$y'-\sin(\ln cx)-\cos(\ln cx)=0\,. \tag{10}$$

Endi (9) va (10) tengliklardan c ni yo‘qotamiz. Buning uchun (9) dan

$$\frac{y}{x}=\sin(\ln cx) \tag{11}$$

tenglikni topib, uni (10) ga qo‘yamiz:

$$y'-\frac{y}{x}=\cos(\ln cx)\,. \tag{12}$$

Topilgan (11) va (12) tengliklarni kvadratga ko‘tarib hadma-had qo‘shamiz:

$$\left(y'-\frac{y}{x}\right)^2+\left(\frac{y}{x}\right)^2=1\,.$$

Oxirgi tenlikda ixchamlashlarni bajarib, izlangan differensial tenglamani hosil qilamiz:

$$x(x-2y)y'+2y^2-x^2=0\,.$$ ☝

Misol 9. (x,y) tekisligidagi barcha aylanalar oilasi qanoatlantiruvchi differensial tenglamani tuzing.

🗝Bu aylanalar oilasining tenglamasi:

$$(x-a)^2+(y-b)^2=r^2; \tag{13}$$

bu yerda 3 ta parametr bor: a,b,r. Bu tenglikni $y=y(x)$ ekanligini hisobga olib x bo‘yicha uch marta ketma-ket differensiallaymiz

$$\begin{aligned} &x-a+(y-b)y'=0,\\ &1+y'^2+(y-b)y''=0,\\ &3y'y''+(y-b)y'''=0. \end{aligned} \tag{14}$$

Endi (13) va (14) tengliklardan a, b, r parametrlarni yo‘qotish kerak. Buning uchun (14) dagi uchinchi tenglikni y'' ga ko‘paytirish va ikkinchi tenglikni hisobga olish kifoya:

$$(1+y'^2)y'''-3y'y''^2=0.$$

Bu $-$ izlangan differensial tenglama. U uchinchi tartibli differensial tenglamadir.

Masalalar

Berilgan funksiya berilgan differensial tenglamaning qaysi I oraliqda yechimi bo‘lishini aniqlang (**1**– **4**):

1. $y=\sqrt{1-2x}$, $y'+xy^3-1=0$; **2.** $y=\ln(x^2-3x-1)$, $xy'-2y-\ln(x^2-3x-1)=0$;

3. $y=\sin^2(2x-1)$, $y'^2+y^2+15\cos^4(2x-1)-14\cos^2(2x-1)=1$;

4. $y=e^{\sqrt[3]{3x-1}}$, $\frac{1}{3}y'^2-\sqrt[3]{3x-1}\,y-x=0$.

Oshkormas ko‘rinishda berilgan funksiya berilgan differensial tenglamaning qaysi I oraliqda yechimi bo‘lishini aniqlang (**5**–**8**):

5. $x^2+y^2-2y=0$, $(1-y)y'=x$; **6.** $(y+2)e^y-x^2=0$, $(y+3)e^y y'=2x$;

7. $x^2-x+y^2=0$, $2yy'+2x=1$; **8.** $xy^3-e^{-y}-1=0$, $(xy^3+3xy^2-1)y'+y^3=0$.

Egri chiziqlar oilasi qanoatlantiruvchi differensial tenglamani tuzing (**9**–**12**):

9. $y-\sin(x+c)=0$; **10.** $y=c_1x^2+c_2x$;

11. $c_1x^2-xy+c_2=0$; **12.** $(y+c)e^y-x^3=0$.

Mustaqil ish № 1 topshiriqlari:

I. Berilgan funksiya biror I oraliqda berilgan differensial tenglamaning yechimi bo‘la oladimi (α va β parametrlar qiymatlarini o‘qituvchidan oling)?

1. $y=\sqrt{\alpha x^{\beta+1}}-\beta x$; $y'-\beta y^2-\alpha x+\beta=0$.

2. $y=\ln(\alpha x-\beta);\ y'+\beta xy+\alpha x-\beta=0$.

3. $y=\sqrt[3]{\alpha x^2+\beta x+1}$; $xy'+\alpha\sqrt{\beta x+y}-1=0$.

4. $y=\sqrt[4]{\alpha x+\beta};\ \alpha y'^4+\beta xy^2-2=0$.

5. $y=e^{\sqrt[3]{\alpha x+\beta}};\ y'+\alpha x^2y^2-x=0$.

6. $y=\sin(\alpha x+\beta);\ \alpha xy'+\beta y-1=0$.

7. $y=\cos(\alpha x-\beta);\ \alpha xy'^2+\beta y-x=0$.

8. $y=\alpha x-\beta\sqrt{x};\ \sqrt{\alpha x}y'^2-\beta y^2+1=0$.

9. $y=\mathrm{arctg}(\alpha x-\beta);\ \beta y'^2+\alpha xy-\beta=0$.

10. $y=\mathrm{arcctg}(\alpha x^2-\beta);\ \alpha xy'^2+\beta x^3y-1=0$.

11. $y=\alpha\sqrt{x+\beta};\ \beta xy'^2-\alpha y+\beta x=0$.

12. $y=\alpha\sqrt[3]{x-\beta};\ \beta y'+\alpha x^2y^3-1=0$.

13. $y=x\sqrt{\alpha x-\beta};\ y'^2-\alpha xy^2+\beta x=0$.

14. $y=\ln(\alpha x^2-\beta);\ y'^2+\alpha y+\beta x=0$.

15. $y=e^{\sqrt{\beta x-\alpha}};\ y'+\alpha xy^2+\beta x=0$.

II. Differensial tenglama berilgan. Oshkor yoki oshkormas ko‘rinishda berilgan funksiyaning qaysi I oraliqda shu tenglama yechimi ekanligini aniqlang.

1. $y''-2y'=e^x(x^2+x-1),\quad y=e^{2x}-(x^2+x+1)e^x$.

2. $(xy-x^3)dx+dy=0,\ y=x^2-2+c\cdot\exp(-x^2/2)$.

3. $xy'-xy-y=x^2,\ y=x(ce^x-1)$.

4. $y''+9y=x\sin 3x,\quad y=-\dfrac{x^2}{12}\cos 3x+(1+\dfrac{x}{36})\sin 3x$

5. $(x^2y-1)y'+xy^2-1=0,\qquad 2x-x^2y^2+2y=c$

6. $(4y^2+x^2)y'-xy=0,\ \ 8y^2\ln|y|=cy^2+x^2$

7. $(x^2-\sin^2 y)dx+x\sin 2y\cdot dy=0,\ x^2+\sin^2 y=cx.$

8. $y''' + y' = \sin x,\ \ y = 2\cos x + \sin x - \dfrac{x}{2}\sin x$

9. $y''' - y = 2x,\ \ y = \dfrac{4}{3}e^{x} - \dfrac{4}{3}e^{-x/2}\cos\dfrac{\sqrt{3}}{2}x - 2x$

10. $y''' - y' = 5e^{-x}(\cos x - \sin x),\ \ y = 2e^{x} - 4 + (2\cos x - \sin x)e^{-x}$

11. $x(3y+2x)y' + 3(y+x)^2 = 0,\ \ 6x^2y^2 + 8x^3y + 3x^4 = c$

12. $xydx + \left(\dfrac{x^2}{2} + \dfrac{1}{y}\right)dy = 0,\ \ x^2y + 2\ln|y| = c$

13. $2ydy + (2x + x^2 + y^2)dx = 0,\ \ x^2 + y^2 = ce^{-x}$

14. $(xy^2 + x)dx + (y - x^2y)dy = 0,\ \ c(1+y^2) = 1 - x^2$

15. $(2xy^2 - y)dx + (y^2 + x + y)dy = 0,\ \ x^2 - \dfrac{x}{y} + y + \ln|y| = c$

16. $y' = y(xy' - 2y),\ \ y^3 = c(3xy - 1)^2$

17. $(1+x^2)y'' - 2y = 0,\ \ y = 1 + x^2 + x + (1+x^2)\mathrm{arctg}x$

18. $xy' = x^2y^2 - y + 1,\ \ xy = \mathrm{tg}(x+c)$

19. $y' - y - 2x + 3 = 0,\ \ y = 1 - 2x + ce^{x}$

20. $x^2y' - \cos 2y - 1 = 0,\ \ y = \mathrm{arctg}\left(c - \dfrac{2}{x}\right)$

21. $3y^2y' - 2xy^3 - 16x = 0,\ \ y^3 = ce^{x^2} - 8$

22. $xy' = y(1 + \ln y - \ln x),\ \ y = xe^{cx}.$

23. $xy' = y\cos\ln\dfrac{y}{x},\ \ \ln cx = \mathrm{ctg}\ln\sqrt{\dfrac{y}{x}}.$

24. $y' = 2\left(\dfrac{y+2}{x+y-1}\right)^2,\ \ y + 2 = c\exp\left(-2\mathrm{arctg}\dfrac{y+2}{x-3}\right)$

25. $y' = \dfrac{y+2}{x+1} + \mathrm{tg}\dfrac{y-2x}{x+1},\ \ c(x+1) + \sin\dfrac{y-2x}{x+1} = 0$

26. $xy' - xy - e^x = 0, \quad y = e^x(c + \ln|x|).$

27. $xydx + (x^2 + y^2 + 1)dy = 0, \ y^4 + 2y^2(x^2 + 1) = c$

28. $x(x^2 \sin y - 1)y' + 2y = 0, \quad y + x^2 \cos y = cx^2$

29. $(e^y - y')x = 2, \ \ y + \ln(x + cx^2) = 0$

30. $(y - y^3)dx + (x + xy^2)dy = 0, \ \ y^2 + cxy - 1 = 0$

31. $(x^2 + y^2 + x)dx + ydy = 0, \quad (x^2 + y^2)e^{2x} = c$

32. $x^2y^2(xy' + y) = 1, \quad x^3y^3 - 3x = c$

33. $y^2dx + (xy + \mathrm{tg}xy)dy = 0, \ \ y \sin xy = c$

34. $(xy + x^3)y' = y^2, \ \ y^2 + 2x^2y - cx^2 = 0$

35. $(x^2 - \sin^2 y)dx + x\sin 2y \cdot dy = 0, \ \sin^2 y + x^2 = cx$

36. $xdx + (xy - y^3)dy = 0, \ (2x - y^2)(x + y^2)^2 = c$

37. $y^{\mathrm{IV}} = e^x - 1, \quad y = e^x - \dfrac{x^4}{24} + cx^3 + x + 1$

38. $(1 + x^2)y' - 2xy = 4\sqrt{y(1 + x^2)}\mathrm{arctg}x, \ y = (1 + x^2)(\mathrm{arctg}^2 x + c)^2$

39. $x^3 \sin y \cdot y' - xy' + 2y = 0, \ y = (c - \cos y) \cdot x^2$

40. $(2y + x + 7)y' + 2x - y + 4 = 0, \qquad \ln\left((y + 2)^2 + (x + 3)^2\right) + \mathrm{arctg}\dfrac{y + 2}{x + 3} = c.$

III. Berilgan egri chiziqlar oilasi qanoatlantiruvchi differensial tenglamani tuzing.

1. $\arcsin x - cxy + e^y = 0$.

2. $c_1x^2 + (y - c_2)^2 = 1.$

3. $c_1x^2 + c_2x + c_3 - y = 0.$

4. $\dfrac{x^2}{1 + c_1} + \dfrac{y^2}{2 + c_2} - 1 = 0.$

5. $c_1x + c_2e^x + c_3(x + x^2) - y = 0.$

6. $y = c_1x + c_2.$

7. $(x - c)^2 + (y + 2c)^2 = 1.$

8. $(1 + x^2)(1 + y^2) = cy.$

9. $y=(x+c)e^{-\sin x}$.

10. $\frac{xy}{x-y}+\ln\frac{x}{y}=cx$.

11. $x^2+y^2+x+y=cy$.

12. $(x-c_1)^2+(y-c_2)^2=1$.

13. $c(x^2+y^2)=xy$.

14. $y=c_1\cos x+c_2\sin x$.

15. $y=c_1\cos(x+c_2)$.

16. $y=c_1\sin(x+c_2)$.

17. $y=c_1e^x+c_2e^{-x}$.

18. $\sin y+x-cxy=0$.

19. $\ln(1+x^2)+\ln(1+y^2)=cx$.

20. $c\,\mathrm{arctg}x+\mathrm{arctg}(1+y^2)=x$.

21. $x^2y^2+\ln(x^2+y^2)=cy$.

22. $c\arcsin x+2xy=y$.

23. $\sqrt{x^2+1}+c\sqrt{y^2+1}=x$.

24. $y\left(\frac{x^3}{6}+c\right)-x^3=0$.

25. $y-(x+c)\sin x=0$.

26. $(y^3+x)^3=c(y^3-x)$.

27. $y-x\cos(\ln cx)=0$.

28. $(1+x^2)^3-c(1+y^3)^2=0$.

29. $y^2-2\ln c(1+e^x)=0$.

30. $3x^2+xy^2+3y-cx=0$.

31. $x^3+cy^3-x^2y-y^2x=0$.

32. $x^2-cy^2-xy-1=0$.

33. $c(\cos x+y)+y^2-1=0$.

34. $cy-\cos cx=0$.

35. $xy-cy^3-6y+2x-1=0$.

36. $(x+c)^2+c(y+x)-3=0$.

37. $y-(x-c)^3=0$.

38. $y^2-c(x+2)^3=0$.

39. $y^2-(x+c)^5=0$.

40. $y^3-(x-c)^4=0$.

2. O‘ZGARUVCHILARI AJRALADIGAN DIFFERENSIAL TENGLAMALAR

Maqsad – o‘zgaruvchilari ajraladigan differensial tenglamalarni yechishni o‘rganish.

Yordamchi ma’lumotlar:

1^0. *O‘zgaruvchilari ajraladigan differensial tenglama* deb

$$M(x)N(y)dx+P(x)Q(y)dy=0 \qquad (1)$$

yoki

$$y'=f(x)g(y) \qquad (2)$$

ko‘rinishdagi tenglamaga aytiladi; bunda berilgan M,N,P,Q,f,g funksiyalar argumentlari biror mos intervalda o‘zgarganda uzluksiz.

$N(y)$ va $P(x)$ funksiyalar nolga aylanmaydi deb hisoblab, (1) tenglamaning har ikkala tomonini $N(y)\cdot P(x)$ ga bo‘lib, o‘zgaruvchilari ajralgan tenglamaga kelamiz:

$$\frac{M(x)}{P(x)}dx+\frac{Q(y)}{N(y)}dy=0$$

Oxirgi tenglamani integrallab, berilgan (1) tenglamaning barcha yechimlarini oshkormas ko‘rinishda topamiz:

$$\int\frac{M(x)}{P(x)}dx+\int\frac{Q(y)}{N(y)}dy=c. \qquad (c=\text{const}) \qquad (3)$$

Agar $N(y)$ yoki $P(x)$ nolga aylanadigan bo‘lsa, (1) dan (3) ga kelish jarayonida yechim yo‘qolishi mumkin. Masalan, agar $N(y_0)=0$ bo‘lsa, u holda (1) tenglama $y=y_0$ yechimga ega. Bu yechim (3) yechimlar orasida bo‘lmasligi mumkin. Shuning uchun bunaqa yechimlarni alohida (qo‘shimcha) topish kerak.

(2) tenglamada ham o‘zgaruvchilar yuqoridagiga o‘xshash ajratiladi. (2) ni

$$dy=f(x)g(y)dx$$

ko‘rinishda yozib olib, uni

$$\frac{dy}{g(y)}=f(x)dx \qquad (g(y)\neq 0)$$

ko‘rinishga keltiramiz. Oxirgi tenglamadan

$$\int\frac{dy}{g(y)}=\int f(x)dx+c$$

yechimlarini hosil qilamiz. $g(y)$ funksiya 0 ga aylanadigan har bir $y=y_0$ qiymat ham yechimni aniqlaydi.

Misol 1. Differensial tenglamani yeching

$$3xy^2y'+x-1=0.$$

🗝 Tenglamada $y'=\dfrac{dy}{dx}$ deymiz va uni quyidagi ko‘rinishga keltiramiz:

$$3xy^2dy=(1-x)dx.$$

Bu tenglamaning har ikkala tomonini x ga bo‘lib, o‘zgaruvchilarni ajratamiz va integrallashlarni bajarib, yechimni topamiz:

$$\int 3y^2dy=\int\frac{1-x}{x}dx+c,\ c=\text{const},$$

$$y^3=\int\Big(\frac{1}{x}-1\Big)dx+c,\ \ y^3=\ln|x|-x+c,$$

$$y=\sqrt[3]{\ln|x|-x+c}.$$

Demak, barcha yechimlar majmuasi $y=\sqrt[3]{\ln|x|-x+c}$ ($x>0$ va $x<0$ oraliqlarda) formula bilan ifodalanadi; bunda $c-$ixtiyoriy o‘zgarmas. 👍

Misol 2. Differensial tenglamani yeching

$$y'\cos^2 x+\sqrt{1+e^{-y}}\,e^y\sin x=0.$$

🗝 Tenglamaning har ikkala tomonini $\dfrac{1}{\sqrt{1+e^{-y}}\,e^y\cos^2 x}dx$ ga ko‘paytirib $\left(y'=\dfrac{dy}{dx}\right)$, o‘zgaruvchilarni ajratamiz va integrallashlarni bajaramiz:

$$\frac{1}{\sqrt{1+e^{-y}}\,e^y}dy+\frac{\sin x}{\cos^2 x}dx=0,\ \int\frac{dy}{\sqrt{1+e^{-y}}\,e^y}+\int\frac{\sin x}{\cos^2 x}dx=-c,$$

$$\int\frac{d(1+e^{-y})}{\sqrt{1+e^{-y}}}+\int\frac{d(\cos x)}{\cos^2 x}=c,\ 2\sqrt{1+e^{-y}}-\frac{1}{\cos x}=c.$$

Yechimlarni oshkormas ko‘rinishda topdik (ularni $y = y(x,c)$ oshkor ko‘rinishda topish qiyin emas). ☝

Misol 3. a) Ushbu

$$xy(y+1)dx - dy = 0$$

differensial tenglamani yeching.

b) Berilgan tenglamaning $x = 0$ da $y = 1$ qiymat qabul qiluvchi $(y(0) = 1)$ yechimini toping, ya’ni ushbu

$$\begin{cases} xy(y+1)dx - dy = 0 \\ y(0) = 1 \end{cases}$$

Koshi masalasini yeching.

🗝 a) Bu o‘zgaruvchilari ajraladigan tenglama. O‘zgaruvchilarni ajratish uchun uning har ikkala tomonini $y(y+1)$ ga bo‘lamiz:

$$xdx - \frac{dy}{y(y+1)} = 0.$$

Bundan

$$\int xdx - \int \frac{dy}{y(y+1)} = c_1. \tag{4}$$

Lekin

$$\int xdx = \frac{x^2}{2} \;,\; \int \frac{dy}{y(y+1)} = \int \left(\frac{1}{y} - \frac{1}{y+1} \right) dy = \int \frac{dy}{y} - \int \frac{dy}{y+1} = \ln|y| - \ln|y+1| .$$

Shuning uchun (4) ga ko‘ra

$$\frac{x^2}{2} - \left(\ln|y| - \ln|y+1| \right) = c_1 \;;\; \frac{x^2}{2} - \ln\frac{|y|}{|y+1|} = \ln c \;\; (c_1 = \ln c,\, c > 0) \;;$$

$$\frac{x^2}{2} = \ln\frac{c|y|}{|y+1|} \;\; (c > 0) \;; \quad e^{\frac{x^2}{2}} = c\frac{|y|}{|y+1|} \;\; (c > 0) \;; \quad e^{\frac{x^2}{2}} = c\frac{y}{y+1} \;\; (c \neq 0),$$

$$y = \frac{e^{\frac{x^2}{2}}}{c - e^{\frac{x^2}{2}}} \quad (c \neq 0) \ . \tag{5}$$

Biz tenglamani $y(y+1)$ ga bo‘lganda $y=0$ va $y=-1$ yechimlarni yo‘qotishimiz mumkin edi. Lekin (5) formuladan $y=-1$ yechim $c=0$ da hosil bo‘ladi, $y=0$ yechim esa undan hosil bo‘lmaydi.

Javob: $y=0, \quad y = \dfrac{e^{\frac{x^2}{2}}}{c - e^{\frac{x^2}{2}}}$ ($c-$ixtiyoriy o‘zgarmas).

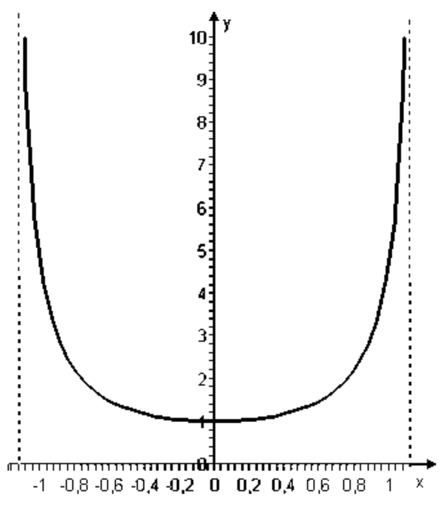
b) Topilgan (5) yechimda $x=0$ va $y=1$ deymiz. U holda

$$1 = \frac{1}{c-1}, \ c = 2$$

hosil bo‘ladi. Demak, qo‘yilgan Koshi masalasining izlangan yechimi

$$y = \frac{e^{\frac{x^2}{2}}}{2 - e^{\frac{x^2}{2}}}, \quad |x| < \sqrt{\ln 4} \ \left(\sqrt{\ln 4} \approx 1{,}177\right), \tag{*}$$

ko‘rinishda beriladi. Bu yechim grafigi (integral chiziq) 2.1- rasmda ko‘rsatilgan.



2.1- rasm. (*) funksiya (yechim) grafigi.

2⁰. *O‘zgaruvchilari ajraladigan tenglamaga keltiriluvchi tenglamalar.*

Ushbu

$$y' = f(ax+by+c) \qquad (b \neq 0)$$

ko‘rinishdagi tenglama $u = ax+by+c$ formula bilan yangi $u = u(x)$ noma’lum funksiyani kiritish yordamida o‘zgaruvchilari ajraladigan tenglamaga keltiriladi:

$$u = ax+by+c, \qquad u' = a+by' \quad \Rightarrow \quad u' = a+bf(u).$$

Misol 4. Ushbu

$$y' = \sqrt{x+4y+1}$$

tenglamani yeching.

$u = x+4y+1$ deb, yangi $u = u(x)$ noma’lum funksiyani kiritamiz. U holda $u' = 1+4y'$ va berilgan tenglama

$$\frac{u'-1}{4} = \sqrt{u}\text{, ya’ni } u' = 4\sqrt{u}+1$$

ko‘rinishga keladi. Bundan

$$\frac{du}{4\sqrt{u}+1} = dx$$

(yechim yo‘qolmaydi, chunki $u \geq 0$ да $4\sqrt{u}+1 \geq 1 > 0$), yoki

$$\int \frac{du}{4\sqrt{u}+1} = x+c. \tag{6}$$

Bu yerdagi integralni hisoblash uchun $\sqrt{u} = t$ $(t \geq 0)$, $u = t^2$ almashtirishni bajaramiz. U holda $du = 2tdt$ va

$$\int \frac{du}{4\sqrt{u}+1} = \int \frac{2tdt}{4t+1} = \frac{1}{2}\int \frac{4tdt}{4t+1} = \frac{1}{2}\int \frac{4t+1-1}{4t+1}dt = \frac{1}{2}\int dt - \frac{1}{2}\int \frac{dt}{4t+1} = \frac{1}{2}t - \frac{1}{8}\ln(4t+1) =$$

$$= \frac{1}{2}\sqrt{u} - \frac{1}{8}\ln(4\sqrt{u}+1).$$

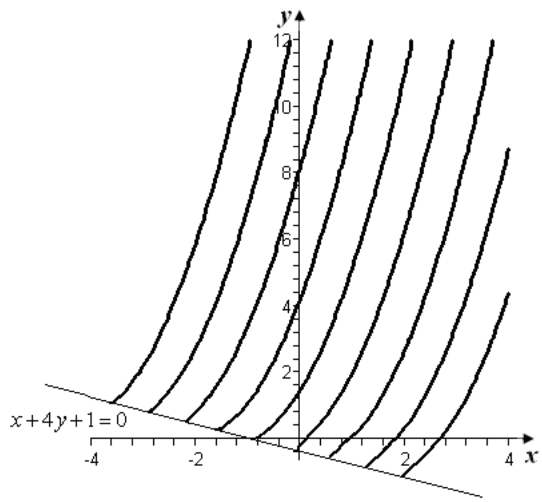
Demak, (6) ga ko‘ra

$$\frac{1}{2}\sqrt{u} - \frac{1}{8}\ln(4\sqrt{u}+1) = x+c.$$

Bu yerda y noma’lumga $u = x+4y+1$ formulaga ko‘ra qaytib, berilgan tenglamaning oshkormas ko‘rinishdagi yechimlarini hosil qilamiz:

$$4\sqrt{x+4y+1}-\ln(4\sqrt{x+4y+1}+1)=8x+c.$$

Yechimning $c=32,26,20,14,8,2,-4,-10,-16,$ bo‘lgandagi grafiklari (chapdan o‘ngga) 2.2- rasmda keltirilgan. Yechimlar grafigi $x+4y+1\geq 0$ yarim tekislikda joylashadi. ☝



2.2- rasm. Misol 4 dagi tenglamaning yechimlari.

Endi o‘zgaruvchilari ajraladigan tenglamalarga keltiriluvchi ayrim amaliy masalalarni keltiraylik.

Misol 5. *Choyning sovishi.* Issiq modda sovuq xonaga kiritilganda, modda o‘zidagi issiqlikni tashqi muhitga bera boshlaydi va, demak, uning temperaturasi (harorati) kamayadi. $T(t)$ bilan moddaning t paytdagi temperaturasini T_0 bilan tashqi muhitning temperaturasini belgilasak, u holda sovish uchun Nyuton qonuniga ko‘ra predmet temperaturasining kamayish tezligi tashqi muhit va predmet temperaturalari farqiga to‘g‘ri proporsional:

$$\frac{dT}{dt}=-k(T-T_0)\quad (k>0)\,. \tag{7}$$

$T_0=\text{const}$ deb faraz qilamiz. U holda (2) – o‘zgaruvchilari ajraladigan tenglama. $T_0<T$ (xona moddaga qaraganda sovuq) bo‘lgani uchun $\frac{dT}{dt}<0$ va, demak, vaqt o‘tishi bilan modda temperaturasi kamayadi. (7) tenglamada o‘zgaruvchilarni ajratamiz va zarur integrallashlarni bajarib, uning umumiy yechimini topamiz:

$$T = T_0 + ce^{-kt}.$$

Bu yerdagi k sovish konstantasi deb ataladi. Konkret misol qarab chiqaylik.

Misol. Temperaturasi $60^\circ C$ bo‘lgan bir piyola choy temperaturasi $20^\circ C$ bo‘lgan xonaga qo‘yildi. 5 minutdan so‘ng choyning temperaturasi $50^\circ C$ ga tushdi. Qancha vaqtdan so‘ng choy temperaturasi $30^\circ C$ bo‘ladi?

Choyning sovishi Nyutonning qonuniga bo‘yso‘nadi deb faraz qilamiz. U holda choyning t paytdagi temperaturasi yuqoridagiga ko‘ra

$$T(t) = T_0 + ce^{-kt},\ T_0 = 20^\circ C$$

formula bilan aniqlanadi.(t -minutlarda o‘lchanadi).

Vaqtni choy xonaga qo‘yilgan paytdan boshlab hisoblaymiz. U holda $T(0) = 60^\circ C$ bo‘ladi. Bu shartdan

$$T(0) = 60^\circ = 20^\circ + c \Rightarrow c = 40^\circ.$$

Demak, $T(t) = 20 + 40e^{-kt}$. $T(5) = 50^\circ C$ bo‘lgani uchun sovish konstantasi k ni topish mumkin:

$$T(5) = 50 = 20 + 40e^{-k\cdot 5} \Rightarrow k = \frac{1}{5}\ln\frac{4}{3}.$$

Demak, $T(t) = 20 + 40e^{-\left(\frac{1}{5}\ln\frac{4}{3}\right)t}$.

Shunday qilib, choy xonaga qo‘yilgandan so‘ng t vaqt o‘tgach, uning temperaturasi $T(t) = 30^\circ C$ bo‘lishini aniqlash uchun

$$20 + 40e^{-\left(\frac{1}{5}\ln\frac{4}{3}\right)t} = 30$$

tenglamadan t ni topish kerak. Oxirgi tenglamani yechib,

$$t = 5\cdot\ln 4/(\ln 4 - \ln 3) \approx 24$$

minut ekanligini topamiz.

Shunday qilib, choy 5 minut ichida $60^\circ C$ dan $50^\circ C$ gacha sovigandan so‘ng uning temperaturasi $30^\circ C$ gacha tushishi uchun yana 24 - 5 =19 minut kerak. 👍

Misol 6. *Parashutchi tezligini topish.* Samolyotdan (tayoradan) sakragan Parashutchi tezligi v_0 ga yetgandan so‘ng Parashutini ochdi. Havoning Parashutchi harakatiga qarshilik kuchi Parashutchining tezligi v ga to‘g‘ri proporsional deb hisoblab, Parashutchi tezligining o‘zgarish qonunini toping.

Parashutchiga havoning yuqoriga yo‘nalgan $F_q = kv$ – qarshilik kuchi ($k > 0$ – proporsionallik koeffitsienti) va pastga yo‘nalgan $P = mg$ – og‘irlik kuchi ta’sir etadi (m – Parashutchi massasi, g – erkin tushish tezlanishi). Nyutonning ikkinchi qonuniga ko‘ra

$$ma = P - F_q \text{ yoki } ma = mg - kv,$$

bu yerda $a = \dfrac{dv}{dt}$ – Parashutchining tezlanishi. Demak, Parashutchining harakat tenglamasi

$$\frac{dv}{dt} = g - Bv, B = \frac{k}{m}.$$

Bu tenglamada o‘zgaruvchilarni ajratamiz va integrallashlarni bajaramiz:

$$\int \frac{dv}{g - Bv} = \int dt, \quad -\frac{1}{B}\ln|g - Bv| = t + c_1, \quad g - Bv = ce^{-Bt}.$$

$v(0) = v_0$ bo‘lishi uchun $g - Bv_0 = c$ bo‘lishi kerak. Demak,

$$g - Bv = (g - Bv_0)e^{-Bt}, \quad v = \frac{g}{B}\left[1 - \left(1 - \frac{B}{g}v_0\right)e^{-Bt}\right],$$

$$v = \frac{mg}{k}\left[1 + \left(\frac{k}{mg}v_0 - 1\right)e^{-\frac{k}{m}t}\right].$$

Vaqt o‘tishi bilan $v = v(t)$ tezlik o‘zgarmas $\frac{mg}{k}$ limit tezlikka intiladi: $\lim\limits_{t\to+\infty} v(t) = \frac{mg}{k}$. Bu intilish eksponensial tezlik bilan bo‘ladi. Bundan tashqari, limit tezlik boshlang‘ich tezlik v_0 ga bog‘liq emas.

Agar g = 0,98 m/s^2, m = 70 kg, $k =$ 110 kg/s bo‘lsa, Parashutchining limit tezligi $\frac{mg}{k} \approx 6,2$ m/s bo‘ladi. Bu tezlik bilan yerga urilish Parashutchi uchun xavfsiz. ☝

Misol 7. *Aholining ko‘payishi.* Aholining ko‘payishini o‘rganish katta amaliy ahamiyatga ega. t paytdagi aholining sonini $N(t)$ bilan belgilaylik. Biz $N = N(t)$ funksiyani uzluksiz differensiallanuvchi deb faraz qilamiz. Aslida u faqat butun qiymatlar qabul qiladi va uzluksiz emas. Lekin katta N lar uchun biz uni silliq funksiya deb hisoblashimiz mumkin.

Aholining $\frac{dN}{dt}$ o‘sish tezligi tug‘ilish tezligi bilan o‘lish tezligining ayirmasiga teng. Umumiy holda bu tezliklar qaralayotgan payt va shu paytdagi aholining soniga bog‘liq. Tug‘ilish tezligini mavjud aholi soni N ga to‘g‘ri proporsional, o‘lish tezligini esa N^2 ga to‘g‘ri proporsional deb hisoblaymiz (katta N larda o‘lish tezligi tug‘ilish tezligidan ortiq bo‘ladi). Demak, bizning farazlarimizga ko‘ra aholi sonining o‘zgarishi ushbu

$$\frac{dN}{dt} = aN - bN^2 \tag{8}$$

differensial tenglama bilan ifodalanadi. Bu yerda a va b lar — musbat konstantalar, (8) o‘zgaruvchilari ajraladigan tenglamani $N(0) = N_0$ boshlang‘ich shartda yechishimiz kerak. Tushunarliki, agar dastlabki aholining soni $N_0 = a/b$ bo‘lsa, $N(t) = a/b$ o‘zgarmas funksiya tenglamaning yechimi bo‘ladi (vaqt o‘tishi bilan aholi soni o‘zgarmaydi). Endi faraz qilaylik, $N(0) = N_0 \neq a/b$ bo‘lsin. U holda $N(t)$

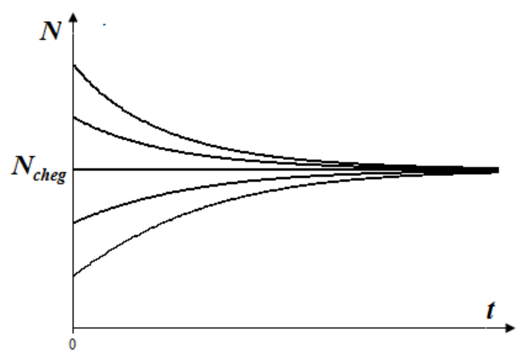
yechim kichik t lar uchun a/b ga teng bo‘lolmaydi. (1) tenglamada o‘zgaruvchilarni ajratamiz, integrallashni va soddalashtirishlarni bajaramiz:

$$\frac{dN}{N(a-bN)}=dt,\quad \frac{b}{a}\frac{dN}{a-bN}+\frac{1}{a}\frac{dN}{N}=dt,\quad \frac{b}{a}\int_0^{N_0}\frac{dN}{a-bN}+\frac{1}{a}\int_0^{N_0}\frac{dN}{N}=\int_0^t dt,$$

$$\ln\left|\frac{N}{a-bN}\right|-\ln\left|\frac{N_0}{a-bN_0}\right|=at,\quad \frac{N}{a-bN}=\frac{N_0}{a-bN_0}e^{at},$$

$$N=\frac{N_{cheg}N_0}{N_0+(N_{cheg}-N_0)e^{-at}},\ N_{cheg}=a/b.$$

Topilgan bu yechimdan vaqt o‘tishi bilan aholining soni, agar $N(0)=N_0<N_{cheg}$ bo‘lsa, ortib, $N(0)=N_0>N_{cheg}$ bo‘lganda esa kamayib, N_{cheg} ga yaqinlashishini ko‘ramiz: $N(t)\xrightarrow[t\to+\infty]{}N_{cheg}$ (2.3- rasm).



2.3-rasm. Aholi sonining o‘zgarishi.

Jahon aholisi uchun $a\approx 0,3$ va $b\approx 3\times 10^{-12}$ deyish mumkin. Bunda vaqt o‘tishi bilan jahon aholisining soni 10^{11} (yuz milliard) kishiga yaqinlashadi (albatta, agar bizning farazlarimiz to‘g‘ri bo‘lsa!). ☝

Misol 8 (kimyoviy). Faraz qilaylik, A va B moddalardan C moddaning hosil bo‘lishi ushbu

$$2A+B=2C$$

kimyoviy reaksiya tenglamasi bilan ifodalansin. Kimyoviy reaksiya tezligi C moddaning hosil bo‘lish tezligini xaraterlaydi va temperatura o‘zgarmas bo‘lganda bu tezlik massalar qonuniga ko‘ra reaksiyaga kiruvchi A va B moddalar konsentratsiyalari ko‘paytmasiga to‘g‘ri proporsional bo‘ladi. Moddaning konsentratsiyasi uning molekulalari soni bilan aniqlanadi. Reaksiya tenglamasidan ravshanki, C moddaning $x = x(t)$ dona molekulasi hosil bo‘lishi uchun A moddaning x ta molekulasi B moddaning esa $x/2$ ta molekulasi kerak. Aytaylik, $t = 0$ paytda A va B moddalarning konsentratsiyalari mos ravishda a va b $(a > 0, b > 0)$, C moddaning konsentratsiyasi esa $x(0) = 0$ bo‘lsin. t paytdagi reaksiya tezligi $\frac{dx}{dt}$ shu paytdagi Ava B moddalarning $a - x$ va $b - x/2$ konsentratsiyalari ko‘paytmasiga to‘g‘ri proporsional, ya’ni

$$\frac{dx}{dt} = k(a - x)\left(b - \frac{x}{2}\right); \tag{9}$$

bu yerdagi $k = \text{const} > 0$proporsionallik koeffitsienti reaksiya tezligi konstantasi deb ataladi. Bu o‘zgaruvchilari ajraladigan tenglamani yechamiz. Dastlab $a \neq 2b$ deb faraz qilamiz. Quyidagilarni topamiz $(x \neq a, x \neq 2b)$:

$$\ln\left|\frac{2b - x}{a - x}\right| = \frac{2kt}{2b - a} + \ln c \ \ (c > 0),$$

$$\frac{2b - x}{a - x} = \pm c \exp\left(\frac{2kt}{2b - a}\right) \ (c > 0).$$

Bu yerdagi c ga 0 qiymat qabul qilishga ”ruxsat” berib, yo‘qolgan $x = 2b$yechimni hosil qilamiz. $x = a$ yechim ham bor. Barcha yechimlar quyidagicha ($a \neq 2b$):

$$x = \frac{2b - cae^{\sigma t}}{1 - ce^{\sigma t}}, \ \sigma = \frac{2k}{2b - a}, \text{ va } x = a.$$

Endi $x(0) = 0$ boshlang‘ich shartdan c o‘zgarmasning qiymatini aniqlaymiz va x ning o‘zgarish qonunini hosil qilamiz ($a \neq 2b$):

$$x = 2ab\frac{1 - e^{\sigma t}}{a - 2be^{\sigma t}}, \ \sigma = \frac{2k}{2b - a}.$$

Yechimni analiz qilaylik. Agar $a > 2b$ $(\sigma < 0)$ bo‘lsa, $t \to +\infty$ da (reaksiya tugaganda) $x \to 2b$. Agar $a < 2b$ $(\sigma > 0)$ bo‘lsa, $t \to +\infty$ da $x = 2ab\dfrac{e^{-\sigma t} - 1}{2be^{-\sigma t} - a} \to a$.

Demak, x ning qiymati monoton ortadi va $t \to +\infty$ da $x_{\max} = \min\{a, 2b\}$ ga intiladi.

Endi $a = 2b$ holni qaraymiz. Bu holda (9) tenglama

$$\frac{dx}{dt} = \frac{k}{2}(a - x)^2$$

ko‘rinishni oladi. Bu tenglamaning $x(0) = 0$ boshlang‘ich shartni qanoatlantiruvchi yechimi

$$x = \frac{a^2kt}{2 + akt}. \; t \to +\infty \text{ da } x \to a = 2b. \; ☝$$

Masalalar

Tenglamalarni yeching (**1** – **10**):

1. $y' = y^2 + y$.

2. $y'\sin x + y(y+1)\cos x = 0$.

3. $2xdx + 3(1 + x^4)\sqrt{1 + 2y}\,dy = 0$.

4. $x\sqrt{2 + y^2}\,dx - y\sqrt{1 + x^2}\,dy = 0$.

5. $y(1 + e^x)dy - 2e^x dx = 0.$

6. $xy' + y(1 + \ln y) = 0.$

7 . $xy' + y(1 - y)\sin x = 0$.

8. $(3x + 3y - 1)dy + 2(x + y)dx = 0$.

9. $x^2y'^2 + y^2 = 2x(2 - yy')$.

10. $y' + \sqrt{x + y} = 0$.

Tenglamalarning ko‘rsatilgan shartni qanoatlantiruvchi yechimini toping (**11**–**14**):

11. $xy' = y^2 + 1,\ y(1) = 1.$

12. $(1 + y)y' - \cos 2x = 0,\ y(0) = 1.$

13. $xy' + y(1 + y)\cos x = 0,\ y(0) = -1.$

14. $(e^x + 1)y' + (e^x - 1)y = 0,\ y(0) = 1.$

Masalalarni yeching **(15** – **25)**:

15. Shunday $y = y(x)$ egri chiziqni topingki, uning bilan , Ox o‘qi bilan, $x = 0$ va

$x = 'x'$ chiziqlar bilan chegaralangan yuza $a^2 \ln \frac{y(x)}{a}$ $(a = const > 0)$ ga teng bo‘lsin. Shunday $y = y(x)$ egri chiziqni topingki, uning bilan , Ox o‘qi bilan, $x = 0$ va $x = 'x'$ chiziqlar bilan chegaralangan yuza $a^2 \ln \frac{y(x)}{a}$ $(a = const > 0)$ ga teng bo‘lsin.

16. To‘g‘ri chiziq bo‘ylab harakat qiluvchi $m = 1(kg)$ massali moddiy nuqtaga uning tezligining kvadratiga va harakat boshlanishidan o‘tgan vaqt t ga to‘g‘ri proporsional bo‘lgan qarshilik kuchi ta’sir etadi. Agar $t = 10\,(s)$ paytda nuqtaning tezligi $2\,(m/s)$ ga, ta’sir etuvchi kuch esa $0,5\,(N)$ ga teng bo‘lsa, nuqtaning tezligi $t = 20(s)$ paytda qancha bo‘ladi?

17. Kema tezligini suv qarshiligi tufayli yo‘qotmoqda. Suvning qarshilik kuchi kemaning tezligiga to‘g‘ri proporsional. Kemaning dastlabki tezligi 10 (m/s), 5 sekunddan so‘ng uning tezligi 8 (m/s) ga tushdi. Qachon kemaning tezligi 1 (m/s) gacha kamayadi?

18. Idishda 10 (kg) tuz bor. Bu idishga 90 (kg) suv quyilsa, 1 soatda dastlabki tuzning yarmi erib ketishi ma’lum bo‘ldi. Agar $2 \cdot 90 = 180$ (kg) suv quyilganda edi, o‘sha 1 soatda qancha miqdordagi tuz erigan bo‘lardi? Tuzning erish tezligi mavjud tuz miqdoriga hamda aralashma konsentratsiyasi bilan to‘yingan aralashma konsentratsiyasi (3 kg aralashmada 1 kg tuz) orasidagi farqqa to‘g‘ri proporsional deb hisoblang.

19. Shunday $y = f(x)$ silliq funksiyani topingki, tekislikda yuqoridan $y = f(x)$, quyidan $y = 0$, chapdan $x = x_0$ va o‘ngdan $x = 'x'$ chiziqlar bilan chegaralangan egri chiziqli trapetsiya yuzi ordinatalar ayirmasi $f(x) - f(x_0)$ ning kubiga proporsional bo‘lsin.

20. Devorga o‘q v_0 tezlik bilan kirib, undan v_1 tezlik bilan t_1 paytda chiqib ketdi. Agar devorning o‘q harakatiga qarshilik kuchi o‘q tezligining kvadratiga to‘g‘ri proporsional bo‘lsa, devorning qalinligi d ni aniqlang.

21. Shunday yassi egri chiziqni topingki, u $(2;1)$ nutadan o‘tsin va uning ixtiyoriy nuqtasidan o‘tkazilgan urinmaning koordinatalar o‘qi orasida qolgan qismi shu urinish nuqtasida teng ikkiga bo‘linsin.

22. Ko‘ndalang kesimi ichki radiusi r_1 tashqi radiusi r_2 ($r_2 > r_1$) bo‘lgan doiraviy xalqadan iborat bo‘lgan teshik cheksiz silindrsimon temir sterjen statsionar issiqlik holatida (bu holatda barcha nuqtalardagi temperatura vaqt bo‘yicha o‘zgarmaydi, har xil nuqtalarda har xil temperatura bo‘lishi mumkin). Agar ichki sirtda temperatura T_1, tashqi sirtda esa T_2 bo‘lsa, temperaturaning sterjenda taqsimot qonunini toping.

Ko‘rsatma. Fur’ye qonunidan fodalaning. Bu qonunga ko‘ra bir vaqt birligida ΔS yuza orqali unga tashqi normal $\boldsymbol{n}$ ($|\boldsymbol{n}| = 1$) yo‘nalishida oqib o‘tgan issiqlik miqdori $-k\Delta S \dfrac{\partial T}{\partial \boldsymbol{n}}$ ga teng; bunda $k = const > 0 -$ proporsionallik koeffitsienti (u issiqlik o‘tkazuvchanlik koeffitsienti deb ataladi), $\dfrac{\partial T}{\partial \boldsymbol{n}} -$ temperaturalar maydoni T dan $\boldsymbol{n}$ yo‘nalishida olingan hosila, ya’ni agar $\boldsymbol{n} = (\cos\alpha, \cos\beta, \cos\gamma)$ bo‘lsa, $\dfrac{\partial T}{\partial \boldsymbol{n}} = \dfrac{\partial T}{\partial x}\cos\alpha + \dfrac{\partial T}{\partial y}\cos\beta + \dfrac{\partial T}{\partial z}\cos\gamma$ bo‘ladi.

23. Idishdagi suyuqlikning sath balandligi h, idishning z balandikdagi ko‘ndalang kesim yuzi $S(z)$. $t = 0$ paytdan boshlab idish tubidagi σ kichik yuzali teshik orqali suyuqlik oqib keta boshlaydi. Suyuqlik sath balandligi z ning vaqt t bo‘yicha o‘zgarish qonunini aniqlang. Qancha vaqtda idish bo‘shaydi?

Ko‘rsatma. Suyuqlik balandligi z bo‘lganda uning teshikdan oqib chiqish tezligi $v = k\sqrt{2gz}$ Torichelli formulasiga ko‘ra aniqlanadi. Bu yerda $k > 0 -$ proporsionallik koeffitsienti, u suyuqlikning yopishqoqligiga bog‘liq (tajriba asosida aniqlanadi); $g -$ erkin tushish tezlanishi.

24. Yer ustida v_0 boshlang‘ich tezlik bilan vertikal yo‘nalishda yuqoriga jism otildi. Havoning qarshiligini hisobga olmay jism tezligining o‘zgarish qonuniyatini toping. v_0 qanday bo‘lganda jism yer tortishini yengib, cheksizlikka ketib qoladi?

25. Mijoz $t = 0$ paytda bankka Q_0 miqdorda foyizli omonat pul qo‘ydi. Bank omonat pulni uzluksiz ravishda p (%) tezlik bilan orttirib boradi. Mijoz hisobidagi pulni w tezlik bilan oladi. Agar mijozning t paytdagi omonat puli miqdorini $Q=Q(t)$ bilan belgilasak, uni aniqlash uchun quyidagi boshlang‘ich masala hosil bo‘ladi:

$$\frac{dQ}{dt}=pQ-w,\ Q\big|_{t=0}=Q_0\,.$$

Bu masalani yeching va $Q=Q(t)$ pul miqdorining o‘zgarish qonuniyatini tahlil qiling. Qaysi hollarda omonat pul ortadi, o‘zgarmaydi, kamayadi?

Mustaqil ish №2 topshiriqlari:

I. Dastlab berilgan differensial tenglamani yeching, so‘ngra uning ko‘rsatilgan boshlang‘ich shartni qanoatlantiruvchi yechimini toping.

1. $y'=\cos(2x+y-1), y(0)=1.$

2. $y'=\mathrm{tg}(y-x)$, $y(\pi/4)=\pi/2$.

3. $y'=\cos(x+y)$, $y(0)=\pi$.

4. $(x^2-1)y'+2xy^2=0$, $y(2)=1$.

5. $y'=\sqrt[3]{x+y}$, $y(0)=1$.

6. $2yy'+xe^y=0$, $y(1)=0$.

7. $tx'-x=x^2,\ x(1)=2$.

8. $(1+e^x)yy'=e^x$, $y(0)=1$.

9. $xy'-1=\cos y$, $y(1)=0$.

10. $xy'-1=\sin 2y$, $y(1)=\pi/4$.

11. $y'=x-2y+3$, $y(1)=1$.

12. . $y'=(y+1)\ln(x+1),\ y(0)=1$

13. $\frac{dx}{dt}+x^2=1$, $x(0)=2$.

14. $y'=\frac{xy}{x^2+1}$, $y(1)=1$.

15. $\frac{dx}{dt}=k(a-x)(b-x)\ \ (k,a,b-const).\ k>0\ \ x(0)=1.$

16. $\frac{d\rho}{d\phi}=\rho(\rho-1)(\rho-2)\ \ \rho(0)=\frac{1}{2}, (\rho(0)=\frac{3}{2}$ yoki $\rho(0)=3$).

17. $y\cos x\cdot dx-y(1+2\sin x)dy=0,\ y(0)=2$.

18. $(xy^2+x)dx+(y^2x^4+y^2)dy=0, y(0)=1$.

19. $2y\sqrt{1-y}dx-(1-x^2)dy=0, y(0)=0$.

20. $(1+y)xdx-\sqrt{x+1}dy=0, y(0)=-1$.

21. $\sqrt{y^2+1}\,dx = x^2 y dy,\ \ y(1)=1.$

22. $y' - y = x+2,\ y(0)=1.$

23. $y' = y\operatorname{tg}x,\ \ y(\pi/4)=1.$

24. $(1+x^2)y' = 4xy,\ \ y(1)=1.$

25. $ydx + (\sqrt{xy}-\sqrt{x})dy = 0, y(1)=2.$

26. $x^2y' + 1 = \sin^2 y,\ \ y(1)=0.$

27. $e^x dy + (1+e^x)ydx = 0,\ \ y(0)=1.$

28. $\ln x \cdot dy + (1+y)dx = 0,\ \ y(e)=1.$

29. $y' = \sqrt[3]{x+y+1},\ \ y(0)=-1.$

30. $\sqrt{t+2x}\,dx = dt,\ \ x(0)=1/2.$

31. $ydx + (1+x^2)dy = 0,\ \ y(1)=1.$

32. $\sqrt{1+y}\,dx + e^x ydy = 0,\ \ y(0)=0.$

33. $yx\sin x - y'\ln y = 0, y(\pi/2)=1.$

34. $y' = \sin(x-y-1)\ ,\ y(0)=1.$

35. $y' = \sqrt[3]{x-y-1},\ y(2)=-1.$

36. $y' = (y-1)(2y-1),\ \ y(0)=2$

37. $ydy + x(1+y^4)dx = 0,\ \ y(0)=1.$

38. $\dfrac{dy}{dx} + \sqrt{1-x^2}\cdot y = 0,\ \ y(0)=1.$

39. $u' = \sqrt{x+2u+2},\ \ u(0)=0.$

40. $y^2\operatorname{tg}x \cdot dx + dy = 0,\ \ y\left(\pi/4\right)=1.$

II. Matematik modeli o‘zgaruvchilari ajraladigan differensial tenglamalar bilan ifodalanuvchi masalalardan namunalar keltiring.

3. O‘ZGARUVCHILARIGA NISBATAN BIR JINSLI DIFFERENSIAL TENGLAMALAR

Maqsad – o‘zgaruvchilariga nisbatan bir jinsli va unga keltiriluvchi differensial tenglamalarni yechishni o‘rganish

Yordamchi ma’lumotlar:

1⁰. Ushbu

$$y' = f(x,y) \tag{1}$$

ko‘rinishdagi differensial tenglama berilgan bo‘lsin; bunda $f \in C(D)$ ($D-\mathbb{R}^2$даги soha). Agar $f(x,y)$ funksiyani bir o‘zgaruvchining biror $g(t)$ funksiyasi orqali

$$f(x,y)=g\left(\frac{y}{x}\right)\ \Big(\text{yoki}\quad f(x,y)=g\left(\frac{x}{y}\right)\Big) \tag{2}$$

kabi ifodalash mumkin bo‘lsa, u holda (1) tenglama ***o‘zgaruvchilariga nisbatan bir jinsli differensial tenglama*** deyiladi. Shunday qilib, bir jinsli tenglama

$$y'=g\left(\frac{y}{x}\right)\ \Big(\text{yoki}\quad y'=g\left(\frac{x}{y}\right)\Big) \tag{3}$$

ko‘rinishga keltiriladi.

Masalan,

$$f(x,y)=\frac{xy}{x^2-y^2}$$

funksiyani (2) ko‘rinishda tasvirlash mumkin:

$$f(x,y)=\frac{xy}{x^2-y^2}=\frac{\frac{xy}{x^2}}{\frac{x^2-y^2}{x^2}}=\frac{\frac{y}{x}}{1-\left(\frac{y}{x}\right)^2}=g\left(\frac{y}{x}\right),\quad g(t)=\frac{t}{1-t^2}.$$

Bir jinsli tenglama (3) ni yechish uchun $\frac{y}{x}=u,\ y=xu$ deb yangi $u=u(x)$ noma’lum funksiyani kiritamiz. Natijada o‘zgaruvchilari ajraladigan tenglamaga kelamiz:

$$y'=(xu)'=x'u+xu'=u+xu',\ \ u+xu'=g(u),\ \ xdu=(g(u)-u)dx,$$

$$\frac{du}{g(u)-u}=\frac{dx}{x}.$$

Oxirgi tenglamani integrallab, va dastlabki $y=xu$ noma’lumga qaytib, (3) tenglamaning bir parametrli yechimlar oilasini hosil qilamiz:

$$\varphi(u)=\ln|x|+\ln c_1,\ \text{bunda}\ \ \varphi(u)=\int\frac{du}{g(u)-u},\ c_1>0.$$

Bundan

$$|x|c_1=e^{\varphi(u)}\quad \text{yoki}\quad xc=e^{\varphi(\frac{y}{x})}\ (c\neq 0).$$

(3) tenglamani yechish jarayonida $g(u)-u=0$ tenglamadan topiladigan yechimlar yo‘qolishi mumkin. Bu holni alohida tekshirish lozim.

Misol 1. Ushbu

$$(x^2-y^2)dy-2xydx=0 \tag{4}$$

tenglamani yeching.

🗝 Berilgan tenglama o‘zgaruvchilariga nisbatan bir jinslidir, chunki u

$$y'=\frac{2xy}{x^2-y^2} \quad \text{yoki} \quad y'=\frac{2\frac{y}{x}}{1-\left(\frac{y}{x}\right)^2} \tag{5}$$

ko‘rinishda yoziladi. Yuqorida aytilgan $u=\frac{y}{x}$, $y=xu$ almashtirishni bajaramiz.

$y'=u+xu'$ bo‘lgani uchun (5) tenglamani

$$u+xu'=\frac{2u}{1-u^2},\ xu'=\frac{u+u^3}{1-u^2}$$

ko‘rinishga keltiramiz. Oxirgi tenglamada o‘zgaruvchilarni ajratamiz va integrallashlarni bajaramiz:

$$\frac{1-u^2}{u(1+u^2)}du=\frac{dx}{x} \quad , \quad \int\frac{1-u^2}{u(1+u^2)}du=\int\frac{dx}{x}-\ln c_1 \quad (c_1>0) \tag{6}$$

Endi $\int\frac{1-u^2}{u(1+u^2)}du$ integralni hisoblaymiz. Integral ostidagi funksiyani sodda kasrlar yig‘indisi sifatida tasvirlaymiz va integrallashlarni bajaramiz:

$$\int\frac{1-u^2}{u(1+u^2)}du=\int\left(\frac{-2u}{1+u^2}+\frac{1}{u}\right)du=\int\frac{du}{u}-\int\frac{2udu}{1+u^2}=$$

$$=\ln|u|-\int\frac{d(1+u^2)}{1+u^2}=\ln|u|-\ln(1+u^2).$$

Demak, (6) tenglik

$$\ln|u|-\ln(1+u^2)=\ln|x|-\ln c_1 \ \text{yoki} \quad |x|(1+u^2)=|u|c_1$$

yoki

$$x(1+u^2)=uc \quad (c\neq 0)$$

ko‘rinishni oladi. Bu yerda $u=\dfrac{y}{x}$ deb eski noma’lumga qaytamiz:

$$x\left(1+\frac{y^2}{x^2}\right)=\frac{y}{x}c \quad \text{yoki} \quad x^2+y^2=cy. \tag{7}$$

Biz (6) tenglamani $u(1+u^2)$ ga bo‘lganda $u=0,$ ya’ni (4) ning $y=0$ yechim yo‘qolgan bo‘lishi mumkin edi. Bu yechim (7) formuladan c ning hech qanday qiymatida hosil bo‘lmaydi. Lekin, agar (7) dagi c ni $\dfrac{1}{c}$ bilan almashtirib uni $y=c(x^2+y^2)$ ko‘rinishda yozsak, barcha yechimlarni ifodalagan bo‘lamiz.

Javob: $y=c(x^2+y^2)$, $c-$ixtiyoriy o‘zgarmas. ☝

Misol 2. Ushbu

$$yy'=x(2+\ln y-\ln x)$$

tenglamani yeching.

🗝 Tenglama (1) (birinchi chorak) da berilgan. Uni

$$y'=\frac{x}{y}\left(2+\ln\frac{y}{x}\right) \qquad (x>0,\ y>0)$$

ko‘rinishda yozib, bir jinsli tenglama ekanligini ko‘ramiz.

Tenglamada $u=\dfrac{y}{x}$, $\quad y=xu, \quad y'=u+xu'$ deb, zaruriy ixchamlashlarni bajaramiz va ushbu

$$xu'=\frac{2+\ln u-u^2}{u} \qquad (x>0, u>0) \tag{8}$$

o‘zgaruvchilari ajraladigan tenglamaga kelamiz.

$2+\ln u-u^2=0$ tenglama ikkita u_1 va u_2 $(u_1<u_2)$ yechimlarga ega. Bu tasdiq $w=u^2-2$ va $w=\ln u$ funksiyalar grafiklarining ikkita nuqtada kesishishidan kelib chiqadi (grafiklarni quring). Hisoblashlar $u_1\approx 0{,}138,\ u_2\approx 1{,}564$ ekanligini ko‘rsatadi. Ravshanki, (8) differensial tenglama $u=u_1$ va $u=u_2$ o‘zgarmas yechimlarga ega. Boshqa yechimlarni topish uchun (8) tenglamada o‘zgaruvchilarni ajratamiz va integrallashlarni bajaramiz. Ushbu

$$\varphi(u)=\int\frac{udu}{2+\ln u-u^2}$$

belgilashni kiritib ($\varphi(u)-$elementar funksiya emas), yechimlarni

$$\varphi(u)=\ln x+\ln c \quad (c>0)$$

oshkormas ko‘rinishda ifodalaymiz. Bu yerda dastlabki noma’lum *y* ga qaytib, berilgan $yy'=x(2+\ln y-\ln x)$ differensial tenglamaning oshkormas ko‘rinishdagi yechimlarini hosil qilamiz:

$$\varphi\left(\frac{y}{x}\right)=\ln x+\ln c \quad (c>0) \quad .$$

Bundan tashqari, berilgan tenglama $y=u_1x$ va $y=u_2x$ yechimlarga ham ega.

Izoh. kiritilgan $\varphi(u)$ funksiya (8) differensial tenglamaning uchta yechimini aniqlaydi. Ular mos ravishda $0<u<u_1$, $u_1<u<u_2$, $u_2<u$ intervallarida aniqlangan bo‘ladi. ☝

2^0.*Bir jinsli tenglama almashtirishidan foydalanib yechiladigan tenglamalar*

Ushbu

$$y'=\frac{y}{x}+g\left(\frac{y}{x}\right)h(x)$$

ko‘rinishdagi tenglama o‘zgaruvchilariga nisbatan bir jinsli bo‘lmasada, uni ham $y=xu,\ u=u(x),$ almashtirish yordamida ushbu $xu'=g(u)h(x)$ o‘zgaruvchilari ajraladigan tenglamaga keltirib yechish mumkin.

Misol 3. Differensial tenglamani yeching:

$$xy' = y + x^2 e^{y/x}.$$

🗝 Bu tenglamani

$$y' = \frac{y}{x} + e^{y/x} x$$

ko‘rinishda yozib, $y = xu$, $u = u(x)$, almashtirishni bajarish kerakligini ko‘ramiz. Yangi $u = u(x)$ noma’lum funksiyaga nisbatan ushbu

$$u' = e^u$$

tenglamani hosil qilamiz. Bu tenglamada o‘zgaruvchilarni ajratamiz va inegrallashni bajarib, topamiz:

$$u = -\ln(c - x).$$

Buni $y = xu$ almashtirish formulasiga qo‘yib, izlangan yechimlarni topamiz:

$$y = -x\ln(c - x). \text{ 👍}$$

3⁰. *Bir jinsli tenglamaga keltiriluvchi tenglamalar*

Ushbu

$$y' = g\left(\frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}\right) \qquad (a_1, b_1, c_1, a_2, b_2, c_2 - \text{const}) \tag{9}$$

ko‘rinishdagi tenglama, agar $a_1 b_2 - a_2 b_1 \neq 0$ bo‘lsa, bir jinsli tenglamaga keltiriladi. Buning uchun $x = \xi + \alpha$, $y = \eta + \beta$ deb, α va β o‘zgarmaslarni shunday tanlash kerakki, natijada (8) tenglamaning o‘ng tomoni

$$g\left(\frac{a_1\xi + b_1\eta}{a_2\xi + b_2\eta}\right) = g\left(\frac{a_1 + b_1\dfrac{\eta}{\xi}}{a_2 + b_2\dfrac{\eta}{\xi}}\right) = h\left(\frac{\eta}{\xi}\right), \quad h(t) = g\left(\frac{a_1 + b_1 t}{a_2 + b_2 t}\right),$$

ko‘rinishini olsin; bunda α va β lar uchun ushbu

$$\begin{cases} a_1\alpha + b_1\beta + c_1 = 0 \\ a_2\alpha + b_2\beta + c_2 = 0 \end{cases}$$

tenglamalar sistemasi hosil bo‘ladi. Bu holda

$$y' = \frac{dy}{dx} = \frac{d\eta}{d\xi}$$

bo‘lgani uchun berilgan tenglama

$$\frac{d\eta}{d\xi} = h\left(\frac{\eta}{\xi}\right)$$

ko‘rinishdagi bir jinsli tenglamaga keladi.

Agar $a_1b_2 - a_2b_1 = 0$ bo‘lsa, u holda $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k$, $a_1x + b_1y = k(a_2x + b_2y)$ va (9) tenglama $z = a_2x + b_2y$ almashtirish yordamida quyidagi o‘zgaruvchilari ajraladigan tenglamaga keltiriladi:

$$\frac{dz}{dx} = a_2 + b_2\, g\left(\frac{kz + c_1}{z + c_2}\right).$$

Eslatma. (9) tenglama $u = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ almashtirish bilan to‘g‘ridan-to‘g‘ri o‘zgaruvchilari ajraladigan tenglamaga keltirilishi mumkin.

Misol 4. Ushbu

$$(x - y + 1)dy - (2x + y - 4)dx = 0$$

tenglamani yeching.

🗝 Berilgan tenglamani

$$y' = \frac{2x + y - 4}{x - y + 1} \tag{10}$$

ko‘rinishda yozaylik. Bizda $a_1 = 2,\ b_1 = 1,\ c_1 = -4,\ a_2 = 1,\ b_2 = -1,\ c_2 = 1,$ $a_1b_2 - a_2b_1 = -3 \neq 0$ va $x = \xi + \alpha,\ y = \eta + \beta$ almashtirishda

$$x - y + 1 = \xi - \eta + (\alpha - \beta + 1)$$

$$2x + y - 4 = 2\xi + \eta + (2\alpha + \beta - 4)$$

bo‘ladi. Biz α va β larni ushbu

$$\begin{cases} \alpha - \beta + 1 = 0 \\ 2\alpha + \beta - 4 = 0 \end{cases}$$

shartlardan tanlaymiz. Oxirgi sistemani yechib $\alpha = 1,\ \beta = 2$ ekanligini topamiz. Shunday qilib,

$$x = \xi + 1, \quad y = \eta + 2 \qquad \left(\frac{dy}{dx} = \frac{d\eta}{d\xi} \right)$$

almashtirishda (10) tenglama

$$\frac{d\eta}{d\xi} = \frac{2\xi + \eta}{\xi - \eta} \quad \text{yoki} \quad \frac{d\eta}{d\xi} = \frac{2 + \dfrac{\eta}{\xi}}{1 - \dfrac{\eta}{\xi}} \tag{11}$$

ko‘rinishga (bir jinsli tenglamaga)keladi. Endi (11) tenglamada

$$u = \frac{\eta}{\xi}, \ \ \eta = u\xi, \qquad \frac{d\eta}{d\xi} = u + \xi \frac{du}{d\xi}$$

deymiz. U holda

$$u + \xi \frac{du}{d\xi} = \frac{2 + u}{1 - u} \quad \text{yoki} \quad \xi \frac{du}{d\xi} = \frac{2 + u^2}{1 - u}$$

tenglama hosil bo‘ladi. Oxirgi tenglamani o‘zgaruvchilarni ajratib integrallaymiz $(2 + u^2 > 0)$:

$$\frac{1-u}{2+u^2} du = \frac{d\xi}{\xi} \ , \quad \int \frac{1-u}{2+u^2} du = \int \frac{d\xi}{\xi} + \ln c \quad (c > 0)$$

$$\frac{1}{\sqrt{2}} \operatorname{arctg} \frac{u}{\sqrt{2}} - \frac{1}{2} \ln(2 + u^2) = \ln|\xi| + \ln c \quad (c > 0)$$

$$\frac{1}{\sqrt{2}} \operatorname{arctg} \frac{u}{\sqrt{2}} = \ln\left(\sqrt{2 + u^2} \cdot |\xi| \cdot c \right) \quad (c > 0)$$

$$\exp\left(\frac{1}{\sqrt{2}} \operatorname{arctg} \frac{u}{\sqrt{2}} \right) = \sqrt{2 + u^2}\, |\xi| \cdot c \quad (c > 0)$$

Oxirgi tenglikda eski o‘zgaruvchilarga qaytib, berilgan tenglamaning oshkormas ko‘rinishdagi yechimlarini hosil qilamiz:

$$\exp\Big(\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{y-2}{\sqrt{2}(x-1)}\Big)=c\sqrt{2(x-1)^2+(y-2)^2}\quad (c>0)\,.$$ ☝

Misol 5. Ushbu

$$y'=\frac{2x+y+1}{4x+2y-3}$$

tenglamani yeching.

Qaralayotgan tenglama uchun $a_1=2, b_1=1,\ \ a_2=4, b_2=2$ va $a_1b_2-a_2b_1=0,$ ya’ni $4x+2y$ ifoda $2x+y$ ga proporsional: $4x+2y=2(2x+y)$ Shuning uchun $z=2x+y$ deymiz. U holda $z'=2+y'$ va demak, berilgan tenglama

$$z'=2+\frac{z+1}{2z-3}\quad \text{yoki}\quad z'=\frac{5(z-1)}{2z-3}$$

ko‘rinishga keladi. Oxirgi tenglamani o‘zgaruvchilarni ajratib integrallaymiz:

$$\frac{2z-3}{5(z-1)}dz=dx\,,\ \int\frac{2z-3}{5(z-1)}dz=\int dx+\ln c_1\quad (c_1>0)$$

Bu yerda integrallashlarni bajaramiz, eski o‘zgaruvchilarga qaytamiz va kerakli ixchamlashtirishlardan so‘ng berilgan tenglamaning oshkormas ko‘rinishdagi yechimlarini hosil qilamiz:

$$e^{x-2y}(2x+y-1)=c\,.$$ ☝

4⁰. *Umumlashgan bir jinsli tenglamalar*

Ba’zi tenglamalarni $y=z^{\alpha}$ yoki $y=-z^{\alpha}$ almashtirish yordamida bir jinsli tenglamaga keltirish mumkin. Bunday tenglamalar ***umumlashgan bir jinsli tenglamalar*** deyiladi. Masalan,

$$y'=\frac{y}{x}g\Big(\frac{y}{x^{\alpha}}\Big)$$

ko‘rinishdagi tenglama umumlashgan bir jinsli.

Izoh. $y = x^{\alpha}u$ almashtirish umumlashgan bir jinsli tenglamani to‘g‘ridan-to‘g‘ri o‘zgaruvchilari ajraladigan tenglamaga olib keladi.

Misol 6. Ushbu

$$y' = 4\sqrt{y} - 2x$$

differensial tenglamani yeching.

🗝 Bu tenglama bir jinsli emas. Tenglamada $y = z^{\alpha}$ almashtirish bajaraylik:

$$\alpha z^{\alpha-1} z' = 4z^{\frac{\alpha}{2}} - 2x, \quad z' = \frac{4z^{\frac{\alpha}{2}} - 2x}{\alpha z^{\alpha-1}}.$$

Tushunarliki, agar $\frac{\alpha}{2} = 1 = \alpha - 1$, ya’ni $\alpha = 2$ bo‘lsa, oxirgi tenglama bir jinsli bo‘ladi. Shunday qilib, $y = z^2$ almashtirish natijasida

$$z' = \frac{2z - x}{z} \text{ yoki } z' = 2 - \frac{x}{z}$$

bir jinsli tenglamaga kelamiz. Oxirgi tenglamada $z = x \cdot u$ deymiz va

$$u + xu' = 2 - \frac{1}{u} \quad \text{yoki} \quad xu' = -\frac{(u-1)^2}{u}$$

tenglamani hosil qilamiz. Oxirgi tenglamada o‘zgaruvchilarni ajratamiz, integrallashlarni bajaramiz va y noma’lumga qaytamiz:

$$\frac{udu}{(u-1)^2} = -\frac{dx}{x} \quad (u \neq 1), \quad \int \frac{udu}{(u-1)^2} = -\int \frac{dx}{x} ,$$

$$\ln|u-1| - \frac{1}{u-1} = -\ln|x| - \ln c_1 , \quad |u-1| \cdot |x| \cdot c_1 = e^{\frac{1}{u-1}} ,$$

$$(u-1)xc = e^{\frac{1}{u-1}} \quad (u = \frac{z}{x}) , \quad \left(\frac{z}{x} - 1\right)xc = \exp\left(1 / \left(\frac{z}{x} - 1\right)\right) \quad (z = \sqrt{y}) ,$$

$$(\sqrt{y} - x)c = \exp\left(\frac{x}{\sqrt{y} - x}\right). \tag{12}$$

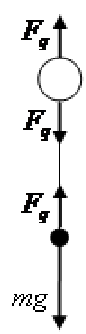
$u = 1$ ga mos keluvchi $z = x$, ya’ni yo‘qolgan $y = x^2$ yechim (12) dan c ning hech qanday qiymatida hosil bo‘lmaydi.

Javob: $y = x^2$, $(\sqrt{y} - x)\, c = \exp\left(\frac{x}{\sqrt{y} - x}\right)$.

Matematik modellari o‘zgaruvchilariga nisbatan bir jinsli birinchi tartibli differensial tenglamalar orqali ifodalanuvchi masalalardan namunalar keltiraylik.

Misol 7. Havo shariga cho‘zilmas va vaznsiz ip bilan osilgan m massali yuk v_0 tezlik bilan yerga tushib kelmoqda. Havoning sharga ta’sir etuvchi qarshilik kuchi tezlikning kvadratiga va shar sirti yuzasiga to‘g‘ri proporsional. t_0 payt shar teshiladi va uning sirt yuzasi $S = \frac{S_0}{(t_0 + t)^2}$ qonunga ko‘ra kamaya boshlaydi (shar sirtining sferaligi saqlanadi). Shar teshilgandan so‘ng yuk tezligi v ning o‘zgarish qonunini toping. Havoning yukka ta’sir (qarshilik) kuchini, sharning massasini va unga ta’sir etuvchi Arximed ko‘tarish kuchini hisobga olmang.

Masalaning shartiga ko‘ra sharga havoning qarshilik kuchi $F_q = kSv^2$ $(k = \text{const} > 0)$ (yuqoriga yo‘nalgan) va ipning taranglik kuchi T (pastga yo‘nalgan) ta’sir etadi (3.1- rasm).



3.1- rasm. Havo shari va yukka ta’sir etuvchi kuchlar.

Sharning massasi nolga teng bo‘lganligi sababli Nyutonning ikkinchi qonuniga ko‘ra $T = F_q$. Ip cho‘zilmas va vaznsiz bo‘lgani uchun u shu $T = F_q$ kuch bilan yukni yuqoriga tortadi. Yukka uning og‘irligi mg (yerning tortish kuchi) ham ta’sir etadi. Nyutonning ikkinchi qonuniga ko‘ra

$$m\frac{dv}{dt} = mg - kSv^2 \text{ yoki } m\frac{dv}{dt} = mg - k\frac{S_0}{(t_0+t)^2}v^2.$$

Boshlang‘ich shart $v\big|_{t=0} = v_0$. Oxirgi tenglamada $\tau = t_0 + t$ desak, ushbu

$$\frac{dv}{d\tau} = g - \frac{kS_0}{m}\frac{v^2}{\tau^2}$$

o‘zgaruvchilariga nisbatan bir jinsli tenglamaga kelamiz. Uni yechish uchun $v = u\tau$ deb yangi u noma’lum funksiyani kiritamiz. O‘zgaruvchilari ajraladigan tenglama hosil bo‘ladi:

$$\frac{du}{d\tau}\tau = g - u - \frac{kS_0}{m}u^2.$$

Bu tenglamada o‘zgaruvchilarni ajratamiz va $u\big|_{\tau=0} = v_0 / t_0$ boshlang‘ich shartni hisobga olib uni yechamiz. Natijada $v = u\tau,\ \tau = t_0 + t$ formulalar bilan v ga qaytamiz va izlangan qonuniyatni topamiz:

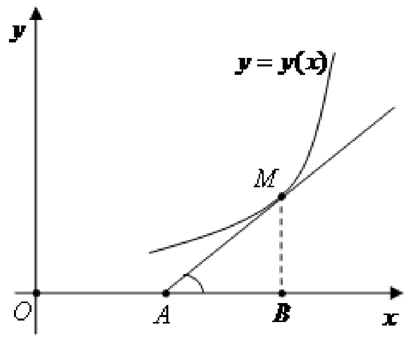
$$v = \frac{m(t_0+t)}{2kS_0}\left(\alpha - 1 + \frac{2\alpha}{c(t_0+t)^{\alpha} - 1}\right).$$

Bu yerda

$$\alpha = \sqrt{1 + \frac{4kS_0 g}{m}},\ c = \frac{1}{t_0^{\alpha}}\frac{2kS_0v_0 + (\alpha+1)mt_0}{2kS_0v_0 - (\alpha-1)mt_0}.$$ 👍

Misol 8. Shunday egri chiziqlarni topingki, ularning ixtiyoriy urinmasining absissalar o‘qi bilan kesishish nuqtasidan koordinatalar boshi va urinish nuqtasigacha bo‘lgan masofalar teng bo‘lsin.

🗝 Izlanayotgan chiziq tenglamasi $y = y(x)$ bo‘lsin. Uning ixtiyoriy $M(x, y)$ nuqtasini olaylik (3.2- rasm).



3.2- rasm.

Shu nuqtadan urinma o‘tkazib uning abssissalar o‘qi bilan kesishish nuqtasi A ni topaylik. Ravshanki,

$$|OB| = x, |BM| = y, \operatorname{tg}\angle MAB = y', |AO| = |OB| - |AB| = x - \frac{y}{y'}, \qquad |AM| = \sqrt{y^2 + \left(\frac{y}{y'}\right)^2}.$$

Masalaning shartiga ko‘ra $|AO| = |AM|$, ya’ni $x - \frac{y}{y'} = \sqrt{y^2 + \left(\frac{y}{y'}\right)^2}$. Buni soddalashtirib, o‘zgaruvchilariga nisbatan bir jinsli tenglamani hosil qilamiz:

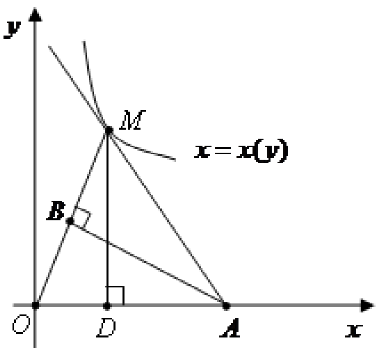
$$y' = \frac{2xy}{x^2 - y^2} \text{ yoki } y' = \frac{2y/x}{1-(y/x)^2}.$$

$u = y/x$ almashtirishni bajarib, o‘zgaruvchilari ajraladigan tenglamaga kelamiz:

$$xu' = \frac{2u}{1-u^2} - u, \; xu' = \frac{u(1+u^2)}{1-u^2}.$$

Oxirgi tenglamani yechib, $\ln\left|\frac{u}{1+u^2}\right| = \ln|x| + \ln c$ $(c > 0)$ munosabatni topamiz. Bu tenglamadan u ni aniqlaymiz va $y = xu$ ga qaytamiz. Natijada ushbu $y = c(x^2 + y^2)$ $(c \neq 0)$ ellipslarni hosil qilamiz. Demak, izlangan chiziqlar ana shu ellipslardan iborat. ♁

Misol 9. Shunday egri chiziqlarni topingki, agar ularning ixtiyoriy M nuqtasidan urinma o‘tkazilsa, bu nuqtadan kordinatalar boshigacha bo‘lgan $|MO|$ masofa urinmaning abssissalar o‘qi bilan kesishish nuqtasidan MO gacha bo‘lgan masofaga teng bo‘lsin (3.3-rasm).



3.3- rasm.

🗝 Izlanayotgan chiziq tenglamasi $x = x(y)$ bo‘lsin, teskari funksiya $y = y(x)$. Uning ixtiyoriy $M(x, y)$ nuqtasini olaylik (3.3- rasm). Shu nuqtadan urinma o‘tkazib uning abssissalar o‘qi bilan kesishish nuqtasi A ni topaylik. Ravshanki,

$$|OD| = x,\ |DM| = y,\ |MO| = \sqrt{x^2 + y^2}\,,$$

$$|OA| = |OD| + |DA| = x + y\operatorname{ctg}\angle MAD = x - y\frac{1}{y'} = x - y\frac{dx}{dy}\,.$$

Masalaning shartiga ko‘ra $|MO| = |AB|$. Endi ushbu $S_{\triangle OAM} = \frac{1}{2}|OA|\cdot|DM| = \frac{1}{2}|MO|\cdot|AB|$

tenglikka ko‘ra $\left(x - y\frac{dx}{dy}\right)\cdot y = x^2 + y^2$ yoki

$$y^2\frac{dx}{dy} = xy - x^2 - y^2\,.$$

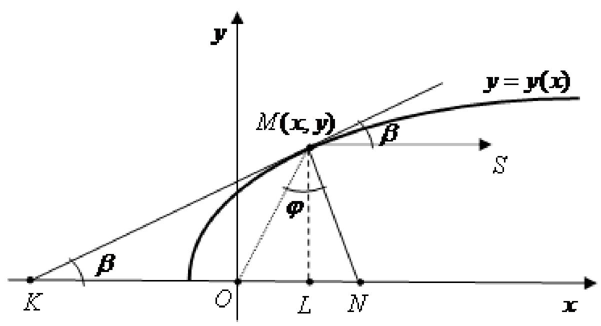
Bu o‘zgaruvchilariga nisbatan bir jinsli tenglamada $u = x/y$ almashtirish bajaramiz va ushbu

$$y\frac{du}{dy} = -1 - u^2$$

tenglamani hosil qilamiz. Bu tenglamani o‘zgaruvchilarni ajratib yechamiz va $y = ce^{-\operatorname{arctg} u}$ yechimni topamiz. Endi $u = x/y$ deb, izlangan egri chiziqlarning oshkormas ko‘rinishdagi tenglamasi $y = ce^{-\operatorname{arctg}\frac{x}{y}}$ ga kelamiz. Bu chiziqlarni $x = y\operatorname{tg}(\ln\frac{c}{y})$ ko‘rinishda ham ifodalash mumkin. ☝

M**isol 10.** Shunday ko‘zgu shaklini topingki, berilgan nuqtaviy yorug‘lik manbasidan chiquvchi barcha nurlar bu ko‘zgudan berilgan yo‘nalishga parallel yo‘nalishda qaytsin.

🗝Yorug‘lik manbasi O nuqtada joylashgan bo‘lsin. Shu nuqta orqali berilgan yo‘nalishga parallel tekislik o‘tkazamiz. Bu tekislikda koordinatalar boshini O nuqtada joylashtiramiz, x o‘qini berilgan yo‘nalish bo‘ylab, y o‘qini esa unga perpendikukyar qilib yo‘naltiramiz va izlanayotgan ko‘zguning $y > 0$ yarim tekislikdagi tenglamasini $y = y(x)$ bilan belgilaymiz (3.4- rasm). Ravshanki, $y < 0$ holida bu tenglama $y = -y(x)$ bo‘ladi .



3.4- rasm.

O nuqtadan chiqqan nur $M(x, y)$ nuqtada ko‘zgudan Ox yo‘nalishiga parallel holda qaytadi va MS bo‘ylab tarqaladi. Fizikadan ma’lumki, bunda tushish burchagi qaytish burchagiga teng bo‘ladi. Agar MN bilan ko‘zguga uning M nuqtasidagi normalini belgilasak, tushish butchagi $\angle OMN$ qaytish burchagi $\angle NMS$ ga teng bo‘lishi kerak, ya’ni $\angle OMN = \angle NMS = \varphi$. Demak, agar KM bilan M nuqtada o‘tkazilgan urinmani ($KM \perp MN$) belgilasak, u holda $\beta = \angle OKM = \angle OMK = \frac{\pi}{2} - \varphi$ va

$|OM|=|OK|$ bo‘ladi. Endi $|OM|$ va $|OK|$ larni alohida-alohida hisoblaymiz. Ravshanki, $|OM|=\sqrt{x^2+y^2}$. To‘g‘ri burchakli $\triangle KML$ dan

$$|OK|=|KL|-|OL|=\frac{|ML|}{tg\beta}-|OL|=\frac{y}{y'}-x,$$

chunki $tg\beta=y'=y'(x)$, $|OL|=x,\ |ML|=y$. Demak,

$$\sqrt{x^2+y^2}=\frac{y}{y'}-x \text{ yoki } y'=\frac{y}{x+\sqrt{x^2+y^2}}.$$

Bu bir jinsli tenglamani yechib, $y^2=c^2+2cx$ ekanligini topamiz. Bu tenglama parabolani ifodalaydi. Ko‘zgu shakli ana shu parabolani Ox o‘qi atrofida aylantirishdan hosil bo‘ladi. Demak, izlangan ko‘zgu aylanma paraboloiddan iborat ekan. ☝

Masalalar

Tenglamalarni yeching (**1** – **6**):

1. $2(x+y)y'=3x+y$. **2.** $(x^3+y^3)y'+x^2y=0$.

3. $(x-1)y'=3x+2y+3$. **4.** $(x+y-2)y'+x-y-1=0$.

5. $yy'=xy+x^3$. **6.** $x^3y'-x^4y^2=2$.

Koshi masalalarini yeching (**7** – **10**):

7. $(2x+y)y'=x-2y,\ y(-1)=0$. **8.** $(x^2-3y^2)y'+xy=0,\ y(1)=1$.

9. $xy'=y+x\mathrm{tg}\dfrac{y}{x},\ y|_{\pi/2}=1$. **10.** $x^4y'=4x^6-y^2,\ y|_1=1$.

Mustaqil ish №3 topshiriqlari:

I. Differensial tenglamalarni yeching.

1. $xy'=y(2-\ln y+\ln x)$; $(x+y-2)dx-(3x-y-2)dy=0$.

2. $xydx=(x^2-y^2)dy$; $(9x-y-8)y'=x+7y-8$.

3. $xy'=y\sin\ln\dfrac{y}{x}$; $y'=\dfrac{x+2y-3}{4x-y-3}$.

4. $(x^2+y^2)dy=2xydx$; $(5x-y-4)y'=x+3y-4$.

5. $xy' = \sqrt{x^2 + y^2} + y.$; $y' - x = \sqrt{y}$.

6. $y' = \dfrac{x+2y}{2x-y}$; $2y' = 4\sqrt{y} - x$.

7. $y^2 + (x^2 - xy)y' = 0$; $x^2 y' = xy + y^3$.

8. $(y-x)dy = (y+x)dx$; $x(xy+2)y' + y = 0$.

9. $(x - y\cos\dfrac{y}{x})dx + x\cos\dfrac{y}{x}dy = 0$; $y' = \dfrac{5y+5}{4x+3y-1}$.

10. $(x-2y)y' = x + 9y$; $y' = \left(\dfrac{y+2}{x+y-1}\right)^2$.

11. $(x+y)y' = 2x - y$; $x^3 y' = x^4 - y^2$.

12. $(4x+2y-3)y' = 2x + y + 1$; $x^3 y' = 1 - 8x^4 y^2$.

13. $(2y + x + 7)dx + (2x - y + 4)dy = 0$; $y' = 4x^2 - \dfrac{y^2}{x^4}$.

14. $\dfrac{dy}{dx} = \dfrac{2x - y + 4}{2y - x - 5}$; $\left(2x^4 y^2 - 1\right)dx + x^3 dy = 0$.

15. $xy' = y(2 - \ln y + \ln x)$; $y' = \dfrac{y^3 + xy}{x^2}$.

16. $(x^3 - y^3)dy = x^2 y dx$; $y' = x + \dfrac{x^3}{y}$.

17. $xy' = y(1 + \ln x - \ln y)$; $(2xy + x^4)y' = 2x^3 y + y^2$.

18. $yy' = 4x + 3y - 2$; $2\left(\sqrt{x^4 y^2 + 1} - x^2 y\right)dx - x^3 dy = 0$.

19. $xy' - y = x\,\mathrm{tg}\dfrac{y}{x}$; $y' = \dfrac{x + 4y - 5}{6x - y - 5}$.

20. $xy' = \dfrac{3y^3 + 2yx^2}{2y^2 + x^2}$.; $(5x + 2y - 1)y' + 2x + y = 0$.

21. $y' = \dfrac{x+2y}{2x-y}$.; $y' = 2x^2 + y^{\frac{2}{3}}$.

22. $xy' = 2\sqrt{x^2+y^2} + y$.; $y' = x^3 + 3y^{\frac{3}{4}}$.

23. $x(x-y)y' = y^2$; $(y'+1)\ln\dfrac{y+x}{x+1} = \dfrac{y+x}{x+1}$.

24. $y' = \dfrac{x-y}{x+y}$; $y' = \dfrac{y+1}{x-1} + \operatorname{tg}\dfrac{y+x}{x-1}$.

25. $xy' = \dfrac{3y^3+2yx^2}{2y^2+x^2}$; $x^3y' = y^2 + x^4$.

26. $(y^2+xy)dx - 2x^2dy = 0$; $2x^2y' = y^3 - 2xy$.

27. $(x+4y)y' + y = 0$; $y' = \dfrac{y}{x} + e^{y/x}\ln x$.

28. $(y-2x)dy = (2y+x)dx$; $x^3y' = 2(\sqrt{x^4y^2+1} - x^2y)$.

29. $(x+\sqrt{x^2+y^2})y' = y$; $4x^2y' = y^3 + xy$.

30. $(x-\sqrt{x^2+y^2})y' = y$; $x^2y' = y^3 - xy$.

31. $x\sin\dfrac{y}{x}dx + \left(x - y\sin\dfrac{y}{x}\right)dy = 0$; $y' = \dfrac{x+5y-6}{7x-y-6}$.

32. $xy' = y + \operatorname{ctg}\dfrac{y}{x}$; $y' = \dfrac{2x+y-1}{2x-2}$.

33. $(x+y)dy = (2x-y)dx$; $x^2y' = y^3 - 4xy$.

34. $(2y-3x)dy = (y+x)dx$; $6xy^5y' = x^3 + 4y^6$.

35. $(xy'-y)\operatorname{arctg}\dfrac{y}{x} = x$; $y' = \dfrac{6y-6}{5x+4y-9}$.

36. $y' = \dfrac{y^2-2xy-x^2}{y^2+2xy-x^2}$; $x^2y' = 3y^3 + 2xy$.

37. $(2xy^3 - x^4)y' = y^4 - 2x^3y$; $(x-y+3)y' = x+y+1$.

38. $xydx-(4y^2+x^2)dy=0$; $(4x^2+y^4)y'=2xy$.

39. $xy'=y+\sin\dfrac{y}{x}$; $y'=\dfrac{x+6y-7}{8x-y-7}$.

40. $(x+2y)dx-(2x+y)dy=0$; $y'=\dfrac{y+2}{x+1}+\operatorname{tg}\dfrac{y-2x}{x+1}$.

II. Matematik modeli o‘zgaruvchilariga nisbatan bir jinsli differensial tenglamalar bilan ifodalanuvchi masalalardan namunalar keltiring.

4. BIRINCHI TARTIBLI CHIZIQLI DIFFERENSIAL TENGLAMALAR

Maqsad – birinchi tartibli chiziqli differensial tenglamalarni hamda ularga keltiriluvchi tenglamalarni yechishni o‘rganish

Yordamchi ma’lumotlar:

Birinchi tartibli chiziqli differensial tenglama deb

$$y'+p(x)y=q(x) \tag{1}$$

ko‘rinishdagi tenglamaga aytiladi; bunda $p(x)$ va $q(x)$ funksiyalari $\in C(I)$.

Tushunarliki, $r(x)y'+p(x)y=q(x),\ r(x)\neq 0,$ ko‘rinishdagi tenglama ham (1) ko‘rinishga keladi.

Ushbu

$$y'+p(x)y=0 \tag{2}$$

tenglama (1) ga mos bir jinsli differensial tenglama deyiladi.

Jumla. Bir jinsli tenglama (2) ning har qanday yechimi

$$y=c\cdot e^{-\int_{x_0}^{x}p(s)ds} \tag{3}$$

formula bilan beriladi; bunda $x_0\in I-$ tayinlangan, $x\in I-$ o‘zgaruvchi nuqtalar, $c-$ ixtiyoriy o‘zgarmas $(c=y(x_0))$.

Chiziqli tenglama (1) ni yechishning uch usulini keltiramiz.

1. Lagranj usuli. (1) ga mos bir jinsli tenglama yechimidagi ixtiyoriy o‘zgarmasni variyatsiyalab (o‘zgaruvchi deb qarab) (1) ning yechimini izlaymiz, ya’ni uning yechimini

$$y = c(x)e^{-\int_{x_0}^{x} p(s)ds} \tag{4}$$

ko‘rinishda izlaymiz bunda $c(x)$ hozircha noma’lum funksiya. (4) ni (1) ga qo‘yib uning qanoatlanishi shartidan $c(x)$ ga nisbatan tenglama hosil qilamiz:

$$\left(c(x)e^{-\int_{x_0}^{x} p(s)ds}\right)' + p(x)c(x)e^{-\int_{x_0}^{x} p(s)ds} = q(x)$$

yoki

$$c'(x)e^{-\int_{x_0}^{x} p(s)ds} + c(x)e^{-\int_{x_0}^{x} p(s)ds} \cdot (-p(x)) + p(x)c(x)e^{-\int_{x_0}^{x} p(s)ds} = q(x),$$

ya’ni

$$c'(x)e^{-\int_{x_0}^{x} p(s)ds} = q(x).$$

Oxirgi tenglamadan dastlab $c'(x)$ ni, so‘ngra esa $c(x)$ ni topamiz:

$$c'(x) = q(x)e^{\int_{x_0}^{x} p(s)ds}, \quad c(x) = c(x_0) + \int_{x_0}^{x} q(t)e^{\int_{x_0}^{t} p(s)ds} dt \tag{5}$$

bu yerda $c(x_0)-$ ixtiyoriy o‘zgarmas; biz uni c harfi bilan belgilaymiz: $c = c(x_0)$ (5) ni (4) ga qo‘yib (1) ning barcha yechimlarini (umumiy yechimini ifodalovchi formulani) topamiz:

$$y = ce^{-\int_{x_0}^{x} p(s)ds} + e^{-\int_{x_0}^{x} p(s)ds} \cdot \int_{x_0}^{x} q(t)e^{\int_{x_0}^{t} p(s)ds} dt \tag{6}$$

(1)ning umumiy yechimini beruvchi (6) formuladagi birinchi qo‘shiluvchi mos bir jinsli tenglama (2) ning umumiy yechimidan, ikkinchi qo‘shiluvchi esa (1) ning xususiy (aniqrog‘i $x = x_0$ da nolga aylanuvchi) yechimidan iborat.

2. Eyler-Bernulli usuli. (1) tenglamaning yechimini $y = u \cdot v$ ko‘rinishda izlaymiz; bunda $u = u(x)$ va $v = v(x)$ hozircha noma’lum funksiyalar. Biz ularni shunday tanlaymizki, natijada $y = u \cdot v$ funksiya (1) ni qanoatlantirsin.

Quyidagilarga egamiz:

$$(uv)' + p(x)uv = q(x), \quad u'v + uv' + p(x)uv = q(x),$$

$$(u' + p(x)u)v + uv' = q(x)\,. \tag{7}$$

Endi u ni ushbu

$$u' + p(x)u = 0 \tag{8}$$

shartdan tanlaymiz. U holda (7) dan

$$uv' = q(x) \tag{9}$$

bo‘lishi kerakligini topamiz. (8) tenglamaning ushbu

$$u = e^{-\int_{x_0}^{x} p(s)ds} \tag{10}$$

yechimini olaylik. U holda (9) ga ko‘ra

$$e^{-\int_{x_0}^{x} p(s)ds} \cdot v' = q(x)\,, \text{ ya’ni } \quad v' = q(x)e^{\int_{x_0}^{x} p(s)ds}$$

bo‘lishini topamiz. Bundan

$$v = c + \int_{x_0}^{x} q(t)e^{\int_{x_0}^{t} p(s)ds} dt. \tag{11}$$

Endi (10) va (11) ga ko‘ra (1) ning $y = u \cdot v$ yechimini topamiz. Bunda yana (6) formula hosil bo‘ladi.

3. Integrallovchi ko‘paytuvchi usuli. (1) tenglamani integrallovchi ko‘paytuvchi deb ataluvchi ushbu

$$\mu(x) = e^{\int_{x_0}^{x} p(s)ds} \tag{12}$$

funksiyaga ko‘paytiramiz. Quyidagilarga ega bo‘lamiz:

$$y'e^{\int_{x_0}^{x} p(s)ds} + p(x)e^{\int_{x_0}^{x} p(s)ds} \cdot y = q(x)e^{\int_{x_0}^{x} p(s)ds}; \quad \left(ye^{\int_{x_0}^{x} p(s)ds} \right)' = q(x)e^{\int_{x_0}^{x} p(s)ds};$$

$$ye^{\int_{x_0}^{x} p(s)ds} = c + \int_{x_0}^{x} q(t)e^{\int_{x_0}^{t} p(s)ds} dt; \quad y = ce^{-\int_{x_0}^{x} p(s)ds} + e^{-\int_{x_0}^{x} p(s)ds} \cdot \int_{x_o}^{x} q(t)e^{\int_{x_0}^{t} p(s)ds} dt.$$

Yana (6) formula bilan ifodalangan yechimni hosil qildik.

Ba'zi tenglamalar chiziqli bo'lmasada, ulardagi ***funksiya va argument rollarini almashtirish*** yordamida chiziqli tenglamalar hosil qilish mumkin. Ba'zi tenglamalarni esa ***noma'lum funksiyani almashtirish*** yordamida chiziqli tenglamaga olib kelinadi.

Ushbu

$$y' + p(x)y = q(x)y^m \quad (m = \text{const}) \tag{13}$$

Bernulli tenglamasi deb ataluvchi tenglama

$$u = \frac{1}{y^{m-1}}$$

almashtirish yordamida $u = u(x)$ ga nisbatan chiziqli tenglamaga keltiriladi ($m \neq 0$, $m \neq 1$ deb hisoblaymiz, chunki aks holda biz chiziqli tenglamaga ega bo'lardik).

Bernulli tenglamasini Eyler-Bernulli usuli bilan ham yechish mumkin. Bunda yechimni $y = u \cdot v$ ko'rinishda izlab, u ni $u' + p(x)u = 0$ shartdan tanlab (ya'ni $u = \exp\left(-\int p(x)dx\right)$ deb), v uchun o'zgaruvchilari ajraladigan tenglama hosil qilinadi.

Ushbu

$$y' + p(x)y = q(x)e^{\alpha y} \quad (\alpha = \text{const}) \tag{14}$$

tenglama ham $u = e^{\alpha y}$ almashtirish yordamida $u = u(x)$ ga nisbatan chiziqli tenglamaga keltiriladi.

Rikkati tenglamasi ushbu

$$y' = a(x)y^2 + b(x)y + c(x)$$

ko‘rinishga ega. U – (umumiy holda) nochiziqli tenglama. Agar Rikkati tenglamasining $y = y_1(x)$ yechimi ma’lum bo‘lsa, $y = y_1(x) + \dfrac{1}{u}$ almashtirish yordamida uni $u = u(x)$ ga nisbatan chiziqli differensial tenglamaga keltirish mumkin.

Misol 1. Ushbu

$$(x+1)y' = y + x^2, y(0) = 2$$

Koshi masalasi yechilsin.

Dastlab berilgan

$$(x+1)y' = y + x^2 \tag{15}$$

chiziqli differensial tenglamani Lagranj usuli bilan yechamiz. So‘ngra topilgan yechimlar orasidan berilgan boshlang‘ich shartni qanoatlantiradiganini ajratamiz.

Mos bir jinsli tenglama

$$(x+1)y' = y$$

ko‘rinishga ega. Bu tenglamaning umumiy yechimi (3) ga ko‘ra quyidagicha:

$$y = c \cdot (x+1).$$

Endi (15) tenglamaning yechimini

$$y = c(x) \cdot (x+1) \tag{16}$$

ko‘rinishda izlaymiz. (16) ni berilgan tenglama (15) ga qo‘yamiz va quyidagilarni bajaramiz:

$$(x+1)(c'(x) \cdot (x+1) + c(x)) = c(x) \cdot (x+1) + x^2; \;\; (x+1)^2 \cdot c'(x) = x^2;$$

$$c'(x) = \frac{x^2}{(x+1)^2}; \;\; c(x) = c + \int_0^x \frac{s^2}{(s+1)^2} ds, \;\; x > -1; \; c(x) = c + \int_0^x \left(1 - \frac{2}{s+1} + \frac{1}{(s+1)^2}\right) ds;$$

$$c(x) = c + x - 2\ln(x+1) - \frac{1}{x+1}, \;\; x > -1.$$

Oxirgi ifodani (16) ga qo‘yib berilgan tenglamaning $x > -1$ oraliqda umumiy yechimini hosil qilamiz:

$$y = \left(c + x - 2\ln(x+1) - \frac{1}{x+1}\right)(x+1)$$

yoki

$$y = c\cdot(x+1) + x(x+1) - 2(x+1)\ln(x+1) - 1,\ \ x > -1. \qquad (17)$$

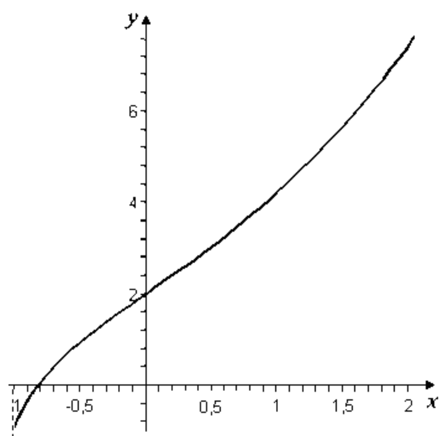
Endi berilgan Koshi masalasini yechish uchun (17) umumiy yechimdan $y(0)=2$ shartni qanoatlantiradiganini ajratamiz. (17) da $x=0$ va $y=y(0)=2$ deymiz. Natijada

$$2 = c - 1, \text{ ya'ni } c = 3$$

ekanligini topamiz. Buni (17) ga qo‘yib va ixchamlashlarni bajarib, izlangan yechimni hosil qilamiz

$$y = x^2 + 4x + 2 - 2(x+1)\ln(x+1)\ \ (x > -1)\,.$$

Bu yechimning grafigi (integral chiziq) 4.1- rasmda ko‘rsatilgan ($\lim\limits_{x\to -1+} y = -1$).



4.1- rasm. $y = x^2 + 4x + 2 - 2(x+1)\ln(x+1)$ funksiya grafigi.

Misol 2. Ushbu

$$xy' - 2y = 2x^4 \qquad (x > 0). \qquad (18)$$

tenglamaning umumiy yechimini toping.

Tenglamani integrallovchi ko‘paytuvchi metodi bilan yechamiz.

Berilgan tenglama $x > 0$ da

$$y' - \frac{2}{x}y = 2x^3 \qquad \left(\ p(x) = -\frac{2}{x}\ \right) \qquad (19)$$

tenglamaga teng kuchli. Bu tenglama uchun (12) formulaga ko‘ra integrallovchi ko‘paytuvchi

$$\mu(x)=e^{\int_1^x p(s)ds}=e^{-\int_1^x \frac{2}{s}ds}=e^{-2\ln s\big|_1^x}=\frac{1}{x^2}.$$

Endi (19) tenglamani $\mu(x)=\dfrac{1}{x^2}$ ga ko‘paytiramiz va quyidagilarni bajaramiz:

$$\frac{1}{x^2}y'-\frac{2}{x^3}y=2x,\ \left(\frac{1}{x^2}y\right)'=2x,\ \frac{1}{x^2}y=c_1+\int_1^x 2sds,$$

$$\frac{1}{x^2}y=c+x^2,\ \ y=cx^2+x^4,\ \ x>0.$$

Javob: berilgan tenglamaning umumiy yechimi $x>0$ oraliqda

$$y=cx^2+x^4$$

formula bilan beriladi. 👍

Misol 3. Ushbu

$$(x+y^2)dy=ydx$$

tenglamani yeching.

🗝 Bu tenglama $y=0$ yechimga ega. Tenglama $y>0$ (yoki $y<0$) bo‘lganda $x=x(y)$ funksiyaga nisbatan chiziqli:

$$\frac{dx}{dy}-\frac{1}{y}x=y\,.$$

Oxirgi tenglamani Eyler-Bernulli usuli bilan yechamiz. $x=u\cdot v$ $(u=u(y),v=v(y))$ deymiz. U holda

$$\frac{dx}{dy}=u\frac{dv}{dy}+v\frac{du}{dy}\quad \text{va}\quad u\frac{dv}{dy}+v\frac{du}{dy}-\frac{1}{y}u\cdot v=y$$

yoki

$$u\frac{dv}{dy}+v\left(\frac{du}{dy}-\frac{1}{y}u\right)=y$$

Endi $u=u(y)$ funksiyani

$$\frac{du}{dy}-\frac{1}{y}u=0$$

shartdan tanlaymiz. Bu tenglamaning $u=y$ yechimini olamiz. U holda $v=v(y)$ funksiya uchun

$$u\frac{dv}{dy}=y,\ \text{ ya'ni }\ y\frac{dv}{dy}=y,\ \frac{dv}{dy}=1,\ v=y+c\,.$$

hosil bo‘ladi.

Shunday qilib, berilgan tenglamaning $x=x(y)$ yechimlari $(x=u\cdot v)$

$$x=y\cdot(y+c)$$

formula bilan beriladi. Bundan tashqari $y(x)=0$ yechim ham bor.

Misol 4. Ushbu

$$(e^{y}+x)y'=1;\ y(0)=0$$

Koshi masalasini yeching.

Berilgan tenglama $y=y(x)$ noma’lum funksiyaning teskarisi $x=x(y)$ ga nisbatan chiziqli ($y'=\frac{dy}{dx}=1\Big/\frac{dx}{dy}$):

$$\frac{dx}{dy}=x+e^{y}\,.$$

Bu chiziqli differensial tenglamani integrallovchi ko‘paytuvchi usuli bilan yechamiz:

$$\frac{dx}{dy}-x=e^{y},\ \frac{dx}{dy}e^{-y}-xe^{-y}=1,\ \frac{d}{dy}\left(xe^{-y}\right)=1,$$

$$xe^{-y}=y+c\,,\ x=(y+c)e^{y}\,.$$

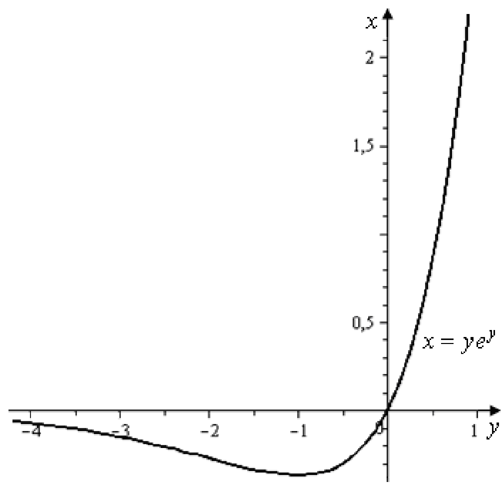
Oxirgi formula berilgan tenglamaning umumiy yechimini ifodalaydi. Boshlang‘ich shartga ko‘ra o‘zgarmas c ni aniqlaymiz:

$$y(0)=0\Rightarrow 0=(0+c)e^{0}\Rightarrow c=0\,.$$

Demak, yechim $x = ye^y$ tenglikdan topiladi. Izlangan $y = y(x)$ yechim $x(y) = ye^y$ funksiyaga teskari funksiyadan iborat. y noma'lumga nisbatan $x = ye^y$ tenglamani yechaylik.

$$\frac{d}{dy} ye^y = e^y + ye^y = (1+y)e^y$$

tenglikdan ravshanki, $y < -1$ bo'lganda $x = ye^y$ funksiya kamayuvchi, $y > -1$ bo'lganda esa – o'suvchi. Funksiya $y = -1$ da minimumga ega: $x_{\min} = x\big|_{y=-1} = e^{-1}$. $x = ye^y$ funksiyaning grafigi 4.2- rasmda tasvirlangan.



4.2- rasm. $x = ye^y$ funksiya grafigi.

Rasmdan ravshanki, $x = ye^y$ tenglama ikkita uzluksiz funksiyani aniqlaydi: $[-e^{-1};+\infty)$ oraliqda uzluksizini $y = W_0(x)$ bilan, $[-e^{-1};0)$ oraliqda uzluksizini esa $y = W_{-1}(x)$ bilan belgilaylik. Ular Lambert funksiyalari deb ataladi. Demak, ta'rifga ko'ra $W_0(x)e^{W_0(x)} = x,\ x \in [-e^{-1};+\infty)$, va $W_{-1}(x)e^{W_{-1}(x)} = x,\ x \in [-e^{-1};0)$. Tushunarliki, $W_0(x) \in C^1(-e^{-1};+\infty)$ va $W_{-1}(x) \in C^1(-e^{-1};0)$ ham bo'ladi. Bizni $y(0) = 0$ shartni qanoatlantiruvchi yechim qiziqtiradi. Bu yechim $y = W_0(x)$, $x \in (-e^{-1};+\infty)$. Uning grafigi 4.3- rasmda keltirilgan.

4.3- rasm. $y=W_0(x)$ funksiya grafigi. ☝

Misol 5. Ushbu

$$(x+1)(y'+y^2)=-y.$$

tenglamani yeching.

🗝 Berilgan tenglama $x>-1$ (yoki $x<-1$) intervalda ushbu

$$y'+\frac{y}{x+1}+y^2=0$$

Bernulli tenglamasiga teng kuchli. Ravshanki, $y=0-$ yechim. Boshqa yechimlarni topish uchun oxirgi tenglamani y^2 ga bo‘lib, ushbu

$$\frac{y'}{y^2}+\frac{1}{y}\cdot\frac{1}{x+1}+1=0$$

tenglamani hosil qilamiz. $u=\dfrac{1}{y}$ deb yangi noma’lum kiritamiz. U holda

$$u'-\frac{1}{x+1}u=1$$

chiziqli tenglama hosil bo‘ladi. Uni integrallovchi ko‘paytuvchi usuli bilan yechamiz. Bu tenglama uchun integrallovchi ko‘paytuvchi

$$\mu=e^{-\int\frac{dx}{x+1}}=\frac{1}{x+1}\ .$$

Tenglamani shu μ ga ko‘paytirib, quyidagilarni bajaramiz:

$$\frac{1}{x+1}u'-\frac{1}{(x+1)^2}u=\frac{1}{x+1},\ \left(\frac{1}{x+1}\cdot u\right)'=\frac{1}{x+1},$$

$$\frac{1}{x+1}u=\int\frac{dx}{x+1}+c,\ \ u=(x+1)(c+\ln|x+1|).$$

Endi dastlabki noma’lum $y=\frac{1}{u}$ ga qaytamiz:

$$y=\frac{1}{(x+1)(c+\ln|x+1|)}.$$

Javob: $y=0,\ \ y=\frac{1}{(x+1)(c+\ln|x+1|)}$ $(x>-1$ yoki $x<-1)$. ☝

Misol 6. Ushbu

$$x^2y'-x^2y^2+xy+1=0$$

tenglamani yeching.

🗝Berilgan tenglama $x>0$ (yoki $x<0$) intervalda

$$y'=\frac{2}{x^2}-\frac{2y}{x}-y^2$$

Rikkati tenglamasiga aylanadi. Oxirgi tenglama $y_1=\frac{1}{x}$ (xususiy) yechimga ega (tekshirib ko‘ring). Tenglamada $y=\frac{1}{x}+\frac{1}{u}$ almashtirishni bajarib, ushbu

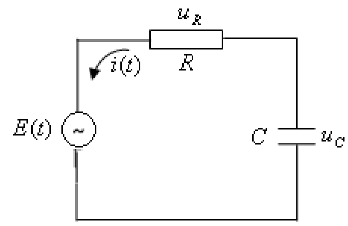
$$xu'+u+x=0$$

tenglamaga kelamiz. Bundan $u=\frac{c}{2x}-\frac{x}{2}$. Demak, berilgan tenglama yechimi

$$y=\frac{1}{x}+\frac{1}{\frac{c}{2x}-\frac{x}{2}}=\frac{c+x^2}{x(c-x^2)}\ .$$ ☝

Endi matematik modeli birinchi tartibli chiziqli differensial tenglamaga keltiriluvchi masalalar bilan tanishaylik.

Misol 7. Ushbu (4.4- rasm)



4.4- rasm. $C, R, E(t)$ elektr zanjiri.

elektr zanjir kondensatoridagi kuchlanish $u_C$ning vaqt bo'yicha o'zgarish qonunini aniqlang. Kondensator sig'imi C, qarshilik qiymati R va elektr yurituvchi kuch (lanish) $E(t)$ berilgan. Boshlang'ich shart $u_C(0) = u_0$.

🗝Ma'lumki, berilgan zanjir uchun R qarshilikdagi u_R kuchlanish, C kondensatordagi u_C kuchlanish va i tok kuchi orasida $u_R = iR,\ i = C\dfrac{du_C}{dt}$ munosabatlar mavjud (4.4- rasm). To'la zanjir uchun Om qonuniga ko'ra

$$u_R + u_C = E(t),\ \text{ ya'ni }\ Ri + u_C = E(t)\ \text{ yoki }\ RC\frac{du_C}{dt} + u_C = E(t).$$

Demak, $u_C = u_C(t)$ noma'lun funksiya birinchi tartibli chiziqli differensial tenglamani qanoatlantiradi. Uning uchun quyidagi boshlang'ich masala hosil bo'ldi:

$$\begin{cases} \dfrac{du_C}{dt} + \dfrac{1}{RC}u_C = E(t), \\ u_C(0) = u_0. \end{cases}$$

Bu yerdagi differensial tenglamani $e^{\frac{1}{RC}t}$ integrallovchi ko'paytuvchiga ko'paytirib, uni ushbu

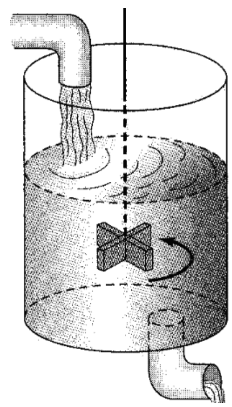
$$\left(e^{\frac{1}{RC}t}u_C\right)' = e^{\frac{1}{RC}t}E(t)$$

ko'rinishga keltiramiz. Oxirgi tenglikni 0dan t gacha intrgrallab, va boshlang'ich shartni hisobga olib, kuchlanishning ushbu

$$u_C = u_0 e^{-\frac{1}{RC}t} + \int_0^t e^{\frac{1}{RC}(s-t)}E(s)ds.$$

o‘zgarish qonunini topamiz. ☝

Misol 8. Idishda μ moddaning σ suyuqlikdagi v_0(m^3) hajmli ρ_0 (kg / m^3) konsentratsiyali eritmasi (aralashmasi) bor edi. Unga o‘sha μ moddaning o‘sha σ suyuqlikdagi ρ (kg / m^3) konsentratsiyali eritmasi (aralashmasi) r_1 (m^3 / s) tezlik bilan $t=0$ paytdan boshlab quyila boshladi. Eritmalar (aralashmalar) bir onda bir jinsligacha aralashib, r_2 (m^3 / s) tezlik bilan idishdan oqib chiqib ketadi. Idishning hajmini yetarlicha katta deb hisoblab, undagi eritma (aralashma)dagi μ moddaning miqdorini t (s) vaqt bo‘yicha o‘zgarish qonuniyatini aniqlang.



🗝Idishdagi μ moddaning t paytdagi miqdorini $m=m(t)$(kg) bilan belgilaylik. Vaqt boshida $m(0)=\rho_0 v_0$. Idishga t vaqt ichida $r_1 t$ hajmli eritma qo‘shilib, $r_2 t$ hajmli eritma esa chiqib ketadi. Demak, idishdagi eritmaning t paytdagi hajmi $v=v(t)=v_0+(r_1-r_2)t$ ga teng bo‘ladi. Δt ni yetarlicha kichik deb hisoblab, modda miqdorining $m(t+\Delta t)-m(t)$ o‘zgarishini hisoblaymiz. Δt vaqt ichida $r_1\Delta t$ hajmli eritma quyiladi, uning bilan $\rho r_1\Delta t$ miqdorli μ modda qo‘shiladi. Lekin shu vaqt ichida μ moddaning $\approx \dfrac{m(t)}{v(t)} r_2\Delta t$ miqdori chiqib ketadi. Endi tushunarliki,

$$m(t+\Delta t)-m(t)=\rho r_1\Delta t-\frac{m(t)}{v(t)}r_2\Delta t+o(\Delta t),\ \Delta t\to 0$$

yoki

$$m(t+\Delta t)-m(t)=\rho r_1\Delta t-\frac{m(t)}{v_0+(r_1-r_2)t}r_2\Delta t+o(\Delta t),\ \Delta t\to 0.$$

Bu tenglikning har ikkala tomonini Δt ga bo‘lamiz:

$$\frac{m(t+\Delta t)-m(t)}{\Delta t}=\rho r_1-\frac{m(t)}{v_0+(r_1-r_2)t}r_2+\frac{o(\Delta t)}{\Delta t},\ \Delta t\to 0.$$

Oxirgi tenglikda $\Delta t\to 0$ deb limitga o‘tamiz va ushbu

$$\frac{dm(t)}{dt}+\frac{r_2}{v_0+(r_1-r_2)t}m(t)=\rho r_1$$

chiziqli differensial tenglamani hosil qilamiz. Shunday qilib, $m=m(t)$ noma’lum funksiyani topish uchun ushbu

$$\begin{cases}\dfrac{dm}{dt}+\dfrac{r_2}{v_0+(r_1-r_2)t}m=\rho r_1\\ m(0)=\rho_0 v_0\end{cases}$$

Koshi masalasi hosil bo‘ldi. Uni yechib, μ modda miqdorining

$$m=\big(v_0+(r_1-r_2)t\big)\rho+(\rho_0-\rho)v_0^{\frac{r_1}{r_1-r_2}}\big(v_0+(r_1-r_2)t\big)^{-\frac{r_1}{r_1-r_2}}$$

o‘zgarish qonunini topamiz. ☝

Masalalar

Tenglamalarni yeching (**1**–**10**):

1. $y'=y+2xe^x$. **2.** $xy'-y=x+1$. **3.** $x^3y'+1+x=2x^2y$.

4. $(x+y^2\sin y)y'=y$. **5.** $x^2y'+3xy=\dfrac{\ln x}{x}$. **6.** $(y\operatorname{tg}x-e^x\sin x)dx+dy=0$.

7. $(4x+3e^y)y'=1$. **8.** $(e^x+1)dy=(y+x)dx$. **9.** $2xy'+4y=x^2y^2,\ x>0$.

10. $y'\cos x+y\sin x+3y^2\cos x=0$.

Boshlang‘ich masalalarni yeching (**11**–**14**):

11. $xy'+3y=x^3,\ y(1)=-1$. **12.** $xy'+2xy^2=2y,\ y(-1)=-1$.

13. $(x^4\ln y-xy^2)y'+y^3=0,\ y(-1)=1$. **14.** $xyy'-y^2+x^3=0,\ y(1)=1$.

Rikkati tenglamasining xususiy yechimni tanlab, umumiy yechimini quring (**15** – **17**):

15. $x^2y'+x^2y^2+3xy=3$. **16.** $y'+e^{-x}y^2+2y=10e^x$.

17. $y'=y^2+y\sin x+\cos x-2\sin^2 x$.

Mustaqil ish № 4 topshiriqlari:

I. Berilgan tenglama yoki berilgan Koshi masalasini yeching. Bunda berilgan yoki hosil bo‘luvchi chiziqli differensial tenglamani uch: Lagranj, Eyler-Bernulli va integrallovchi ko‘paytuvchi usullari bilan yeching.

1. $y'+y\mathrm{tg}x=\dfrac{x}{\cos x},\ y(\dfrac{\pi}{4})=\sqrt{2}$.

2. $2yy'-x=\dfrac{x}{x^2-1}y^2\ \ (u=y^2)$.

3. $(\sin^2 y+x\mathrm{ctg}y)y'=1\ \ (x=x(y))$.

4. $(xy+e^x)dx-xdy=0$.

5. $(2e^y-x)y'=2$.

6. $e^{-x}y'-e^{-x}=e^y,\quad y(0)=1$.

7. $(x+2y^3)y'=y\ ,\ (x=x(y))$.

8. $y'(x^3\sin y-x)+2y=0\ \ (x=x(y))$.

9. $(xy'-1)\ln x=2y$.

10. $(3x-y^2)y'=y\quad (x=x(y))$.

11. $.\,y'\sin x-y\cos x=\dfrac{\sin^2 x}{x^2},y(\dfrac{\pi}{2})=0$.

12. $y'+y\mathrm{tg}x=\dfrac{1}{\sin x},\ \ y(\dfrac{\pi}{4})=\sqrt{2}$.

13. $2(x-y^2)dy-ydx=0$.

14. $(x+\cos y+\sin 2y)y'=1$.

15. $(x\mathrm{tg}y+1)y'=1/\cos y$.

16. $yy'+y^2\mathrm{ctg}x=\cos x$.

17. $xy'-y=e^x\cdot x,\quad y(1)=0.$

18. $y'-y=e^x+1,\quad y(0)=e.$

19. $y'-y=e^x+e^{-x},\quad y(0)=1$.

20. $dy+(xy+x^2)dx=0$.

21. $y'+\cos x\cdot y=\dfrac{1}{2}\sin 2x$.

22. $\dfrac{x}{\sin^2 y}y'+\mathrm{ctg}y-x^2=0\ \ (u=\mathrm{ctg}y)$

23. $e^{-x}y'-e^{-x}=2e^y,\quad y(0)=1.$

24. $xy'+y\ln y+xy=0\quad (u=\ln y)$.

25. $(y+\sin x)dx+xdy=0$.

26. $(\sqrt{x}+1+y)dx+xdy=0$.

27. $xy'-y=\sqrt{x},\quad y(1)=1$.

28. $y'-y\cdot\mathrm{ctg}x=\sqrt{x+1}\sin x$.

29. $xe^y y' + e^y = x^2$ $(u = e^y)$.

30. $y' + \dfrac{2y}{x} + 1 = 0, \quad y(1) = 1.$

31. $y' - 2xy = x^2 - x^4, \quad y(0) = 1.$

32. $y' + 2y = y^2 e^{2x}$

33. $y' \cdot (x^3 \sin y - x) + 2y = 0$ $(x = x(y))$.

34. $y' + 2y = y^2 e^x, \quad y(0) = 1/2.$

35. $xy' - 4y = x^2 \sqrt{y}, \quad y(1) = 1.$

36. $xy' + y = y^2 x \ln x, \quad y(1) = 1.$

37. $(x^2 + y^2 + 1)y' + xy = 0$ $(x = x(y),\ u = u(y) = x^2(y))$.

38. $xdx + (2 + 2y - x^2)dy = 0$ $(u = 2 + 2y - x^2)$.

39. $xy' = 2x^2 \sqrt{y} + 4y$ ($\sqrt{y} = u$ almashtirish bajaring).

40. $x^2 y' + xy + x^2 y^2 = 9$ (xususiy yechim tanlang, $y = y_1 = a / x$).

II. Matematik modeli birinchi tartibli chiziqli differensial tenglamalar bilan ifodalanuvchi masalalardan namunalar keltiring.

5. TO‘LA DIFFERENSIALLI VA UNGA KELTIRILUVCHI TENGLAMALAR

Maqsad – to‘la differensialli tenglamani yechishni va integrallovchi ko‘paytuvchini topish usullarini o‘rganish.

Yordamchi ma’lumotlar:

To‘la differensialli tenglamalar. Agar

$$M(x,y)dx+N(x,y)dy=0 \qquad (1)$$

tenglamaning chap tomonidagi differensial ifoda potensial deb ataluvchi biror $u=u(x,y)$ funksiyaning to‘la differensialidan iborat, ya’ni

$$du(x,y)=M(x,y)dx+N(x,y)dy \qquad (2)$$

bo‘lsa, u holda (1) tenglama ***to‘la (to‘liq) differensialli tenglama*** deyiladi. Bu yerdagi M va N funksiyalar tekislikdagi biror D sohada aniqlangan. Biz ularni hamda $\dfrac{\partial M}{\partial y}$ va $\dfrac{\partial N}{\partial x}$ xususiy hosilalarni uzluksiz deb hisoblaymiz.

Aytaylik, (1) tenglama to‘la differensialli bo‘lsin. U holda uni

$$du(x,y)=0 \qquad (3)$$

ko‘rinishda yozishimiz mumkin. Bu tenglik sohada (ochiq va bog‘lanishli to‘plamda) o‘rinli bo‘lgani uchun undan

$$u(x,y)=c \quad (c=\text{const}) \qquad (4)$$

munosabat kelib chiqadi. Demak, (1) tenglamaning $y=y(x)$ yoki $x=x(y)$ yechimi (4) tenglamani qanoatlantiradi. Aksincha, (4) tenglik aniqlovchi $y=y(x)$ yoki $x=x(y)$ uzluksiz differensiallanuvchi funksiya (1) ni qanoatlantiradi. Shunday qilib, (1) tenglamaning yechimlari (4) munosabat bilan oshkormas ko‘rinishda beriladi. Demak, to‘la differensialli tenglama (1) ni yechish mos potensial u ni topishga keltiriladi.

(2) tenglikka ko‘ra

$$du\equiv\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=Mdx+Ndy.$$

Bundan differensialning yagonalik xossasiga ko‘ra

$$\frac{\partial u}{\partial x} = M, \qquad \frac{\partial u}{\partial y} = N \, . \tag{5}$$

$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}$ va $\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$ xususiy hosilalar uzluksiz bo‘lgani uchun analizdan ma’lum teoremaga ko‘ra $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ va (5) tengliklardan ushbu

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{6}$$

munosabatni topamiz. Bu (6) tenglik (1) ning to‘la differensialli tenglama bo‘lishi uchun zaruriy shartni ifodalaydi.

Eslatma. Agar bir bog‘lamli D sohada $M, N, M'_y, N'_x \in C(D)$ va (6) shartlar o‘rinli bo‘lsa, u potensialni egri chiziqli integral yordamida topsa bo‘ladi. Bu tasdiq matematik analiz kursida isbotlanadi.

Ba’zan u potensialni topishda (5) sistema tenglamalarini ketma-ket yechish qulayroq bo‘ladi.

Misol 1. Ushbu

$$2xydx + (x^2 + y^2)dy = 0 \tag{8}$$

tenglamani yeching.

🗝 Qaralayotgan misolda

$$M(x, y) = 2xy, \quad N(x, y) = x^2 + y^2, \;\; D = \mathbb{R}^2, \;\; \frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

Demak, berilgan tenglama $D = \mathbb{R}^2$ sohada to‘la differensialli. u potensial uchun (5) shartlar qanoatlanishi kerak:

$$\frac{\partial u}{\partial x} = 2xy, \tag{9}$$

$$\frac{\partial u}{\partial y} = x^2 + y^2 \, . \tag{10}$$

(9) tenglikni (x bo‘yicha) 0 dan x gacha integrallaymiz (y tayinlangan):

$$u = \int_0^x 2syds + \varphi(y) = x^2 y + \varphi(y); \tag{11}$$

bunda $\varphi(y)$ hozircha noma'lum funksiya . (11) ni (10) ga qo'yamiz:

$$x^2 + \varphi'(y) = x^2 + y^2. \tag{12}$$

Bu tenglikni 0 dan y gacha integrallaymiz (x tayinlangan):

$$\varphi(y) = \frac{y^3}{3}. \tag{13}$$

(integrallash doimiysini nolga teng deb oldik; bizga biror potensial kerak!). Endi (13) ni (11) ga qo'yib u potensialni topamiz:

$$u = x^2 y + \frac{y^3}{3}.$$

Nihoyat, (4) ga ko'ra (8) tenglamaning yechimlarini yozamiz:

$$x^2 y + \frac{y^3}{3} = \frac{c}{3} \quad \text{yoki} \quad 3x^2 y + y^3 = c.$$

Izoh. u potensialni differensial xossalaridan foydalanib to'g'ridan-to'g'ri topish ham mumkin. Bunda quyidagicha shakl almashtirishlarni bajarish kerak:

$$2xydx + (x^2 + y^2)dy = y \cdot (2xdx) + x^2 dy + y^2 dy =$$

$$= yd(x^2) + x^2 dy + d(\frac{y^3}{3}) = d(x^2 y) + d(\frac{y^3}{3}) =$$

$$= d\left(x^2 y + \frac{y^3}{3}\right) = du.$$

Bunda biz differensialning quyidagi xossalaridan foydalandik:

$$f'(x)dx = df(x), \quad \text{yoki} \quad f(x)dx = d(\int f(x)dx);$$

$$wdv + vdw = d(wv), \; dw + dv = d(w+v).$$

Integrallovchi ko'paytuvchi. Aytaylik, (1) tenglama uchun bir bog'lamli D sohada (6) shart bajarilmasin. U holda bu tenglama to'la differensialli bo'lmaydi. Lekin biz uni D sohada nolga aylanmaydigan biror $\mu = \mu(x, y)$ silliq funksiyaga ko'paytirib, to'la differensialli tenglamani hosil qilishga harakat qilishimiz mumkin. Ushbu

$$\mu M dx + \mu N dy = 0 \qquad (14)$$

tenglamaning to‘la differensialli bo‘lishi uchun D sohada

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \qquad (15)$$

shart bajarilishi kerak. Bu yerdagi μ funksiya (1) tenglama uchun integrallovchi ko‘paytuvchi deb ataladi. Oxirgi shartni

$$N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y} = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu \qquad (16)$$

ko‘rinishga keltiraylik. Bu tenglik μ funksiyaga nisbatan xususiy hosilali tenglamani ifodalaydi. (16) tenglamaning yechimlarini ba`zi hollarda osongina topish mumkin.

Faraz qilaylik, μ ni biror ma’lum $\omega(x, y)$ funksiyaning funksiyasi sifatida topish mumkin bo‘lsin, ya’ni $\mu = \mu(\omega)\,, \omega = \omega(x, y)$, bo‘lsin. Bu holda (16) tenglama

$$\frac{d\mu}{d\omega} = \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N\dfrac{\partial \omega}{\partial x} - M\dfrac{\partial \omega}{\partial y}} \cdot \mu$$

ko‘rinishga keladi. Bu tenglama $\mu = \mu(\omega)$ ko‘rinishdagi yechimga ega bo‘lishi uchun

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N\dfrac{\partial \omega}{\partial x} - M\dfrac{\partial \omega}{\partial y}} = \psi \qquad (17)$$

ifoda ωning funksiyasi kabi ifodalanishi, ya’ni $\psi = \psi(\omega)$ bo‘lishi kerak. Bu holda integrallovchi ko‘paytuvchi uchun tenglama

$$\frac{d\mu}{d\omega} = \psi(\omega)\mu$$

ko‘rinishni olib, u

$$\mu = e^{\int \psi(\omega)d\omega} \qquad (18)$$

yechimga ega.

Shunday qilib, agar (17) ifoda faqat ω ning funksiyasi sifatida ifodalansa, u holda (1) tenglama uchun (18) ko‘rinishdagi integrallovchi ko‘paytuvchi mavjud.

Masalan, $\mu=\mu(x)$ $(\mu=\mu(y))$ ko‘rinishdagi integrallovchi ko‘paytuvchi mavjud bo‘lishi uchun

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\psi \qquad \left(\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{-M}=\psi\right)$$

ifoda x o‘zgaruvchiga (y o‘zgaruvchiga) bog‘liq bo‘lmasligi, ya’ni u faqat x ga (y ga) bog‘liq bo‘lishi kerak. Bu holda

$$\mu(x)=e^{\int\psi(x)dx} \quad (\mu(y)=e^{\int\psi(y)dy})$$

ko‘rinishdagi integrallovchi ko‘paytuvchi mavjud.

Ba’zi hollarda berilgan differensial tenglama uchun integral ko‘paytuvchini tenglamani qismlarga ajratib, har bir qism uchun integrallovchi ko‘paytuvchini topish yordamida qurish mumkin. Aytaylik, berilgan tenglamani ushbu

$$(M_1dx+N_1dy)+(M_2dx+N_2dy)=0 \tag{19}$$

har bir qismining integrallovchi ko‘paytuvchisi osongina topiladigan ko‘rinishda yozish mumkin bo‘lsin:

$$\mu_1\cdot(M_1dx+N_1dy)=du_1,$$
$$\mu_2\cdot(M_2dx+N_2dy)=du_2.$$

Ravshanki, μ_1 bilan birgalikda $\mu_1\varphi(u_1)$ ham mos tenglamaning integrallovchi ko‘paytuvchisidir, chunki

$$\mu_1\varphi(u_1)\cdot(M_1dx+N_1dy)=\varphi(u_1)du_1=d\left(\varphi(u_1)du_1\right).$$

Buni hisobga olgan holda shunday $\varphi(u_1)$ va $\psi(u_2)$ funksiyalarni topaylikki, ular uchun

$$\mu_1\varphi(u_1)=\mu_2\psi(u_2)$$

munosabat o‘rinli bo‘lsin. U holda berilgan (19) tenglamaning har ikkala tomonini

$$\mu = \mu_1 \varphi(u_1) = \mu_2 \psi(u_2)$$

integrallovchi ko‘paytuvchiga ko‘paytirib,

$$\varphi(u_1)du_1 + \psi(u_2)du_2 = 0$$

tenglamani hosil qilamiz. Bu to‘la differensialli tenglama osongina yechiladi va berilgan (19) tenglamaning yechimlari topiladi:

$$\int \varphi(u_1)du_1 + \int \psi(u_2)du_2 = c\,.$$

Ba’zi hollarda differensial ifodani to‘la differensialga keltirish uchun quyidagi jadvaldan foydalanish mumkin:

Differensial ifoda	Integrallovchi ko‘paytuvchi $\mu = \mu(x,y)$	To‘la differensial
$ydx - xdy$	$-\frac{1}{x^2}$	$-\frac{ydx - xdy}{x^2} = d\left(\frac{y}{x}\right)$
$ydx - xdy$	$\frac{1}{y^2}$	$\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$
$ydx - xdy$	$-\frac{1}{xy}$	$-\frac{ydx - xdy}{xy} = d\left(\ln\lvert\frac{y}{x}\rvert\right)$
$ydx - xdy$	$-\frac{1}{x^2+y^2}$	$-\frac{ydx - xdy}{x^2+y^2} = d\left(\mathrm{arctg}\left(\frac{y}{x}\right)\right)$
$ydx + xdy$	$\frac{1}{xy}$	$\frac{ydx + xdy}{xy} = d\left(\ln\lvert xy\rvert\right)$
$ydx + xdy$	$\frac{1}{(xy)^a}$, $a>1$	$\frac{ydx + xdy}{(xy)^a} = d\left(\frac{-1}{(a-1)(xy)^{a-1}}\right)$
$ydx + xdy$	$\frac{1}{x^2+y^2}$	$\frac{ydx + xdy}{x^2+y^2} = d\left(\frac{1}{2}\ln(x^2+y^2)\right)$
$ydx + xdy$	$\frac{1}{(x^2+y^2)^a}$, $a>1$	$\frac{ydx + xdy}{(x^2+y^2)^a} = d\left(\frac{-1}{2(a-1)(x^2+y^2)^{a-1}}\right)$
$\alpha ydx + \beta xdy$ $(\alpha,\beta - \mathrm{const})$	$x^{\alpha-1}y^{\beta-1}$	$x^{\alpha-1}y^{\beta-1}(\alpha ydx + \beta xdy) = d(x^{\alpha}y^{\beta})$

Misol 2. Ushbu

$$(x^3+y^2)dx+ydy=0 \tag{20}$$

dfferensial tenglamani yeching.

🗝 Berilgan tenglamada $M=x^3+y^2,\ N=y,\ D=\mathbb{R}^2$. Demak,

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=2y.$$

Ushbu

$$\psi=\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=2$$

ifoda y ga bog‘liq emas. Demak, berilgan tenglama $\mu=\mu(x)$ ko‘rinishdagi integrallovchi ko‘paytuvchiga ega. Bu $\mu=\mu(x)$ funksiya

$$\frac{d\mu}{dx}=\psi(x)\mu,\ \text{ ya'ni }\ \frac{d\mu}{dx}=2\mu$$

tenglamani qanoatlantiradi. Oxirgi tenglamadan

$$\mu=e^{2x}$$

integrallovchi ko‘paytuvchini topamiz. Endi berilgan tenglamani $\mu=e^{2x}$ ga ko‘paytirib, to‘la differensialli tenglama hosil qilamiz:

$$e^{2x}(x^3+y^2)dx+e^{2x}ydy=0\,.$$

Bu tenglamaning yechimi

$$u(x,y)=c$$

ko‘rinishga ega. Bunda u potensial uchun

$$\frac{\partial u}{\partial x}=e^{2x}(x^3+y^2), \tag{21}$$

$$\frac{\partial u}{\partial y}=e^{2x}y \tag{22}$$

bo‘lishi kerak. (22) tenglikni y bo‘yicha integrallab topamiz:

$$u=\frac{1}{2}e^{2x}y^2+\varphi(x), \tag{23}$$

bunda $\varphi(x)$ hozircha noma'lum funksiya . (23) ni (21) ga qo'yib, $\varphi(x)$ uchun tenglama hosil qilamiz:

$$e^{2x}y^2+\varphi'(x)=e^{2x}(x^3+y^2) \text{ yoki } \varphi'(x)=e^{2x}x^3.$$

Bu munosabatdan bo'laklab integrallash yordamida noma'lum funksiya $\varphi(x)$ ni topamiz:

$$\varphi(x)=\int e^{2x}x^3dx=\frac{1}{2}\int x^3d(e^{2x})=\frac{1}{2}e^{2x}x^3-\frac{1}{2}\int e^{2x}d(x^3)=\frac{1}{2}e^{2x}x^3-\frac{3}{2}\int x^2e^{2x}dx=$$

$$=\cdots=\frac{1}{2}e^{2x}x^3-\frac{3}{4}e^{2x}x^2+\frac{3}{4}e^{2x}x-\frac{3}{8}e^{2x}.$$

Buni (23) ga qo'yib, u potensialni aniqlaymiz

$$u=\frac{1}{2}e^{2x}y^2+\frac{1}{2}e^{2x}x^3-\frac{3}{4}e^{2x}x^2+\frac{3}{4}e^{2x}x-\frac{3}{8}e^{2x}.$$

Demak, berilgan (20) tenglamaning yechimlari

$$e^{2x}(4y^2+4x^3-6x^2+6x-3)=c$$

tenglik bilan oshkormas ko'rinishda ifodalanadi. ♙

Misol 3. Differensial tenglamani yeching

$$(x-xy)dx+(y+x^2)dy=0. \tag{24}$$

Berilgan tenglamada

$$M=x-xy,\ \ N=y+x^2,\ \ -x=\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}=2x$$

va u to'la differensialli emas. Integrallovchi ko'paytuvchi μ uchun quyidagi shartni topamiz:

$$\big((x-xy)\mu\big)'_y=\big((y+x^2)\mu\big)'_x\ ,\ x(1-y)\mu'_y=(y+x^2)\mu'_x+3x\mu\,.$$

Oxirgi tenglikdan faqat y ga bog'liq $\mu=\mu(y)$ integrallovchi ko'paytuvchi mavjudligini ko'ramiz:

$$(1-y)\frac{d\mu}{dy}=3\mu\ \Rightarrow\ \mu=\frac{1}{(1-y)^3}\quad (y\neq 1).$$

Berilgan tenglamani $\mu = \dfrac{1}{(1-y)^3}$ ga ko‘paytirib, to‘la differensialli tenglamani hosil qilamiz:

$$\frac{x}{(1-y)^2}dx + \frac{y+x^2}{(1-y)^3}dy = 0.$$

Bu tenglamaning $u = u(x,y)$ potensiali uchun

$$\frac{\partial u}{\partial x} = \frac{x}{(1-y)^2}, \quad \frac{\partial u}{\partial y} = \frac{y+x^2}{(1-y)^3}$$

shartlardan u ni topamiz:

$$u = \frac{x^2+1}{2(y-1)^2} + \frac{1}{y-1}.$$

Demak, qaralayotgan (24) tenglamaning yechimi

$$u = c \text{ , ya’ni } \frac{x^2+1}{2(y-1)^2} + \frac{1}{y-1} = c.$$

munosabat bilan oshkormas ko‘rinishda beriladi. Oxirgi tenglikda quyidagi shakl almashtirishlarni bajaramiz:

$$x^2+1+2y-2 = 2c(y-1)^2, \quad x^2+y^2 = 2c(y-1)^2 + y^2 - 2y + 1,$$

$$x^2 + y^2 = (2c+1)(y-1)^2.$$

Bu yerdagi $2c+1$ni yangi o‘zgarmas $1/c$ bilan belgilab, yechimni $c(x^2+y^2) = (y-1)^2$ tenglama bilan oshkormas ko‘rinishda ifodalaymiz. Oxirgi tenglik yo‘qolgan $y=1$ yechimni ham o‘z ichiga olgan ($c=0$ da).

Javob: $(y-1)^2 = c(x^2+y^2)$. 👍

Misol 4. Ushbu

$$(2x^2y^2 + y)dx + (x^3y - x)dy = 0$$

tenglamani yeching

🗝 Berilgan tenglama to‘la differensialli emas. Uning integrallovchi ko‘paytuvchisi μ uchun tenglama

$$x(x^2y-1)\mu'_x - y(2x^2y+1)\mu'_y = (x^2y+2)\mu.$$

Ravshanki, bu tenglama $\mu(x)$ yoki $\mu(y)$ ko‘rinishdagi yechimga ega emas. Uning $\mu=\mu(\omega),\ \omega=\omega(x,y)$, ko‘rinishdagi yechimini izlab ko‘ramiz ($\mu=\mu(\omega)$ va $\omega=\omega(x,y)$ hozircha noma’lum funksiyalar). $\mu=\mu(\omega)$ uchun

$$\left(x(x^2y-1)\frac{\partial\omega}{\partial x}-y(2x^2y+1)\frac{\partial\omega}{\partial y}\right)\frac{d\mu}{d\omega}=(x^2y+2)\mu$$

tenglama hosil bo‘ladi. Chap tomondagi $\left(x(x^2y-1)\frac{\partial\omega}{\partial x}-y(2x^2y+1)\frac{\partial\omega}{\partial y}\right)$ ifodadan (x^2y+2) qavsdan tashqariga chiqishi va o‘ng tomondagi (x^2y+2) bilan qisqarishi kerak. Buning uchun $\omega=xy$ deymiz. Natijada

$$-\omega\frac{d\mu}{d\omega}=\mu$$

tenglamaga kelamiz. Oxirgi tenglama $\mu=\frac{1}{\omega}$ yechimga ega. Demak, berilgan differensial tenglama $\mu=\frac{1}{\omega}=\frac{1}{xy}$ integrallovchi ko‘paytuvchiga ega. Berilgan tenglamani $\mu=\frac{1}{xy}$ ga ($x\neq 0, y\neq 0$) ko‘paytirib, quyidagi to‘la differensialli tenglamaga kelamiz:

$$\left(2xy+\frac{1}{x}\right)dx+\left(x^2-\frac{1}{y}\right)dy=0.$$

Bu tenglamani, differensiallash qoidalaridan foydalanib, quyidagicha yechamiz:

$$2xydx+\frac{1}{x}dx+x^2dy-\frac{1}{y}dy=0,\ \ yd(x^2)+d\ln|x|+x^2dy-d\ln|y|=0,$$

$$d(x^2y)+d\left(\ln|x|-\ln|y|\right)=0,\ \ x^2y+\ln\frac{|x|}{|y|}=c.$$

Ravshanki, berilgan tenglamaning $y=0$ va $x=0$ yechimlari ham bor. ✍

Misol 5. Tenglamani yeching

$$(x^3-xy^2-y)dx+(x^2y-y^3+x)dy=0. \tag{26}$$

🗝 Berilgan tenglamada hadlarni quyidagicha qismlarga ajratamiz (guruhlaymiz):

$$\left[x(x^2-y^2)dx+y(x^2-y^2)dy\right]+\left[xdy-ydx\right]=0 \tag{27}$$

Bu yerdagi birinchi qismning integrallovchi ko‘paytuvchisi $\mu_1=\dfrac{2}{x^2-y^2}$:

$$\frac{2}{x^2-y^2}\left[x(x^2-y^2)\,dx+y(x^2-y^2)\,dy\right]=2xdx+2ydy=du_1,\quad u_1=x^2+y^2. \tag{28}$$

Ikkinchi qismning integrallovchi ko‘paytuvchisi esa $\mu_2=\dfrac{1}{x^2}$ (yuqoridagi jadvalga qarang):

$$\frac{1}{x^2}\cdot\left[xdy-ydx\right]=\frac{xdy-ydx}{x^2}=du_2,\quad u_2=\frac{y}{x}. \tag{29}$$

Endi $\varphi(u_1)=\varphi(x^2+y^2)$ va $\psi(u_2)=\psi(\frac{y}{x})$ funksiyalarni

$$\mu_1\cdot\varphi(u_1)=\mu_2\cdot\psi(u_2),$$

ya’ni

$$\frac{2}{x^2-y^2}\varphi(x^2+y^2)=\frac{1}{x^2}\psi(\frac{y}{x})$$

munosabatdan topamiz. Oxirgi tenglikni

$$2\varphi(x^2+y^2)=\left(1-(\frac{y}{x})^2\right)\psi(\frac{y}{x}) \tag{30}$$

Ko‘rinishda yozib olamiz. (30) dan ravshanki,

$$\varphi(x^2+y^2)=\frac{1}{2},\qquad \psi(\frac{y}{x})=\frac{1}{1-(\frac{y}{x})^2}$$

deyish mumkin. Shunday qilib, biz berilgan (26) tenglamaning

$$\mu=\mu_1\varphi(u_1)=\mu_2\psi(u_2)=\frac{1}{x^2-y^2} \tag{31}$$

integrallovchi ko‘paytuvchisini topdik. (26)ni topilgan (31) integrallovchi ko‘paytuvchiga ko‘paytirib, quyidagi to‘la differensialli tenglamaga kelamiz:

$$\left(x-\frac{y}{x^2-y^2}\right)dx+\left(y+\frac{x}{x^2-y^2}\right)dy=0.$$

Bu tenglamada quyidagicha shakl almashtirishlarni bajaramiz:

$$xdx-\frac{ydx}{x^2-y^2}+ydy+\frac{xdy}{x^2-y^2}=0, \quad d\left(\frac{x^2}{2}\right)+d\left(\frac{y^2}{2}\right)+\frac{xdy-ydx}{x^2-y^2}=0,$$

$$d\left(\frac{x^2+y^2}{2}\right)+\frac{x^2}{x^2-y^2}\cdot\frac{xdy-ydx}{x^2}=0, \quad d\left(\frac{x^2+y^2}{2}\right)+\frac{1}{1-(\frac{y}{x})^2}d\left(\frac{y}{x}\right)=0.$$

Oxirgi tenglamani quyidagicha yozish mumkin:

$$d\left(\frac{x^2+y^2}{2}+\frac{1}{2}\ln\left|1-\frac{y}{x}\right|-\frac{1}{2}\ln\left|1+\frac{y}{x}\right|\right)=0.$$

Demak, berilgan (26) tenglamaning yechimi ushbu

$$x^2+y^2+\ln\left|\frac{x-y}{x+y}\right|=C$$

tenglama bilan oshkormas ko‘rinishda beriladi. 👍

Misol 6. Ushbu

$$3ydx+x(\ln x+\ln y+1)dy=0 \quad (x>0, y>0)$$

differensial tenglamani yeching.

🗝 Bu tenglama to‘la differensialli emas. $\mu=\mu(\omega)$, $\omega=\omega(x,y)$, ko‘rinishdagi integrallovchi ko‘paytuvchini topishga harakat qilamiz. Integrallovchi ko‘paytuvchini ushbu

$$(3y\mu)'_y=(x(\ln x+\ln y+1)\mu)'_x$$

tenglamadan topamiz. Zarur hisoblashlarni bajarib, oxirgi tenglamani quyidagi ko‘rinishga keltiramiz:

$$(3y\omega'_y-x(\ln x+\ln y+1)\omega'_x)\frac{d\mu}{d\omega}=(\ln x+\ln y-1)\mu\,.$$

Oxirgi tenglamaning chap tomonidagi $\frac{d\mu}{d\omega}$ oldidagi koeffitsiyentni o‘ng tomondagi qavsga tenglashtirish maqsadida $\omega=\omega(x,y)$ funksiya uchun

$$\omega'_x=-\frac{1}{x} \quad \text{va} \quad 3y\omega'_y+2=0$$

shartlarni qo‘yamiz va $\mu=\mu(\omega)$ funksiya uchun $\mu'=\mu$, ya’ni $\mu=e^{\omega}$ ekanligini topamiz. Endi zarur integrallashlarni va ixchamlashtirishlarni bajarib, integrallovchi ko‘paytuvchini aniqlaymiz:

$$\omega'_x=-\frac{1}{x}\Rightarrow\omega=-\ln x+\psi(y),\ \omega'_y=\psi'(y);$$

$$3y\omega'_y+2=0\Rightarrow\omega'_y=-\frac{2}{3y}=\psi'(y)\Rightarrow\psi(y)=-\frac{2}{3}\ln y;$$

$$\omega=-\ln x+\psi(y)=-\ln x-\frac{2}{3}\ln y=\ln\frac{1}{xy^{2/3}};\ \mu=e^{\omega}=\exp\left(\ln\frac{1}{xy^{2/3}}\right)=\frac{1}{xy^{2/3}}.$$

Endi berilgan differensial tenglamani topilgan $\mu=\dfrac{1}{xy^{2/3}}$ integrallovchi ko‘paytuvchiga ko‘paytirib, ushbu

$$\frac{3}{x}y^{1/3}dx+\frac{1}{y^{2/3}}(\ln x+\ln y+1)dy=0$$

to‘la differensialli tenglamani hosil qilamiz. Bu tenglamaning $u=u(x,y)$ potensiali uchun

$$u'_x=\frac{3}{x}y^{1/3},\ u'_y=\frac{1}{y^{2/3}}(\ln x+\ln y+1)$$

bo‘lishi kerak. Oxirgi ikki shartdan $u=u(x,y)$ funksiyani odatdagicha topamiz:

$$u=3y^{1/3}\left(\ln x+\ln y-2\right).$$

Demak, berilgan tenglamaning yechimi

$$u(x,y)=c/3,\ \text{ya'ni}\ \ y^{1/3}(\ln x+\ln y-2)=c$$

tenglama bilan oshkormas ko‘rinishda beriladi. 👍

Misol 7. Tenglamani yeching

$$xydx+(x^2y+y^3-x^2)dy=0.$$

🗝 Berilgan tenglamadan kelib chiqib, quyidagicha almashtirishlar bajaramiz:

$$y(x^2+y^2)dy=x(xdy-ydx),\ \frac{y}{x}dy=\frac{xdy-ydx}{x^2+y^2},$$

$$\frac{y}{x}dy = d\left(\text{arctg}\frac{y}{x}\right) \text{ (jadvalga qarang),}$$

$\frac{y}{x} = u$ deymiz. U holda $udy = d(\text{arctg}u)$ va $dy = \frac{du}{u\left(1+u^2\right)}$,

$$2y + \ln c = \ln\frac{u^2}{1+u^2}, \quad e^{2y}c = \frac{y^2}{x^2+y^2} \quad (c > 0).$$

Yo‘qolgan $y = 0$ yechim oxirgi formuladan $c = 0$ da hosil bo‘ladi.

Javob. $\left(x^2+y^2\right)e^{2y}c = y^2$, $c = \text{const} \geq 0$. ✍

Misol 8. Agar tekislikdagi egri chiziqqa o‘tkazilgan normallarning barchasi bitta tayin nuqtadan o‘tsa, bu chiziq aylana ekanligini isbotlang.

Koordinatalar boshini normallar o‘tuvchi nuqtada joylashtiramiz. Egri chiziqning ixtiyoriy $M(x, y)$ nuqtasini qaraylik, $x = x(t), y = y(t)$ ($t -$egri chiziqdagi parametr). Shu nuqtadagi urinmaning yo‘naltiruvchi vektori $\vec{\tau} = \left(x'(t), y'(t)\right)$. Masalaning shartiga ko‘ra bu vektor $M(x(t), y(t))$ nuqtaning radius-vektori $\vec{r} = (x(t), y(t))$ ga perpendikulyar, ya’ni $x \cdot x' + y \cdot y' = 0$ yoki $xdx + ydy = 0$. Oxirgi tenglamadan $x^2 + y^2 = c$ $(c > 0)$. Demak, barcha normallari koordinatalar boshidan o‘tgan egri chiziq – bu markazi koordinatalar boshida joylashgan aylana ekan. ✍

Masalalar

Tenglamalarni yeching (**1-11**):

1. $(2 - 4x^3 - y)dx + (4y^3 - x)dy = 0$.

2. $(y-1)dx + (x + \cos y)dy = 0$.

3. $(y^2 + e^x)dx = (e^y - 2xy)dy$.

4. $(1 + 2x\ln y)dx + \left(3y^2 + \frac{x^2}{y}\right)dy = 0.$

5. $(x + \ln y)dy + (y - 2x)dx = 0$.

6. $(x + y^2)dx + (2xy - \sin y)dy = 0$.

7. $\left(2x + \frac{y}{x}\right)dx + \left(1 + e^y + \ln(xy)\right)dy = 0$.

8. $e^y dx + (\cos y + xe^y)dy = 0.$

9. $\left(1 + \frac{2y}{x}\right)dx + (2\ln x + y^3)dy = 0$.

10. $\left(1 + \frac{x}{\sqrt{x^2+y^2}}\right)dx + \left(1 + \frac{y}{\sqrt{x^2+y^2}}\right)dy = 0$.

11. $(\sin 2x - 2\cos(x+y))dx - 2\cos(x+y)dy = 0.$

Tenglamalarni integrallovchi ko‘paytuvchi yoki kerakli almashtirish yordamida yeching (**12-26**) :

12. $(\ln y - 4x^3)ydx + xdy = 0.$

13. $(x^2y + 2x)y' = 2xy^2 + y$.

14. $(y + e^x)dy = y^2dx$.

15. $xy' = (x^2 + \ln y)y$.

16. $y^2dx = (e^x + 2y)dy$.

17. $xdy = \left(y + 2x\sqrt{x^2 + y^2}\right)dx$.

18. $(2xy^2 - y)dx + (y^2\ln y - y + x)dy = 0$.

19. $(y^3 + x^2y)dx - (x^3 + xy^2 + 2x^2y^3)dy = 0$.

20. $(x^2 + 1)(\cos y dy + 2xdx) + 2x\sin y dx = 0$.

21. $x(\ln y + 3\ln x + 1)dy = 3ydx$.

22. $\left(x + y\sqrt{x^2 + y^2}\right)dx + \left(x\sqrt{x^2 + y^2} + y\right)dy = 0$.

23. $y(1 + 3x^2\ln y)dx + (2y^2 + x^3)dy = 0$.

24. $(y^4 - 4xy)dx + (2xy^3 - 3x^2)dy = 0$

25. $x^2(1 - y)dx + (x^3 + y^2)dy = 0$.

26. $(2x + y^2)dx = (x^2 - 2y)dy$.

Mustaqil ish №5 topshiriqlari:

I. Koshi masalasini yeching:

1. $(x^3 + xy^2)dx + (x^2y + y^3)dy = 0, \quad y(1) = 1.$

2. $(\sin xy + xy\cos xy)dx + x^2\cos xy \cdot dy = 0, \ y(\pi/4) = 1.$

3. $(x + y^2)dx + 2xydy = 0, \quad y(1) = 1.$

4. $(x^2 + y)dx + xdy = 0, \quad y(1) = 2.$

5. $\left(\frac{x}{y} + 1\right)dx + \left(1 - \frac{x^2}{2y^2}\right)dy = 0, \ y(1) = 2.$

6. $\left(\frac{\sin 2x}{y} + x\right)dx + \left(y - \frac{\sin^2 x}{y^2}\right)dy = 0, \ y(\frac{\pi}{4}) = 1.$

7. $\left(2x+\dfrac{x^2+y^2}{x^2y}\right)dx-\dfrac{x^2+y^2}{xy^2}dy=0,\ \ y(1)=1.$

8. $(3x^2-2x-y)dx+(2y-x+3y^2)dy=0,\ y(0)=1$**.**

9. $(x^2-y)dx-(x+1)dy=0,\quad y(1)=2$.

10. $2xe^y dx+(x^2e^y-1)dy=0,\ y(0)=0$.

11. $(3x^2+4y^2)dx+(8xy+e^y)dy=0.\ y(1)=0$.

12. $(x-1-\dfrac{y}{x^2})dx+(2y+\dfrac{1}{x})dy=0,\ y(1)=0$.

13. $\left(3x^2+\dfrac{2}{y}\cos\dfrac{2x}{y}\right)dx-\dfrac{2x}{y^2}\cos\dfrac{2x}{y}dy=0,\ y(0)=1$.

14. $(x\cos y-y\sin y)dy+\left(x^2+\sin y\right)dx=0,\ y(0)=\dfrac{\pi}{4}$.

15. $(\sin 2x-2\cos(x+y))dx-2\cos(x+y)dy=0,\ \ y(0)=\pi$.

16. $\left(x+6xy^2\right)dx+\left(6x^2y+3y^2\right)dy=0,\ \ y(1)=1.$

17. $\left(x^2y^3+y\right)dx+\left(x^3y^2+x\right)dy=0,\quad y(1)=e$.

18. $\left(\dfrac{1}{x^2}+\dfrac{3y^2}{x^4}\right)dx-\dfrac{2y}{x^3}dy=0,\ y(1)=1.$

19. $\dfrac{y}{x^2}\cos\dfrac{y}{x}dx-\left(\dfrac{1}{x}\cos\dfrac{y}{x}+2y\right)dy=0,\ y(\pi)=0$.

20. $\left(\dfrac{x}{\sqrt{x^2+y^2}}+y\right)dx+\left(x+\dfrac{y}{\sqrt{x^2+y^2}}\right)dy=0,\ y(0)=1.$

21. $(y^2+\ln x)dx+\left(2xy+y\right)dy=0,\ y(1)=1.$

22. $2xydx+(2y+x^2)dy=0,\quad y(1)=1.$

23. $y^2dx+\left(2xy+tgy\right)dy=0,\quad y(1)=\dfrac{\pi}{4}$.

24. $\left(x^2+y^2-y\right)dx+x\left(2y-1\right)dy=0,\quad y(1)=2$.

25. $\dfrac{dx}{y}-\dfrac{x+y^2}{y^2}dy=0,\ y(0)=1.$

26. $\frac{y}{x^2}dx-\frac{xy+1}{x}dy=0,\ y(1)=1$.

27. $\left(\sin x+\frac{y}{x^2}\right)dx-\frac{1}{x}dy=0,\ y(1)=0$.

28. $\left(\frac{y}{x^2+y^2}+e^x\right)dx+\frac{x}{x^2+y^2}dy=0,\ y(1)=0$.

29. $(ye^x+1)dx+(e^x-y)dy=0,\quad y(0)=1$.

30. $(3y^2-x)dx+(2y+6xy)dy=0,\quad y(1)=1$.

31. $e^ydx+(1-\cos y+xe^y)dy=0,\ y(0)=1$.

32. $(1+y^2\sin 2x)dx-2y\cos^2 x\cdot dy=0,\quad y(\frac{\pi}{4})=1$.

33. $\frac{xdx+ydy}{\sqrt{x^2+y^2}}+\frac{ydx-xdy}{x^2}=0,\quad y(1)=0$.

34. $(y^3+\cos x)dx+(3xy^2+e^y)dy=0,\ y(0)=1$.

35. $(xy^2+x)dx+(y+x^2y)dy=0,\quad y(1)=2$

36. $(xy^2-x^2)dx+(x^2y-y)dy=0,\ y(0)=1$.

37. $(\cos(x+y^2)+\sin x)dx+2y\cos(x+y^2)dy=0,\ y(0)=1$.

38. $(\sin y+y\sin x)dx+\left(x\cos y-\cos x+\frac{1}{y}\right)dy=0,\ y(1)=1$

39. $\left(1+\frac{1}{y}e^{\frac{x}{y}}\right)dx+\left(1-\frac{x}{y^2}e^{\frac{x}{y}}\right)dy=0,\ y(0)=1$.

40. $(xy^2+1)dx+y(x^2+y)dy=0,\ \ y(0)=1$.

II. Differensial tenglamalarni yeching:

1. $(x^2-xy^3)dy+(y^4+xy)dx=0$. [qismlarga ajrating]

2. $x(\ln y+2\ln x-1)dy-2ydx=0$. $\left[\mu=\mu(xy^2)\right]$

3. $(x^3+3\ln y)ydx-xdy=0$. [$u=\ln y$ belgilash kiriting]

4. $y(x^2-y^2)dx+x(xdy-ydx)=0$. [$x^3$ ga bo‘ling va shakl almashtiring]

5. $(x^5\ln x-2xy^3)dx+3x^2y^2dy=0$. $[\mu=\mu(x)]$

6. $y(x^2+y)dx+(xy+4)dy=0$. $[\mu=\mu(y)]$

7. $(x+\sin x+\sin y)dx+\cos y\cdot dy=0$. $[\mu=\mu(x)]$

8. $\left(x^3\sin x-2y^3\right)dx+3xy^2dy=0$. $[\mu=\mu(x)]$

9. $\left(x\cos y-y\sin y\right)dy+\left(x\sin y+y\cos y\right)dx=0$.$[\mu=\mu(x)]$

10. $2xy\ln y\cdot dx+\left(x^2+y^2\sqrt{y^2+1}\right)dy=0$. $[\mu=\mu(y)]$

11. $\left(2x^2y^2-y\right)dx+\left(x^2y^2+x\right)dy=0$. $[\mu=\mu(xy)]$

12. $\left(x^2y^3+y\right)dx-\left(x^3y^2-x\right)dy=0$. $[\mu=\mu(xy)]$

13. $x\left(4+\dfrac{1}{x^2-y^2}\right)dx-y\left(4-\dfrac{1}{x^2-y^2}\right)dy=0$. $\left[\mu=\mu(x^2-y^2)\right]$

14. $\dfrac{y}{\sqrt{x^2+y^2}}dx-\left(\dfrac{x}{\sqrt{x^2+y^2}}+y\right)dy=0$. $[\mu=\mu(y)]$

15. $\left(xy^2-y\right)dx+(y+x)dy=0$. $[\mu=\mu(y)]$

16. $y^2dx+(3xy+\mathrm{tg}xy)dy=0$. $[$ $(2xy+\mathrm{tg}xy)$ ga bo‘ling va $\mu=\mu(y)$?

17. $y^2dx-(xy+2x^3)dy=0$. $[\mu=\mu(x)]$

18. $\left(x^2+2\ln y\right)ydx=xdy$. [$u=\ln y$ belgilash kiriting va $\mu=\mu(x)$?]

19. $y(x^2+y)dx+(xy+4)dy=0$.$[\mu=\mu(y)]$

20. $\left(1+x^2-y^2\right)dx+xydy=0$. $[\mu=\mu(x)]$

21. $\left(6x-2y-2y^2\right)dx+\left(5y^2-8xy-x\right)dy=0$. **[**qismlarga ajrating]

22. $(3y^2-x)dx+(2y^3-6xy)dy=0$. $\left[\mu=\mu(x+ay^2)\right]$

23. $(x^2+y^2+1)dx-2xydy=0$. $[\mu=\mu(x)]$

24. $\left(xy^3+2y\right)dx+(x-x^2y^2)dy=0$. $\left[\mu=\mu(x^3y^2)\right]$

25. $ydx+\left(2x+\dfrac{\mathrm{tg}xy}{y}\right)dy=0$. $[\mu=\mu(xy)]$

26. $(2x-\sqrt{x^2-y})dx-dy=0$. $\left[\mu=\mu(x^2-y)\right]$

27. $(x^2y^3+2y)dx+(x^3y^2-x)dy=0$. $[\mu=\mu(xy)]$

28. $ydx+\left(3x-\dfrac{\cos y}{y}\right)dy=0$. $[\mu=\mu(y)]$

29. $(y-\ln x)dx+(2yx^2-x)dy=0$. $[\mu=\mu(x)]$

30. $2(x^4-\sin^2 y)dx+x\sin 2y\,dy=0$. $[\mu=\mu(x)]$

31. $y(xy+1)dx+(y^2-x)dy=0$. $[\mu=\mu(y)]$

32. $y(x+x^2y+y)dx+x(y^3-x)dy=0$. $\mu=\mu(xy^2)$

33. $(x^2+2x+y)dx+(3x^2y-x)dy=0$. $[\mu=\mu(x)]$

34. $\left(xy^2+\dfrac{x\ln x}{y}\right)dy+dx=0$. $[\ \mu(y/x)\ ]$

35. $(x^2+x+y)dx+(x^2y-x)dy=0$. $[\mu=\mu(x)]$

36. $ydx+\left(\dfrac{e^x}{y}-4\right)dy=0$. $\left[\mu=\mu(e^x-3y)\right]$

37. $(x^2-y^2-y)dx+x(2y+1)dy=0$. $[\mu=\mu(x)]$

38. $ydx+x(2+\ln x+\ln y)dy=0$. $[\mu=\mu(x)]$

39. $ydx+(2x-e^{2y})dy=0$. $[\mu=\mu(y)]$

40. $(xy-x^3)dx+(x^2+xy^2)dy=0$. $[\mu=\mu(x)]$

6. BIRINCHI TARTIBLI NORMAL KO‘RINISHDAGI DIFFERENSIAL TENGLAMA UCHUN KOSHI MASALASI YECHIMINING MAVJUDLIGI VA YAGONALIGI

Maqsad – birinchi tartibli normal ko‘rinishdagi differensial tenglama uchun Koshi masalasi yechimining mavjudligi va yagonaligi haqidagi teoremani o‘rganish

Yordamchi ma’lumotlar:

Ushbu

$$\frac{dy}{dx}=f(x,y) \quad \text{yoki} \quad y'=f(x,y)$$

ko‘rinishdagi tenglama ***birinchi tartibli normal ko‘rinishdagi differensial tenglama*** (yoki qisqacha: ***normal tenglama***) deyiladi.

$f(x,y)$ funksiyani biror $E\subset\mathbb{R}^2$ to‘plamda qaraylik ($E\subset D(f)$). Agar

$$\exists L>0\ \ \forall\{(x,y_1),(x,y_2)\}\subset E\quad |f(x,y_1)-f(x,y_2)|\le L|y_1-y_2|$$

shart bajarilsa, u holda $f(x,y)$ funksiya E to‘plamda y ga nisbatan (yoki y bo‘yicha) ***Lipshits shartini qanoatlantiradi*** deyiladi.

Ushbu

$$\begin{cases} y'=f(x,y)\\ y|_{x_0}=y_0\end{cases}\tag{K}$$

Koshi masalasini qaraylik.

Teorema (to‘rtburchakda MYaT). Faraz qilaylik, $f(x,y)$ funksiya

$$T=\{(x,y)\in\mathbb{R}^2\,\big|\,|x-x_0|\le a,\ |y-y_0|\le b\}\ \ (a>0,\,b>0)$$

to‘rtburchakda uzluksiz va y ga nisbatan Lipshits shartini qanoatlantiruvchi bo‘lsin. $|f(x,y)|$ funksiya T da $M>0$ soni bilan yuqoridan chegaralangan deylik:

$$\forall(x,y)\in T\ \ |f(x,y)|\le M\ \ (M>0).$$

h bilan $h=\min\left\{a,\dfrac{b}{M}\right\}$ musbat sonni belgilaylik.

U holda (K) Koshi masalasi $[x_0-h;x_0+h]$ oraliqda yagona $y=\phi(x)$ yechimga ega. Bu yechim ushbu

$$y_0(x)=y_0,\ \ y_n(x)=y_0+\int_{x_0}^{x}f(s,y_{n-1}(s))ds,\quad n\in\mathbb{N},\tag{1}$$

funksional ketma-ketlikning $n\to\infty$ да $|x-x_0|\le h$ oraliqdagi tekis limitidan iborat. $y=\varphi(x)$ yechim bilan $y_n(x)$ $n-$yaqinlashish orasidagi farq uchun

$$|\varphi(x)-y_n(x)|\le\frac{ML^n}{(n+1)!}h^{n+1},\quad x\in[x_0-h;x_0+h],\tag{2}$$

baho o‘rinli.

Quyidagi jumla funksiyani y ga nisbatan Lipshits shartini qanoatlantiruvchi bo‘lishi uchun yetarli shartni beradi.

Tushunarliki, agar $f(x,y)$ funksiya T to‘rtburchakda uzluksiz bo‘lib, uning ichki nuqtalarida chegaralangan $\frac{\partial f}{\partial y}$ xususiy hosilaga ega bo‘lsa, u holda $f(x,y)$ funksiya T da to‘rtburchakda MYaT ning shartlarini qanoatlantiradi.

Teorema (soha uchun MYaT). Agar $f(x,y)$ funksiya va uning $f_y'(x,y)$ hosilasi D sohada uzluksiz bo‘lsa, D sohaning har bir nuqtasidan $y' = f(x,y)$ tenglamaning bittadan yechimi (integral chizig‘i) o‘tadi.

Bu D soha tenglama uchun yechimning yagonalik sohasi deyiladi.

Teorema (davomsiz yechim to‘g‘risidagi). Agar $f(x,y)$ funksiya D sohadagi har bir nuqtaning biror atrofida MYaT shartlarini qanoatlantirsa (buning uchun, masalan, $f \in C(D)$ va $f_y' \in C(D)$ bo‘lishi yetarli), u holda ixtiyoriy $(x_0, y_0) \in D$ uchun (K) Koshi masalasi aniqlanish sohasi eng katta bo‘lgan yagona yechimga ega; bu yechim davomsiz yechim deyiladi. Davomsiz yechim D sohaning chegarasidan chegarasigacha boradi (davom etadi).

Yechimning mavjudligini (yagonaligini emas!) quyidagi teorema ta’minlaydi.

Teorema (Peano). Agar $f(x,y)$ funksiya T to‘rtburchakda uzluksiz bo‘lsa, u holda (K) Koshi masalasi yuqorida aytilgan $|x - x_0| \le h$ oraliqda kamida bitta yechimga ega.

Misol 1. $f(x,y) = ye^x + y^2 x$ bo‘lsin. $|f(x,y)|$ ni

$T = \{(x;y) \,|\, |x-1| \le 2, |y| \le 2\}$ to‘rtbuchakda yuqoridan baholang.

⊶ Ravshanki, T da $|x-1| \le 2$ va, demak, $|x| \le 3$. $\forall (x,y) \in T$ uchun quyidagilarga egamiz:

$$|f(x,y)| = |ye^x + y^2 x| \le |ye^x| + |y^2 x| = |y| e^x + |y|^2 \cdot |x| \le 2e^3 + 12 .$$

Demak, $|f(x,y)|$ funksiya T da $M = 2e^3 + 12$ soni bilan yuqoridan chegaralangan.

Ko‘rsatish mumknki, aslida

$$\max_{(x,y)\in T} |f(x,y)| = f(3;2) = 2e^3 + 12 = M .$$

Demak, M ni yuqorida topilgan qiymatdan kichiklashtirib, ya’ni

$$|f(x,y)| \le 2e^3 + 12, \quad (x,y) \in T$$

bahoni kuchaytirib (yaxshilab) bo‘lmaydi. ✍

Misol 2. $f(x,y) = x\sin y + y^4$ funksiya ixtiyoriy $T = \{(x,y) \mid |x| \le a,\ |y| \le b\}$ to‘rtburchakda y bo‘yicha Lipshits shartini qanoatlantirishini isbotlang.

⊶ *Birinchi usul:* ta’rifga ko‘ra bevosita isbot. $T = \{(x,y) \mid |x| \le a,\ |y| \le b\}$ ning ixtiyoriy (x, y_1) va (x, y_2) nuqtalarini olaylik ($|x| \le a$, $|y_1| \le b$, $|y_2| \le b$). Quyidagi baholashlarni bajaramiz:

$$|f(x,y_1) - f(x,y_2)| = |x\sin y_1 + y_1^4 - (x\sin y_2 + y_2^4)| \le |x(\sin y_1 - \sin y_2) + (y_1^4 - y_2^4)| \le$$

$$\le |x| \cdot |\sin y_1 - \sin y_2| + |y_1^4 - y_2^4| \le a \cdot |y_1 - y_2| + |y_1 - y_2| \cdot |y_1 + y_2| \cdot |y_1^2 + y_2^2| \le$$

$$\le a \cdot |y_1 - y_2| + |y_1 - y_2| \cdot 2b \cdot 2b^2 = (a + 4b^3) \cdot |y_1 - y_2| =$$

$$= L\,|y_1 - y_2|\ ,\ L = a + 4b^3\ .$$

Ikkinchi usul: Berilgan funksiya, ravshanki, $\mathbb{R}^2$ da uzluksiz. Demak, u $T \subset \mathbb{R}^2$ da ham uzluksiz. $\dfrac{\partial f}{\partial y} = x\cos y + 4y^3$ xususiy hosila $\mathbb{R}^2$ da (xususan uning qismi T da) mavjud va u T da chegaralangan:

$$(x,y) \in T \Rightarrow \left|\frac{\partial f}{\partial y}\right| = |x\cos y + 4y^3| \le |x| \cdot |\cos y| + 4|y|^3 \le a \cdot 1 + 4 \cdot b^3.$$

Chekli orttirmalar haqidagi Lagranj teoremasiga ko‘ra

$$|f(x,y_1) - f(x,y_2)| = \left|\frac{\partial f(x,y^*)}{\partial y}(y_1 - y_2)\right| \le L|y_1 - y_2|,\ L = a + 4b^3.$$

Demak, berilgan funksiya berilgan to‘rtburchakda y bo‘yicha Lipshits shartini qanoatlantiradi. ✍

Misol 3. Ushbu

$$f(x,y) = x\sqrt{y}$$

funksiya $T = \{(x,y) \mid 0 < x < 1, 0 < y < 1\}$ to‘rtburchakda y bo‘yicha Lipshits shartini qanoatlantirmasligini ko‘rsating.

🗝 Teskarisini faraz qilaylik, ya'ni berilgan funksiya berilgan T da y bo'yicha Lipshits shartini qanoatlantirsin:

$$\exists L>0\ \forall\{(x;y_1),(x;y_2)\}\subset T \quad \left|x\sqrt{y_1}-x\sqrt{y_2}\right|\le L|y_1-y_2|.$$

Oxirgi tengsizlikni

$$x\left|\sqrt{y_1}-\sqrt{y_2}\right|\le L|y_1-y_2|$$

ko'rinishda yozaylik; bunda $x\in(0;1)$, $y_1\in(0;1)$ va $y_2\in(0;1)$. Bu tengsizlikda $x=\frac{1}{2}$, $y_2\to 0+$ deymiz. U holda

$$\sqrt{y_1}\le 2Ly_1, \qquad (y_1\in(0;1)),$$

tengsizlik hosil bo'ladi. Bundan

$$1\le 2L\sqrt{y_1} \qquad (y_1\in(0;1)).$$

Bu yerda $y_1\to 0+$ deb limitga o'tsak, ziddiyat hosil bo'ladi:

$$1\le 0\ ?!$$

Demak, farazimiz noto'g'ri, ya'ni $f(x,y)=x\sqrt{y}$ funksiya $T=\{(x,y)|\,0<x<1, 0<y<1\}$ to'rtburchakda y bo'yicha Lipshits shartini qanoatlantirmaydi. 👍

Misol 4. Ushbu

$$y'=2x+y^3,\ y(0)=0$$

Koshi masalasi berilgan.

a) Bu masala yechimiga uchinchi yaqinlashishni toping.

b)Tenglamani $|x|\le 1,\ |y|\le 1$ kvadratda qarang. MYaT ketma-ket yaqinlashishlarning qaysi oraliqda yechimga tekis yaqinlashishini ta'minlaydi?

v) Koshi masalasi yechimi bilan topilgan uchinchi yaqinlashish orasidagi xatolikni baholang.

g) MYaT ni qaysi $\{|x|\le a, |y|\le b\}$ to‘rtburchakda qo‘llasak, bu teoremaga ko‘ra topilgan h eng katta bo‘ladi (teorema ta’minlovchi yechimning eng katta aniqlanish oralig‘i topiladi) ?

🗝 a) Ketma-ket yaqinlashishlar (1) formula bilan hisoblanadi. Bizning misolda

$$y_0(x)=0,$$

$$y_1(x)=\int_0^x (2s+y_0^3(s))ds=\int_0^x 2sds = x^2,$$

$$y_2(x)=\int_0^x (2s+y_1^3(s))ds=\int_0^x (2s+s^6)ds = x^2+\frac{x^7}{7},$$

$$y_3(x)=\int_0^x (2s+y_2^3(s))ds=\int_0^x (2s+(s^2+\frac{s^7}{7})^3)ds =$$

$$=x^2+\frac{x^7}{7}+\frac{x^{12}}{28}+\frac{3x^{17}}{17\cdot 49}+\frac{x^{22}}{22\cdot 343}.$$

b) Dastlab $T_1=\{(x,y)\,\big|\; |x|\le 1,\ |y|\le 1\}$ kvadratda $|f(x,y)|=\left|2x+y^3\right|$ ning maksimumini topamiz.

$$M=\max_{(x,y)\in T_1}\left|2x+y^3\right| = 2\cdot 1+1^3=3$$

ekanligini ko‘rish qiyin emas. MYaT ga ko‘ra yechimga ketma-ket yaqinlashishlar $|x|\le h,\ \ h=\min(a,\frac{b}{M})=\min(1,\frac{1}{3})=\frac{1}{3},$ oraliqda tekis yaqinlashuvchi bo‘ladi.

v) Berilgan masala yechimi $\phi(x)$ bilan unga uchinchi yaqinlashish orasidagi xatolik (2) tengsizlik bilan baholanadi:

$$|\varphi(x)-y_3(x)|\le\frac{ML^3}{4!}\cdot\left(\frac{1}{3}\right)^4,\ |x|\le\frac{1}{3}, \tag{3}$$

bunda $M=3,\ L-$Lipshits doimiysi. Biz L ni topishimiz kerak. $(x,y_1)\in T_1$ va $(x,y_2)\in T_1$ nuqtalar uchun $|x|\le 1,\ |y_1|\le 1,\ |y_2|\le 1$ va

$$|f(x,y_1)-f(x,y_2)|=\left|y_1^3-y_2^3\right|=|y_1-y_2|\cdot\left|y_1^2+y_1y_2+y_2^2\right|\le$$

$$\le|y_1-y_2|\left(|y_1|^2+|y_1|\cdot|y_2|\cdot|y_2|^2\right)\le|y_1-y_2|\cdot 3.$$

Demak, $L=3$ deb olish mumkin. Shuning uchun (3) ga ko‘ra

$$\left|\varphi(x)-y_3(x)\right|\le\frac{1}{4!}=\frac{1}{24}\approx 0{,}042,\qquad |x|\le\frac{1}{3}.$$

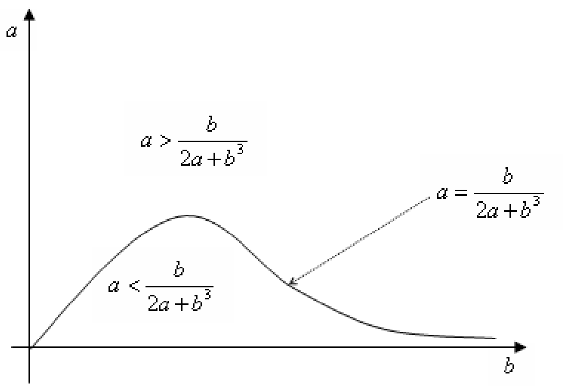
g) Dastlab $T=\left\{(x,y)\big||x|\le a,\ |y|\le b\right\}$ to‘rtburchakda $|f(x,y)|=\left|2x+y^3\right|$ ning maksimumini topaylik. Analiz vositalaridan foydalanib ko‘rsatish mumkinki,

$$M=\max_{(x,y)\in T}|f(x,y)|=2a+b^3.$$

Qo‘yilgan Koshi masalasining yechimi $|x|\le h$ oraliqda mavjud va yagona (MYaTga ko‘ra); bunda

$$h=\min\left\{a;\frac{b}{M}\right\}=\min\left\{a,\frac{b}{2a+b^3}\right\}.$$

Biz $a>0$ va $b>0$ larni shunday tanlashimiz kerakki, natijada $h=h(a,b)$ eng katta qiymat qabul qilsin. Shunda biz bir marta qo‘llanilgan MYaT ta`minlovchi yechim mavjud bo‘lgan eng katta $-h\le x\le h$ oraliqni topgan bo‘lamiz (Albatta, MyaT ni qayta-qayta qo‘llab, yechimni mumkin bo‘lgan eng katta oraliqqacha davom ettirsa bo‘ladi; bu – boshqa masala).



6.1- rasm. $a=b/(2a+b^3)$ funksiya grafigi.

$a=\dfrac{b}{2a+b^3}$ egri chiziq birinchi chorakni ikki sohaga ajratadi (6.1- rasm). Ravshanki, $a<\dfrac{b}{2a+b^3}$ bo‘lgan sohada $h=a$, $a>\dfrac{b}{2a+b^3}$ bo‘lgan sohada esa $h=\dfrac{b}{2a+b^3}$. Bu sohalarda h funksiya ekstremumga erisha olmaydi, chunki

$a < \dfrac{b}{2a+b^3}$ sohada $h = a$ funksiya a ga nisbatan o‘suvchi $\left(\dfrac{\partial h}{\partial a} = 1 \neq 0\right)$, $a > \dfrac{b}{2a+b^3}$ sohada esa $b > 0$ soni tayinlanganda $a \in (0;+\infty)$ oraliqda $h = \dfrac{b}{2a+b^3}$ funksiya kamayuvchi $\left(\dfrac{\partial h}{\partial a} < 0\right)$. Demak, $h = h(a,b)$ funksiyaning ekstremum nuqtasida

$$a = \frac{b}{2a+b^3} \tag{4}$$

bo‘lishi kerak (6.1- rasm.). (4) egri chiziq ustida $h = a = \dfrac{b}{2a+b^3}$. $b > 0$ ni erkli o‘zgaruvchi deb hisoblab, (4) tenglikdan $a = a(b)$ bog‘lanishni topamiz:

$$a(b) = \frac{b}{2a(b)+b^3} \quad \left(a = a(b) = -\frac{1}{4}b^3 + \frac{1}{4}\sqrt{b^6+8b}\right).$$

va $h = a(b)$ funksiya uchun ekstremum shartini yozamiz:

$$\frac{da(b)}{db} = 0, \quad \text{ya’ni} \quad \frac{\partial}{\partial b}\left(\frac{b}{2a+b^3}\right) = 0.$$

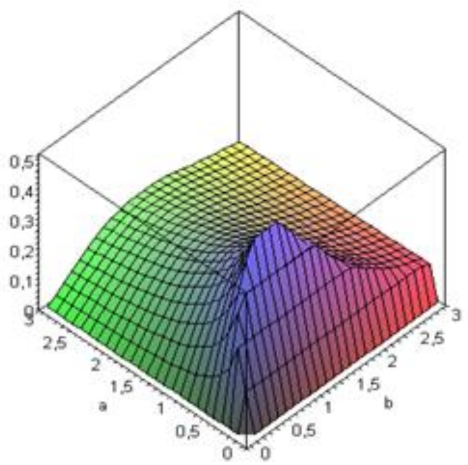
Oxirgi tenglikdan

$$2a + b^3 - 3b^2 b = 0 \quad \text{yoki} \quad b^3 = a$$

bo‘lishini topamiz. Buni (4) ga qo‘yib

$$b^3 = \frac{b}{3b^3}, \quad b^5 = \frac{1}{3}, \; b = \sqrt[5]{\frac{1}{3}} \approx 0{,}80 \; ; \; a = b^3, \; a \approx 0{,}51$$

ekanligini topamiz (6.2- rasm).



6.2- rasm. $h = \min\{a, b/(2a+b^3)\}$ funksiya grafigi.

Demak, berilgan Koshi masalasiga MyaTni bir marta

$$T=\{\,|x|\le a,|y|\le b\}\approx\{\,|x|\le 0{,}51;\,|y|\le 0{,}80\}$$

to‘rtburchakda qo‘llab, yechimni eng katta $-0{,}51\le x\le 0{,}51$ oraliqda mavjudligini topamiz.

Tushunarliki, qaralayotgan Koshi masalasining davomsiz yechimi biz topgan $-0{,}51\le x\le 0{,}51$ oraliqqa qaraganda kengroq oraliqda aniqlangan bo‘ladi (yechim o‘ngga va chapga davom etadi). ✍

Misol 5. Ushbu

$$xy'=2xy^2+\sqrt{x^3-y^3}$$

differensial tenglama uchun yechimning yagonalik sohalarini toping.

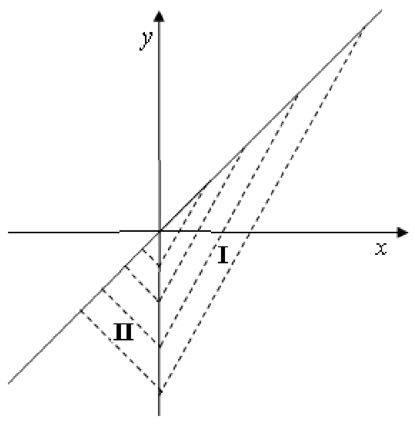
🗝Berilgan tenglamani

$$y'=\frac{2xy^2+\sqrt{x^3-y^3}}{x}$$

ko‘rinishda yozib olaylik. Bu tenglama uchun

$$f(x,y)=\frac{2xy^2+\sqrt{x^3-y^3}}{x},\quad \frac{\partial f(x,y)}{\partial y}=\frac{1}{x}\Big(4xy-\frac{3y^2}{2\sqrt{x^3-y^3}}\Big).$$

Bundan ravshanki, tenglamaning o‘ng tomoni $f(x,y)$ va uning $f_y'(x,y)$ hosilasi x,y tekisligining $\mathcal{O}=\{(x,y)\,|\,x\ne 0, x>y\}$ qismida ($\mathcal{O}$ - ochiq to‘plam, nega?) uzluksiz.



6.3-rasm. $\mathcal{O}=\{(x,y)\,|\,x\ne 0, x>y\}=\mathbf{I}\cup\mathbf{II}$ to‘plam.

Demak, soha uchun MYaTga ko‘ra tekislikning $\mathcal{O}$ ochiq qismi berilgan differensial tenglama uchun yechimning yagonalik to‘plami, ya’ni $\mathcal{O}$ to‘plamning har bir nuqtasidan tenglamaning bittadan yechimi o‘tadi. $\mathcal{O}$ to‘plam ikki sohaning birlashmasidan iborat: $\mathcal{O}=\mathrm{I}\cup\mathrm{II}$,

$$\mathbf{I}=\left\{(x;y)\big| x>y, x>0\right\},\ \mathbf{II}=\left\{(x;y)\big| x>y, x<0\right\}.$$ ☝

Masalalar

1. $f(x,y)=x^2\sqrt[3]{y}$ funksiya $(x,y)\in T_1\overset{def}{=}\left\{(x,y)\in\mathbb{R}^2\big|\,|x|\le 1,\ |y|\le 1\right\}$ to‘plamda y bo‘yicha Lipshits shartini qanoatlantiradimi?

$(x,y)\in T_2\overset{def}{=}\left\{(x,y)\in\mathbb{R}^2\big|\,|x|\le 1,\ \varepsilon\le y\le 1\right\}$ $(\varepsilon>0)$ to‘plamda-chi?

Koshi masalalari (2-4) uchun MYaTga ko‘ra a) uchinchi yaqinlashishni toping, b) h ni aniqlang, c) uchinchi yaqinlashish va aniq yechim orasidagi xatolikni baholang:

2. $y'=2x+y^2, y(1)=0$; $(x,y)\in T\overset{def}{=}\left\{(x,y)\in\mathbb{R}^2\big|\,|x-1|\le 1,\ |y|\le 1\right\}$.

3. $y'=-x+y^3, y(1)=-1$; $(x,y)\in T\overset{def}{=}\left\{(x,y)\in\mathbb{R}^2\big|\,|x-1|\le 1, |y+1|\le 2\right\}$.

4. $y'=1+x\ln y$, $y(0)=e$; $(x,y)\in T\overset{def}{=}\left\{(x,y)\in\mathbb{R}^2\big|\,|x|\le 1, |y-e|\le 1\right\}$.

Differensial tenglamalar uchun yagonalik sohalarini toping (**5-10**):

5. $y'=\sqrt{2x+y}-1$. **6.** $y'=\sqrt[5]{3y-2x}-y^2$. **7.** $xy'=2\ln x-\sin y+1$.

8. $e^y y'=2\arcsin(x-y)+1$. **9.** $y'=e^{\sqrt[3]{x+2y}}-1$.

10. $e^x y'=3\ln(2x-y)-\sqrt{x-y^2}$.

Mustaqil ish № 6 topshiriqlari:

I. Koshi masalasi berilgan (x_0, y_0, α, β parametrlarning qiymatlarini o‘qituvchidan oling).

a) Berilgan masala yechimiga uchinchi yaqinlashishni toping.

b) Tenglamani $|x-x_0|\le 1,\ |y-y_0|\le 1$ kvadratda qarang. Mavjudlik va yagonalik teoremasi ketma-ket yaqinlashishlarning qaysi oraliqda yechimga tekis yaqinlashishini ta'minlaydi?

v) Koshi masalasi yechimi bilan topilgan uchinchi yaqinlashish orasidagi xatolikni baholang.

g) MYaTni $\{|x-x_0|\le a, |y-y_0|\le b\}$ to'rtburchaklarning qaysisi biriga qo'llasak, eng katta oraliqda aniqlangan yechimni topamiz?

1. $y'=\alpha y^3+\beta xy+1,\ \ y(x_0)=y_0;$ **2**. $y'=\alpha x+\beta e^y,\ \ y(x_0)=y_0;$

3. $y'=\alpha xy+\beta y^2+1,\ \ y(x_0)=y_0;$ **4**. $y'=\alpha x+\beta y^3+\alpha,\ \ y(x_0)=y_0;$

5. $y'=x^2+\alpha y^3+\beta,\ \ y(x_0)=y_0;$ **6**. $y'=\alpha x^2\sin x+\beta y^2+1,\ \ y(x_0)=y_0;$

7. $y'=\beta+\alpha\sqrt{x+1}+\beta y^2,\ \ y(x_0)=y_0;$ **8**. $y'=\beta y^2+\sqrt[4]{\alpha x+\beta},\ \ y(x_0)=y_0;$

9. $y'=\alpha y^3+\sqrt[3]{\beta x}+\alpha,\ \ y(x_0)=y_0;$ **10**. $y'=\beta y^2+\sqrt[3]{\alpha x+1}+1,\ \ y(x_0)=y_0;$

11. $y'=\beta xy^2+\alpha+\cos x,\ \ y(x_0)=y_0;$

12. $y'=\beta xy+y^2\sin x+\alpha,\ \ y(x_0)=y_0;$

13. $y'=\alpha\ln(x^2+1)+\beta y^2+1,\ \ y(x_0)=y_0;$

14. $y'=\alpha\sqrt{x^2+1}+\beta xy^2-2,\ \ y(x_0)=y_0;$

15. $y'=\alpha y^3+\beta x\cos x-\alpha,\ \ y(x_0)=y_0;$

II. Berilgan differensial tenglama uchun yechimning yagonalik sohalarini toping va chizmada tasvirlang.

1. $(x-1)y'=x\sqrt{y+x}-y^2$ **2.** $(y-1)y'-y\ln x+\sqrt{x-2y}=0$

3. $xy'=\sqrt{2x-y}+\ln x+1$ **4.** $e^x y'=x\,\mathrm{tg}\,y-\sqrt[3]{y}+2$

5. $xy'-y^3+\sqrt{y-x}=0$ **6.** $x^2y'=y+\sqrt{y^2-x^2}+1$

7. $yy'=\sqrt[3]{y-x}+xy^2\sin x$ **8.** $y'\ln x=\sqrt[4]{y-3x}+2\cos x$

9. $y'\sqrt{x-1}=e^{2x}y+\ln(y-x)-x$ **10.** $(y-2x)y'=y^2\ln x+\sin x$

11. $xy'-\mathrm{ctg}\,y+e^x\ln x=0$ **12.** $x^3y'=1-y^2\cos x+\sqrt{y-2x}$

13. $(x-2)y' = e^x - \sqrt{y^2 - x^2} + 1$

14. $y' \ln x = \sqrt[3]{y+2x} - y^4 + 1$

15. $x^3 y' = y + \ln(y+2x) - \sqrt{x}$

16. $(x-1)y' = \sqrt{y^2 - x^2} - 3xy$

17. $\sqrt{x-1}y' - e^{-x}y + \ln(y-x) = 0$

18. $(y-2)y' = e^x y - x\sqrt{y+x} - 1$

19. $y' \ln x = \sqrt{y^2 - x^2} + 2xy - 1$

20. $xy' = y^2 - \sqrt[4]{y-2x} + \cos x$

21. $(x-1)y' - \sqrt{y-3x} + \ln x = 0$

22. $y'\sqrt{x-1} = \mathrm{tg}y + 2xy^3$

23. $y' \ln x = \sqrt{y^2 - x^2} + y$

24. $(x-1)y' = x\sqrt{y-3x} - y^4$

25. $x^2 y' = \sqrt{y+2x} - y^2 + 1$

26. $x^2 y' = \mathrm{ctg}y + x\ln x$

27. $xy' = \mathrm{tg}y + \ln x - \sqrt{x-1}$

28. $(x-2y)y' - y^3 + \sqrt{\ln x} = 0$

29. $(x-1)y' - \sqrt{y^3 - x^3} + e^{-x} = 0$

30. $x^2 y' = y^3 + \ln(y-x) - \sqrt{x}$

7. HOSILAGA NISBATAN YECHILMAGAN BIRINCHI TARTIBLI DIFFERENSIAL TENGLAMALAR. MAXSUS YECHIMLAR

Maqsad – birinchi tartibli differensial tenglamalarning maxsus yechimlarini topishni, $F(x,y,y') = 0$ ko‘rinishdagi tenglamalarni yechish usullarini o‘rganish.

Yordamchi ma’lumotlar:

I. ***Birinchi tartibli hosilaga nisbatan yechilmagan differensial tenglama*** ushbu

$$F(x,y,y') = 0 \ \text{(yoki } F_1(x,y,y') = F_2(x,y,y')\text{)} \qquad (1)$$

ko‘rinishda yoziladi; bu yerda $F(x,y,p)$ ($F_1(x,y,p)$ va $F_2(x,y,p)$) – uchta x,y,p haqiqiy o‘zgaruvchilarning berilgan haqiqiy funksiyasi (funksiyalari) $y = y(x)$- noma’lum funksiya va $y' = y'(x)$- uning hosilasi; y' hosila (1) tenglamada qatnashgan deb hisoblanadi, ya’ni (x,y) tayinlanganda F funksiya p ga nisbatan o‘zgarmasga aylanmaydi: p o‘zgarishi bilan F ham o‘zgaradi. (1) tenglama uchun qo‘yilgan ushbu

$$\begin{cases} F(x,y,y')=0 \\ y(x_0)=y_0 \end{cases} \qquad (2)$$

Koshi masalasi yechimi $y(x)$ ning $x=x_0$ dagi $y'(x_0)=p_0$ hosilasi bir nechta (hatto, cheksiz ko‘p) bo‘lishi mumkin. Bu p_0 qiymatlar, ravshanki, p ga nisbatan $F(x_0,y_0,p)=0$ tenglamaning yechimlaridan iborat bo‘ladi. Agar $y'(x_0)=p_0$ bo‘lsa, $y=y(x)$ integral chiziq (x_0,y_0) nuqtadan $p_0=y'(x_0)$ yo‘nalishida o‘tadi (chiqadi) deyiladi. Agar (x_0,y_0) nuqtadan p_0 yo‘nalishida bitta yechim o‘tsa, bu yechim yagona deyiladi; bu holda shu nuqtadan boshqa yo‘nalishlarda yechimlar o‘tishi mumkin.

Quyidagi teorema (1) tenglama uchun Koshi masalasi yechimining mavjudligi va yagonaligini ta’minlovchi yetarli shartlarni beradi.

Teorema. Agar

1. $F(x_0,y_0,p_0)=0$,

2. $F(x,y,p)$ funksiya $(x_0,y_0,p_0)\in\mathbb{R}^3$ nuqtaning biror atrofida o‘zining birinchi tartibli xususiy hosilalari bilan birgalikda uzluksiz ($\in C^1$),

3. $\dfrac{\partial F(x_0,y_0,p_0)}{\partial p}\neq 0$

shartlar bajarilsa , u holda (2) Koshi masalasi (x_0,y_0) nuqtadan p_0 yo‘nalishda o‘tuvchi yagona $y=y(x)$ yechimga $(y_0=y(x_0),\ \ p_0=y'(x_0))$ ega; bu yechim $x=x_0$ nuqtaning yetarli kichik atrofida aniqlangan bo‘ladi.

Misol 1. Ushbu

$$\begin{cases} y'^2-(2x+y)y'+2xy=0 \\ y(0)=1 \end{cases}$$

Koshi masalasini tekshiraylik. Berilgan misolda $F(x,y,p)=p^2-(2x+y)p+2xy$. Yechimning $x=0$ nuqtadagi $p=p_0=y'(0)$ yo‘nalishi $F(0,1,p)=p^2-p=0 \Rightarrow p(p-1)=0$ tenglamadan topiladi. Bundan p_0 uchun

$p_0 = 0$ va $p_0 = 1$ qiymatlarni topamiz. $F \in C^1(\mathbb{R}^3)$ ekanligi ravshan. $\dfrac{\partial F}{\partial p} = 2p - 2x - y$ hosila $(x_0, y_0, p_0) = (0,1,0)$ va $(x_0, y_0, p_0) = (0,1,1)$ nuqtalarda nolga aylanmaydi:

$$\frac{\partial F(0,1,0)}{\partial p} = -1, \quad \frac{\partial F(0,1,1)}{\partial p} = 1.$$

Demak, berilgan Koshi masalasi $y'(0) = 0$ va $y'(0) = 1$ yo‘nalishlarda o‘tuvchi (har bir yo‘nalishda bittadan) ikkita yechimga ega. Qaralgan Koshi masalasini elementar funksiyalarda yechish mumkin. Berilgan differensial tenglamaning chap tomonini ko‘paytuvchilarga ajratib, quyidagi ikki masalaga kelamiz:

$$\begin{cases} y' = y \\ y(0) = 1 \end{cases}, \quad \begin{cases} y' = 2x \\ y(0) = 1 \end{cases}$$

Bularni yechib, qo‘yilgan Koshi masalasining ikki dona $y = e^x$ va $y = x^2 + 1$ yechimlarini hosil qilamiz.

II. (1) ***tenglamaning maxsus yechimi*** deb uning shunday yechimiga aytiladiki, bu yechim grafigining har bir nuqtasidan shu integral chiziqqa (grafikka) urinib, boshqa bir yechim grafigi ham o‘tadi; maxsus yechim grafigining har bir nuqtasida Koshi masalasi yechimining yagonalik xossasi buziladi.

II.1. Dastlab (1) tenglamaning xususiy holi bo‘lgan ushbu

$$y' - f(x, y) = 0 \tag{1*}$$

tenglamaning maxsus yechimini topishda to‘xtalaylik. (1*) tenglama uchun Koshi masalasi yechimining mavjudligi va yagonaligi haqidagi teoremadan quyidagi tasdiq kelib chiqadi.

Teorema. (maxsus yechim uchun zaruriy shart). Faraz qilaylik, (1*) dagi $f(x, y)$ funksiya $D \subset \mathbb{R}^2$ ochiq yoki yopiq sohada uzluksiz bo‘lsin. Agar (1*) tenglamaning maxsus yechimi mavjud bo‘lsa, u holda bu yechim grafigi (maxsus integral chiziq) atrofida $f(x, y)$ funksiya y bo‘yicha Lipshits shartini qanoatlantirmaydi. Bunday nuqtalar atrofida $\frac{\partial f}{\partial y}$ xususiy hosila chegaralanmagan bo‘ladi.

Shunday qilib, y bo‘yicha Lipshits sharti qanoatlanmaydigan $\left(\frac{\partial f}{\partial y}=\infty\right)$ egri chiziqni topib, uning

a) integral chiziqligini

b) har bir nuqtasidan boshqa integral chiziq unga urinib o‘tishini asoslasak, u holda topilgan egri chiziq (1*) ning maxsus integral chizig‘ini beradi.

Misol 2. Ushbu $y' = \sqrt[3]{y^2}$

differensial tenglamaning maxsus yechimlarini toping.

🗝 Berilgan tenglama uchun

$$f(x,y) = \sqrt[3]{y^2}$$

funksiya $\mathbb{R}^2$ da uzluksiz. Ushbu

$$\frac{\partial f}{\partial y} = \frac{2}{3\sqrt[3]{y}}$$

xususiy hosila esa $y=0$ to‘g‘ri chiziq atrofida chegaralanmagan. Demak, agar maxsus yechim mavjud bo‘lsa, u $y=0$ to‘g‘ri chiziq (yoki uning qismi) dan iborat bo‘ladi.

a) $y=0$ ning berilgan tenglama yechimi ekanligi ravshan.

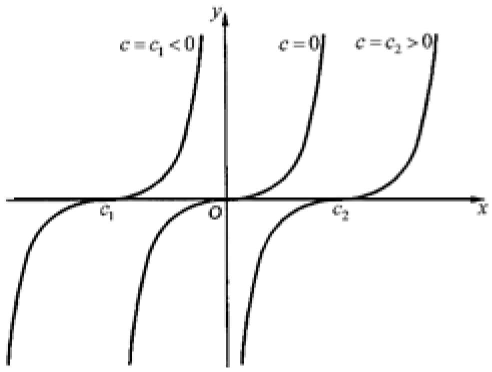
b) endi bu yechimni maxsuslikka tekshiramiz, ya’ni $y=0$ to‘g‘ri chiziq nuqtalaridan berilgan tenglamaning boshqa integral chiziqlari unga urinib o‘tishini (yoki o‘tmasligini) aniqlaymiz. Buning uchun berilgan tenglamada o‘zgaruvchilarni ajratib, uni integrallaymiz: $\frac{dy}{\sqrt[3]{y^2}} = dx$ (tenglamaning har ikkala tomonini $\sqrt[3]{y^2}$ ga bo‘lishda $y=0$ yechim yo‘qoladi, lekin bu yechimni biz bilamiz).

$$3y^{\frac{1}{3}} = x - c,$$

$$y = \frac{(x-c)^3}{27}.$$

Oxirgi tenglik, ravshanki, berilgan tenglamaning $(c,0)$ nuqtadan o‘tuvchi yechimini ifodalaydi. Bu yechim shu $(c,0)$ nuqtadan o‘tuvchi $y=0$ yechimga urinadi:

$$\left(\frac{1}{27}(x-c)^3\right)' = \frac{1}{9}(x-c)^2 \text{ hosila } x=c \text{ da } 0 \text{ ga teng.}$$



7.1- rasm. $y' = \sqrt[3]{y^2}$ tenglamaning yechimlari.

Demak, $y=0$ berilgan tenglamaning maxsus yechimi (7.1- rasm). 👍

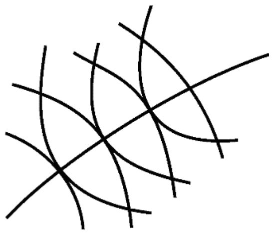
Misol 3. Ushbu $y' = \sqrt[3]{y^2} + 1$ tenglamaning maxsus yechimlarini toping.

🗝 Berilgan tenglamada $f(x,y) = \sqrt[3]{y^2} + 1$ va $\frac{\partial f}{\partial y} = \frac{2}{3\sqrt[3]{y}}$. Demak, agar maxsus yechim mavjud bo‘lsa, y, albatta $y=0$ dan (yoki uning qismidan) iborat bo‘lishi kerak. Lekin $y=0$ (har qanday I oraliqda), berilgan tenglamaning yechimi emas. Shuning uchun berilgan tenglamaning maxsus yechimi yo‘q. 👍

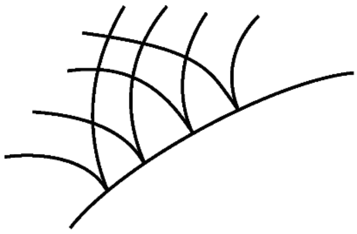
II.2. Endi $F(x,y,y')=0$ ko‘rinishda berilgan differensial tenglamaning maxsus yechimlarini topish yo‘llari bilan tanishaylik. Faraz qilaylik, $F(x,y,p)$ funksiya $D \subset \mathbb{R}^3$ sohada C^1 sinfga tegishli bo‘lsin, u holda yuqorida keltirilgan teoremaga ko‘ra (1) tenglamaning maxsus yechimlari

$$\begin{cases} F(x,y,p)=0 \\ \dfrac{\partial F(x,y,p)}{\partial p}=0 \end{cases} \tag{3}$$

shartlarni qanoatlantiradi. (3) sistemani qanoatlantiruvchi $(x,y)\in\mathbb{R}^2$ (tekislikdagi) nuqtalar to‘plami (1) tenglamaning p-diskriminant chizig‘i deyiladi. U bir nechta chiziqlar (shoxlar) birlashmasidan iborat bo‘lishi mumkin. p-diskriminant chiziq tarkibida, umumiy holda, maxsus yechim nuqtalaridan tashqari yechimlar grafiklarining urinish nuqtalari (7.2- rasm) hamda qaytish nuqtalari (7.3- rasm) ham bo‘lishi mumkin.



7.2- rasm. Yechimlar grafiklarining urinish nuqtalari.



7.3- rasm. Qaytish nuqtalari

Shunday qilib, agar $F\in C^1(D)$ bo‘lsa, u holda $F(x,y,y')=0$ differensial tenglama maxsus yechimlarini bu tenglamaning $p-$diskriminant chizig‘i orasidan qidirish kerak. Bundan maxsus yechimni topish uchun quyidagi qoida kelib chiqadi:

1^0. Dastlab (3) sistemadan p-diskriminant chiziqni aniqlash .

2^0. p-diskriminant chiziqni (uning shoxlarini) (1) tenglamaning yechimi bo‘lishga tekshirish.

3^0. Yechim bo‘lgan p-diskriminant chiziq shoxlarini (1) tenglamaning maxsus yechimi bo‘lishga (bo‘lmaslikka) tekshirish.

Misol 4. Ushbu

$$y'^2-4xy'-y=0$$

differensial tenglamaning maxsus yechimlarini toping.

⊶Berilgan tenglama uchun p-diskriminant chiziqni topamiz:

$$\begin{cases} F(x,y,p) \equiv p^2 - 4xp - y = 0 \\ \dfrac{\partial F(x,y,p)}{\partial p} \equiv 2p - 4x = 0, \end{cases}; \quad \begin{cases} p^2 - 4xp - y = 0 \\ p = 2x, \end{cases}; \quad \begin{cases} x = p/2 \\ y = -p^2. \end{cases}$$

Oxirgi sistema p-diskriminant chiziqning parametrik tenglamasidir. Bu yerda p ni yo‘qotib p -diskriminant chiziqning ushbu

$$y = -4x^2$$

oshkor ko‘rinishdagi tenglamasini hosil qilamiz. Lekin bu funksiya berilgan tenglamaning yechimi emas:

$$y'^2 - 4xy' - y \equiv (-8x)^2 - 4x(-8x) - (-4x^2) \equiv 100 \cdot x^2 \neq 0$$

Demak, berilgan differensial tenglamaning maxsus yechimi yo‘q. ☝

Misol 5. Differensial tenglamaning maxsus yechimlarini toping

$$xy'^2 - 2yy' + x = 0.$$

🗝 p -diskriminant chiziq ushbu

$$\begin{cases} F(x,y,p) \equiv xp^2 - 2yp + x = 0 \\ \dfrac{\partial F(x,y,p)}{\partial p} \equiv 2xp - 2y = 0 \end{cases}$$

sistemadan topiladi. Sistemaning ikkinchi tenglamasidan $p = y/x$ ni topib birinchisiga qo‘yamiz va $y^2 - x^2 = 0$ tenglikni hosil qilamiz. Demak, maxsus yechim bo‘lishi mumkin (maxsus yechimlikka nomzod) bo‘lgan funksiyalar $y = \pm x$. Bu funksiyalarni berilgan tenglamaga qo‘yib, ularning yechim ekanligiga ishonch hosil qilamiz. Endi biz bu yechimlarni maxsuslikka tekshirishimiz kerak. Buning uchun $y = \pm x$ yechimlar grafiklari nuqtalaridan ularga urinib boshqa yechim o‘tishini (yoki o‘tmasligini) aniqlashimiz lozim. Berilgan tenglamadan y' ni topaylik:

$$y' = \frac{y \pm \sqrt{y^2 - x^2}}{x}$$

Bu tenglama o‘zgaruvchilariga nisbatan bir jinsli. Uni $y = xu$ almashtirish yordamida yechamiz:

$$y' = u + xu',\ \ u + xu' = u \pm \sqrt{u^2 - 1},\ \ \frac{du}{\pm\sqrt{u^2-1}} = \frac{dx}{x},$$

$$\pm\ln\left|u + \sqrt{u^2-1}\right| = \ln|x| - \ln|c|\ \ (c>0),\ \ u \pm \sqrt{u^2-1} = x/c\ (c \neq 0).$$

Bu yerdan radikalni yo‘qotamiz:

$$\frac{1}{u \pm \sqrt{u^2-1}} = \frac{c}{x},\ \ u \mp \sqrt{u^2-1} = \frac{c}{x},\ \ 2u = \frac{x}{c} + \frac{c}{x}.$$

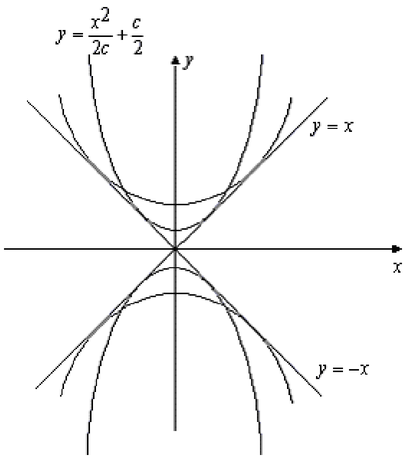
Oxirgi tenglikda $u = y/x$ deb, dastlabki noma’lum y ga qaytamiz:

$$2\frac{y}{x} = \frac{c}{x} + \frac{x}{c},\ \ \ y = \frac{x^2}{2c} + \frac{c}{2}.$$

Bir parametrli yechimlar oilasi hosil bo‘ldi. $y = x$ to‘g‘ri chiziq nuqtasi bo‘lmish (a,a) $\left(a \neq 0\right)$ dan o‘tuvchi yechimni aniqlaylik:

$$y = \frac{x^2}{2c} + \frac{c}{2} \Rightarrow c = a$$

Demak, $y = x$ yechimning ixtiyoriy (a,a) $\left(a \neq 0\right)$ nuqtasidan $y = \frac{x^2}{2a} + \frac{a}{2}$ yechim o‘tadi (bu yechim $y = x$ yechimdan farqli) $y'(a) = 1$ bo‘lgani uchun bu yechim $y = x$ yechimga urinadi. Shuning uchun $y = x$ $(x > 0$ yoki $x < 0)$ berilgan tenglamalarning maxsus yechimi.



7.4-rasm. $y = \dfrac{x^2}{2c} + \dfrac{c}{2}$ oila va $y = \pm x$ chiziqlar.

Ravshanki, $y=-x$ yechim grafigining ixtiyoriy $(a,-a)$ $\left(a\neq 0\right)$ nuqtasidan berilgan tenglamaning bu yechimdan boshqa $y=-\dfrac{x^2}{2a}-\dfrac{a}{2}$ yechimi $y=-x$ ga urinib o‘tadi. Demak, $y=-x$ $(x>0$ yoki $x<0)$ ham berilgan tenglamaning maxsus yechimi (7.4- rasm). ☝

Izoh. $y=x$ va $y=-x$ maxsus yechim emas, chunki $x=0, y=0$ nuqtada unga urinib boshqa yechim o‘tmaydi. To‘rtta maxsus yechim bor: $y=x$, $x>0$; $y=x$, $x<0$; $y=-x$, $x>0$ va $y=-x$, $x<0$.

II.3. Maxsus yechimni bir parametrli yechimlar oilasining o‘ramasi sifatida ham topish mumkin. Aytaylik, egri chiziqlarning ushbu

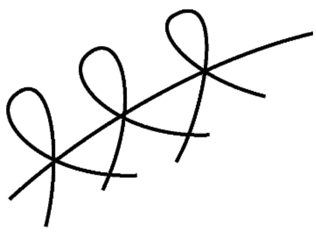
$$\Phi(x,y,c)=0 \tag{4}$$

bir parametrli oilasi berilgan bo‘lsin (c parametrli oila, $\Phi\in C^1$). Agar berilgan egri chiziq (4) oilada bo‘lmasa va uning har bir nuqtasidan unga urinib (4) oilaning biror chizig‘i o‘tsa, u holda berilgan egri chiziq (4) oilaning o‘rama egri chizig‘i (o‘ramasi) deyiladi. Differensial geometriyadan ma’lumki, (4) ning o‘ramasi ushbu

$$\begin{cases}\Phi(x,y,c)=0\\ \dfrac{\partial\Phi(x,y,c)}{\partial c}=0\end{cases} \tag{5}$$

sistemani qanoatlantiradi.

Bu sistemaning (x,y) yechimlari to‘plami (4) oilaning $c-$diskriminant chizig‘i deyiladi. Umumiy holda $c-$diskriminant chiziq tarkibida o‘ramadan tashqari qaytish nuqtalari (7.3- rasm) hamda tugun nuqtalari (7.5- rasm) ham bo‘lishi mumkin.



7.5- ram. Tugun nuqtalari

Diskriminant chiziqdan o‘ramani ajratishda differensial geometriya kursidan ma’lum bo‘lgan quyidagi tasdiqdan foydalanish mumkin: agar $c-$diskriminant chiziqning biror silliq qismida

$$\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial y}\right)^2\neq 0\ ,$$

bo‘lsa, u holda bu silliq qism o‘ramadan iborat bo‘ladi.

Misol 6. Ushbu

$$y'^2(2-3y)^2=4(1-y)$$

differensial tenglamani yeching. Maxsus yechimlarini toping.

🗝 Berilgan tenglamani ushbu

$$y'\cdot(2-3y)=\pm 2\sqrt{1-y}$$

o‘zgaruvchilari ajraladigan tenglamalar ko‘rinishida yozib olib, yechamiz:

$$\frac{(2-3y)dy}{2\sqrt{1-y}}=\pm\,dx\ ,\ y\sqrt{1-y}=\pm(x-c)\,.$$

Demak, yechimlar oilasi

$$\Phi(x,y,c)\overset{def}{=}y^2(1-y)-(x-c)^2=0\,.$$

tenglama bilan beriladi. Maxsus yechimlarni topish maqsadida bu yechimlar oilasining o‘ramasini aniqlaymiz. Daslab $c-$diskriminant chiziqni topamiz:

$$\begin{cases} y^2(1-y)-(x-c)^2=0 \\ 2(x-c)=0 \end{cases}$$

Bu sistemadan $c-$diskriminant chiziq uchun ushbu

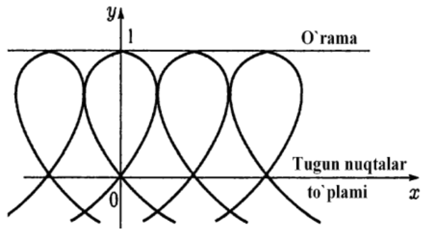
$$y^2(1-y)=0\,.$$

tenglamani hosil qilamiz. Demak, $y=1$ va $y=0$ to‘g‘ri chiziqlar $c-$diskriminant chiziqni tashkil etadi. $y=1$ o‘rama bo‘ladi, chunki

$$\frac{\partial\Phi}{\partial x}=\frac{\partial}{\partial x}\left(y^2(1-y)-(x-c)^2\right)=-2(x-c),\ \frac{\partial\Phi}{\partial y}=2y-3y^2\,;$$

$$\left.\left(\frac{\partial\Phi}{\partial x}\right)\right|_{y=1}^2+\left.\left(\frac{\partial\Phi}{\partial y}\right)\right|_{y=1}^2=4(x-c)^2+(2y-3y^2)^2\Big|_{y=1}\geq 1>0\,.$$

$c-$diskriminant chiziqdagi $y=0$ to‘g‘ri chiziq, ravshanki, berilgan tenglamaning yechimi emas. U yechimlarning tugun nuqtalari to‘plamini beradi (7.6- rasmga qarang). ✍



7.6- rasm. O‘rama va tugun nuqtalar.

III. Endi (1) tenglamani yechish (integrallash) metodlarini keltiramiz.

III.1. Agar (1) tenglama y' ga nisbatan yechilib, bir yoki bir nechta $y'=f_j(x,y)$ $(j=1,2,\ldots)$ ko‘rinishdagi tenglamaga keltirilsa, oxirgi tenglamalarni integrallab va yechimlarini birlashtirib (1) ning yechimlarini topamiz.

Misol 7. Quyidagi differensial tenglamani yeching:

$$y'^3-(y+1)y'^2+(x+y-x^2)y'+xy(x-1)=0.$$

🗝 Berilgan tenglamani

$$(y'-y)(y'-x)(y'+x-1)=0$$

ko‘rinishda yozamiz. Endi

$$y'=y,\quad y'=x,\quad y'=-x+1$$

tenglamalarni yechib,

$$y=c_1e^x,\ y=\frac{x^2}{2}+c_2,\ y=-\frac{x^2}{2}+x+c_3 \tag{6}$$

yechimlar oilasini topamiz. Tushunarliki, berilgan tenglamaning barcha yechimlari (6) formulalar bilan beriladi. ✍

III.2. (1) tenglamani y o‘zgaruvchiga nisbatan yechish mumkin bo‘lsin:

$$y=f(x,y')\quad (f\in C^1,\ f'_{y'}\neq 0 \text{ deb faraz qilinadi }). \tag{7}$$

Bu yerda

$$y'=p\quad (dy=pdx) \tag{8}$$

deb p parametr kiritamiz va $y=f(x,p)$ tenglikka kelamiz. Agar biz x ni ham p ning funksiyasi sifatida topsak, u holda (7) differensial tenglamaning parametrik ko‘rinishdagi yechimi osongina yoziladi: $x=x(p), y=f(x(p),p)$. Quyidagicha ish tutamiz:

$$dy=f_x{}'dx+f_p{}'dp,\ pdx=f_x{}'dx+f_p{}'dp$$

Oxirgi tenglikdan

$$\left(p-f_x'\right)dx=f_p'dp \tag{9}$$

munosabatni topamiz. Endi yana qo‘shimcha ravishda $f_x'\neq p$ deb faraz qilamiz va oxirgi tenglamani

$$\frac{dx}{dp}=\frac{f_p'}{p-f_x'}$$

ko‘rinishda yozamiz. Uni yechib, $x=\varphi(p,c)$, $x(p,c)=\int\frac{f_p'}{p-f_x'}dp+c$, bog‘lanishni topamiz Shunday qilib, biz qo‘yilgn shartlarda (7) differensial tenglamaning parametrik ko‘rinishdagi bir parametrli yechimlar oilasini topdik:

$$\begin{cases}x=x(p,c)\\ y=f(x(p,c),p)\end{cases}$$

($p-$egri chiziqdagi parametr, $c-$egri chiziqlarni belgilovchi parametr)

Agar $f_x'(x,p)=p$ tenglik $x=x(p)\in C^1$ funksiyani aniqlasa va $f_p'(x(p),p)\equiv 0$ ham bo‘lsa, u holda (9) tenglama, ravshanki, $x=x(p)$ yechimga ega. Buni $y=f(x,p)$ ga qo‘yib, qaralayotgan (7) differensial tenglamaning { $x=x(p)$, $y=f(x(p),p)$ } parametrik yechimini hosil qilamiz. Endi (9) tenglamani $x-x(p)$ ga “qisqartirib”, so‘ngra yuqoridagidek ish tutib, (7) tenglamaning bir parametrli yechimlar oilasini (agar mavjud bo‘lsa) topamiz. (III.4 va III.5 bandlarga qarang).

Misol 8. Ushbu

$$3x^3y'^4-x^2yy'^3+1=0$$

differensial tenglamani yeching.

Tenglamadan y osongina topiladi:

$$y = \frac{1}{x^2 y'^3} + 3xy'.$$

$y' = p$ $(dy = pdx)$ parametrni kiritamiz. U holda

$$y = \frac{1}{x^2 p^3} + 3xp \tag{10}$$

tenglikni hosil qilamiz. $x = x(p)$ bog‘lanishni topish uchun oxirgi tenglikni differensiallaymiz va ixchamlashtirishlarni bajaraniz:

$$dy = d\left(\frac{1}{x^2 p^3}\right) + 3d(xp), \ \ pdx = -\frac{2}{x^3 p^3}dx - \frac{3}{x^2 p^4}dp + 3xdp + 3pdx,$$

$$\frac{p}{x^2}\left(\frac{1}{p^4} - x^3\right)\left(\frac{2}{x}dx + \frac{3}{p}dp\right) = 0.$$

Bu yerdan

$$\frac{1}{p^4} - x^3 = 0 \ \text{va} \ \ \frac{2}{x}dx + \frac{3}{p}dp = 0$$

tenglamalarni hosil qilamiz. Ularni yechib quyidagilarni topamiz:

$$x = p^{-4/3} \ \text{va} \quad x = cp^{-3/2}. \tag{11}$$

(11) ni (10) ga qo‘yib, berilgan differensial tenglama yechimlarini parametrik ko‘rinishda hosil qilamiz:

$$\begin{cases} x = p^{-4/3} \\ y = 4p^{-1/3} \end{cases} \ \text{va} \ \ \begin{cases} x = cp^{-3/2} \\ y = c^{-2} + 3c^{2/3}p^{-1/2} \end{cases}.$$

Bu yerda p parametrni yo‘qotib, yechimlarning oshkor ko‘rinishini topish ham mumkin ($c^{2/3}$ o‘rniga c qo‘yilgan):

$$y = \pm 4\sqrt[4]{x} \ \ (x > 0) \ \ \text{va} \ \ y = c^{-3} + 3cx^{1/3}.$$

Bu yerdagi $y = \pm 4\sqrt[4]{x}$ $(x > 0)$ yechimlar maxsus yechimlardir, chunki ularning $\left(a, \pm 4\sqrt[4]{a}\right)$ $(a > 0)$ nuqtalari orqali ularga urinib $y = c^{-3} + 3cx^{1/3}$ yechimlar oilasidagi $c = \pm a^{-1/12}$ bo‘lgandagi (mos ravishda) $y = \pm\left(c^{1/4} + 3a^{-1/12}x^{1/3}\right)$ yechimlar o‘tadi:

$$y = \pm\varphi(x) \overset{def}{=} \pm 4\sqrt[4]{x}\ ,\ y = \pm\psi(x) \overset{def}{=} \pm\left(c^{1/4} + 3a^{-1/12}x^{1/3}\right),$$

$$\varphi(a) = \psi(a) = 4\sqrt[4]{a}\ ,\ \varphi'(a) = \psi'(a) = a^{-3/4}\ .$$

III.3. (1) tenglamani x ga nisbatan yechish mumkin bo‘lsin:

$$x = f(y, y') \quad \left(f \in C^1,\ f'_{y'} \neq 0,\ f'_y \neq y' \text{ deb faraz qilinadi}\right).$$

Bu tenglamaning parametrik umumiy yechimi yuqoridagi usul bilan topiladi.

Misol 9. Ushbu

$$y' = \exp\left(\frac{xy'}{y}\right)$$

tenglamani yeching.

Berilgan tenglamani x ga nisbatan osongina yechamiz:

$$x = \frac{y}{y'}\ln y'\ .$$

$y' = p\ \ (dx = dy/p)$ parametrni kiritamiz va oxirgi tenglamani

$$x = \frac{y}{p}\ln p \qquad (*)$$

ko‘rinishda yozamiz. $y = y(p)$ bog‘lanishni topish uchun oxirgi tenglikni differensiallaymiz, $dx = dy/p$ dan foydalanamiz va zarur shakl almashtirishlarni bajaramiz:

$$dx = d\left(\frac{y}{p}\ln p\right),\ \frac{dy}{p} = \frac{\ln p}{p}dy + \frac{y}{p^2}dp - \frac{y}{p^2}\ln p\, dp\ ,$$

$$(1 - \ln p)\left(\frac{1}{p}dy - \frac{y}{p^2}dp\right) = 0\ .$$

Oxirgi tenglikdan ushbu

$$1 - \ln p = 0 \quad \text{va} \quad \frac{1}{p}dy - \frac{y}{p^2}dp = 0$$

tenglamalarni topamiz. Bu tenglamalarni yechib,

$$p = e \ \text{ va } \ y = cp$$

munosabatlarga kelamiz. Ularni (*) ga qo‘yib, quyidagi yechimlarni topamiz:

$$x=\frac{y}{e} \quad \text{va} \quad \begin{cases} x=c\ln p \\ y=cp \end{cases}.$$

Bundan yechimlarning oshkor ko‘rinishini hosil qilamiz:

$$y=ex \text{ va } y=c\exp\left(\frac{x}{c}\right).$$

Bu yerdagi $y=ex$ maxsus yechimdir (tekshiring).☝

III.4. *Parametr kiritishning umumiy usuli*.

Faraz qilaylik,

$$\left\{(x,y,y')\in\mathbb{R}^3 \,|\, F(x,y,y')=0\right\}$$

to‘plamni

$$\begin{cases} x=\varphi(u,v) \\ y=\psi(u,v) \\ y'=\chi(u,v) \end{cases} \tag{12}$$

parametrik ko‘rinishda ifodalash mumkin bo‘lsin; bu yerda φ,ψ,χ funksiyalari u,v parametrlarning biror $O\subset\mathbb{R}^2$ ochiq to‘plamida C^1 sinfga tegishli. Ma’lumki, φ va ψ funksiyalarning differensiallari

$$dx=\frac{\partial\varphi}{\partial u}du+\frac{\partial\varphi}{\partial v}dv, \quad dy=\frac{\partial\psi}{\partial u}du+\frac{\partial\psi}{\partial v}dv$$

ko‘rinishga ega. Lekin $\frac{dy}{dx}=\chi(u,v)$, ya’ni $dy=\chi(u,v)dx$ bo‘lgani uchun

$$\frac{\partial\psi}{\partial u}du+\frac{\partial\psi}{\partial v}dv=\chi\cdot\left(\frac{\partial\varphi}{\partial u}du+\frac{\partial\varphi}{\partial v}dv\right)$$

yoki

$$\left(\frac{\partial\psi}{\partial u}-\chi\frac{\partial\varphi}{\partial u}\right)du=\left(\chi\frac{\partial\varphi}{\partial v}-\frac{\partial\psi}{\partial v}\right)dv. \tag{13}$$

Oxirgi (13) tenglik u va v parametrlar orasidagi differensial bog‘lanishni ifodalaydi.

Agar $\frac{\partial\psi}{\partial u}-\chi\frac{\partial\varphi}{\partial u}\neq 0$ bo‘lsa, u ni noma’lum funksiya, v ni esa erkli o‘zgaruvchi deb, (13) tenglikdan ushbu

$$\frac{du}{dv}=\frac{\chi\frac{\partial\varphi}{\partial v}-\frac{\partial\psi}{\partial v}}{\frac{\partial\psi}{\partial u}-\chi\frac{\partial\varphi}{\partial u}} \qquad (14)$$

hosilaga nisbatan yechilgan differensial tenglamaga kelamiz.

Shunga o‘xshash , agar $\chi\frac{\partial\varphi}{\partial v}-\frac{\partial\psi}{\partial v}\neq 0$ bo‘lsa, $v=v(u)$ funksiya uchun

$$\frac{dv}{du}=\frac{\frac{\partial\psi}{\partial u}-\chi\frac{\partial\varphi}{\partial u}}{\chi\frac{\partial\varphi}{\partial v}-\frac{\partial\psi}{\partial v}} \qquad (15)$$

differensial tenglamani hosil qilamiz.

Agar (14) (yoki (15)) differensial tenglamaning bir parametrli yechimlar oilasi $u=u(v,c)$ ($v=v(u,c)$) topilgan bo‘lsa, u holda (12) dan (1) differensial tenglamaning ushbu $x=\varphi(u(v,c),v)),\ \ y=\psi(u(v,c),v))$ ($x=\varphi(u,v(u,c)),\ \ y=\psi(u,v(u,c))$) bir parametrli yechimlar oilasini hosil qilamiz.

III.4. *Lagranj tenglamasi* deb ataluvchi ushbu

$$y=xf(y')+g(y') \qquad (16)$$

differensial tenglamani yechaylik; bu yerda $f(p),\ g(p)-$ biror oraliqda uzluksiz differensiallanuvchi funksiyalar.

⊶ Lagranj tenglamasi (16) da $y'=p$ deb parametr kiritamiz. U holda

$$y=xf(p)+g(p) \qquad (17)$$

munosabat hosil bo‘ladi. Endi x ni p ning funksiyasi sifatida topamiz. (17) ni differensiallab, $dy=y'dx=pdx$ ekanligini hisobga olib topamiz:

$$pdx=f(p)dx+xf'(p)dp+g'(p)dp,$$

$$p-f(p)=(xf'(p)+g'(p))\frac{dp}{dx},$$

$$\frac{dx}{dp}=\frac{f'(p)}{p-f(p)}x+\frac{g'(p)}{p-f(p)} \qquad (p-f(p)\neq 0) \qquad (18)$$

Oxirgi tenglama $x=x(p)$ noma’lum funksiyaga nisbatan chiziqli differensial tenglamadir. Uning umumiy yechimi

$$x = A(p)c + B(p) \tag{19}$$

ko‘rinishda bo‘ladi; bunda $A(p)$, $B(p)$ funksiyalar (18) dan topiladi, $c-$ixtiyoriy o‘zgarmas. Endi (19) ni (17) ga qo‘yib, y ni p ning funksiyasi sifatida topamiz:

$$y = \tilde{A}(p)c + \tilde{B}(p) \ \left(\tilde{A}(p) = A(p)\psi(p),\ \tilde{B}(p) = B(p)\psi(p) + \chi(p)\right). \tag{20}$$

(19) va (20) formulalar Lagranj tenglamasining bir parametrli yechimlar oilasini beradi. (18) tenglamani hosil qilishda $p - f(p) \neq 0$ deb faraz qilingan. Demak, biz $p - f(p) = 0$ tenglamaning p_i $(i = 1,2,3\ldots)$ ildizlari orqali topiluvchi $p = p_i$ yechimlarini yo‘qotgan bo‘lishimiz mumkin. $p = p_i$, $f(p_i) = p_i$ ni (17) qo‘yib Lagranj tenglamasining yana ushbu

$$y = p_i x + g(p_i) \quad (i = 1,2,3\ldots) \tag{21}$$

ko‘rinishdagi yechimlarini - to‘g‘ri chiziqlarni - topamiz. Ba’zi hollarda (21) yechimlar maxsus yechimlarni ifodalaydi. ☝

III.5. Lagranj tenglamasida $f(p) \equiv p$ bo‘lgan holni qaraylik. Bu holda ***Klero tenglamasi*** deb ataluvchi ushbu

$$y = xy' + g(y') \tag{22}$$

differensial tenglama hosil bo‘ladi. Bu tenglamani yechish uchun ham $y' = p$ deb parametr kiritamiz. U holda

$$y = xp + g(p) \tag{23}$$

munosabat hosil bo‘ladi. Endi x ni p ning funksiyasi ko‘rinishidagi ifodasini topamiz. $dy = y'dx = pdx$ va (23) tengliklardan

$$pdx = pdx + xdp + g'(p)dp \text{ yoki } (x + g'(p))dp = 0. \tag{24}$$

Oxirgi (24) tenglama

$$dp = 0 \quad \text{va} \quad x + g'(p) = 0 \tag{25}$$

tenglamalarga ajraladi. Ularning birinchisidan $p = c$ ($c-$const), va buni (23) ga qo‘yib, Klero tenglamasi (22) ning quyidagi bir parametrli yechimlar oilasini hosil qilamiz:

$$y = cx + g(c). \tag{26}$$

(25) dagi ikkinchi tenglamadan x ni p parametrning funksiyasi sifatida topiladi:

$$x = -g'(p) \quad . \tag{27}$$

(27) ni (22) ga qo‘yib, y ni p ning funksiyasi sifatida topamiz. Natijada Klero tenglamasining yana bir yechimi (parametrik ko‘rinishdagi) topiladi:

$$\begin{cases} x = -g'(p) \\ y = -pg'(p) + g(p). \end{cases} \tag{28}$$

Ko‘rsatish mumkinki, agar $g''(p)$ ikkinchi tartibli hosila uzluksiz bo‘lsa va u nolga aylanmasa, u holda (28) yechim (26) yechimlarning o‘ramasidan iborat bo‘ladi. Shunday qilib, bu holda parametrik ko‘rinishda berilgan (28) yechim Klero tenglamasining maxsus yechimini beradi. 👍

Masalalar

Tenglamalarni yeching. Maxsus yechimlarni toping (**1 - 12**):

1. $y'^2 - 4xy' + 4y = 0$.

2. $y'^3 - xy' + y = 0$.

3. $xy'^2 - 2yy' - 2y + 3x = 0$.

4. $2y^2y'^2 - 2xy'^2 + 4yy' + 1 = 0$.

5. $xy'^2 - y' - e^{-2y} = 0$.

6. $2xy' - y(2 + \ln y') = 0$.

7. $2y' - \ln y' + 2y - x = 0$.

8. $y'^2 \cos^4 x - y' \dfrac{\sin 2x}{2} + y = 0, \ |x| < \dfrac{\pi}{2}$

9. $xy' + \dfrac{y'}{\sqrt{1 + y'^2}} - y = 0$.

10. $2y'^3 - 3y'^2 - y + x = 0$.

11. $y(y - 2xy')^3 - y'^2 = 0$.

12. $y'^4 - 8y(2y - xy')^2 = 0$.

Mustaqil ish № 7 topshiriqlari:

I. Berilgan differensial tenglamadan y' ni toping va uni yeching. Maxsus yechimlarni aniqlang.

1. $yy'^2 - 2xy' + y = 0$.

2. $xy'^2 + 2xy' - y = 0$.

3. $y'^2 - 2yy' - y^2(e^x - 1) = 0$.

4. $4y'^2 - 4y' + \sin 2x = 0$.

5. $y'^3 + xe^y = 0$.

6. $xy'^2 - 2yy' + 4x = 0$.

7. $4y^3y'^2 - 4xy' + y = 0$

8. $(y' + 1)^3 - (x + y)^2 = 0$.

9. $y'^2-(2x+y)y'+(x^2+xy)=0$.

10. $yy'^2+(x-y)y'-x=0$.

11. $yy'^2-(xy+1)y'+x=0$.

12. $xy'^2-2yy'+2y=0$.

13. $y^2(1+y'^2)=1$.

14. $y'^2-yy'+xy-x^2=0$.

15. $x^3y'^2+x^2yy'+y^2=0$.

16. $y'^2+(y-x)y'=2y^2+2xy$.

17. $y'^3-y'^2-2yy'+2y=0$.

18. $y'^2-4xy'+5x^2-y=0$.

19. $y'^2-2yy'-x^2y^2=0$.

20. $xy'^2-4yy'+x=0$.

21. $y'^3-yy'=0$.

22. $y'^2-(y^2+1)y'+y^2=0$.

23. $y'^3-yy'^2-x^2y'+x^2y=0$.

24. $y'^3-(x+1)e^y=0$.

25. $yy'^2+(x+1)(1+y)=0$.

26. $(y'+2)^3-27(y+2x)^2=0$.

27. $y'^2+4x^3(y-1)=0$.

28. $y'^3-(x-1)y^2=0$.

29. $(1+x^2)y'^2+4y^2-4y^3=0$.

30. $xy'(xy'+y)-2y^2=0$.

31. $y'^3-(2x+1)e^{3y}=0$.

32. $y'^2+2yy'-y^2(e^x-1)=0$.

33. $(xy'+y)^2-4x=0$.

34. $y'(2y-y')-y^2\sin^2 x=0$.

35. $y'^2+y^2-y^4=0$.

36. $(y-2xy')^2-y^2=0$.

37. $y'^2-2y^2y'-x^4-2x^2y^2=0$.

38. $y'^2-2xy'+2xy-y^2=0$.

39. $y'^2+(\cos x-2xy)y'-$

$-2xy\cos x=0$.

40. $(xy'-y)^2-y^4=0$

II. Quyidagi differensial tenglamalarni parametr kiritish metodi yordamida yeching. Maxsus yechimlarni toping.

1. $5y=\dfrac{x^3}{5y'^2}+2xy'$.

2. $xy'-2y=\sqrt{1+y'^2}$.

3. $y'^3-x^3(1-y')=0$ $(y'=tx)$.

4. $y'^3+y^3-3yy'=0$ $(y=y't)$.

5. $y=x^3y'^2-2xy'$.

6. $y=2xy'+\ln y'$.

7. $y=xy'+\dfrac{1}{2y'}$.

8. $y=y'(1+y'\cos y')$.

9. $y = xy'^2 - \dfrac{1}{y'}$.

10. $y^4 - y'^4 - yy'^2 = 0$.

11. $xyy' = 1 - x^2y'^2 \left(y' = p, \mu = \mu(xp)\right)$

12. $y^2 + y'^2 = 1\ (y = \cos t, y' = \sin t)$.

13. $x(1 + y'^2)^{3/2} = a$.

14. $xy'^2 = \exp(1/y')$.

15. $x = \dfrac{y}{y'} + \dfrac{1}{y'^2}$.

16. $y + y'^2 = x(x + y')$.

17. $x = y'^2 - 2y' + 2$.

18. $y'^3 = xyy' - y^2$.

19. $y = 3xy' + \ln y'$.

20. $y = 5xy' + \sin y'$.

21. $x = y' + \sin y'$.

22. $y = \dfrac{3}{2}xy' + e^{y'}$.

23. $y = (y' - 5)\exp(y')$.

24. $y = xy'\sqrt{1 + y'^2}$.

25. $y = xy' + y'^2$.

26. $y = xy' + \dfrac{1}{y'^2}$.

27. $xy' = y + x\sqrt{1 + y'^2}$.

28. $y' + y = 2xy'^2$.

29. $x = y' + \cos y'$.

30. $y = \dfrac{xy'}{2} + \dfrac{y'^2}{x^2}$.

31. $x = y' \cos y'$.

32. $x^3y'^2 + x^2yy' + 1 = 0$.

33. $y = (1 + y)\sqrt{1 - y'}$.

34. $y = xy' - 2y'^2$.

35. $2y'^2(y - xy') = 1$.

36. $y = \arcsin y' + \ln(1 + y'^2)$.

37. $y'^2 - y'^3 - y^2 = 0$.

38. $x = \ln y' + \sin y'$.

39. $y^{\frac{2}{7}} + y'^{\frac{2}{7}} = 1\ (y' = \cos^7 t, y' = \sin^7 t)$.

40. $y^{\frac{2}{3}} + y'^{\frac{2}{3}} = 1\ (y = \cos^3 t, y' = p = \sin^3 t)$

8. BIRINCHI TARTIBLI ARALASH DIFFERENSIAL TENGLAMALAR

Maqsad – differensial tenglamalarning qaysi sinfga taaluqli ekanligini aniqlash va uni yechish ko‘nikmalarini mustahkamlash.

Yordamchi ma'lumotlar

Birinchi tartibli differensial tenglamalarning quyidagi sinflariga tegishli bo'lgan tenglamalarni yechish usullari bilan tanishdik:

O'zgaruvchilari ajraladigan differensial tenglamalar:

$$M(x)N(y)dx + P(x)Q(y)dy = 0 \qquad (1)$$

$$y' = f(x)g(y) \qquad (2)$$

Bunday tenglamalarni o'zgaruvchilarni ajratib yechishni o'rgandik.

O'zgaruvchilari ajraladigan tenglamaga keltiriluvchi tenglamalar:

$$y' = f(ax + by + c) \qquad (b \neq 0) \qquad (3)$$

Bu yerda $y = y(x)$ noma'lum funksiya o'rniga yangi $u = u(x)$ noma'lum funksiyaga $u = ax + by + c$ formula bilan o'tib, uni yechish mumkin.

O'zgaruvchilariga nisbatan bir jinsli differensial tenglamalar:

$$y' = g\left(\frac{y}{x}\right) \left(\text{yoki } \; y' = g\left(\frac{x}{y}\right)\right) \qquad (4)$$

Bu tenglamani $y = xu, u = u(x)$, belgilash yordamida o'zgaruvchilari ajraladigan tenglamaga olib kelib yechish mumkin.

Bir jinsli tenglama almashtirishidan foydalanib yechiladigan tenglamalar:

$$y' = \frac{y}{x} + g\left(\frac{y}{x}\right)h(x) \qquad (5)$$

tenglamani ham $y = xu, u = u(x)$, almashtirish yordamida

$$xu' = g(u)h(x)$$

o'zgaruvchilari ajraladigan tenglamaga keltirib yechish mumkin.

Bir jinsli tenglamaga keltiriluvchi tenglamalar:

$$y' = g\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right) \qquad (a_1, b_1, c_1, a_2, b_2, c_2 - \text{const}) \qquad (6)$$

ko'rinishdagi tenglama, agar $a_1b_2 - a_2b_1 \neq 0$ bo'lsa, bir jinsli tenglamaga keltiriladi.

Umumlashgan bir jinsli tenglamalar.

Ba’zi tenglamalarni $y=z^{\alpha}$ yoki $y=-z^{\alpha}$ almashtirish yordamida bir jinsli tenglamaga keltirish mumkin bo‘ladi. Bunday tenglamalar ***umumlashgan bir jinsli tenglamalar*** ded ataladi.

Birinchi tartibli chiziqli differensial tenglama ushbu

$$y'+p(x)y=q(x) \quad \text{yoki} \quad a(x)y'+b(x)y=c(x) \tag{7}$$

ko‘rinishga ega. Bunday tenglamalarni Eyler-Bernulli, Lagranj, integrallovchi ko‘paytuvchi usullari bilan yechishni o‘rgandik.

Bernulli tenglamasi:

$$y'+p(x)y=q(x)y^{m} \ (m=\text{const}\neq 1)\,. \tag{8}$$

Bu tenglamani Eyler-Bernulli usuli yoki $u=1/y^{m-1}$ almashtirish yordamida yechish mumkin.

To‘la differensialli tenglama:

$$M(x,y)dx+N(x,y)dy=0\,, \tag{9}$$

bunda $M(x,y)dx+N(x,y)dy=du(x,y)$, ko‘rinishga ega. Uning yechimi $u(x,y)=c$ formula bilan oshkormas ko‘rinishda aniqlanadi.

Integrallovchi ko‘paytuvchi.

Ba’zi tenglamalarni integrallovchi ko‘paytuvchiga ko‘paytirib yechish mumkin.

Rikkati tenglamasi

$$y'=p(x)y^{2}+q(x)y+r(x) \tag{10}$$

ko‘rinishga ega. Umumiy holda bu tenglama kvadraturalarda yechilmaydi. Agar Rikkati tenglamasining biror (xususiy) yechimi $y=y_1(x)$ topilgan bo‘lsa, $y=u+y_1(x)$ almashtirish yordamida bu tenglama $u=u(x)$ noma’lum funksiyaga nisbatan Bernulli tenglamasiga keltiriladi.

Ushbu

$$y=f(x,y') \quad (y \text{ ga nisbatan yechilgan})$$

va

$$x=f(y,y') \quad (x \text{ ga nisbatan yechilgan})$$

ko‘rinishdagi differensial tenglamalarning yechimini $p = y'$ parametr kiritish yordamida parametrik ko‘rinishda topish mumkin.

Xususan, Lagranj $(y = xf(y') + g(y'))$ va Klero $(y = xy' + g(y'))$ tenglamalarining yechimlari ham $p = y'$ parametr yordamida ifodalanishi mumkin.

Berilgan birinchi tartibli differensial tenglamani yuqorida sanab o‘tilgan sinflarning qaysi biriga mansub ekanligini aniqlash va uni ma’lum algoritmga ko‘ra yechish kerak. Agar berilgan tenglama bu sinflarning birortasiga ham tegishli bo‘lmasa, u yoki bu almashtirishni bajarib, uni ”yechiladigan” ko‘rinishga olib kelish va yechish kerak.

Ba’zan berilgan bitta tenglamalarni turli usullar bilan yechish mumkin bo‘ladi.

Misol 1. $xdy - ydx = 0$ tenglamani yeching.

1- usul. Tenglamani $\mu = \dfrac{1}{x^2}$ integrallovchi ko‘paytruvchiga ko‘paytirib yechamiz:

$$\frac{xdy - ydx}{x^2} = 0\,,\ d\frac{y}{x} = 0\,,\ \frac{y}{x} = c\,.$$

Bunda $x = 0$ yechim yo‘qoldi.

2- usul. Tenglamani $\mu = \dfrac{1}{xy}$ integrallovchi ko‘paytruvchiga ko‘paytirib, o‘zgaruvchilarni ajratib yechish:

$$\frac{xdy - ydx}{xy} = 0\,,\ \frac{dy}{y} - \frac{dx}{x} = 0\,,\ d\big(\ln|y| - \ln|x|\big) = 0\,,\ \ln\left|\frac{y}{x}\right| = c_1\,,\ y = cx\,.$$

$x = 0 -$ yo‘qolgan yechim.

3- usul. Tenglamani chiziqli tenglamaga keltirib yechish ham mumkin:

$$\frac{dy}{dx} - \frac{1}{x}y = 0,\ \dots\,.$$

Ba’zan kerakli (muvaffaqiyatli) shakl almashtirishlar yordamida tenglamani osongina yechiladigan ko‘rinishga keltirish va uni yechish mumkin.

Misol 2. Tenglamani yeching. $(x^2 + 2x + y)dx = (x - 2x^2 y)dy$.

🗝Tenglamada quyidagicha shakl almashtirishni bajaraylik:

$$(x^2+2x)dx+\underline{ydx-xdy}+2x^2ydy=0.$$

Agar chap tomondagi 2- va 3- hadlarni guruhlasak, nisbatning differensialiga "o‘xshash" ifodani ko‘ramiz. Boshqa hadlarning ko‘rinishidan kelib chiqib, tenglamani x^2 ga (y^2 ga emas!) bo‘lamiz, ya’ni $\mu=1/x^2$ ga ko‘paytiramiz:

$$\left(1+\frac{2}{x}\right)dx+\frac{ydx-xdy}{x^2}+2ydy=0.$$

Endi bu yog‘i tushunarli:

$$d(x+\ln x^2)-d\left(\frac{y}{x}\right)+d(y^2)=0,\ \ d\left(x+\ln x^2-\frac{y}{x}+y^2\right)=0.$$

Demak, yechim

$$x+\ln x^2-\frac{y}{x}+y^2=c$$

oshkormas ko‘rinishda ifodalanadi.👍

Ba’zi tenglamalarni yechish uchun noma’lum funksiya va uning argumentini almashtirish kerak bo‘ladi.

Misol 3. Tenglamani yeching $ydx-xdy=3y^3\sqrt{x^2+y^2}dy$.

🗝Tenglamani $\mu=1/y^2$ ga ko‘paytirib, quyidagi shakl almashtirishlarni bajaramiz va yechimni topamiz ($y>0$ deb hisoblaymiz):

$$\frac{ydx-xdy}{y^2}=3y\sqrt{x^2+y^2}dy,\ d\frac{x}{y}=3y\sqrt{x^2+y^2}dy,\ d\frac{x}{y}=3y^2\sqrt{\left(\frac{x}{y}\right)^2+1}dy,$$

$$u=\frac{x}{y}\ \text{deymiz},\ \frac{du}{\sqrt{u^2+1}}=3y^2\,dy,\ d\left(\operatorname{arcsh}u\right)=d(y^3)\ ,$$

$$\operatorname{arcsh}u-y^3=c,\ \operatorname{arcsh}\frac{x}{y}-y^3=c.$$ 👍

Masalalar

Tenglamalarni yeching

1. $(x-1)y'-xy=4y^2$.

2. $xdy-(y+2y^2)dx=0$.

3. $xyy'+x^2=2$.

4. $xyy'=3x^2-2y^2$.

5. $(y+2xy\ln y)dx+(1+x^2)dy=0$.

6. $xy'=-3y(1-xy)$.

7. $(x+2y)y'=2x+y$.

8. $(2xy-1)y'+y^2=1$.

9. $(y^2-3x^2)y'+2xy=0$.

10. $x(y^3+\ln x)y'+y=0$.

11. $xy'-2y=x^3, y(0)=0$.

12. $y'+2y=x\sqrt[3]{y}$.

13. $(x^2-y^2+y)dx+x(2y-1)dy=0$.

14. $y'+2xy-x\sqrt[3]{y}=0$.

15. $ydx+x(1+2\ln x+2\ln y)dy=0$.

16. $(x+2y^2)y'+y=0$.

17. $x\sin y\,dx+(1-x)\cos y\,dy=0$.

18. $2xyy'=3y^2-x$.

19. $\cos y\,dx=(x\sin y-y^3)dy$.

20. $y'+2xy=x^3y^3$.

21. Barcha yechimlarni toping

$$y=x\left(y'\right)^2+\left(y'\right)^2.$$

22. $\left(x^2-y^2\right)y'=xy.$

23. $(\cos y+x\sin y)y'=1$.

24. $e^x\operatorname{tg}y+y'(e^x-1)\sec^2 y=0$.

25. $(x+y-2)dx+(x-y-2)dy=0$.

26. $y'=\sqrt{y-2x-1}$.

27. $(x\cos y+\sin 2y)y'=x$.

28. $(x+y+1)^2y'=(y+1)^2$.

29. $y'=\left(\dfrac{3y-1}{x+2y+1}\right)^2$.

30. $y'+3x^2y^2+2xy=0$.

31. $(2x+xy^2)dx+(x^2y+1)dy=0$.

32. $xy'+2y+\sin x=0$.

33. $(x^2y^2-y-1)dx-xdy=0$.

34. $(x^3y^2-x)y'+x^2y^3+y=0$.

35. $y'=x^2-y^2+1$.

36. $y'^2-4xyy'+4y^2=0$.

37. $4y'=x-\arcsin y'$.

38. $y'^2-4xyy'+8y^2=0$.

39. $xy'^2-2yy'+x=0$.

40. $xy'+(x^3+\ln y)y=0$.

41. $y'^2-4y^3=0$.

42. $xy'-y=x^2, y(0)=0$.

43. $2y'^2+x^3y'-2x^2y=0$.

44. $xy'=y+(x-y)\ln\dfrac{x-y}{x}$.

45. $(x+y^2)\sin(2x)dx-2y\cos^2 x\,dy=0$.

46. $x(e^y-y')=2$.

47. $3x^2y(x+\ln y)dx=(2y^2-x^3)dy$.

48. $(2x+\ln y)y'-ydx=0$.

49. $x(xe^y-2)y'=1$.

50. $y'+24x^4y^4+6xy=0$.

51. $x(e^y-y')=1$.

52. $x(y-y')=y\ln y$.

53 $(x^2y^3+xy)y'=1$.

54 $2ydx-(\ln y+2x-1)dy=0$.

55. $y'=y-e^{-x^2}$ tenglamaning $\lim\limits_{x\to-\infty}y(x)=\lim\limits_{x\to+\infty}y(x)=0$ shartlarni qanoatlantiruvchi yechimlarini toping.

56 $y'^2-2y'-y^2=0$.

57. $(y-x^2)y'=x$.

58. $(y^2-3xy-2x^2)dx+x(y-x)dy=0$.

59. $(y^2+x^2+x)dy-ydx=0$.

60. $(y+x)^2y'=1$.

61. $6xy^2y'+2x^3+x=0$.

62. $xy'+y=1+\ln x$.

63. $y'\cos x-y\sin x=\sin 2x$.

64. $y'-xy=-y^3e^{-x^2}$.

65. $(\sin 2x-y\cos x)dx-dy=0$.

66. $x\sqrt{1-y}dx-y\sqrt{1-x^2}dy=0$.

67. $(2x+3y)y'+y=0$.

68. $(2xy+1)dx+\dfrac{2y-x}{y}dy=0$.

69. $(x^3\ln y-xy)y'+y=0$.

70. $y'\cos x+y\sin x+2y^2\cos x=0$.

71. $\sqrt{y'}=xy'+y$.

72. $(x^2+y+1)y'-2x=0$.

73. $2y'=y^2+\dfrac{2}{x^2}$.

74. $(2x^2+x+y)dx=(x-3x^2y)dy$.

75. $y^2dx+(e^x-y)dy=0$.

76. $(4x+y)^2y'=1$.

77. $y'=-y^2+x^2-2x$.

78. $(x^2y\ln y-x)y'=y$.

79. $xdx+(y-x^2-1)dy=0$.

80. $y'x\ln x-y=2x^2\ln^2 x$.

81. $(x\cos y+\sin 2y)y'=1$.

82. $(y'+x)^2=4y^2$.

83. $yy'-2xy'^2=1$.

84. $x(xy-3)dy+y(xy-1)dx=0$.

85. $(x+4y)y'=2x+3y-1$.

86. $x^3(y'-x)=y^2$.

87. $x^3(y'-2x)=y^2$.

88. $(y'+1)\ln\dfrac{y+x}{x+3}=\dfrac{y+x}{x+3}$.

89. $2x^2y'=y^3+xy$.

90. $y\left(1+\sqrt{x^2y^4+1}\right)dx+2xdy=0$.

91. $y'=y/(3x-y^2)$.

92. $(x^2-2y+1)y'=x$.

93. $x^2y'+xy+x^2y^2=4$.

94. $(x^2-1)y'\sin y+2x\cos y=2x-x^2$.

95 $(x+1)(yy'-1)=y^2$.

96. $3y'+2/y^2=y^2$.

97. $xy'=\sqrt{y}\arcsin x-2y$.

98. $y'-2xy+y^2=5-x^2$.

99. $yy'(yy'-1)=x-y^2$.

100. $y=\dfrac{x}{2}+y'-\dfrac{1}{2}\ln y'$.

101. $y^2dx+(e^x-y)dy=0$.

102. $ydx+x(y^2-x)dy=0$.

103. $yy'-2x^2+y^2=0$.

104. $3y\,dx=(5x-y^2)dy$.

105. $3x^2y'-1=y^2$.

106. $(x-4y^5)y'=y$.

107. $2x\cos ydx=(1+x^2\sin y)dy$.

108. $\left(x-\dfrac{4}{y}\right)y'=y$.

109. $\dfrac{dx}{2x}+\left(3x-\dfrac{1}{y}\right)dy=0$.

110. $xy'-1/\cos y=3y'$.

111. $(x-1)y'+2xy=4$.

112. $x(x+1)(y'-1)=2y$.

113. $y'-y=\dfrac{5}{y}\sin x$.

114. $(4xy+1)y'=y^2$.

115. $y'\sqrt{x}=\sqrt{y+x}-\sqrt{x}$.

116. $xy'+4y=\sqrt{y}\,\mathrm{arctg}\,x$.

117. $2y'-x=e^{y'}$.

118. $ydx-xdy=y\sqrt{x^2+y^2}dx$.

119. $y'=y^2+2xy+x^2$.

120. $y'=y^2-2xy+x^2$.

121. $y'=\sin(y-x)$.

122. $(x+1)(y'-1)=2y$.

123. $xy'^2=y'-e^{-2y}$.

124. $y'+2y+2e^{-x}=e^xy^2$.

125. $y'+2y\sin x=y^2+\cos x+\sin^2 x$.

126. $(e^y+2xy)dx+(xe^y+x^2-1)dy=0$.

127. $y(1+3x^2\ln y)dx+(3y^3+x^3)dy=0$.

128. $(x-3x^2y+e^y)dx+(y-x^3+xe^y)dy=0$.

129. $\left(\sin xy + xy\cos xy\right)dx + x^2\cos xydy = 0$.

130. $(x^4 + 2y + 1)y' = 4x^3$.

131. $x(x-1)y' = (y+1)(y+3x^2)$.

132. $y'\sin x = 2x - y\cos x$.

133. $y(y^2 + x^2)dx = x(x^2 + y^2 - 2xy^3dy)$.

134. $y'\mathrm{tg}\, y - 2x = \cos y$.

135. $(y^2 + 2e^x)dx + (2xy - 3e^y)dy = 0$.

136. $y' = \left((6x + 2y^3 + 1)/y\right)^{1/2}$.

137. $(\cos x + y)dx + (x + e^y)dy = 0$.

138. $ydx - (2x + 1 + \ln y)dy = 0$.

139. $2y(x^2 + 1)y' = 3x^2 - 2xy^2 - 1$.

140. $(2x^3 - \sin^2 y)dx + x(2xy - \sin 2y)dy = 0$.

141. $(1 + y^2)(2 + x^2)dx + (1 + x^2)(3 + y^2)dy = 0$.

142. $\frac{1}{\sqrt{1-x^2}}\left(1 - x\sqrt{1-y^2}\right)dx + \frac{1}{\sqrt{1-y^2}}\left(1 - y\sqrt{1-x^2}\right)dy = 0$.

143. $\sqrt{y}(\sqrt{y} + 2x\sqrt{x})dx + \sqrt{x}(\sqrt{x} + 2y\sqrt{y})dy = 0$.

144. $4ye^{x/y}dx + \left(y - 4xe^{x/y}\right)dy = 0$ **(** $y > 0$ **).**

145. $(e^x + y)y' + 2y^2 = 0$.

146. $y'^2 + 2y^2 = 3yy'$.

147. $\sin x + y'\cos^2 x\cos y = 0$.

148. $e^{x-y} + e^{y-x}y' = 0$.

149. $xy^2y' - y^3 = x^4$.

150. $2ydx + (4x + 2x\ln x + x\ln y)dy = 0$.

151. $yy' - 2x\sqrt{y+1} = 0$.

152. $(1 + xy)y' = 1 + y^2$.

153. $(x + ye^y)y' = 2$.

154. $(4x^2y - 2y\ln y)dx - xdy = 0$.

155. $y' = 1 + \frac{1}{3}(x-y)^3$.

156. $xy' - e^{x-y} + 2 = 0$

9. YUQORI TARTIBLI DIFFERENSIAL TENGLAMALAR. ULARNING TARTIBINI PASAYTIRISH VA YECHISH

Maqsad – yuqori tartibli differensial tenglama uchun Koshi masalasi yechimining mavjudligi va yagonaligi to`g`risidagi teoremani va tenglama tartibini pasaytirish metodlarini o`rganish

Yordamchi ma'lumotlar:

I. Koshi masalasi yechimining mavjudligi va yagonaligi.

Ushbu

$$F(x, y, y', ..., y^{(n)}) = 0 \tag{1}$$

n-tartibli differensial tenglamani qaraylik ($n \geq 2$). Biz bu tenglamani yuqori tartibli hosila $y^{(n)}$ ga nisbatan yechiladi deb faraz qilamiz:

$$y^{(n)} = f(x, y, y', ..., y^{(n-1)}); \tag{2}$$

bunda $f(x, y, p_1, ..., p_{n-1})$ funksiya biror $D(f) \subset \mathbb{R}^{1+n}$ to`plamda aniqlangan va uzluksiz deb hisoblanadi..

Lipshits sharti. Agar shunday $L > 0$ mavjud bo`lib, ixtiyoriy $(x, y, p_1, ..., p_{n-1}) \in E$ va $(x, \tilde{y}, \tilde{p}_1, ..., \tilde{p}_{n-1}) \in E$ nuqtalar uchun

$$\left| f(x, y, p_1, ..., p_{n-1}) - f(x, \tilde{y}, \tilde{p}_1, ..., \tilde{p}_{n-1}) \right| \leq L\left(|y - \tilde{y}| + |p_1 - \tilde{p}_1| + ... + |p_{n-1} - \tilde{p}_{n-1}| \right) \tag{3}$$

bo`lsa, $f(x, y, p_1, ..., p_{n-1})$ funksiya $E$ to`plamda $y, p_1, ..., p_{n-1}$ o`zgaruvchilarga nisbatan Lipshits shartini qanoatlantiradi deyiladi.

Teorema (Lipshits sharti uchun yetarli shart). Agar

1. $G \subset \mathbb{R}^n$ sohaga uning ixtiyoriy $(x, y, p_1, ..., p_{n-1}) \in G$ va $(x, \tilde{y}, \tilde{p}_1, ..., \tilde{p}_{n-1}) \in G$ nuqtalar bilan birgalikda ularni tutashtiruvchi kesma $\left\{ (x, y + \theta(\tilde{y} - y), p_1 + \theta(\tilde{p}_1 - p_1), ..., p_{n-1} + \theta(\tilde{p}_{n-1} - p_{n-1})) \,\middle|\, 0 < \theta < 1 \right\}$ ham tegishli bo`lsa;

2. G sohada barcha

$$\frac{\partial f}{\partial y}, \frac{\partial f}{\partial p_1}, ..., \frac{\partial f}{\partial p_{n-1}}$$

xususiy hosilalar mavjud va chegaralangan bo`lsa, u holda $f$ funksiya $G$ sohada $y, p_1, ..., p_{n-1}$ o`zgaruvchilarga nisbatan Lipshits shartini qanoatlantiradi.

Koshi masalasi**.** Ushbu $x_0, y_0, y_0', ..., y_0^{(n-1)}$ haqiqiy sonlar berilgan va $(x_0, y_0, y_0', ..., y_0^{(n-1)}) \in D(f)$ bo`lsin. n-tartibli (1) tenglamaning ushbu

$$y(x_0) = y_0, y'(x_0) = y_0', \ldots, y^{(n-1)}(x_0) = y_0^{(n-1)}$$

Koshi shartlar (boshlang`ich shartlar)ni qanoatlantiradigan yechimini biror $I$ $(x_0 \in I)$ oraliqda topish Koshi masalasi (yoki boshlang`ich masala) deyiladi va bu masala ushbu

$$\begin{cases} y^{(n)} = f(x, y, y', ..., y^{(n-1)}) \\ y(x_0) = y_0, y'(x_0) = y'_0, \ldots, y^{(n-1)}(x_0) = y_0^{(n-1)} \end{cases} \tag{4}$$

ko`rinishda yoziladi. E'tirof etish kerakki, boshlang`ich shartlar (Koshi shartlari) ning barchasi bitta x_0 nuqtada qo`yilgan.

Teorema (mavjudlik va yagonalik teoremasi - MYaT). Ushbu

$П = \left\{(x, y, p_1, ..., p_{n-1}) \in \mathbb{R}^{1+n} \,\middle|\, |x - x_0| \le a, |y - y_0| \le b, |p_1 - y'_0| \le b, ..., |p_{n-1} - y_0^{(n-1)}| \le b\right\}$

$(a > 0, b > 0)$

parallelipipedda $f(x, y, p_1, ..., p_{n-1})$ funksiya birato`la barcha argumentlari bo`yicha uzluksiz va $(y, p_1, ..., p_{n-1})$ o`zgaruvchilarga nisbatan Lipshits shartini ham qanoatlantirsin. U holda (4) Koshi masalasi biror $|x - x_0| \le h$ oraliqda aniqlangan $y = y(x)$ yagona yechimga ega bo`ladi.

Izoh. Teoremadagi h musbat sonni quyidagicha tanlash mumkin:

$$h = \min\left\{a, \frac{b}{\max\limits_{П}\{m, |p_1|, ..., |p_{n-1}|\}}\right\};$$

bu yerda

$|f(x, y, p_1, ..., p_{n-1})| \le m$ $((x, y, p_1, ..., p_{n-1}) \in П$, $m-$ o`zgarmas musbat son (m son mavjud, chunki $П-$ chegaralangan va yopiq, ya'ni u kompakt va $f : П \to \mathbb{R}$ uzluksiz).

Misol 1. Ushbu

$$\begin{cases} y''' = \cos(y'^2) + 2y''y - x^2 \\ y(1) = 0, y'(1) = -1, y''(1) = 2 \end{cases} \tag{5}$$

masalaga Koshi masalasi yechimining mavjudligi va yagonaligi to`g`risidagi teoremani tatbiq etaylik.

⊶ Berilgan tenglamaning o`ng tomonidagi $f(x, y, p_1, p_2) = \cos(p_1^2) + 2p_2 y - x^2$ funksiyaning $C^1(\mathbb{R}^4)$ sinfga tegishliligi ravshan. Biz $П$ -parallelipiped sifatida

$П = \{(x, y, p_1, p_2) \in \mathbb{R}^4 \mid |x-1| \le 2, |y| \le 1, |p_1 + 1| \le 1, |p_2 - 2| \le 1\}$ ni olaylik.

Ushbu

$$\left|\frac{\partial f}{\partial y}\right| = |2p_2| \le 6, \ \left|\frac{\partial f}{\partial p_1}\right| = \left|-\sin(p_1^2) \cdot 2 \cdot p_1\right| \le 4, \ \left|\frac{\partial f}{\partial p_2}\right| = |2y| \le 2.$$

baholashlardan berilgan tenglamaning o`ng tomoni Lipshits shartini qanoatlantirishi kelib chiqadi. U holda MYaT ga ko`ra biror $|x-1| \le h$ oraliqda yagona yechim mavjud.

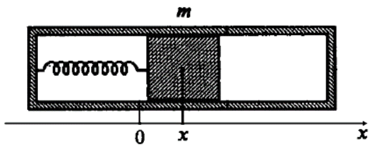
Izoh. Ravshanki, $П$ da $|p_1| \le 2$, $|p_2| \le 3$ bo`lgani uchun

$$\max_{П}(16, |p_1|, |p_2|) = 16 \text{ va } \quad h = \min\left\{2, \frac{1}{\max\limits_{П}(16, |p_1|, |p_2|)}\right\} = \min\left\{2, \frac{1}{16}\right\} = \frac{1}{16}.$$

Demak, (5) masala $|x-1| \le \frac{1}{16}$ oraliqda yagona yechimga ega.

Yuqori tartibli differensial tenglama uchun Koshi masalasiga keltiriluvchi bir misol qaraylik.

Masala. m massali yuk (moddiy nuqta) bikirlik koeffitsienti k bo`lgan prujina yordamida konteyner devoriga bog`langan. Konteyner yopishqoqlik koeffitsienti λ bo`lgan suyuqlik bilan to`ldirilgan (8.1- rasm). Yukka x o`qining $x_0$ nuqtasida $v_0$ boshlang`ich tezlik berildi ($x=0$ yukning tinch (muvozanat) holatiga mos keladi). Yukning x o`qi bo`ylab harakatini ifodalovchi tenglama va boshlang`ich shartlarni yozing.



8.1- rasm. Konteynerdagi yuk.

Yuk markazi $x = x(t)$ nuqtada bo`lganda unga $x$ o`qi bo`ylab prujinaning elastiklik kuchi $F_{el} = -kx$ (Guk qonuni) va suyuqlikning qarshilik kuchi $F_{qar} = -\lambda \frac{dx}{dt}$ ta'sir etadi. Nyutonning ikkinchi qonuniga ko`ra $ma = F$, bunda $a = \frac{d^2x}{dt^2} -$ tezlanish, $F -$ ta'sir etuvchi kuch., $F = F_{el} + F_{qar} = -kx - \lambda \frac{dx}{dt}$. Demak, harakat tenglamasi

$$m\frac{d^2x}{dt^2} = -kx - \lambda \frac{dx}{dt},$$

ya'ni

$$m\frac{d^2x}{dt^2} + \lambda \frac{dx}{dt} + kx = 0$$

ko`rinishga ega. Harakat $x_0$ nuqtadan $v_0$ tezlik bilan boshlangan bolsin. Bunga quyidagi boshlang`ich shartlar mos keladi:

$$x(0) = x_0, \ \frac{dx(0)}{dt} = v_0.$$

II. Differensial tenglama tartibini pasaytirish va uni yechish usullari

II.1. $n-$**tartibli differensial tenglama ushbu**

$$y^{(n)} = f(x), \quad f(x) \in C\big((a,b)\big),$$

**ko`rinishda** bo`lsin. U holda uning barcha yechimlari majmuasini ketma-ket n marta integrallash yordamida topish mumkin. Integrallash tartibini o`zgartirib, yechimni ushbu

$$y=\frac{1}{(n-1)!}\int_{x_0}^{x} f(t)(x-t)^{n-1}dt+\frac{y_0^{(n-1)}}{(n-1)!}\cdot(x-x_0)^{n-1}+$$
$$+\frac{y_0^{(n-2)}}{(n-2)!}\cdot(x-x_0)^{n-2}+\ldots+y_0(x-x_0)+y_0 \tag{6}$$

formula bilan bersa ham bo`ladi. Bu yerda $x_0\in(a,b),\ y_0,y_0',...,y_0^{(n-1)}$ lar esa ixtiyoriy o`zgarmas sonlar.

Misol 2. Ushbu

$$\begin{cases}\dfrac{d^2x}{dt^2}=a\\ x(t_0)=x_0,\dfrac{dx(t_0)}{dt}=v_0\end{cases}$$

Koshi masalasi yechilsin $(a,t_0,x_0,v_0-$o`zgarmaslar).

(6) formulaga ko`ra bu yechim mana bu

$$x=\int_0^t a(t-y)dy+v_0t+x_0=\frac{at^2}{2}+v_0t+x_0$$

to`g`ri chiziqli tekis o`zgaruvchan harakat formulasini beradi ($t-$ vaqt, $x=x(t)$ – harakatlanuvchi nuqta koordinatasi).

II.2. Agar (1) tenglamada $y,y',...,y^{(k-1)}$ **lar bevosita qatnashmasa**, ya'ni tenglama

$$F(x,y^{(k)},y^{(k+1)},...,y^{(n)})=0 \qquad (1\le k\le n) \tag{7}$$

ko`rinishda bo'lsa (k -tartibli hosila tenglamada qatnashgan), u holda yangi noma'lum z funksiyani

$$z=y^{(k)}$$

formula bilan kiritib, (7) tenglamani $(n-k)$-tartibli tenglamaga keltirish mumkin:

$$F(x,z,z',...,z^{(n-k)})=0. \tag{8}$$

Faraz qilaylik, hosil qilingan (8) tenglamaning

$$z=\omega(x,c,...,c_{n-k})$$

yechimi topilgan bo`lsin ($c_1,...,c_{n-k}$ -ixtiyoriy o`zgarmaslar). Endi dastlabki y noma'lumga qaytib, 1.bandda o`rganilgan ushbu

$$y^{(k)}=\omega(x,c,...,c_{n-k})$$

tenglamadan $y=\omega(x,c,...,c_{n-k},...,c_n)$ $(c_1,c_2,..,c_n$- ixtiyoriy o`zgarmaslar) yechimni hosil qilamiz.

Misol 3. $y'''=2xy''$ tenglamani yeching.

Bu tenglamada y,y' qatnashmagan. $z=y''$ deb $z=z(x)$ ga nisbatan ushbu

$$z'=2xz$$

birinchi tartibli chiziqli bir jinsli tenglamani hosil qilamiz. Uning umumiy yechimi $z=c_1e^{x^2}$.

Demak, $z = y''$ belgilashga ko`ra $y'' = c_1 e^{x^2}$, va (6) formulaga ko`ra esa berilgan tenglamaning yechimini topamiz

$$y = \int_0^x c_1 e^{t^2}(x-t)dt + c_2 x + \tilde{c}_3$$

yoki

$$y = c_1 x \int_0^x e^{t^2} dt - \frac{c_1}{2} e^{x^2} + c_2 x + c_3 .$$

II.2.1. Ushbu

$$F(y^{(n-1)}, y^n) = 0 \qquad (n > 1) \tag{9}$$

tenglama berilgan bo`lsin. Faraz qilaylik, bu tenglama $y^{(n)}$ ga nisbatan yechilsin:

$$y^{(n)} = f(y^{(n-1)}).$$

Bu yerda $y^{(n-1)} = z$ deb, yangi noma'lum funksiya kiritamiz va $z' = f(z)$ o`zgaruvchilari ajraladigan tenglamani hosil qilamiz.

Demak, $\dfrac{dz}{f(z)} = dx,\ z = \omega(x, c_1),\ y^{(n-1)} = \omega(x, c_1)$ tenglamani integrallab izlangan yechimni hosil qilamiz. $f(z) = 0$ bo`lishi mumkin bo`lgan hollar alohida tekshirilishi kerak, chunki bu holda yechim yo`qolishi mumkin.

Agar (9) tenglama $y^{(n)}$ ga nisbatan oddiygina yechilmasa, lekin uni

$$y^{(n-1)} = \varphi(t), \quad y^{(n)} = \psi(t) \tag{10}$$

ko`rinishda parametrlashtirish mumkin bo`lsa, u holda $dy^{(n-1)} = y^{(n)} dx$ va (10) ga ko`ra

$$dx = \frac{dy^{(n-1)}}{y^{(n)}} = \frac{\varphi'(t)dt}{\psi(t)}$$

tenglikni va bu yerdan

$$x = \int \frac{\varphi'(t)dt}{\psi(t)} + c_1 \tag{11}$$

bog`lanishni topamiz. Agar (10) dagi birinchi munosabatda parametr $t$ ni $y-$ noma'lum funksiya o`zgaruvchisi sifatida qaralsa 1.bandga ko`ra uning yechimini

$$y = \varphi_n(t, c_2, c_3, ..., c_n) \tag{12}$$

ko`rinishda yozib olish mumkin. (11) va (12) lardan $x$ va $y$ larning ($n$ parametrli) yechimlar oilasini parametrik ko`rinishda hosil qilamiz (t-parametr; $c_1, ..., c_n$- ixtiyoriy o`zgarmaslar).

Misol 4. Ushbu $y''' = y''^2$ tenglamani yeching.

Tenglamada y, y' oshkor ko`rinishda (bevosita) qatnashmagan. $y'' = z$ yangi o`zgaruvchini kiritamiz. Demak, $z' = z^2$. Bu tenglamaning $z = 0$ yechimi mavjudligi ravshan. Qolgan yechimlarni o`zgaruvchilarni ajratish yordamida topamiz:

$$\frac{dz}{z^2} = dx, \quad \int \frac{dz}{z^2} = \int dx, \quad -\frac{1}{z} = x + c_1, \quad z = -\frac{1}{x + c_1}, \quad c_1 \in \mathbb{R} .$$

Endi dastlabki noma’lumga qaytamiz. Ushbu

1) $y''=0$ va 2) $y''=-\dfrac{1}{x+c_1}$ tenglamalardan $y=c_1x+c_2$ va

$y=c_3-(x+c_1)\ln(c_2(x+c_1))$

yechimlar majmuasini hosil qilamiz.

(7) tenglamaning muhim ikki holini qaraylik.

II.2.2. Tenglama ushbu

$$F(y^{(n-2)},y^{(n)})=0 \tag{13}$$

**ko`rinishda** bo`lsin. Faraz qilaylik, bu tenglama $y^{(n)}$ ga nisbatan yechilsin:

$$y^{(n)} = f(y^{(n-2)})\,. \tag{18}$$

$z=y^{(n-2)}$ yangi noma’lum funksiyani kiritaylik. U holda

$$z''=f(z)$$

ikkinchi tartibli tenglamani hosil qilamiz. Bu tenglamaning ikkala tomonini $2z'\cdot dx$ ga ko`paytiramiz:

$$2z'\cdot z''dx=2f(z)\cdot z'dx;$$

Hosil bo`lgan bu tenglamaning shaklini o`zgartiramiz:

$$(z'^2)'=2f(z)dz;$$

Buni integrallab,

$$z'^2=2\int f(z)dz+c_1,\quad z'=\pm\sqrt{2\int f(z)dz+c_1}\;,$$

$$\pm\frac{dz}{\sqrt{2\int f(z)dz+c_1}}=dx,\quad z=\varphi(x,c_1,c_2)$$

yechimni topamiz. Demak, $y^{(n-2)}=\varphi(x,c_1,c_2)$. Oxirgi ko`rinishdagi tenglamani biz 1.bandda o`rgandik.

Faraz qilaylik, (13) tenglama $y^{(n)}$ ga nisbatan oddiygina yechilmasin, ammo uni parametrlashtirish mumkin bo`lsin:

$$y^{(n-2)}=\varphi(t),\quad y^{(n)}=\psi(t).\quad F(\varphi(t),\psi(t))\equiv 0\,. \tag{19}$$

Quyidagilarga egamiz:

$$dy^{(n-1)}=y^{(n)}dx,\quad dy^{(n-2)}=y^{(n-1)}dx$$

Bu yerdagi birinchi ifodaning har ikkala tomonini $y^{(n-1)}$ ga ko`paytirib, hamda ikkinchi ifodani hisobga olib topamiz:

$$y^{(n-1)}dy^{(n-1)}=y^{(n)}y^{(n-1)}dx=y^{(n)}dy^{(n-2)}$$

yoki

$$d\left([y^{(n-1)}]^2\right)=2\psi(t)\cdot\varphi'(t)dt\,.$$

Bundan ushbu

$$y^{(n-1)}=\pm\sqrt{2\int\psi(t)\varphi'(t)dt+c_1}=\pm\chi(t,c_1)$$

tenglikni hosil qilamiz va bunga $y^{(n-2)}=\varphi(t)$ tenglikni birlashtirib II.1 bandda o`rganilgan holga kelamiz.

II.3. Erkli o`zgaruvchini bevosita o`z ichiga olmagan, ya'ni avtonom tenglama.

Avtonom tenglama

$$F(y, y', y'', ..., y^{(n)}) = 0 \qquad (20)$$

ko`rinishda bo`ladi. Unda x erkli o`zgaruvchi oshkor ko`rinishda qatnashmagan.

$y' = z$ yangi noma'lum funksiyani kiritamiz va erkli o`zgaruvchi sifatida $y$ ni qabul qilamiz, ya'ni $z = z(y)$ deymiz. (20) tenglama $z = z(y)$ noma'lum funksiyaga nisbatan $n-1$- tartibli tenglamaga keladi. Buning uchun $y'', y'''..., y^{(n)}$ larni $z = z(y)$ va uning $y$ bo`yicha hosilalari orqali ifodalaymiz:

$$y'' = \frac{dy'}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = \frac{dz}{dy} \cdot z$$

$$y''' = \frac{dy''}{dx} = \frac{d}{dx}\left(\frac{dz}{dy} \cdot z\right) = \frac{d}{dy}\left(\frac{dz}{dy} \cdot z\right) \cdot \frac{dy}{dx} = \left(\frac{d^2 z}{dy^2} z + \left(\frac{dz}{dy}\right)^2\right) z$$

$$\dots\dots\dots\dots\dots\dots\dots$$

$$y^{(n)} = \omega\left(z, \frac{dz}{dy}, ..., \frac{d^{n-1}z}{dy^{n-1}}\right)$$

va ularni (20) tenglamaga qo`yamiz:

$$F\left(y, z, \frac{dz}{dy} \cdot z, ..., \omega\left(z, \frac{dz}{dy}, ..., \frac{d^{n-1}z}{dy^{n-1}}\right)\right) = 0. \qquad (21)$$

Bu (21) tenglama esa $n-1$ tartiblidir. Agar (21) ni yechib,

$$z = \varphi(y_1, c_1, ..., c_{n-1})$$

yechimni topsak, dastlabki noma'lum y ga qaytib, ushbu

$$y' = \varphi(y, c_1, ..., c_{n-1})$$

birinchi tartibli tenglamani hosil qilamiz.

Biz yuqorida y noma'lum funksiyani o`zgaruvchi sifatida qaradik, bunda $y = const$ yechimni yo`qotishimiz mumkin. Shuning uchun (20) ga $y = b$ qo`yib, hosil bo`lgan ushbu

$$F(b, 0, ..., 0) = 0$$

tenglamani yechib, $b = b_i$ ildizlarni topsak, u holda (20) tenglamaning $y = b_i$ ko`rinishdagi o`zgarmas yechimlariga ega bo`lamiz.

Misol 5. $y \cdot y'' + y = y'^2$ tenglamani yeching.

🗝 Bu tenglama avtonom, ya'ni unda x erkli o`zgaruvchi oshkor ko`rinishda qatnashmagan. Tenglamaning tartibini pasaytirish uchun $y' = z$ deymiz va $z = z(y)$ noma'lum funksiyani izlaymiz. Bunda $y = b -$ const yechim yo`qolishi mumkin. Berilgan tenglamada $y = b$ deb, $b = 0$ ekanligini topamiz. Demak, $y = 0 -$yechim.

Endi $y'' = \frac{dy'}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = \frac{dz}{dy} \cdot z$ va $y' = z$ larni berilgan avtonom tenglamaga qo`yib, uni ushbu $y\cdot\dfrac{dz}{dy}z+y=z^2$ ko`rinishga keltiramiz. Uni $z^2=u$ deb chiziqli tenglamaga keltirib yechish mumkin. Biz boshqacha ish tutamiz: differensiallash qoidalaridan foydalanib, quyidagilarni yozamiz:

$$y\cdot\frac{dz}{dy}z+y=z^2,\quad yzdz+ydy=z^2dy,\ \ 2yzdz-2z^2dy=-2ydy,$$

$$yd(z^2)-\frac{1}{y}z^2d(y^2)=-2ydy\quad(:y^3),\qquad \frac{y^2d(z^2)-z^2d(y^2)}{y^4}=-\frac{2dy}{y^2},$$

$$d\frac{z^2}{y^2}=d\frac{2}{y},\qquad d\left(\frac{z^2}{y^2}-\frac{2}{y}\right)=0.$$

Bundan

$\dfrac{z^2}{y^2}-\dfrac{2}{y}=c_1$, ya’ni $z^2=2y+y^2c_1$, yoki $(y')^2=2y+y^2c_1$ $(c_1\in\mathbb{R})$ ni hosil qilamiz.

Oxirgi differensial tenglamani c_1 o`zgarmasning turli qiymatlarida yechib, ushbu

$$c_1^{\,2}y+1=\pm\mathrm{ch}(c_1x+c_2)\text{ va } c_1^{\,2}y-1=\pm\cos(c_1x+c_2),\ y=0,\ y=\frac{1}{2}(x+c_1^{\,2})$$

yechimlar oilasini hosil qilamiz. ☝

II.4. Noma’lum funksiya va uning hosilalariga nisbatan bir jinsli tenglamalar.

Agar ushbu

$$F(x,y,y'...,y^{(n)})=0 \tag{18}$$

tenglamada ixtiyoriy $\lambda>0$ (yoki $\lambda<0$) va biror α uchun

$$F(x,\lambda y,\lambda y',...,\lambda y^{(n)})=\lambda^{\alpha}F(x,y,y',...,y^{(n)}) \tag{19}$$

ayniyat (shart) bajarilsa, (18) tenglama $y,y',...,y^{(n)}$ larga nisbatan bir jinsli tenglama deyiladi. Bu tenglama $y(x)\to\lambda\cdot y(x)$ $(\lambda=\mathrm{const})$ almashtirishga nisbatan invariant. Bunday tenglamani yechish uchun

$$\frac{y'}{y}=z$$

deb, yangi noma’lum $z=z(x)$ funksiyani kiritamiz.

Quyidagi hisoblashlarni bajaramiz:

$$y'=yz,\quad y''=y'z+yz'=(yz)z+yz'=y(z^2+z')$$

$$y'''=y(z^3+3zz'+z''),\ ...,\ y^{(n)}=y\omega(z,z',...,z^{(n-1)})$$

Endi bu munosabatlarni (18)ga qo`yib, va $F$ funksiyaning bir jinsli ekanligini e’tiborga olsak (bu yerda $\lambda$ o`rniga y noma’lum funksiya kelyapti), ushbu

$$y^{\alpha}F(x,1,z,z^2+z',...,\omega(z,z',...,z^{(n-1)}))=0$$

tenglikka kelamiz. Oxirgi tenglikni y^{α} ga qisqartirib, z ga nisbatan $(n-1)$ tartibli tenglamani hosil qilamiz. Agar hosil bo`lgan tenglamaning ushbu

$$z=\varphi(x,c_1,c_2,...,c_{n-1})$$

yechimini topa olsak, bu yerda z ni $\frac{y'}{y}$ ga almashtirib

$$\frac{y'}{y}=\varphi(x,c_1,c_2,...,c_{n-1})$$

tenglamaga ega bo`lamiz.

Nihoyat, oxirgi tenglamadan

$$y=c_n\exp\left(\int\varphi(x,c_1,c_2,...,c_{n-1})dx\right)$$

yechimni hosil qilamiz. Bu yechim $c_n=0$ bo`lganda $y=0$ yechimni o`z ichiga oladi, ya'ni yuqorida $\alpha>0$ bo`lganda $y^{\alpha}$ ga qisqartirish bajarilganda yechim yo`qolmaydi.

Misol 6. Ushbu $(x^2+1)(y'^2-yy'')=xyy'$ tenglamani yeching.

Berilgan tenglamada

$$F(x,y,y',y'')=(x^2+1)(y'^2-yy'')-xyy'.$$

Osongina tekshirib ko`rish mumkinki, (19) bir jinslilik sharti $\alpha=2$ bilan bajariladi. Shuning uchun $y'=yz$ $(z=z(x))$ almashtirishni bajarib, berilgan tenglamani birinchi tartibli tenglamaga keltirish mumkin.

$$y''=y(z^2+z')$$

bo`lgani uchun berilgan tenglama quyidagicha o`zgaradi:

$$(x^2+1)(y^2z^2-yy(z^2+z'))=xyyz,$$

$$y^2(x^2+1)(z^2-z^2-z')=xy^2z.$$

Oxigi tenglamani $y^2(x^2+1)$ ga bo`lib, ushbu

$$z'=-\frac{x}{x^2+1}\cdot z$$

chiziqli tenglamani hosil qilamiz. Bundan

$$z=c_1\exp\left(\int\left(-\frac{x}{x^2+1}\right)dx\right)=c_1\exp\left(-\frac{1}{2}\ln(x^2+1)\right)=\frac{c_1}{\sqrt{x^2+1}}$$

yechimni topamiz. Endi $y'=yz$, ya'ni $y'=\frac{c_1}{\sqrt{x^2+1}}y$ ekanligidan

$$y=c_2\exp\left(\int\frac{c_1}{\sqrt{x^2+1}}dx\right)=c_2\exp\left(c_1\ln\left|x+\sqrt{x^2+1}\right|\right)=$$

$$=c_2\left|x+\sqrt{x^2+1}\right|^{c_1}$$

izlangan yechimga ega bo`lamiz.

II.5. x **argumentga nisbatan ekvio`lchamli tenglamalar.**

Agar (18) tenglama uchun ushbu

$$F(\lambda x,y,\lambda^{-1}y',...,\lambda^{-n}y^{(n)})=\lambda^{\alpha}F(x,y,y',...,y^{(n)})\ \left(\lambda=\text{const}>0\right)$$

shart o`rinli bo`lsa, bu differensial tenglama x **argumentga nisbatan ekvio`lchamli tenglama** deyiladi. Bunday tenglama erkli o`zgaruvchini $x=\lambda\xi$ ($\lambda - const$) almashtirishga nisbatan invariant. Haqiqatan ham, aytilgan almashtirishda

$$x=\lambda\xi,\ \frac{d}{dx}=\frac{1}{\lambda}\frac{d}{d\xi},\ \frac{d^2}{dx^2}=\frac{1}{\lambda^2}\frac{d^2}{d\xi^2},\ \dots,\ \frac{d^n}{dx^n}=\frac{1}{\lambda^n}\frac{d^n}{d\xi^n},$$

$$0=F(x,y,y',...,y^{(n)})=F\Big(\lambda\xi,y,\frac{1}{\lambda}\frac{dy}{d\xi},...,\frac{1}{\lambda^n}\frac{d^ny}{d\xi^n}\Big)=\lambda^{\alpha}F\Big(\xi,y,\frac{dy}{d\xi},...,\frac{d^ny}{d\xi^n}\Big),$$

$$F\Big(\xi,y,\frac{dy}{d\xi},...,\frac{d^ny}{d\xi^n}\Big)=0.$$

x argumentga nisbatan ekvio`lchamli tenglamada $x$ o`rniga t erkli o`zgaruvchini $x=e^t$ (bunda $x>0$; agar $x<0$ bo`lsa, $x=-e^t$ deyish kerak) formula bilan kiritib, va y noma’lum funksiyani t ning funksiyasi deb hisoblab, tenglamani o`sha tartibli avtonom tenglamaga (y ga nisbatan) keltirish mumkin:

$$x=e^t,\ \frac{d\bullet}{dx}=\frac{dt}{dx}\frac{d\bullet}{dt}=e^{-t}\frac{d\bullet}{dt},\ \ x\frac{d\bullet}{dx}=\frac{d\bullet}{dt},$$

$$x^2\frac{d^2\bullet}{dx^2}=x\frac{d\bullet}{dx}\Big(x\frac{d\bullet}{dx}\Big)-x\frac{d\bullet}{dx}=\frac{d^2\bullet}{dt^2}-\frac{d\bullet}{dt},\ \ \frac{d^2\bullet}{dx^2}=e^{-2t}\Big(\frac{d^2\bullet}{dt^2}-\frac{d\bullet}{dt}\Big),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$0=F(x,y,y',y'',...)=F\Big(e^t,y,e^{-t}\frac{dy}{dt},e^{-2t}\Big(\frac{d^2y}{dt^2}-\frac{dy}{dt}\Big),...\Big)=$$

$$=e^{t\alpha}F\Big(1,y,\frac{dy}{dt},\frac{d^2y}{dt^2}-\frac{dy}{dt},...\Big),$$

$$F\Big(1,y,\frac{dy}{dt},\frac{d^2y}{dt^2}-\frac{dy}{dt},...\Big)=0.$$

x argumentga nisbatan ekvio`lchamli tenglamaga misol sifatida ushbu

$$a_nx^ny^{(n)}+a_{n-1}x^{n-1}y^{(n-1)}+\ldots+a_1xy'+a_0y=0$$

Eyler tenglamasini keltirish mumkin; bu tenglama uchun

$$F(x,y,y',...,y^{(n)})=a_nx^ny^{(n)}+a_{n-1}x^{n-1}y^{(n-1)}+\ldots+a_1xy'+a_0y=$$

$$=F(\lambda x,y,\lambda^{-1}y',...,\lambda^{-n}y^{(n)})\ \ (\alpha=0).$$

x argumentga nisbatan ekvio`lchamli bo`lgan nochiziqli tenglamaga misol sifatida

$xy''+y^2y'=0$ tenglamani ko`rsatish mumkin. Bu tenglama uchun

$$F(x,y,y',y'')=xy''+y^2y',$$

$$F(\lambda x,y,\lambda^{-1}y',\lambda^{-2}y'')=\lambda x\lambda^{-2}y''+y^2\lambda^{-1}y'=$$

$$=\lambda^{-1}(xy''+y^2y')=\lambda^{-1}F(x,y,y',y'')\ \ (\alpha=-1).$$

Misol 7. Ushbu

$$xy'' + y^2 y' = 0$$

tenglamani yeching.

Berilgan tenglama x argumentga nisbatan ekvio`lchamli bo`lgani uchun tenglamada $x = e^t$ ($x > 0$) almashtirish bajaramiz. Bunda

$$x = e^t, \quad \frac{dy}{dx} = e^{-t}\frac{dy}{dt}, \quad \frac{d^2y}{dx^2} = e^{-2t}\Big(\frac{d^2y}{dt^2} - \frac{dy}{dt}\Big)$$

tengliklarni berilgan tenglamaga qo`yib va e^{-t} ga qisqartirib, ushbu

$$\frac{d^2y}{dt^2} + (y^2 - 1)\frac{dy}{dt} = 0$$

avtonom tenglamaga kelamiz. Bu avtonom tenglamani yuqorida aytilgan usul yordamida yechamiz. Ushbu

$$\frac{dy}{dt} = z, \; z = z(y); \; \frac{d^2y}{dt^2} = \frac{dz}{dt} = \frac{dz}{dy}\cdot\frac{dy}{dt} = z\frac{dz}{dy}$$

almashtirishlarga ko`ra

$$z\Big(\frac{dz}{dy} + y^2 - 1\Big) = 0$$

tenglamani hosil qilamiz. Bundan $z = 0$, ya'ni $y = c_1$ va

$$\frac{dz}{dy} + y^2 - 1 = 0.$$

Oxirgi birinchi tartibli differensial tenglamaning yechimi, ravshanki,

$$z + \frac{y^3}{3} - y = \frac{c_1}{3}.$$

formula bilan beriladi. Demak, $z = \frac{dy}{dt}$ belgilashga ko`ra

$$\frac{dy}{dt} + \frac{y^3}{3} - y = \frac{c_1}{3}.$$

Bundan

$$t + 3\int\frac{dy}{y^3 - 3y - c_1} = c_2$$

va $x = e^t$ o`zgaruvchiga qaytib, berilgan differensial tenglamaning ushbu

$$\ln x + 3\int\frac{dy}{y^3 - 3y - c_1} = c_2, \; y = c_1$$

oshkormas ko`rinishdagi yechimlarini hosil qilamiz.

II.6. Umumlashgan bir jinsli tenglamalar.

Agar ushbu

$$F(x, y, y'..., y^{(n)}) = 0 \tag{20}$$

tenglamada $x, y, y'..., y^{(n)}$ larni mos ravishda $\lambda, \lambda^k, \lambda^{k-1}, ..., \lambda^{k-n}$ larga ko`paytirilganda tenglamaning chap tomoni uchun

$$F(\lambda x, \lambda^k y, \lambda^{k-1} y', ..., \lambda^{k-n} y^{(n)}) = \lambda^\alpha F(x, y, y', ..., y^{(n)}) \tag{21}$$

shart bajarilsa, (20) tenglama **umumlashgan bir jinsli tenglama** deyiladi. Bu turdagi tenglamalarni yechish uchun x va y o`rniga

$$x = e^t, \quad y = ze^{kt} \tag{22_0}$$

deb, yangi t erkli o`zgaruvchi va $z = z(t)$ noma'lum funksiyalarni kiritamiz (bu almashtirish $x > 0$ bo`lganda qo`llaniladi, $x < 0$ bo`lganda esa $x = -e^t$ $y = ze^{kt}$ almashtirishdan foydalanish kerak) va avtonom tenglamaga kelamiz. Haqiqatan ham, y ning x bo`yicha hosilalarini yangi noma'lum funksiya $z$ ning $t$ argument bo`yicha hosilasi orqali ifodalaymiz. Ushbu

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{\frac{dx}{dt}} = \frac{dy}{dt} \cdot \frac{1}{e^t},$$

yoki

$$y' = \frac{dy}{dt} e^{-t} \tag{23}$$

munosabatlar ravshandir. (22_0) tengliklardagi ikkinchi formulani t bo`yicha differensiallab,

$$\frac{dy}{dt} = \left(\frac{dz}{dt} + kz \right) e^{kt}$$

tenglikka ega bo`lamiz. Buni (23)ga qo`yib

$$y' = \left(\frac{dz}{dt} + kz \right) e^{(k-1)t} \tag{22_1}$$

munosabatni hosil qilamiz. Biz y ning x bo`yicha hosilasini $z$ ning $t$ bo`yicha hosilasi orqali ifodalovchi munosabatga ega bo`ldik. Xuddi shu yo`sinda ishni davom ettirib, y ning x bo`yicha yuqori tartibli hosilalarini ham $z$ ning $t$ bo`yicha hosilalari orqali ifodalab chiqamiz.

$$y'' = \frac{dy'}{dx} = \frac{dy'}{dt} e^{-t} = \left(\frac{d^2 z}{dt^2} + (2k-1)\frac{dz}{dt} + k(k-1)z \right) e^{(k-2)t}, \tag{22_2}$$

$$y''' = \frac{dy''}{dx} = \frac{dy''}{dt} e^{-t} == \left(\frac{d^3 z}{dt^3} + (3k-3)\frac{d^2 t}{dt^2} + \big(k(k-1) + \right.$$

$$\left. +(k-2)(2k-1) \big) \frac{dz}{dt} + k(k-1)(k-2)z \right) e^{(k-3)t} \tag{22_3}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

va, nihoyat,

$$y^{(n)} = \omega\left(z, \frac{dz}{dt}, ..., \frac{d^n z}{dt^n} \right) e^{(k-n)t} \quad . \tag{22_n}$$

Endi (20) tenglamada (22_0), (22_1), (22_2), ..., (22_n) almashtirishlarni bajarib, ushbu

$$F(e^t, ze^{kt}, (\frac{dz}{dt}+kz)e^{(k-1)t}, ..., \omega(z, \frac{dz}{dt}, ..., \frac{d^n z}{dt^n})e^{(k-n)t}) = 0$$

tenglamani hosil qilamiz. (21) tenglikdagi t o`rniga $e^t$ ni qo`yib, $e^{\alpha t}$ ni F funksiya ishorasi oldiga chiqarib, va unga qisqartirib,

$$F\left(1, z, \frac{dz}{dt}+kz, ..., \omega\left(z, \frac{dz}{dt}, ..., \frac{d^n z}{dt^n}\right)\right) = 0$$

n-tartibli tenglamani hosil qilamiz. Bu avtonom tenglamadir, ya`ni tenglamada erkli o`zgaruvchi t oshkor ko`rinishda qatnashmagan; shuning uchun **5.** bandga ko`ra uning tartibi bittaga pasayadi.

Misol 8. Ushbu

$$yy' + xyy'' - xy'^2 = x^3$$

tenglamani yeching.

🗝 Qaralayotgan tenglamaning chap tomonidagi birinchi had darajasi $k+k-1$ ga teng, chunki birinchi ko`paytuvchi $y$ ni $\lambda^k y$ ga ikkinchi ko`paytuvchini esa $\lambda^{k-1}y'$ ga almashtirganda ko`paytmada $\lambda^{k+k-1}yy'$ hosil bo`ladi, shunga o`xshash, ikkinchi had darajasi $1+k+k-2$, uchinchi had darajasi $1+2(k-1)$ va nihoyat, oxirgi had darajasi 3 ga teng. Barcha darajalarni tenglashtirib, ushbu

$$k+k-1 = 1+k+k-2 = 1+2(k-1) = 3$$

k ni aniqlovchi munosabatni topamiz. Bu tengliklardan $k=2$ ga ega bo`lamiz.

Demak, berilgan umumlashgan bir jinsli tenglamada $x=e^t$, $y=ze^{2t}$ almashtirishni bajarib, tenglamani avtonom turga olib kelsak bo`ladi. Buning uchun $y'$, $y''$ larni yangi noma'lum $z$ funksiya $t$ erkli o`zgaruvchi va z ning t bo`yicha hosilalari orqali ifodalaymiz:

$$y' = \frac{dy}{dx} = \frac{dy}{dt}e^{-t} = (z'e^{2t} + 2ze^{2t})e^{-t} = (z'+2z)e^t$$

$$y'' = \frac{dy'}{dx} = \frac{dy'}{dt}e^{-t} = \frac{d((z'+2z)e^t)e^{-t}}{dt} = z'' + 3z' + 2z$$

Endi berilgan tenglamada zarur almashtirishlarni bajaramiz:

$$ze^{2t}(z'+2z)e^t + e^t ze^{2t}(z''+3z'+2z) - e^t(z'+2z)^2 e^{2t} = e^{3t},$$

$$z(z'+2z+z''+3z'+2z) - (z'^2+4z'z+4z^2) = 1,$$

$$zz'' + 4zz' + 4z^2 - z'^2 - 4zz' - 4z^2 = 1,$$

$$z''z - z'^2 - 1 = 0.$$

Oxirgi avtonom tenglamani yechish uchun, odatdagidek, yangi noma'lum funksiya sifatida $p=z'$ ni olib, z ni erkli o`zgaruvchi sifatida qaraymiz. U holda

$$z'' = \frac{dz'}{dt} = \frac{dz'}{dz}\cdot\frac{dz}{dt} = p'\cdot p \quad \left(p' = \frac{dp}{dz}\right)$$

va

$$p' \cdot p \cdot z - p^2 - 1 = 0$$
$$p' \cdot p \cdot z = p^2 + 1$$
$$\frac{p \cdot dp}{p^2 + 1} = \frac{dz}{z}$$
Oxirgi tenglikni integrallab, $\ln(p^2+1) = 2\ln(|z| \cdot c_1)$ $(c_1 > 0)$ ga ega bo`lamiz, yoki $p^2 + 1 = z^2 \cdot c_1^2$. Bu tenglikda $p = z'$ ga qaytib, $z'^2 + 1 = z^2 \cdot c_1^2$ va bundan $z' = \pm\sqrt{(zc_1)^2 - 1}$ o`zgaruvchilari ajraladigan tenglamaga kelamiz. Uni yechamiz va $x = e^t$ va $y = ze^{2t}$ tengliklarga ko`ra ushbu
$$2c_1c_2 y = c_1^{\,2} x^{2+c_1} + x^{2-c_1} \quad (x > 0)$$
yechimni hosil qilamiz. ✍

II.7. Chap tomoni to`la hosiladan iborat bo`lgan tenglama.

Faraz qilaylik,
$$F(x, y, y', ..., y^{(n)}) = 0 \tag{24}$$
tenglamaning chap tomoni $x, y, y', ..., y^{(n-1)}$ o`zgaruvchilarning funksiyasi bo`lmish biror $\Phi(x, y, y', ..., y^{(n-1)})$ ning x bo`yicha to`la hosilasidan iborat bo`lsin, ya'ni,
$$F(x, y, y', ..., y^{(n)}) = \frac{d}{dx}\Phi(x, y, y', ..., y^{(n-1)}) \tag{25}$$
yoki
$$F(x, y, y', ..., y^{(n)}) = \frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial y}y' + \frac{\partial\Phi}{\partial y'}y'' + ... + \frac{\partial\Phi}{\partial y^{(n-1)}}y^{(n)} \tag{25}$$
tenglik barcha joiz $x, y, y', ..., y^{(n)}$ o`zgaruvchilarga nisbatan aynan bajarilsin.

U holda (24) tenglamaning tartibi bittaga kamayadi va u ushbu
$$\Phi(x, y, y', ..., y^{(n-1)}) = c_1$$
ko`rinishni oladi.

Agar (24)tenglamaning chap tomoni to`la hosiladan iborat bo`lmasa, ba'zi hollarda shunday $\mu = \mu(x, y, y', ..., y^{(n-1)})$ funksiyani topish mumkin bo`ladiki, (24) tenglamani har ikkala tomonini shu $\mu$ ga ko`paytirib, to`la hosilali tenglama hosil qilamiz. Bu $\mu$ funksiya integrallovchi ko`paytuvchi deb ataladi.

Masalan, agar berilgan ikkinchi tartibli differensial tenglama
$$y'' = \frac{\dfrac{\partial G(x,y)}{\partial x} + \dfrac{\partial G(x,y)}{\partial y}y'}{\dfrac{dF(y')}{dy'}} \qquad (G(x,y), F(y') - \text{berilgan}$$
funksiyalar),

ko`rnishda bo`lsa, uni $\mu = \dfrac{dF(y')}{dy'}$ integrallovchi ko`paytuvchiga ko`paytirib, ushbu

$$\frac{dF(y')}{dx}=\frac{dG(x,y)}{dx}, \text{ ya'ni } \frac{d}{dx}\big(F(y')-G(x,y)\big)=0$$

tenglamaga kelamiz. Oxirgi tenglamadan $F(y')=G(x,y)+c_1$ birinchi tartibli differensial tenglamani hosil qilamiz.

Agar

$$y'''=\frac{\dfrac{\partial H(x,y,y')}{\partial x}+\dfrac{\partial H(x,y,y')}{\partial y}y'+\dfrac{\partial H(x,y,y')}{\partial y'}y''}{\dfrac{dF(y'')}{dy''}}$$

($H(x,y,y'), F(y'')$ – berilgan funksiyalar).

ko`rinishdagi uchinchi tartibli differensial tenglama berilgan bo`lsa, uni $\mu=\dfrac{dF(y'')}{dy''}$ integrallovchi ko`paytuvchi yordamida ushbu $F(y'')=H(x,y,y')+c_1$ ikkinchi tartibli tenglamaga keltirish mumkin.

Misol 9. Ushbu

$$yy''-y'(y'+1)=0$$

tenglamani yeching.

⊶ Bu tenglamani yechish uchun $\mu=\dfrac{1}{y^2}$ integrallovchi ko`paytuvchini tenglamaning har ikkala tomoniga ko`paytiramiz:

$$\frac{yy''-y'(y'+1)}{y^2}=0. \qquad .$$

Oxirgi tenglikning chap tomoni $\dfrac{y'+1}{y}$ funksiyaning x bo`yicha hosilasidir:

$$\frac{yy''-y'(y'+1)}{y^2}=\frac{1}{dx}\left(\frac{y'+1}{y}\right).$$

Demak,

$$\frac{y'+1}{y}=c_1$$

yoki

$$y'+1=y\cdot c_1, \quad c_1\in\mathbb{R}.$$

Bu tenglamani o`zgaruvchilarni ajratib yechamiz.

$$dy=(yc_1-1)dx, \quad \frac{dy}{c_1y-1}=dx.$$

Integrallashlar va ixchamlashlarni bajaramiz:

$$\ln|c_1y-1|=c_1\cdot x+\tilde{c}_2 \qquad \tilde{c}_2=\ln c_2,\ c_2>0$$

$$c_1y-1=e^{c_1x}\cdot c_2,\ c_2\neq 0.$$

Bundan

$$y = \frac{1}{c_1}(1 + c_2 e^{c_1 x}) \tag{26}$$

yechimni topamiz.

Endi yechish jarayonini analiz qilaylik. Dastlabki tenglamani y^2 ifodaga bo`lishda $y = 0$ yechim yo`qolgan. Bundan tashqari, tenglamani $c_1 y - 1$ga bo`lishda $y = 1/c_1$, $c_1 \neq 0,$ yechimlar (ular $c_2 = 0$ da hosil bo`ladi), $c_1 = 0$ da esa $y' + 1 = 0$ da hosil bo`luvchi $y = -x + c_2$ yechimlar yo`qolgan. (26) formuladan $y = 0$ va $y = -x + c_2$ yechimlar hosil bo`lmaydi.

Javob: $y = \frac{1}{c_1}(1 + c_2 e^{c_1 x})$, $y = 0$, $y = -x + c$. ☝

II.8. Tenglamalarni parametr kiritish usuli bilan yechish.

Ba'zi differensial tenglamalarni parametr kiritish yordamida yechish mumkin bo`ladi. Bu usul bilan misollarda tanishamiz.

Misol 10. Ushbu

$$y''^2 - xy'' + y' = 0$$

tenglamani quyidagichs parametr kiritib yechish mumkin.

🗝 $y'' = t$ deylik. x va y larni t parametrning funksiyasi sifatida topamiz. Berilgan tenglamadan $y' = xt - t^2$ ni hosil qilamiz. $d(y') = y''dx = tdx$ bo`lganligi uchun

$$d(xt - t^2) = tdx, \quad xdt + tdx - 2tdt = tdx, \quad (x - 2t)dt = 0.$$

Oxirgi tenglikdan ikki imkoniyatni topamiz: $x = 2t$ yoki $t = c_1$.

Birinchi imkoniyat. $x = 2t \Rightarrow y' = xt - t^2 = t^2$.

Bundan $dy = y'dx = t^2 d(2t) = \frac{2}{3}d(t^3)$ va $y = \frac{2}{3}t^3 + c_1$. Demak, parametrik yechim $\left\{x = 2t,\ y = \frac{2}{3}t^3 + c_1\right\}$, oshkor ko`rinishi esa $y = \frac{x^3}{12} + c_1$.

Endi ikkinchi imkoniyatni qaraymiz.

$$t = c_1 \Rightarrow y' = xt - t^2 = xc_1 - c_1^2 \Rightarrow y = c_1\frac{x^2}{2} - c_1^2 x + c_2.$$

Javob: $y = \frac{x^3}{12} + c_1$, $y = \frac{c_1}{2}x^2 - c_1^2 x + c_2$. ☝

Misol 11. Ushbu

$$y'' + y''\ln y' = 1$$

tenglamani ham parametr kiritish usuli bilan yechish mumkin.

🗝 $y' = t$ deymiz va x va y larni t parametrning funksiyasi sifatida topamiz. Berilgan tenglamadan $y'' = \frac{1}{1 + \ln t}$. Demak,

$$d(y') = y''dx \Rightarrow dt = \frac{1}{1 + \ln t}dx \Rightarrow x = t\ln t + c_1.$$

$$dy = y'dx = td(t\ln t + c_1) = (t\ln t + t)dt \Rightarrow y = \frac{t^2\ln t}{2} + \frac{t^2}{4} + c_2.$$

Endi parametrik ko`rinishdagi yechimni yozamiz.

Javob: $\left\{ x = t\ln t + c_1\,,\ y = \dfrac{t^2 \ln t}{2} + \dfrac{t^2}{4} + c_2 \right\}.$

II.9. Koshi masalalarini yechish.

Agar berilgan tenglamaning umumiy yechimi topilsa, shu tenglama uchun qo`yilgan Koshi masalasini yechish muammo tug`dirmaydi.

Ba'zan Koshi masalalarini yechish jarayonida boshlang`ich shartlardan foydalanib, yozuvlarni qisqartirish mumkin.

Misol 12. Koshi masalasini yeching

$$yy'' - 2y'^2 + y'^3 = 0,\ y(1) = 1\ ,\ y'(1) = -1\ .$$

Berilgan differensial tenglama avtonom bo`lgani uchun $p = y' = dy/dx,\ p = p(y)$, deymiz. U holda $y'' = \dfrac{dp}{dy}p$ va tenglama quyidagi ko`rinishga keladi:

$$y\frac{dp}{dy}p - 2p^2 + p^3 = 0\ .$$

Boshlang`ich shartga ko`ra $p|_{y=1} = -1\ (\neq 0;2)$. Demak, oxirgi tenglamadan

$$\frac{dp}{p(2-p)} = \frac{dy}{y}$$

munosabat hosil bo`ladi. Bu tenglikni $y=1$ dan $y=y$ gacha integrallaymiz. Bunda $p(1) = -1$ ekanligini hisobga olib, topamiz:

$$\int_{-1}^{p}\frac{dp}{p(2-p)} = \int_{1}^{y}\frac{dy}{y}\,,\ \ \frac{1}{2}\left(\ln\frac{p}{p-2} - \ln\frac{1}{3}\right) = \ln y\ \ ,\ \ \frac{p}{p-2} = \frac{y^2}{3}\ .$$

Belgilashga ko`ra $p = y' = dy/dx$. Demak,

$$\frac{y'}{y'-2} = \frac{y^2}{3}\,,\ \frac{3y'}{y^2} = y' - 2\,,\ \left(\frac{3}{y}\right)' = \left(2x - y\right)'\ .$$

Oxirgi tenglikni $x=1$ dan $x=x$ gacha integrallaymiz. Boshlang`ich shartni hisobga olib, quyidagilarni topamiz:

$$\left.\frac{3}{y}\right|_{y=1}^{y=y} = \left.\left(2x-y\right)\right|_{x=1}^{x=x}\,,\ \frac{3}{y} = 2x - y + 2\,,\ y^2 - 2(x+1)y + 3 = 0\,.$$

Bu kvadrat tenglamani $y(1)=1$ boshlang`ih shartga ko`ra yechib, yechim $y = x + 1 - \sqrt{x^2 + 2x - 2}$ ko`rinishda bo`lishini topamiz.

Javob: $y = x+1-\sqrt{x^2+2x-2}$. 👍

Masalalar

Boshlang`ich masalalarga yechimning mavjudlik va yagonalik teoremasini qo`llang (**1 - 4**):

1. $y'' = 3xy\cos y - e^x y',\ y(0)=1,\ y'(0)=-1$.

2. $y''' = \sqrt[3]{xy+1}+2xy'-1,\ y(1)=1,\ y'(1)=0,\ y''(1)=0$.

3. $y'' = \dfrac{\sin x}{y^2+1}+xyy'+\ln(x+1),\ y(0)=-1,\ y'(0)=1$.

4. $y''' = x - y\cos y + 2xy'^2 + y'',\ y(0)=-2,\ y'(0)=1,\ y''(0)=0$.

Differensial tenglamalarni yeching(**5** – **23**):

5. $xy''' + y'' = 1$.

6. $y'''\mathrm{tg}x = y''$.

7. $4y^3y'' = y^4-1$.

8. $y'' + 2\cos y\sin^3 y = 0$.

9. $y'y''' - 3y''^2 = 0$.

10. $yy'' - y'^2 - 12x^2y^2 = 0$.

11. $yy'' - y'^2 = y^3$.

12. $y^2y''' - 3yy'y'' + 2y'^3 = y^3\cos x$.

13. $(x-1)yy'' + y'^2 = yy'$.

14. $xyy'' + x(\ln x - 1)y'^2 + yy' = 0$.

15. $x^2(y'' + yy') + xy^2 = 2y$.

16. $x^3y''' - y'(2y - 3x^2) = 0$.

17. $x^3y'' + (y - xy')^2 = 0$.

18. $x^4y'' - (y - xy')^2 = 0$.

19. $x^4y'' + (y - xy')^2 = 0$.

20. $y'' - x^2y' = 2xy + 2$.

21. $xy'' - y' = x^2y^2$.

22. $yy'' - y'^2 = y^2$.

23. $y'^2 - yy'' = yy'\sin x + y^3\cos x$.

Koshi masalalarini yeching (**24** – **27**):

24. $yy'' = y'^2 - y'^3 = 0, y(1)=1,\ y'(1)=-1$.

25. $xyy'' + 2x^3y'^2 - yy' = 0,\ y(1)=1,\ y'(1)=2$.

26. $yy'' + y^2y' - y'^2 = 0, y(0)=1,\ y'(0)=-1$.

27. $y'' + y'^2\mathrm{tg}y - 2y'^2\sin 2y = 0, y(0)=0,\ y'(0)=-1$.

Mustaqil ish № 9 topshiriqlari:

I. Koshi masalasiga yechimning mavjudlik va yagonalik teoremasini qo`llang.

1. $y''' = 3y'^2 - \ln y + x;\quad y(0)=1,\quad y'(0)=0,\quad y''(0)=1$.

2. $y'' = 2\ln y' + xy$; $y(1) = 2$, $y'(1) = e$.

3. $y^{IV} = 5y''' - xy''^2 + y^3 + 1$; $y(0) = y'(0) = 1$, $y''(0) = y'''(0) = 0$.

4. $y''' = 5\sqrt{y'+1} + xy^2$; $y(1) = 0$, $y'(1) = 0$, $y''(1) = 1$

5. $y'' = -\dfrac{1}{y^2} + xy^2$; $y(0) = 1$, $y'(0) = -1$

6. $\begin{cases} y^{IV} = y'''^2 - \sqrt{y'' + y' + x} \\ y(0) = 1, y'(0) = -1, y''(0) = 2, y'''(0) = 1 \end{cases}$

7. $y'' = 2\cos y + xy'$; $y(1) = 1$, $y'(1) = -1$.

8. $y''' = \sin^2 y + 5xy'^3 + y''$, $y(0) = y'(0) = 1$, $y''(0) = 2$

9. $y''' = \dfrac{2y'}{1+y^2} - xy''^3$, $y(0) = 0, y'(0) = 1$, $y''(0) = -1$

10. $\begin{cases} y^{IV} = y'''^3 + 2y''^2 + y' + x \\ y(0) = y'(0) = 1, \ y''(0) = y'''(0) = -1 \end{cases}$

11. $y''' = y' - \ln y + 2x$, $y(0) = 1$, $y'(0) = 2$, $y''(0) = 0$.

12. $y'' = 6\ln y' + 3xy + y'$; $y(0) = 3$, $y'(0) = e$.

13. $\begin{cases} y^{IV} = x^2 y''' + 2xy''^3 + y^2 + 5 \\ y(1) = y'(1) = 0, y''(1) = y'''(1) = 3 \end{cases}$

14. $\begin{cases} y''' = \sqrt{xy' + 2} + 3xy^2 + 1 \\ y(1) = 1, y'(1) = 0, y''(1) = 2 \end{cases}$

15. $y'' = -\dfrac{x + \sin x}{y^2 + 1} + xy^2 \cos x$, $y(0) = 0$, $y'(0) = -1$

16. $\begin{cases} y^{IV} = xy'''^3 + 2\sqrt{y'' + 2y' + x} \\ y(0) = -2, y'(0) = 1, y''(0) = 2, y'''(0) = 1 \end{cases}$

17. $y'' = x\cos y + y'$; $y(0) = 1$, $y'(0) = -2$.

18. $y''' = \dfrac{y' + \mathrm{tg}x}{2 + y^2} + xy''^2$, $y(0) = 1, y'(0) = 0$, $y''(0) = -1$

19. $\begin{cases} y^{IV} = xy'''^3 + y''^2 + 3\sin xy' + e^x x \\ y(0) = y'(0) = 1, \ y''(0) = y'''(0) = -1 \end{cases}$

20. $y'' = \dfrac{2x \cdot \cos x}{y^3 + 2} + 3xy^2 \sin x$, $y(0) = 1$, $y'(0) = -1$

21. $\begin{cases} y''' = xyy'^2 + \cos y + xe^x \\ y(0) = -2, y'(0) = 0, y''(0) = 1 \end{cases}$

22. $y'' = 2e^{y'} + xy + 1$; $y(1) = 2$, $y'(1) = e$.

23. $\begin{cases} y^{IV} = y''' - \sin(xy''^2) + y^2 + 1 \\ y(0) = y'(0) = -1, y''(0) = y'''(0) = 0 \end{cases}$

24. $\begin{cases} y''' = \sqrt{y' + 1} - x\cos y^2 + 2 \\ y(1) = -1, y'(1) = y''(1) = 1 \end{cases}$

25. $y'' = -\dfrac{\ln(\sin x + 2)}{y^2 + x} + xy - 1$; $y(0) = 1$, $y'(0) = 2$

26. $\begin{cases} y^{IV} = y''^2 - x\sqrt{y'' + \cos y' + x + 2} + 3 \\ y(0) = 1, y'(0) = y''(0) = -1, y'''(0) = 2 \end{cases}$

27. $y'' = -\cos y + 3e^x y',\ y(1) = 1,\ \ y'(1) = -2$

28. $\begin{cases} y^{IV} = 2\cos xy''' + y''^2 + y^2 + 1 \\ y(0) = y'(0) = 1, y''(0) = y'''(0) = -1 \end{cases}$

29. $y''' = \dfrac{y'}{y^2} + xy''^3,\ y(0) = -1, y'(0) = 1,\ \ y''(0) = -1$

30. $\begin{cases} y^{IV} = -y'''^3 + x^3 y''^2 + y' + \ln(x+1) \\ y(0) = y'(0) = -1,\ \ y''(0) = y'''(0) = 1 \end{cases}$

31. $\begin{cases} y''' = -y' + \ln(y+2) + yx - 2 \\ y(0) = 0, y'(0) = 1, y''(0) = -1 \end{cases}$

32. $y'' = 6\ln y' + 3xy + y'$, $\ y(0) = 3$, $\ y'(0) = e$.

33. $\begin{cases} y^{IV} = y''' - xy'' + y + x^2 + 1 \\ y(1) = y'(1) = -1, y''(1) = y'''(1) = 1 \end{cases}$

34. $y''' = \sqrt{xy+2} + xy - 1$, $y(1) = 1$, $y'(1) = 0$, $y''(1) = 0$

35. $y'' = \dfrac{\sin x}{y^2+1} + xy^2 + \ln(x+1)$, $\ y(0) = 0$, $\ y'(0) = -1$

36. $\begin{cases} y^{IV} = y''' + \sqrt{y'' + y' + \cos x} \\ y(0) = -2, y'(0) = 1, y''(0) = 0, y'''(0) = \cos 1 \end{cases}$

37. $y'' = x + y\cos y + y'^2,\ y(0) = 1,\ \ y'(0) = 0$

38. $y''' = \dfrac{y'}{1+y^2} + xy'',\ y(0) = 1,\ y'(0) = 0,\ y''(0) = 1$

39. $\begin{cases} y^{IV} = \mathrm{tg}\, x \cdot y''' + y''^2 + 3\sin x + e^x + x \\ y(0) = y'(0) = 1,\ \ y''(0) = y'''(0) = 1 \end{cases}$

40. $y'' = \dfrac{x+y}{y^2+2} - 2xy^2$, $y(0) = 1$, $\ y'(0) = 1$

II. Yuqori tartibli differensial tenglamalarni yeching.

1. $xy'' + 2y' = y^2 + xyy'$, $y^3 y'' = 4$.

2. $x^2 y'' = 3y^2 y' - 2xy'$, $xy''' + y'' = \dfrac{1}{2\sqrt{x}}$.

3. $y''' = x + \sin x$, $y'y'' + 2y'^2 - y^2 = 0$.

4. $y''^2 + y'^2 = y'^4$, $x^2 y'' - 2xyy' + x^2 y'^2 + y^2 = 0$.

5. $y''^2 - 2y''y' + 3 = 0$, $x^2 yy'' = (xy' - y')^2$.

6. $y''^2 - 3y' + 2 = 0$, $xyy'' + xy'^2 = yy'$.

7. $y''^2 = x - 1$, $y'' + x^2 y^2 = 0$.

8. $y''^2 + y'''^2 = 4$, $xyy'' - xy'^2 - yy' = 0$.

9. $4y' + y''^2 = 4xy''$, $x^3 y''' + xy' = y$.

10. $y'^2 + yy'' = yy'$, $(x+1)y''' + y'' = 0$.

11. $yy'' - y'^2 = y^2 \ln x$, $y'' = 72y^3$.

12. $xyy'' + xy'^2 = 2yy'$, $2xy'y'' - y'^2 + 1 = 0$.

13. $x^3y'' = (y - xy')^2$, $y'' + y'^2 + 1 = 0$.

14. $x^2yy'' = (y - xy')^2$, $(x-1)y''' + y'' - e^{2x} = 0$.

15. $y'' + y'^2 = e^{-y}$, $yy'' - y'^2 - 15y^2\sqrt{x} = 0$.

16. $2yy'' = 1 + y'^2$, $xy'' + yy' = 0$.

17. $2yy'' = x + y'^2$, $y'y''' - 2y''^2 = 0$.

18. $xy'y'' = y'^2 + x^3$, $y'y'' + y'^2 - y^2 = 0$.

19. $xyy'' = xy'^2 + yy'$, $x^3y'' - (y - xy')(y - xy' - x) = 0$.

20. $xy'' = 3y^2y' - y$, $yy'' - y'^2 = 0$.

21. $yy'' = y' + \dfrac{yy'}{\sqrt{1+x^2}}$, $x^2(2yy'' - y'^2) + 2xyy'' - 1 = 0$.

22. $(xy' - y)y'' + 4y'^2 = 0$, $4x^2y^3y'' + y^4 - x^2 = 0$.

23. $y'y'' - x^2yy' = xy^2$, $xy'' - 3y^2y' = 0$.

24. $2xyy'' + y'^2 = 0$, $y'y''' - 3y''^2 = 0$.

25. $xyy'' + yy' = x^2y'^3$, $x^4y''' + 3x^3y'' - 1 = 0$.

26. $yy' - xyy'' = x^3 + xy'^2$, $y''' + y' = 1$.

27. , $2yy'' - y'^2 = 0$.

28. $x^2y'' + 4y + x^2 = 3xy'$, $(1 - x^2)y'' - 2xy' = 4x^3$.

29. $y''(y' + 2)e^{y'} = 1$, $x^2yy'' + y'^2 = 0$.

30. $y''(1 + 2\ln y') = 1$, $y'' + 2xy'^2 = 0$.

31. $y'' = (1 + y'^2)^{\frac{3}{2}}$, $xyy' + xy'^2 - 2yy' = 0$.

32. $x^4y'' + (xy' - y)^3 = 0$, $y^3y'' = 4$.

33. $y'' = 4e^y$, $xyy'' + xy'^2 = yy'$.

34. $2yy'' = y^2y' + y'^2$, $x^2y''' + xy'' = 4y'$.

35. $3xy'^2y'' = x^4 + 3y'^3$, $xy''' + y'' = 2\sqrt{x}$.

36. $y''\cos y + y'^2\sin y = y$, $x^3y'' + (xy' - y)^2 = 0$.

37. $4y^3y'' + 1 = 16y^4$, $x^4y'' + (xy' - y)^2 = 0$

38. $xy'(yy'' - y'^2) = yy'^2 + x^4y^3$, $xy'' - 2yy' = 0$.

39. $y''' = x - \ln 2x$, $x^3y'' - 2(xy' - y)^4 = 0$.

40. $$\left[\begin{array}{l}(1+x^2)y'' + y'^2 + 1 = 0,\\ x^2(yy'' - y'^2) + xyy' = y\sqrt{x^2y'^2 + y^2}.\end{array}\right.$$

10. O`ZGARUVCHAN KOEFFITSIENTLI CHIZIQLI DIFFERENSIAL TENGLAMALAR

Maqsad – chiziqli erkli funksiyalar, chiziqli differensial tenglamalar tartibini pasaytirish, ular yechimlarini tanlash usuli bilan topish va umumiy yechimni qurishni o`rganish

Yordamchi ma'lumotlar:

$n-$ **tartibli chiziqli differentsial tenglama**

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+...+a_1(x)y'+a_0(x)y=g(x)$$

**ko`rinishga** ega; bunda $y=y(x)-$ noma'lum funktsiya, berilgan $a_n(x)\neq 0, a_{n-1}(x),...,a_1(x),a_0(x)$ koeffitsientlar va $g(x)$ ozod had (o`ng tomon) biror I oraliqda aniqlangan va uzluksiz deb hisoblanadi. Agar kerak bo`lsa, tenglamaning har ikala tomonini $a_n(x)$ ga bo`lib, uni

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+...+a_1(x)y'+a_0(x)y=g(x) \tag{1}$$

ko`rinishga keltirish mumkin. Ushbu

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+...+a_1(x)y'+a_0(x)y=0 \tag{1_0}$$

temglama (1)ga mos (chiziqli) bir jinsli tenglamani ifodalaydi.

1⁰. Qo`yilgan $\{a_{n-1}(x),...,a_1(x),a_0(x),g(x)\}\subset C(I)$ shartlarda (1) tenglamaning

$y(x_0)=y_0,\ y'(x_0)=y_0',\ldots,\ y^{(n-1)}(x_0)=y_0^{n-1}$

$(x_0\in I, y_0, y_0', \ldots, y_0^{n-1}-\text{berilgan sonlar})$ boshlang`ich shartlarni qanoatlantiruvchi yechimi mavjud va yagona. Bu yechim birato`la I oraliqda aniqlangan bo`ladi.

$y_1(x),\ y_2(x),\ldots,y_n(x),\ x\in I,$ funksiyalar berilgan bo`lsin. Agar kamida bittasi noldan farqli bo`lgan $\lambda_1,\lambda_2,\ldots,\lambda_n$ sonlar mavjud bo`lib, ular uchun

$$\lambda_1 y_1(x)+\lambda_2 y_2(x)+\ldots+\lambda_n y_n(x)\equiv 0,\ x\in I,$$

shart ($x\in I$ ga nisbatan ayniyat) bajarilsa, qaralayotgan funksiyalar chiziqli bog`langan (chiziqli erksiz) deyiladi. Aks holda esa ular chiziqli erkli (chiziqli bog`lanmagan) funksiyalar deyiladi.

2⁰. Agar $y_1(x),\ y_2(x),\ldots,y_n(x)$ funksiyalar I oraliqda $n-1$ marta differensiallanuvchi va ularning

$$W(x)=W[y_1(x),y_2(x),\ldots,y_n(x)]=\begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{vmatrix}$$

Vronckiani I oraliqning biror nuqtasida noldan farqli bo`lsa, bu funksiyalar $I$ oraliqda chiziqli erkli bo`ladi.

3⁰. (1_0) tenglamaning n ta chiziqli erkli yechimlari uning fundamental (bazis) yechimlari deyiladi. (1_0) tenglamaning fundamental (bazis) yechimlari mavjud va uning ixtiyoriy yechimi tayin fundamental (bazis) yechimlarning chiziqli kombinatsiyasi kabi ifodalanadi. Agar $y_1(x),\ y_2(x),\ldots,y_n(x)$ funksiyalar (1_0) tenglamaning fundamental (bazis) yechimlari bo`lsa, uning umumiy yechimi

$$y=c_1y_1(x)+c_2y_2(x)+\ldots+c_ny_n(x) \tag{2}$$

formula bilan beriladi; bunda $c_1, c_2, \ldots, c_n -$ ixtiyoriy o`zgarmaslar. Umumiy yechim tenglamaning barcha yechimlarini va faqat ularni ifodalaydi.

4⁰. (1_0) tenglamaning ixtiyoriy n ta $y_1(x), y_2(x), \ldots, y_n(x)$ yechimlarining $W(x)$ Vronskiani uchun Ostrogradskiy-Liuvill formulasi deb ataluvchi ushbu

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} a_{n-1}(s)ds\right) \quad (x_0 \in I,\ x \in I) \tag{3}$$

formula o`rinli. Chiziqli erkli yechimlarning Vronskiani nolga aylanmaydi.

5⁰. Agar $y_1(x), y_2(x), \ldots, y_n(x)$ funksiyalar $C^n(I)$ sinfga tegishli va ularning $W(x) = W[y_1(x), y_2(x), \ldots, y_n(x)]$ Vronskiani I oraliqda nolga aylanmasa, fundamental (bazis) yechimlari shu funksiyalardan iborat bo`lgan ($1_0$) ko`rinishdagi $n-$tartibli yagona chiziqli differensial tenglama mavjud va bu tenglama ushbu

$$\frac{1}{W(x)}\cdot\begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) & y \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) & y' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) & y^{(n-1)} \\ y_1^{(n)}(x) & y_2^{(n)}(x) & \cdots & y_n^{(n)}(x) & y^{(n)} \end{vmatrix} = 0 \tag{4}$$

ko`rinishda bo`ladi.

6⁰. Umumiy ko`rinishdagi ($1_0$) tenglamaning fundamental (bazis) yechimlarini topish usuli mavjud emas. Ba'zi hollarda berilgan tenglamaning ko`rinishidan kelib chiqib, yechimlarni u yoki bu ($y = e^{\lambda x}$, $y = \lambda_n x^n + \lambda_{n-1}x^{n-1} + \ldots + \lambda_1 x + \lambda_0$ va hokazo) ko`rinishida izlashga urinib ko`rish mumkin.

7**⁰.** Agar (1_0) tenglamaning biror $y = y_1(x),\ y_1(x) \neq 0,$ yechimi ma'lum bo`lsa, bu tenglamaning tartibini bittaga kamaytirsa bo`ladi. Buning ychun tenglamada $y = y_1(x)u$ almashtirishni bajaramiz; bunda $u = u(x)$ yangi noma'lum funksiya. Natijada $v = u'$ ga nisbatan tartibi $n-1$ga teng bo`lgan chiziqli bir jinsli tenglamaga kelamiz.

8⁰. Agar ikkinchi tartibli chiziqli bir jinsli tenglama $y'' + a_1(x)y' + a_0(x)y = 0$ ning biror $y = y_1(x),\ y_1(x) \neq 0,$ yechimi ma'lum bo`lsa, tenglamaning bu yechimga chiziqli bog`liq bo`lmagan ikkinchi $y = y(x)$ yechimini Ostrogradskiy-Liuvill formulasi (3) dan foydalanib topish ham mumkin:

$$\begin{vmatrix} y_1(x) & y \\ y_1'(x) & y' \end{vmatrix} = c_2\exp\left(-\int_{x_0}^{x} a_1(s)ds\right) \ (c_2 = W(x_0)),$$

$$y_1(x)y' - y_1'(x)y = c_2\exp\left(-\int_{x_0}^{x} a_1(s)ds\right). \tag{5}$$

Oxirgi birinchi tartibli chiziqli differensial tenglamadan $y = y(x)$ noma'lum funksiya (ikkinchi yechim) osongina topiladi. Masalan, (5) ning har ikkala tomonini $y_1^2(x)$ ga bo`lib, uni

$$\left(\frac{y}{y_1(x)}\right)' = c_2\frac{1}{y_1^2(x)}\exp\left(-\int_{x_0}^{x} a_1(s)ds\right)$$

ko`rinishga keltirish va yechish mumkin.

9^0. Bir jinsli bo`lmagan (1) chiziqli tenglamaning umumiy yechimi uning biror xususiy yechimiga mos bir jinsli tenglama ($1_0$) ning umumiy yechimini qo`shishdan hosil bo`ladi. Agar bir jinsli tenglama ($1_0$) ning fundamental (bazis) yechimlari (umumiy yechimi) ma'lum bo`lsa, (1) tenglamaning xususiy yechimini ixtiyoriy o`zgarmaslarni variatsiyalash usuli (Lagranj usuli) yordamida qurish mumkin.

10^0. Agar (1_0) ning fundamental (bazis) yechimlari $y_1(x), y_2(x), \dots, y_n(x)$ bo`lsa, uning umumiy yechimi (2) formula bilan beriladi. Lagranj metodiga ko`ra bu formuladagi ixtiyoriy o`zgarmaslarni variatsiyalab, (1) ning yechimini topsa bo`ladi. Aniqrog`i, (1) tenglamaning yechimini

$$y = c_1(x)y_1(x) + c_2(x)y_2(x) + \dots + c_n(x)y_n(x) \tag{6}$$

ko`rinishda izlash mumkin. Bu yerdagi $c_1(x), c_2(x), \dots, c_n(x)$ funksiyalar ushbu

$$\begin{cases} c_1'(x)y_1(x) + c_2'(x)y_2(x) + \dots + c_n'(x)y_n(x) = 0, \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) + \dots + c_n'(x)y_n'(x) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ c_1'(x)y_1^{(n-2)}(x) + c_2'(x)y_2^{(n-2)}(x) + \dots + c_n'(x)y_2^{(n-2)}(x) = 0, \\ c_1'(x)y_1^{(n-1)}(x) + c_2'(x)y_2^{(n-1)}(x) + \dots + c_n'(x)y_2^{(n-1)}(x) = g(x) \end{cases} \tag{7}$$

sistemadan aniqlanadi. Oxirgi sistemadan $c_1'(x), c_2'(x), \dots, c_n'(x)$ hosilalar bir qiymatli topiladi, chunki sistemaning bu noma'lumlarga nisbatan determinanti $W[y_1(x), y_2(x), \dots, y_n(x)] \neq 0$ Vronskiandan iborat. $c_1'(x), c_2'(x), \dots, c_n'(x)$ lar topilgach integrallash yordamida $c_1(x), c_2(x), \dots, c_n(x)$ larni va ularni (6) formulaga qo`yib, (1) tenglamaning yechimini topamiz.

Bu yerda shuni e'tirof etaylikki, ba'zan (1) tenglamaning ko`rinishidan kelib chiqib, uning xususiy yechimini u yoki bu ko`rinishda tanlash yo`li bilan topish mumkin bo`ladi.

Misol 1. Ushbu

$$1, \sin x, \cos x \ \ (x \in I \subset \mathbb{R})$$

funksiyalarni chiziqli erklilikka tekshiring.

🗝 Birinchi usul: bevosita ta'rifga ko`ra tekshirish. Quyidagi $x \in I$ ga nisbatan ayniyatni qaraymiz:

$$\lambda_1 \cdot 1 + \lambda_2 \sin x + \lambda_3 \cos x = 0, \ x \in I. \tag{8}$$

U $\lambda_1, \lambda_2, \lambda_3$ larning qanday qiymatlarida o`rinli bo`lishi mumkinligini aniqlaymiz. Ayniyatning har ikkala qismini ketma-ket ikki marta differensiallaymiz:

$$\begin{cases} \lambda_2 \cos x - \lambda_3 \sin x = 0 \\ -\lambda_2 \sin x - \lambda_3 \cos x = 0 \end{cases} \ (x \in I) \ . \tag{9}$$

Oxirgi sistemani λ_2 va λ_3 ga nisbatan yechib, $\lambda_2 = \lambda_3 = 0$ ekanligini topamiz. Endi (8) ga ko`ra $\lambda_1$ ham nolga teng. Demak, (8) ayniyat $\lambda_1 = \lambda_2 = \lambda_3 = 0$ bo`lgandagina o`rinli. Ta'rifga ko`ra berilgan funksiyalar chiziqli erkli.

Ikkinchi usul: Vronskian orqali tekshirish. Berilgan funksiyalarnig Vronskiani (determinantni birinchi ustun bo`yicha yoyamiz)

$$W(x)=\begin{vmatrix}1 & \sin x & \cos x\\ 0 & \cos x & -\sin x\\ 0 & -\sin x & -\cos x\end{vmatrix}=\begin{vmatrix}\cos x & -\sin x\\ -\sin x & -\cos x\end{vmatrix}=-1\neq 0 .$$

Yuqorida keltirilgan $\mathbf{2^0}$ tasdiqqa asosan berilgan funksiyalar chiziqli erkli. ✍

Misol 2. Ushbu

$$1,\ \sin^2 x\ ,\ \cos^2 x\quad (x\in\mathbb{R})$$

funksiyalarni chiziqli erklilikka tekshiring.

⊶Bu funksiyalar ta'rifga ko`ra chiziqli bog`liq, chunki ularning quyidagi notrivial chiziqli kombinatsiyasi aynan nolga teng:

$$(-1)\cdot 1+1\cdot\sin^2 x+1\cdot\cos^2 x\ =0\ \ (x\in\mathbb{R})\ .$$ ✍

Misol 3. Ushbu

$$\sqrt{x-2},\ \sqrt{x-1},\ \sqrt{x}\quad (x\geq 2)$$

funksiyalarni chiziqli erklilikka tekshiring.

⊶Tekshirishni ta'rifga ko`ra bajaramiz. Ushbu

$$\lambda_1\sqrt{x-2}+\lambda_2\sqrt{x-1}+\lambda_3\sqrt{x}=0\quad (x\geq 2) \tag{10}$$

ayniyatda $x=2,\ x=3$ va $x=4$ deb quyidagi teengliklarni hosil qilamiz:

$\lambda_2+\lambda_3\sqrt{2}=0,\ \lambda_1+\lambda_2\sqrt{2}+\lambda_3\sqrt{3}=0,\ \lambda_1\sqrt{2}+\lambda_2\sqrt{3}+\lambda_3 2=0$.

Bu tengliklarni $\lambda_1,\lambda_2,\lambda_3$larga nisbatan chiziqli bir jinsli algebraik tenglamalar sistemasi sifatida qarab, sistemaning determinanti

$$\begin{vmatrix}0 & 1 & \sqrt{2}\\ 1 & \sqrt{2} & \sqrt{3}\\ \sqrt{2} & \sqrt{3} & 2\end{vmatrix}=2\sqrt{2}(\sqrt{3}-1)-2\neq 0$$

bo`lgani uchun, yagona $\lambda_1=\lambda_2=\lambda_3=0$ trivial yechimni topamiz. Demak, (10) ayniyat $\lambda_1=\lambda_2=\lambda_3=0$ bo`lgandagina o`rinli. Bu xulosa berilgan funksiyalarning chiziqli erkli ekanligini (ta'rifga ko`ra) anglatadi. ✍

Misol 4. Ushbu

$$x(x-1)y''+(x^2-2)y'+(x-2)y=0 \tag{11}$$

differensial tenglamaning $y_1(x),y_2(x)$ fundamental (bazis) yechimlari hamda umumiy yechimini toping.

⊶Berilgan ikkinchi tartibli chiziqli differensial tenglamaning yechimini $y=e^{\lambda x}$ $(\lambda=const)$ ko`rinishda izlaymiz. Bu funksiyani berilgan tenglamaga qo`yib, uning qanoatlanishini talab qilamiz:

$$x(x-1)y''+(x^2-2)y'+(x-2)y=\left(x(x-1)\lambda^2+(x^2-2)\lambda+(x-2)\right)e^{\lambda x}=0\ ,$$

$$\lambda(\lambda+1)x^2+(1-\lambda^2)x-2(\lambda+1)=0\ .$$

Oxirgi ayniyatdan $\lambda(\lambda+1)=0,\ 1-\lambda^2=0,\ \lambda+1=0$ bo`lishi kerakligini topamiz. Bu shartlardan $\lambda=-1$ hosil bo`ladi. Demak, berilgan tenglama $y=y_1(x)=e^{-x}$ yechimga ega. Bu yechimga chiziqli bog`liq bo`lmagan ikkinchi $y=y(x)$ yechimni topish uchun Ostrogradskiy-Liuvill formulasi (3) dan foydalanamiz (bu yechimni tenglamada $y = y_1(x)u = e^{-x}u$ almashtirish bajarib ham topsa bo`ladi). Qaralayotgan differensial tenglamaning har ikkala tomonini $x(x-1)$ ga bo`lib,

$a_1(x) = \dfrac{(x^2-2)}{x(x-1)}$ ni topamiz va (5) formulaga ko`ra quyidagilarni hosil qilamiz:

$$y_1(x)y' - y_1'(x)y = c_2 \exp\left(-\int a_1(x)\,dx\right), \quad y' + y = c_2 \frac{x-1}{x^2},$$

$$y'e^x + ye^x = c_2 \frac{x-1}{x^2} e^x, \qquad (ye^x)' = c_2 \frac{xe^x - e^x}{x^2},$$

$$ye^x = c_2 \int \left(\frac{e^x}{x}\right)' dx + c_1, \qquad y = c_1 e^{-x} + c_2 x^{-1}.$$

Endi tushunarliki, ikkinchi yechim sifatida $y_2(x) = x^{-1}$ funksiyani olish mumkin. $y_1(x) = e^{-x}$ va $y_2(x) = x^{-1}$ yechimlar chiziqli erkli bo`lgani uchun ular berilgan (11) tenglamaning fundamental (bazis) yechimlarini tashkil etadi. Demak, uning umumiy yechimi

$$y = c_1 e^{-x} + \frac{c_2}{x}$$

formula bilan beriladi. ☝

Eslatma. Ikkinchi $y_2(x) = x^{-1}$ yechimni, tushunarliki, $y_2(x) = x^{\alpha}$ ko`rinishda izlab ham topish mumkin edi.

Misol 5. Berilgan $y_1(x) = e^x$ va $y_2(x) = x^{-1}$ funksiyalarga ko`ra ular qanoatlantiruvchi eng kichik tartibli chiziqli differensial tenglamani tuzing.

⊶ Berilgan funksiyalarning Vronskiani

$$W(x) = \begin{vmatrix} e^x & x^{-1} \\ e^x & -x^{-2} \end{vmatrix} = -\frac{e^x}{x^2}(x+1)$$

$(-\infty;-1)$, $(-1;0)$ va $(0;+\infty)$ oraliqlarining har birida nolga aylanmaydi. Demak, shu oraliqlarning ixtiyoriy birida fundamental (bazis) yechimlari $y_1(x) = e^x$ va $y_2(x) = x^{-1}$ bo`lgan ikkinchi tartibli chiziqli bir jinsli differensial tenglama (4) formulaga ko`ra qurilishi mumkin:

$$\begin{vmatrix} y_1(x) & y_2(x) & y \\ y_1'(x) & y_2'(x) & y' \\ y_1''(x) & y_2''(x) & y'' \end{vmatrix} = 0, \qquad \begin{vmatrix} e^x & x^{-1} & y \\ e^x & -x^{-2} & y' \\ e^x & 2x^{-3} & y'' \end{vmatrix} = 0,$$

$$y'' \cdot \begin{vmatrix} e^x & x^{-1} \\ e^x & -x^{-2} \end{vmatrix} - y' \cdot \begin{vmatrix} e^x & x^{-1} \\ e^x & 2x^{-3} \end{vmatrix} + y \cdot \begin{vmatrix} e^x & -x^{-2} \\ e^x & 2x^{-3} \end{vmatrix} = 0,$$

$$x(x+1)y'' - (x^2-2)y' - (x+2)y = 0,$$

$$y'' - \frac{x^2-2}{x(x+1)}y' - \frac{x+2}{x(x+1)}y = 0. \text{ ☝}$$

Misol 6. Ushbu

$$(2x+1)y'' - 4(x+1)y' + 4y = 4xe^{2x} \tag{12}$$

tenglamaning umumiy yechimini quring.

🗝 Ma'lumki, chiziqli bir jinsli bo`lmagan differensial tenglamaning umumiy yechimi uning biror xususiy yechimiga mos bir jinsli tenglamaning umumiy yechimini qo`shishdan hosil bo`ladi.

Dastlab mos bir jinsli tenglama

$$(2x+1)y''-4(x+1)y'+4y=0 \qquad (13)$$

ning fundamental (bazis) yechimlarini topishga harakat qilamiz. Yechimna $y=e^{\lambda x}$ ko`rinishida izlab, $\lambda=2$ ni topamiz. Demak, $y_1(x)=e^{2x}$ (13) tenglamaning yechimi. Yana bir yechimni Ostrogradskiy-Liuvill formulasi (3) ga ko`ra yoki tenglamada $y=e^{2x}u$ almashtirishni bajarib topsa bo`ladi. Bu misolda, yaxshisi, yechimni $y=kx+b$ ko`rinishda izlaylik. Bu chiziqli funksiyani tenglamaga qo`yib, yning qanoatlanishi uchun $k=1$, $b=1$ bo`lishi kerakligini aniqlaymiz. Demak, ikkinchi yechim $y_2=x+1$. Topilgan yechimlarning Vronskiani

$$W(x)=\begin{vmatrix} e^{2x} & x+1 \\ 2e^{2x} & 1 \end{vmatrix}=-e^{2x}(2x+1)$$

bo`lgani uchun ular chiziqli erkli. Shuning uchun (13) bir jinsli tenglamaning umumiy yechimi

$$y=c_1e^{2x}+c_2(x+1) \qquad (14)$$

ko`rinishda yoziladi.

Endi berilgan (12) tenglamaning xususiy yechimini topamiz. Buning uchun Lagranjning ixtiyoriy o`zgarmaslarni variatsiyalash metodidan foydalanamiz. Xususiy yechimni

$$y=c_1(x)e^{2x}+c_2(x)(x+1) \qquad (15)$$

ko`rinishda izlaymiz. (15) ni (12) tenglamaga qo`yamiz. Buning uchun kerakli hosilalarni hisoblashimiz lozim. $c_1(x)$ va $c_2(x)$ funksiyalar uchun

$$c_1'(x)e^{2x}+c_2'(x)(x+1)=0 \qquad (16)$$

shart qo`yib, $y'=c_1(x)2e^{2x}+c_2(x)\cdot 1$ hosilani topamiz. Bundan $y''=c_1'(x)2e^{2x}+c_2'(x)+c_1(x)4e^{2x}$ ni hisoblaymiz va yuqoridagilarni (12) tenglamaga qo`yib, soddalashtirishlarni bajarib, ushbu

$$2(2x+1)e^{2x}c_1'(x)+(2x+1)c_2'(x)=4xe^{2x} \qquad (17)$$

tenglikka kelamiz. Endi (16) va (17) tenglamalardan $c_1'(x)$ va $c_2'(x)$ larni topamiz:

$$c_1'(x)=\frac{4x(x+1)}{(2x+1)^2}\,,\ c_2'(x)=-\frac{4xe^{2x}}{(2x+1)^2}\,.$$

Zarur integrallashlarni bajarib, $c_1(x)$ va $c_2(x)$ larni aniqlaymiz:

$$c_1(x)=\int\frac{4x(x+1)}{(2x+1)^2}dx=\int\Big(1-\frac{1}{(2x+1)^2}\Big)dx=x+\frac{1}{2(2x+1)}\,,$$

$$c_2(x)=-\int\frac{4xe^{2x}}{(2x+1)^2}dx=\|t=2x\|=-\int\frac{t}{(t+1)^2}e^t dt=$$

$$=-\int\frac{t+1}{(t+1)^2}e^t dt-\int\frac{1}{(t+1)^2}e^t dt=-\int\frac{1}{t+1}e^t dt+\int e^t d\Big(\frac{1}{t+1}\Big)=$$

$$=-\int\frac{e^t}{t+1}dt+\int e^t d\Big(\frac{1}{t+1}\Big)=-\int\frac{e^t}{t+1}dt+\frac{e^t}{t+1}+\int\frac{1}{t+1}d(e^t)=\frac{e^t}{t+1}=\frac{e^{2x}}{2x+1}\,.$$

Bularni (15) ga qo`yib, (12) ning xususiy yechimni topamiz:

$$y=\left(x+\frac{1}{2(2x+1)}\right)e^{2x}+\frac{e^{2x}}{2x+1}(x+1)=xe^{2x}-\frac{1}{2}e^{2x}.$$

Bu xususiy yechimga mos bir jinsli tenglama (13) ning $e^{2x}/2$ yechimini qo`shib, (12) ning yana xususiy yechimni hosil qilamiz:

$$y=xe^{2x}$$

Shunday qilib, berilgan (12) tenglamaning umumiy yechimi

$$y=xe^{2x}+c_1e^{2x}+c_2\cdot(x+1)$$

formula bilan beriladi. 👍

Masalalar

Berilgan funksiyalar ularning umumiy aniqlanish oralig`ida chiziqli erklilimi (**1 - 8**) ? :

1. $y_1(x)=x,\ y_2(x)=x+1$.

2. $y_1(x)=x-2, y_2(x)=x,\ y_3(x)=x+1$.

3. $y_1(x)=x^2+x+1, y_2(x)=2x^2+1,\ y_3(x)=2x-3$.

4. $y_1(x)=\cos x, y_2(x)=\cos(x+1),\ y_3(x)=\cos(x+2)$.

5. $y_1(x)=\ln(x^3), y_2(x)=\ln 2x+1,\ y_3(x)=3$.

6. $y_1(x)=\sqrt[3]{x}, y_2(x)=\sqrt[3]{x-1},\ y_3(x)=\sqrt[3]{x-2}$.

7. $y_1(x)=1, y_2(x)=\arccos x,\ y_3(x)=\arcsin x$.

8. a) $y_1(x)=x^3,\ y_2(x)=|x|^3\ \ x\in(-\infty;+\infty)$ oraliqda.

b) $y_1(x)=x^3,\ y_2(x)=|x|^3\ \ x\in(-\infty;0)$ oraliqda.

Berilgan funksiyalar qanoatlantiruvchi eng kichik tartibli chiziqli bir jinsli differensial tenglamani tuzing. Bu tenglamani berilgan funksiyalar qaysi oraliq(lar)da qanoatlantiradi (**9 - 16**)?

9. $y_1=x,\ y_2=x^2+1$.

10. $y_1=\sqrt{x},\ y_2=\sqrt{x+1}$.

11. $y_1=e^{2x},\ y_2=x+1$.

12. $y_1=\sin x,\ y_2=\sin 2x$.

13. $y_1=\ln x,\ y_2=x$.

14. $y_1=\sqrt{x},\ y_2=x,\ y_3=x^2$.

15. $y_1=x,\ y_2=\ln x,\ y_3=x^2$.

16. $y_1=x^3,\ y_2=e^x,\ y_3=e^{2x}$.

Mustaqil ish № 10 topshiriqlari:

I. Quyidagi funksiyalarni chiziqli erklilikka tekshiring (barcha funksiyalar aniqlangan oraliqda). $y_1(x)$ va $y_2(x)$ funksiyalar qanoatlantiruvchi eng kichik tartibli chiziqli bir jinsli differensial tenglamani tuzing. Bu tenglamani $y_1(x)$ va $y_2(x)$ funksiyalar qaysi oraliq(lar)da qanoatlantiradi?

1. $y_1(x)=x$, $y_2(x)=\cos^2 x$, $y_3(x)=\cos 2x$.

2. $y_1(x)=x+1,\ y_2(x)=\cos x$, $y_3(x)=\sin 2x$.

3. $y_1(x)=2x-1$, $y_2(x)=x^2$, $y_3(x)=x^2+x-2$.
4. $y_1(x)=e^{-x}$, $y_2(x)=x+1$, $y_3(x)=x^3$.
5. $y_1(x)=e^{-2x}$, $y_2(x)=x+1$, $y_3(x)=xe^{-2x}$.
6. $y_1(x)=\sin x$, $y_2(x)=\cos 2x$, $y_3(x)=1$.
7. $y_1(x)=e^{-2x}$, $y_2(x)=x$, $y_3(x)=2x+3e^{-2x}$.
8. $y_1(x)=\cos x$, $y_2(x)=x+1$, $y_3(x)=x+\cos 2x$.
9. $y_1(x)=\mathrm{arctg}x$, $y_2(x)=1$, $y_3(x)=\mathrm{arcctg}x$
10. $y_1(x)=e^{3x}$, $y_2(x)=xe^{2x}$, $y_3(x)=x$.
11. $y_1(x)=x^2-x-1$, $y_2(x)=x$, $y_3(x)=x+1$.
12. $y_1(x)=x$, $y_2(x)=xe^{2x}$, $y_3(x)=x+1$.
13. $y_1(x)=x$, $y_2(x)=xe^{x}$, $y_3(x)=x+1$.
14. $y_1(x)=\ln x$, $y_2(x)=x\ln x$, $y_3(x)=x$.
15. $y_1(x)=x^2$, $y_2(x)=x-1$, $y_3(x)=e^{2x}$.
16. $y_1(x)=\sin x$, $y_2(x)=x$, $y_3(x)=\cos(x+1)$.
17. $y_1(x)=\cos x$, $y_2(x)=x$, $y_3(x)=2x-1$.
18. $y_1(x)=x$, $y_2(x)=\mathrm{tg}x$, $y_3(x)=2x+1$.
19. $y_1(x)=\sin x$, $y_2(x)=\mathrm{tg}x$, $y_3(x)=x$.
20. $y_1(x)=x^2$, $y_2(x)=e^{x}$, $y_3(x)=\sqrt{x-1}$.
21. $y_1(x)=xe^{x}$, $y_2(x)=x-1$, $y_3(x)=x+1$.
22. $y_1(x)=\ln x$, $y_2(x)=x-1$, $y_3(x)=2x+1$.
23. $y_1(x)=\sin 2x$, $y_2(x)=x$, $y_3(x)=\ln(x-1)$.
24. $y_1(x)=e^{-2x}$, $y_2(x)=1/x$, $y_3(x)=x+1$.
25. $y_1(x)=\cos 2x$, $y_2(x)=x-1$, $y_3(x)=x^2+1$.
26. $y_1(x)=\sin^2 x$, $y_2(x)=x$, $y_3(x)=x-2$.
27. $y_1(x)=x^3$, $y_2(x)=e^{2x}$, $y_3(x)=\cos x$.
28. $y_1(x)=\sin x$, $y_2(x)=x\sin x$, $y_3(x)=\sqrt{x+1}$.
29. $y_1(x)=\ln x$, $y_2(x)=x^2-1$, $y_3(x)=\sqrt{x-1}$.
30. $y_1(x)=e^{-x}$, $y_2(x)=x^2$, $y_3(x)=x+1$.

II. Quyidagi tenglamaning umumiy yechimini quring.

1. $x(x+2)y''+(x^2-2)y'-2(x+1)y=3(2-x^2)/x^2$.
2. $x(2x^2\ln x-x^2+1)y''-(2x^2\ln x+x^2-1)y'+4xy=x$
(mos bir jinsli tenglama yechimi $y_1=x^2-1$) .
3. $\sin^2 x\cdot y''-\sin 2x\cdot y'+(2-\sin^2 x)y=1+\cos^2 x$.
4. $x^2(2x-3)y''-2x(2x^2-3)y'+12x(x-1)y=$
$$=-xe^{x}(2x^3-9x^2+12x-6).$$
5. $\sin x\,(2x\cos x-\sin x)\cdot y''-2x\cos 2x\cdot y'+2\cos 2x\cdot y=$
$$=4\cos^2 x-2$$
(mos bir jinsli tenglama yechimi $y_1=x$).

6. $(2x\sin 2x-2\sin 2x+\cos 2x)\cdot y''-4(x-1)\cos 2x\cdot y'+$

$$+4\cos 2x\cdot y=\sin^2 x-\cos^2 x$$

(mos bir jinsli tenglama yechimi $y_1=x-1$) .

7. $x(2x-1)y''+(4x^2-2)y'+4(x-1)y=8x^2-8x+2$.

8. $(2x\cos 2x-\sin 2x)\cdot y''+4x\sin 2x\cdot y'-4\sin 2x\cdot y=$

$$=4x^2\sin 2x+4x\cos 2x-2\sin 2x.$$

9. $x(x\ln x-x+1)y''-(x-1)y'+y=x(2x\ln x-3x+4)$.

10. $(x^2-x-1)y''-(x^2+x-2)y'+(x+2)y=-x^3+2x^2+3x$.

11. $x(x-2)y''-(x^2-2)y'+2(x-1)y=x^2$.

12. $\cos x(1-\cos^2 x)y''-\sin x(2+\cos^2 x)y'+3\cos x\cdot y=$

$$=2+\cos^2 x$$

(mos bir jinsli tenglama yechimi $y_1=\sin x$) .

13. $\cos x(x-\sin x\cdot\cos x)y''-2x\sin x\cdot y'+2\sin x\cdot y=x\cos x-\sin x$

(mos bir jinsli tenglama yechimi $y_1=x$) .

14. $(x\sin x+\cos x)y''-x\cos x\cdot y'+\cos x\cdot y=x$

(mos bir jinsli tenglama yechimi $y_1=x$) .

15. $(x^2+1)y''-2xy'+2y=e^x(x^3+x+2)$.

16. $x^2\ln^2 x\cdot y''-2x\ln x\cdot y'+(2+\ln x)y=x(2-\ln x)$

(mos bir jinsli tenglama yechimi $y_1=\ln x$) .

17. $x^2y''-x(x+2)y'+(x+2)y=-x^3+x+2$.

18. $2x^2y''-4x(x+1)y'+4(x+1)y=-x^3+x+1$.

19. $(x^2+1)y''-2xy'+2y=e^{2x}(2x^2-2x+3)$.

20. $(x-1)y''+(4-5x)y'+3(2x-1)y=e^{4x}(2x-3)$.

21. $(x^2+1)(x^2\mathrm{arctg}\,x-x+\mathrm{arctg}\,x)y''-2x^2y'+2xy=-2x+2\mathrm{arctg}\,x$

(mos bir jinsli tenglama yechimi $y_1=x$) .

22. $(x\sin x+\sin x+\cos x)y''-(x+1)\cos x\cdot y'+\cos x\cdot y=x+1$.

23. $(2x+1)y''+4xy'-4y=4e^{-2x}(x+1)$.

24. $(\sin x\cos x-x)y''+2\sin^2 x\cdot y'-(\sin x\cos x+x)y=$

$$=2\sin x-2x\sin x$$

(mos bir jinsli tenglama yechimi $y_1=\sin x$) .

25. $(2x+3)y''+4(x+1)y'-4y=4(x+2)e^{-2x}$.

26. $(x+2)y''+(x+1)y'-y=(2x^2+2x-7)e^{-2x}$.

27. $2x(x-1)y''-2(2x-1)y'+4y=(x^3-x^2-x+1)e^x$.

28. $(x\sin x+\sin x+\cos x)y''-(x+1)\cos x\cdot y'+\cos x\cdot y=x+1$.

29. $\cos x(2x\sin x+\cos x)y''-2x\cos 2x\cdot y'+2\cos 2x\cdot y=$

$$=3\sin x\cos^2 x-2\sin x-2x\cos^3 x$$

(mos bir jinsli tenglama yechimi $y_1=x$) .

30. $(\cos 3x-3\cos x)y''+6\sin x\cos 2x\cdot y'-2(\cos 3x+3\cos x)y=$

$$=12\sin x$$

(mos bir jinsli tenglama yechimi $y_1 = \sin x$) .

11. *n* - TARTIBLI CHIZIQLI O`ZGARMAS KOEFFITSIENTLI DIFFERENSIAL TENGLAMALAR

Maqsad – n - tartibli chiziqli o`zgarmas koeffitsientli differensial tenglamaning umumiy yechimini qurishni o`rganish

Yordamchi ma'lumotlar:

n - tartibli chiziqli o`zgarmas koeffitsientli differensial tenglama deb

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = f(x) \tag{1}$$

ko`rinishdagi tenglamaga aytiladi, bunda $a_n \neq 0, a_{n-1}, \ldots, a_1, a_0$ koeffitsientlar berilgan o`zgarmas sonlar, o`ng tomon (ozod had) $f(x)$ biror I oraliqda aniqlangan va uzluksiz funksiya.

1⁰. Ushbu

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0 \tag{2}$$

tenglama (1) ga **mos bir jinsli tenglama** deb ataladi.

Ushbu

$$L(\lambda) \stackrel{def}{=} a_n \lambda^n + a_{n-1} \lambda^{n-1} + ... + a_1 \lambda + a_0 = 0 \tag{3}$$

n – darajali algebraik tenglama **(2) differensial tenglamaning xarakteristik tenglamasi** deyiladi. Xarakteristik tenglama ildizi esa **xarakteristik son** deb ataladi.

Algebradan ma'lumki, (3) n – darajali algebraik tenglama kompleks sohada karraliligini hisobga olgan holda n ta ildizga ega.

Agar $\lambda = \lambda_j$ son xarakteristik tenglama (3) ning $k_j\ (k_j \geq 1)$ karrali ildizi, ya'ni

$$L(\lambda_j) = L'(\lambda_j) = ... = L^{(k_j - 1)}(\lambda_j) = 0,\ L^{(k_j)}(\lambda_j) \neq 0,$$

$$\left(L(\lambda) = (\lambda - \lambda_j)^{k_j} L_1(\lambda),\ L_1(\lambda_j) \neq 0\right)$$

bo`lsa, u holda (2) differensial tenglama ushbu

$$e^{\lambda_j x},\ x e^{\lambda_j x}, ..., x^{k_j - 1} e^{\lambda_j x}$$

k_j dona (λ_j ga mos) chiziqli erkli yechimga ega bo`ladi. Agar xarakteristik tenglama (3) ning barcha ildizlarini topib, ularga mos kelgan (2) differensial tenglama yechimlarini tuzsak, u holda (2) ning $n$ dona chiziqli erkli yechimlari $y_1, y_2, \cdots, y_n$ ni (fundamental (bazis) yechimlarni yoki yechimlarning fundamental sistemasini) topgan bo`lamiz. Bu yechimlarning ixtiyoriy chiziqli kombinatsiyasi

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x) \quad (c_1, c_2, ..., c_n - \text{const})$$

(2) differensial tenglamaning umumiy yechimini beradi.

Faraz qilaylik, (2) differensial tenglamaning koeffitsientlari haqiqiy sonlardan iborat bo`lsin. Bu holda odatda yechimlarni haqiqiy qiymatli funksiyalar orasidan izlanadi. Xarakteristik tenglamaning koeffitsientlari haqiqiy sonlar bo`lgani uchun uning ildizlari qo`shma kompleks sonlar tarzida bo`ladi, ya'ni agar $\lambda_j = \alpha_j + i\beta_j$ ($\alpha_j, \beta_j -$ haqiqiy sonlar, $\beta_j \neq 0$, $i -$ mavhum birlik) soni (3) ning k_j karrali ildizi bo`lsa, u holda unga qo`shma bo`lgan $\overline{\lambda}_j = \alpha_j - i\beta_j$ kompleks son ham (3) ning k_j

karrali ildizi bo`ladi. Bu holda (2) differensial tenglama kompleks yechimining haqiqiy va mavhum qismlari ham (2) ning yechimlari bo`ladi. Ushbu

$$\lambda_j = \alpha_j + i\beta_j \quad \text{ва} \quad \bar{\lambda}_j = \alpha_j - i\beta_j.$$

k_j karrali xarakteristik songa (2) differensial tenglamaning ushbu

$$e^{\alpha_j x}\cos\beta_j x,\ xe^{\alpha_j x}\cos\beta_j x,\ ...,\ x^{k_j-1}e^{\alpha_j x}\cos\beta_j x,$$

$$e^{\alpha_j x}\sin\beta_j x,\ xe^{\alpha_j x}\sin\beta_j x,\ ...,\ x^{k_j-1}e^{\alpha_j x}\sin\beta_j x$$

$2k_j$ ta haqiqiy yechimi mos keladi.

Xarakteristik sonlarga mos keluvchi barcha bunday ko`rinishdagi yechimlar (2) ning fundamental (bazis) haqiqiy yechimlarini (fundamental sistemasini) tashkil etadi. Fundamental (bazis) yechimlarning chiziqli kombinatsiyasi (haqiqiy koeffitsientlar bilan) (2) ning umumiy yechimini ifodalaydi.

Misol 1. Ushbu

$$y''' - 2y'' - y' + 2y = 0$$

tenglamani yeching.

Mos xarakteristik tenglamani tuzamiz:

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0.$$

Bu tenglamani yechamiz

$$\lambda^2(\lambda-2) - (\lambda-2) = 0,\ (\lambda-2)(\lambda^2-1) = 0,\ (\lambda-2)(\lambda-1)(\lambda+1) = 0.$$

Xarakteristik sonlar sodda (bir karrali):

$$\lambda_1 = -1,\ \lambda_2 = 1,\ \lambda_3 = 2.$$

Differensial tenglamaning fundamental yechimlari:

$$e^{-x},\ e^{x},\ e^{2x}.$$

Demak, berilgan tenglamaning umumiy yechimi

$$y = c_1e^{-x} + c_2e^{x} + c_3e^{2x}.$$

Misol 2. Ushbu

$$y^{IV} + 2y'' + y = 0$$

tenglamani yeching.

Xarakteristik tenglama

$$\lambda^4 + 2\lambda^2 + 1 \equiv (\lambda^2+1)^2 \equiv (\lambda+i)^2(\lambda-i)^2 = 0.$$

Demak, $\lambda = i$ va $\bar{\lambda} = -i$ ikki karrali xarakteristik sonlar. Berilgan differensial tenglamaning bu xarakteristik sonlarga mos haqiqiy fundamental yechimlari

$$\cos x,\ x\cos x,\ \sin x,\ x\sin x.$$

Umumiy yechim

$$y = c_1\cos x + c_2x\cos x + c_3\sin x + c_4x\sin x.$$

2^0. *Chiziqli o`zgarmas koeffitsientli bir jinsli bo`lmagan differensial tenglama*ni yechish.

Ma'lumki, (1) tenglamaning umumiy yechimi uning biror xususiy yechimiga mos bir jinsli tenglama (2) ning umumiy yechimini qo`shishdan hosil bo`ladi.

(1) tenglamaning xususiy yechimini topishda Lagranjning ixtiyoriy o`zgarmaslarni variatsiyalash usulidan foydalanish mumkin. Ba'zan bu usul uzoq hisoblashlarni talab qiladi.

O`ng tomon $f(x)$ ning ba'zi maxsus ko`rinishlarida xususiy yechimni noma'lum koeffitsientlar metodidan foydalanib topish mumkin. O`ng tomon

$$f(x)=e^{\alpha x}(P_1(x)\cos\beta x+P_2(x)\sin\beta x)$$

kvaziko`phaddan iborat bo`lsin, ya'ni

$$p_n y^{(n)}+...+p_1y'+p_0y=e^{\alpha x}(P_1(x)\cos\beta x+P_2(x)\sin\beta x) \qquad (4)$$

differensial tenglamani qaraylik, bunda α,β lar haqiqiy sonlar; $P_1(x), P_2(x)-$ haqiqiy koeffitsientli ko`phadlar.

(4) tenglamaning xususiy yechimini qurish uchun quyidagicha ish tutamiz. Qaralayotgan (4) tenglama uchun $\alpha+i\beta$ kompleks sonni tuzaylik.

1) Agar $\lambda=\alpha+i\beta$ son xarakteristik tenglama (3) ning ildizi bo`lmasa, u holda (4) ning xususiy yechimini

$$y=e^{\alpha x}(Q_1(x)\cos\beta x+Q_2(x)\sin\beta x)$$

ko`rinishda izlash mumkin, bunda $Q_1(x), Q_2(x)-$ ko`phadlar hamda $\max(\deg Q_1,\deg Q_2)=\max(\deg P_1,\deg P_2)$, $\deg Q$ bilan Q ko`phadning darajasi belgilangan.

2)Agar $\lambda=\alpha+i\beta$ kompleks son (3) xarakteristik tenglamaning k karrali ildizi bo`lsa (rezonans holi), u holda (4) ning xususiy yechimini

$$y=x^k e^{\alpha x}(Q_1(x)\cos\beta x+Q_2(x)\sin\beta x)$$

ko`rinishda izlash mumkin, bunda $Q_1(x), Q_2(x)-$ ko`phadlar va $\max(\deg Q_1,\deg Q_2)=\max(\deg P_1,\deg P_2)$.

Misol 3. Ushbu

$$y''+4y'+4y=\sin 2x \qquad (5)$$

tenglamaning xususiy yechimini toping.

🗝 Tenglamaning o`ng tomoni kvaziko`phad: $\alpha=0$, $\beta=2$, $P_1(x)=0$, $P_2(x)=1$, $\deg P_1=\deg P_2=0$, $\lambda=\alpha+i\beta=2i$ kompleks son $\lambda^2+4\lambda+4=0$ xarakteristik tenglamaning ildizi emas: $(2i)^2+4\cdot 2i+4\neq 0$. Demak, xususiy yechimni

$$y=A\cos 2x+B\sin 2x \qquad (6)$$

ko`rinishda izlash mumkin, bunda $A,B-$ noma'lum koeffitsientlar ( $\deg Q_1=\deg Q_2=0$ ). (6) ni (5) ga qo`yib A va B larni topamiz:

$$(A\cos 2x+B\sin 2x)''+4(A\cos 2x+B\sin 2x)'+4(A\cos 2x+B\sin 2x)=\sin 2x,$$

$$(-4A+8B+4A)\cos 2x+(-4B-8A+4B)\sin 2x=\sin 2x.$$

Demak, $8B=0,\ -8A=1,$ ya'ni

$$A=-\frac{1}{8},\quad B=0.$$

Buni (6) ga qo`yib, (5) ning

$$y=-\frac{1}{8}\cos 2x$$

xususiy yechimini topamiz. 👍

Misol 4. Ushbu

$$y^{IV}+2y''+y=x\cos x \qquad (7)$$

tenglamaning xususiy yechimini quring.

🗝 Berilgan tenglama (7) ning o`ng tomoni kvaziko`phad:

$$\alpha=0,\ \beta=1,\ \deg P_1=1,\ \deg P_2=0.$$

$\lambda=\alpha+i\beta=i$ kompleks son xarakteristik tenglama

$$\lambda^4+2\lambda^2+1=(\lambda^2+1)^2=(\lambda+i)^2(\lambda-i)^2=0$$

ning ikki karrali ildizi (rezonans holi). Demak, (7) tenglamaning xususiy yechimini

$$y=x^2\big((Ax+B)\cos x+(Cx+D)\sin x\big) \tag{8}$$

ko`rinishda izlash mumkin.

(8) ni (7)ga qo`yib, hosil bo`lgan tenglikning ayniyat bo`lishi kerakligidan A,B,C va D noma’lum koeffitsientlarga nisbatan chiziqli algebraik tenglamalar sistemasini hosil qilamiz va uni yechib,

$$A=-\frac{1}{24},\ B=0,\ C=0,\ D=\frac{1}{8}$$

larni topamiz. Bularni (8) ga qo`yib, (7) ning xususiy yechimini hosil qilamiz: $y=-\dfrac{1}{24}x^3\cos x+\dfrac{1}{8}x^2\sin x$. 👍

Xususiy yechimni topishda ba’zan quyidagi tasdiq qo`l keladi.

Agar $y=y_1(x)$ funksiya

$$a_ny^{(n)}+a_{n-1}y^{(n-1)}+...+a_1y'+a_0y=f_1(x)$$

differensial tenglamaning, $y=y_2(x)$ esa

$$a_ny^{(n)}+a_{n-1}y^{(n-1)}+...+a_1y'+a_0y=f_2(x)$$

differensial tenglamaning yechimi bo`lsa, u holda $y=\lambda_1y_1(x)+\lambda_2y_2(x)$ funksiya ushbu

$$a_ny^{(n)}+a_{n-1}y^{(n-1)}+...+a_1y'+a_0y=\lambda_1f_1(x)+\lambda_2f_2(x)$$

differensial tenglamaning yechimi bo`ladi (superpozitsiya prinsipi).

Masalalar

Chiziqli bir jinsli tenglamalarni yeching (**1 - 10**):

1. $y''-3y'+2y=0$. **2.** $y''-y'-2y=0$. **3.** $y''-2y'+2y=0$.

4. $y''-4y'+13y=0$. **5.** $y''+2y'+y=0$. **6.** $y'''-2y''-y'+2y=0$.

7. $y'''-5y''+8y'-6y=0$. **8.** $y'''-7y''+16y'-12y=0$. **9.** $y^{IV}+8y''+16y=0$.

10. $y^{IV}-2y'''+6y''+22y'+13y=0$.

Bir jinsli bo`lmagan chiziqli tenglamalarni yeching (**11 - 20**):

11. $y''-4y'+3y=(1-2x)e^x$. **12.** $y''+5y'+6y=3(1+x^2)e^{-2x}$.

13. $y''-4y'+4y=8x^3+6xe^{2x}$. **14.** $y''+y'-6y=5\sin x+5e^{2x}$.

15. $y'''+y''=2\sin x$. **16.** $y'''-y''+y'-y=x+1+4\cos x$.

17. $y'''-y''-y'+y=e^{-x}(3\cos x+4\sin x)$.

18. $y'''+3y''+4y'+2y=4x+2e^{-x}(\cos x+\sin x)$.

19. $y^{IV}+y''=2\cos x$. **20.** $y^{IV}-2y''+y=x^2+x+1$.

Boshlang`ich masalalarni yeching (**21 -24**).

21. $y''+y'=2x\sin x$, $y(0)=y'(0)=1$.

22. $y''-3y'+2y=4x+e^x$, $y(0)=1$, $y'(0)=0$.

23. $y'''+5y''+4y'=16x$, $y(0)=1$, $y'(0)=-1$, $y''(0)=0$.

24. $y''' - y = 9e^x$, $y(0) = y'(0) = y''(0) = 1$.

Mustaqil ish № 11 topshiriqlari:

I. *Berilgan Koshi masalasini yeching.*

1. $\begin{cases} y^{IV} + 4y''' - 4y' - 4y = x\cos 2x \\ y(0) = -1,\ y'(0) = 1,\ y''(0) = y'''(0) = 0 \end{cases}$

2. $\begin{cases} y''' - 3y'' + 3y' - y = e^x \sin 2x \\ y(0) = -1,\ y'(0) = 2,\ y''(0) = 1/2 \end{cases}$

3. $\begin{cases} y''' + y'' = 6x^2 \\ y(0) = 1, y'(0) = -1, y''(0) = 0 \end{cases}$

4. $\begin{cases} y^{IV} - 5y'' = x^2 + x \\ y(0) = y'(0) = 1,\ y''(0) = y'''(0) = 0 \end{cases}$

5. $\begin{cases} y^{IV} + y' = x + x^2 + 2\cos x \\ y(0) = -1, y'(0) = 1, y''(0) = y'''(0) = 1/2 \end{cases}$

6. $\begin{cases} y^{IV} + 4y''' + 6y'' + 4y' + y = \cos x \\ y(0) = y'(0) = 1,\ y''(0) = y'''(0) = 2 \end{cases}$

7. $y''' + 2y' + 3y = 2x^2 + 1$, $y(0) = 2$, $y'(0) = y''(0) = 1$

8. $y'' + 16y = x\cos 4x$, $y(\pi/3) = 0$, $y'(\pi/3) = 0,2$

9. $y'' + 4y = x^2 \sin^2 x$, $y(\pi/6) = 0$, $y'(\pi/6) = 1$

10. $y'' + 4y = e^x \sin x - 2x$, $y(\pi/3) = -1$, $y'(\pi/3) = 1$

11. $y'' - 8y' + 16y = xe^{4x}$, $y(0) = 2$, $y'(0) = 1$

12. $y'' + 6y' + 13y = e^{-3x}\cos 2x$, $y(0) = 1$, $y'(0) = -1$

13. $y'' - 3y' = (x+1)^2 + e^x$, $y(0) = 1$, $y'(0) = -2$

14. $y''' + y' = \cos x$, $y(0) = -1$, $y'(0) = 1$, $y''(0) = 2$

15. $y'' + y = 2x\sin x + 1$, $y(\pi/6) = 0$, $y'(\pi/6) = -1$

16. $y'' - y = xe^x - 2$, $y(0) = 1$, $y'(0) = 2$

17. $y'' + 2y = x\cos x$, $y(\pi/6) = -2$, $y'(\pi/6) = 1$

18. $y'' + y = 2x\cos x - 1$, $y(\pi/4) = -1$, $y'(\pi/4) = 1$

19. $y''' - 2y'' + y' = 2x + 2e^x$, $y(3) = 0$, $y'(0) = 1$

20. $\begin{cases} y'' - y' = e^{-x}(\cos x + 2x\sin x), \\ y(\pi/6) = -2,\ y'(\pi/6) = 1 \end{cases}$

21. $y'' - 4y' + 5y = e^x \sin x + x^2$, $y(\pi/6) = 1$, $y'(\pi/6) = -1$

22. $y'' + y = \cos x + 2e^{-x}$, $y(\pi/6) = -2$, $y'(\pi/6) = 2$

23. $y''' - y' = \cos x + 1$, $y(0) = -1$, $y'(0) = 1$, $y''(0) = 0,5$

24. $y'' - y = e^x + e^{-x}$, $y(0) = 0$, $y'(0) = 1$

25. $y'' - 2y' = (1 + 2x^2)e^{2x}$, $y(0) = -1$, $y'(0) = 2$

26. $\begin{cases} y^{IV} + 4y' - y = e^x + 2x \\ y(0) = y'(0) = -1,\ y''(0) = y''(0) = 0 \end{cases}$

27. $y''' - y'' = 6x^2 - 1$, $y(0) = 2$, $y'(0) = 1$, $y''(0) = -1$

28. $y'' - 5y' + 6 = e^x \cos x$, $y(\pi/3) = -0,5$, $y'(\pi/3) = 1,5$

29. $y'' - y' - 2y = 2e^x + e^{-2x}$, $y(0) = 2$, $y'(0) = -2$

30. $\begin{cases} y^{IV} + 4y''' - 4y' - 4y = \sin x + 3 \\ y(0) = y'(0) = 1,\ y''(0) = y'''(0) = -1 \end{cases}$

31. $y'' - 4y' + 5y = e^x \cos x - 1$, $y(\pi/4) = 2$, $y'(\pi/4) = 1$

32. $y'' + 3y' = (1 + 2x)e^x$, $y(0) = 1$, $y'(0) = -1$

33. $\begin{cases} y^{IV} - 4y''' + 6y'' - 4y' + y = 2xe^x \\ y(0) = y'(0) = -1,\ y''(0) = y'''(0) = 1 \end{cases}$

34. $y'' - 2y' = 2xe^x + 1$, $y(0) = -1$, $y'(0) = 1$.

35. $y'' + 4y = x\cos x + 2\sin x$, $y(\pi/3) = 0,5$, $y'(\pi/3) = 2,5$

36. $y''' - y' = xe^x + 1$, $y(0) = 2$, $y'(0) = -1$, $y''(0) = 0$

37. $y'' + 4y = x\sin 2x$, $y(\pi/3) = 0$, $y'(\pi/3) = 1$

38. $y'' + 2y' + y = e^{-x}\cos x$, $y(0) = 0,5$, $y'(0) = 2$

39. $y'' + y = x\sin x + x + 1$, $y(\pi/2) = -1$, $y'(\pi/2) = 1$

40. $y'' + 9y = \cos 3x + x$, $y(\pi/3) = -1$, $y'(\pi/3) = 1$

12. CHEGARAVIY MASALALAR

Maqsad – ikkinchi tartibli chiziqli differensial tenglamalar uchun qo`yilgan chiziqli chegaraviy masalalar va ularni Grin funksiyasi yordamida yechishni o`rganish

Yordamchi ma'lumotlar:

Ushbu

$$L[y] \overset{def}{=} y'' + p_1(x)y' + p_0(x)y = f(x) \qquad (1)$$

2-tartibli chiziqli differensial tenglama berilgan bo`lsin; bu yerda $\{p_1, p_0, f\} \subset C([a,b])$ $(a < b)$ deb hisoblanadi.

Biz (1) tenglama uchun Koshi masalasi (boshlang`ich masala) bilan tanishmiz; bunda tayinlangan $x_0 \in [a,b]$ nuqtada $y(x_0)$ va $y'(x_0)$ qiymatlar beriladi va yechim biror $I \ni x_0$ oraliqda izlanadi.

Bu tenglama uchun berilgan segmentning chegaralari bo`lmish $a$ va $b$ nuqtalarda ham shartlar qo`yish mumkin. (1) tenglamaning $x \in [a;b]$ segmentda aniqlangan va

$$\alpha_1 y'(a) + \alpha_0 y(a) = \alpha, \qquad \beta_1 y'(b) + \beta_0 y(b) = \beta \qquad (2)$$

shartlarni qanoatlantiruvchi $y = y(x)$, $x \in [a;b]$, yechimini topish masalasini qaraylik; bu yerda α_1, α_0, β_1, β_0-o`zgarmas sonlar va $|\alpha_1| + |\alpha_0| \neq 0$, $|\beta_1| + |\beta_0| \neq 0$. (2) shartlar chegaraviy shartlar, qo`yilgan (1),(2) masala esa - chegaraviy masala deb ataladi.

(2) chegaraviy shartlar ajralgan chegaraviy shartlar deb ataladi; ularning bittasi $x = a$ nuqtada ikkinchisi esa $x = b$ nuqtada qo`yilgan. (1) tenglama uchun ajralmagan chegaraviy shartlar ham qo`yilishi mumkin. Masalan, davriylik shartlari:

$$y(a) = y(b)\,, \qquad y'(a) = y'(b)\,.$$

Agar (1) tenglamaning umumiy yechimi

$$y = y_{xus}(x) + c_1 y_1(x) + c_2 y_2(x)$$

ma’lum bo`lsa ( $y_{xus}(x)$ funksiya (1) tenglamaning xususiy yechimi, $y_1(x), y_2(x)$ funksiyalar $L[y]=0$ tenglamaning fundamental yechimlari va $c_1, c_2 -$ ixtiyoriy o`zgarmaslar), (1),(2) chegaraviy masalani yechish uchun (2) chegaraviy shartlarni qanoatlantirish kerak. Bunda c_1, c_2 larga nisbatan chiziqli algebraik sistema hosil bo`ladi. Uni yechib, qo`yilgan masala yechim(lar)i topiladi yoki yechimning yo`qligi aniqlanadi.

(1),(2) chegaraviy masala har doim ham yechimga ega bo`lavermaydi. U cheksiz ko`p yechimga ega bo`lishi ham mumkin.

Bir jinsli bo`lmagan (2) chegaraviy shartlarni noma’lum funksiyani almashtirish ( $y$ o’rniga $y+u(x)$ kiritish va $u(x)$ ni keraklidek tanlash) yordamida bir jinsli ko`rinishga keltirish mumkin. Bunda (1) tenglamaning o`ng tomoni o`zgaradi xolos:

$$L[y] = g(x) \quad \left(y'' + p_1(x)y' + p_0(x)y = g(x),\; g(x) \overset{def}{=} f(x) - L[u(x)] \right). \qquad (1')$$

$$\begin{aligned} l_1(y,a) &\overset{def}{=} \alpha_1 y'(a) + \alpha_0 y(a) = 0, \\ l_2(y,b) &\overset{def}{=} \beta_1 y'(b) + \beta_0 y(b) = 0. \end{aligned} \qquad (2_0)$$

Ushbu

$$\begin{cases} L[y]=0 \quad \left(y'' + p_1(x)y' + p_0(x)y = 0\right), \; (1_0) \\ \begin{cases} l_1(y,a) = 0 \;\; (\alpha_1 y'(a) + \alpha_0 y(a) = 0), \\ l_2(y,b) = 0 \;\; (\beta_1 y'(b) + \beta_0 y(b) = 0). \end{cases} \; (2_0) \end{cases}$$

masala bir jinsli chegaraviy masala deb ataladi (bunda chiziqli differensial tenglama ham, chegaraviy shartlar ham bir jinsli).

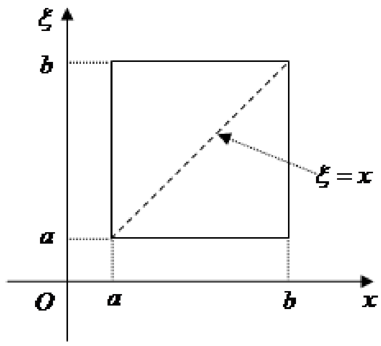
Quyidagi teorema o`rinli.

Teorema 1. (1_0) bir jinsli differensial tenglama (2) ko`rinishdagi ixtiyoriy chegaraviy shartlarda yechimga ega bo`lishi uchun mos bir jinsli masala (1_0), (2_0) ning faqat trivial yechimga ega bo`lishi zarur va yetarlidir.

(1), (2_0) chegaraviy masalaning yechimini Grin funksiyasi orqali yozish mumkin.

Chegaraviy masala (1), (2_0) uchun Grin funksiyasi deb, quyidagi uchta xossaga ega bo`lgan $G(x,\xi)$ funksiyaga aytiladi:

1. $G(x,\xi)$ funksiya $x \in [a,b]$, $\xi \in [a,b]$ bo`lganda aniqlangan va uzluksiz: $G(x,\xi) \in C([a,b]\times[a,b])$.



2. Tayinlangan $\xi \in [a,b]$ uchun $G(x,\xi)$ funksiya x bo`yicha $x \neq \xi$ nuqtalarda mos bir jinsli $L[y]=0$ tenglamani, $x=a$ va $x=b$ nuqtalarda esa (2_0) chegaraviy shartlarni qanoatlantiradi.

3. Tayinlangan $\xi \in (a,b)$ uchun $G(x,\xi)$ funksiyaning birinchi tartibli hosilasi $x=\xi$ nuqtada sakrashga ega va

$$\frac{\partial G(x,\xi)}{\partial x}\Big|_{x=\xi+0} - \frac{\partial G(x,\xi)}{\partial x}\Big|_{x=\xi-0} = 1. \qquad (3)$$

Teorema 2 (Grin funksiyasining mavjudligi haqidagi). Agar (1_0),(2_0) bir jinsli chegaraviy masala faqat trivial yechimga ega bo`lsa, u holda (1),($2_0$) masala uchun Grin funksiyasi mavjud va u quyidagi ko`rinishga ega:

$$G(x,\xi) = \begin{cases} c_1(\xi)y_1(x), \text{ agar } a \le x \le \xi \text{ bo'lsa}, \\ c_2(\xi)y_2(x), \text{ agar } \xi \le x \le b \text{ bo'lsa}, \end{cases} \qquad (4)$$

bu yerda y_1 va y_2 – bir jinsli tenglama (1_0)ning mos ravishda $l_1(y_1,a)=0$ va $l_2(y_2,b)=0$ shartlarni qanoatlantiruvchi notrivial yechimlari, $c_1(\xi)$ va $c_2(\xi)$ lar esa ushbu

$$\begin{cases} c_1(\xi)y_1(\xi) - c_2(\xi)y_2(\xi) = 0 \\ c_2(\xi)y_2'(\xi) - c_1(\xi)y_1'(\xi) = 1 \end{cases} \qquad (5)$$

sistemadan aniqlanadi.

E’tirof etaylikki, keltirilgan teorema Grin funksiyasini qurish usulini ham beradi.

Teorema 3. Faraz qilaylik, (1_0),(2_0) bir jinsli chegaraviy masala faqat trivial yechimga ega, $G(x,\xi)$ uning Grin funksiyasi va $g(x) \in C([a;b])$ bo`lsin. U holda ushbu

$$y'' + p_1(x)y' + p_0(x)y = g(x)$$

$$\alpha_1 y'(a) + \alpha_0 y(a) = 0 \qquad (|\alpha_1| + |\alpha_0| \neq 0)$$

$$\beta_1 y'(b) + \beta_0 y(b) = 0 \qquad (|\beta_1| + |\beta_0| \neq 0)$$

chegaraviy masalaning yagona yechimi

$$y(x) = \int_a^b G(x,\xi)g(\xi)d\xi \qquad (6)$$

formula bilan ifodalanadi.

Misol 1. Ushbu

$$\begin{cases} y'' - y' = x \\ y(0) = 1, \ y'(1) - y(1) = 2 \end{cases} \qquad (7)$$

chegaraviy masalani yeching.

⊶Berilgan (ikkinchi tartibli chiziqli bir jinsli bo`lmagan) tenglamaning umumiy yechimi osongina topiladi:

$$y=-\frac{1}{2}x^2-x+c_1+c_2e^x. \qquad (8)$$

Bu yerdagi c_1 va c_2 o`zgarmaslarni berilgan chegaraviy shartlarning qanoatlanishidan aniqlaymiz. Umumiy yechimni chegaraviy shartlarga qo`yib, c_1 va c_2 noma'lumlarga nasbatan quyidagi tenglamalar sistemasini hosil qilamiz:

$$c_1+c_2=1,\ -1-1+c_2e-\left(-\frac{1}{2}-1+c_1+c_2e\right)=2\ .$$

Bu sistemani yechamiz: $c_1=-\frac{5}{2},\ c_2=\frac{7}{2}$. Topilgan bu qiymatlarni (8) formulaga qo`yib, berilgan chegaraviy masala yechimini hosil qilamiz:

$$y=-\frac{1}{2}x^2-x-\frac{5}{2}+\frac{7}{2}e^x\ .$$ ☝

Misol 2. Ushbu

$$x^2y''+3xy'-3y=0$$

tenglamaning quyidagi chegaraviy shartlarni qanoatlantiruvchi yechimini toping:

1) $y(1)=2$, $y'(+\infty)=0$; 2) $\lim\limits_{x\to 0}y(x)=0$, $y'(1)=1$.

⊶Berilgan tenglama – Eyler tenglamasi. Uning yechimlarini $y=x^\lambda$ ko`rinishida izlaymiz. $\lambda$ ni aniqlash uchun $\lambda(\lambda-1)+3\lambda-3=0$ tenglamani hosi qilamiz. Uning ildizlari: $\lambda_1=1$ va $\lambda_2=-3$. Berilgan differensial tenglamaning umumiy yechimi $y=c_1x+\frac{c_2}{x^3}$. Endi $c_1$ va $c_2$ noma'lum o`zgarmaslarni chegaraviy shartlarning bajarilishidan topamiz.

1) $y'(+\infty)=0$ shartdan $c_1=0$ ekanligini aniqlaymiz. $y(1)=2$ shartdan $c_2=2$ kelib chiqadi. Demak, 1) chegaraviy shartlarni qanoatlantiruvchi yechim $y=\frac{2}{x^3}$.

2) $\lim\limits_{x\to 0}y(x)=0$ shartga ko`ra $c_2=0$. $y'(1)=1$ shartdan $c_1=1$. Demak, 2) chegaraviy shartlarni qanoatlantiruvchi yechim $y=x$. ☝

Misol 3. Ushbu

$$\begin{cases} y''+y=g(x) \quad \left(g(x)\in C([0;\pi])\right) \\ y(0)=0,\ y(\pi)-y'(\pi)=0 \end{cases}$$

chegaraviy masala uchun Grin funksiyasini quring va masala yechimini toping.

⊶Grin funksiyasi teorema 2 ga ko`ra quriladi. $y''+y=0$ tenglamaning $y(0)=0$ va $y(\pi)-y'(\pi)=0$ shartlani qanoatlantiruvchi yechimlarini topamiz. Bu tenglamaning ymumiy yechimi $y=c_1\cos x+c_2\sin x$. $y(0)=0$ shartni qanoatlantiruvchi yechim sifatida $y=y_1=\sin x$ funksiyani, $y(\pi)-y'(\pi)=0$ shartni qanoatlantiruvchi yechim sifatida esa $y=y_2=\cos x+\sin x$ ni olish mumkin. Demak, Grin funksiyasi

$$G(x,\xi)=\begin{cases} c_1(\xi)\sin x, \text{ agar } 0\le x\le\xi \text{ bo'lsa} \\ c_2(\xi)(\cos x+\sin x), \text{ agar } \xi\le x\le\pi \text{ bo'lsa} \end{cases}$$

formula bilan aniqlanadi; bunda (5) ga ko`ra

$$\begin{cases} c_1(\xi)\sin\xi - c_2(\xi)(\cos\xi + \sin\xi) = 0 \\ c_2(\xi)(-\sin\xi + \cos\xi) - c_1(\xi)\cos\xi = 1 \end{cases}$$

bo`lishi kerak. Oxirgi sistemadan $c_1(\xi)$ va $c_2(\xi)$ larni topamiz:
$c_1(\xi) = -(\sin\xi + \cos\xi)$, $c_2(\xi) = -\sin\xi$.

Demak, berilgan chegaraviy masalaning Grin funksiyasi

$$G(x,\xi) = \begin{cases} -(\sin\xi + \cos\xi)\sin x, \text{ agar } 0 \le x \le \xi \text{ bo'lsa} \\ -\sin\xi\cdot(\cos x + \sin x), \text{ agar } \xi \le x \le \pi \text{ bo'lsa} \end{cases},$$

yechimi esa (6) formulaga ko`ra

$$y(x) = \int_0^\pi G(x,\xi)g(\xi)d\xi =$$

$$= -(\cos x + \sin x)\int_0^x \sin\xi\cdot g(\xi)d\xi - \sin x\int_x^\pi (\sin\xi + \cos\xi)\cdot g(\xi)d\xi$$

ko`rinishda bo`ladi.

Misol 4. Ushbu

$$\begin{cases} \cos^2 x\cdot y'' - \sin 2x\cdot y' = g(x) \quad \left(g(x)\in C([0;\pi/4])\right) \\ y(0) + y'(0) = 0,\ y(\pi/4) - y'(\pi/4) = 0 \end{cases}$$

chegaraviy masala uchun Grin funksiyasini quring va masala yechimini toping.

Grin funksiyasini qurish uchun teorema 2 dan foydalanamiz. Ushbu

$$\cos^2 x\cdot y'' - \sin 2x\cdot y' = 0,$$

ya'ni

$$y'' - 2\text{tg}x\cdot y' = 0$$

tenglamaning umumiy yechimi topamiz:

$$\cos^2 x\cdot y'' - \sin 2x\cdot y' = 0,\ (\cos^2 x\cdot y')' = 0,\ \cos^2 x\cdot y' = c_1,\ y = c_1\text{tg}x + c_2$$

Bu umumiy yechimdan $y(0) + y'(0) = 0$ shartni qanoatlantiruvchi $y = y_1 = \text{tg}x - 1$ va $y(\pi/4) - y'(\pi/4) = 0$ shartni qanoatlantiruvchi $y = y_2 = \text{tg}x + 1$ yechimlarni ajratamiz va Grin funksiyasi uchun (4) formulani yozamiz

$$G(x,\xi) = \begin{cases} c_1(\xi)(\text{tg}x - 1), \text{ agar } 0 \le x \le \xi \text{ bo'lsa}, \\ c_2(\xi)(\text{tg}x + 1), \text{ agar } \xi \le x \le \pi/4 \text{ bo'lsa}. \end{cases}$$

Bu yerdagi $c_1(\xi)$ va $c_2(\xi)$ lar (5) ga ko`ra topiladi:

$$\begin{cases} c_1(\xi)(\text{tg}\xi - 1) - c_2(\xi)(\text{tg}\xi + 1) = 0 \\ c_2(\xi)\dfrac{1}{\cos^2\xi} - c_1(\xi)\dfrac{1}{\cos^2\xi} = 1 \end{cases}$$

Oxirgi sistemani yechib,

$$c_1(\xi) = -\frac{\cos^2\xi}{2}(\text{tg}\xi + 1),\ c_2(\xi) = -\frac{\cos^2\xi}{2}(\text{tg}\xi - 1)$$

larni topamiz. Demak,

$$\begin{cases} y'' - 2\text{tg}x\cdot y' = g(x)/\cos^2 x \\ y(0) + y'(0) = 0,\ y(\pi/4) - y'(\pi/4) = 0 \end{cases}$$

Chegaraviy masala uchun Grin funksiyasi

$$G(x,\xi)=\begin{cases}-\dfrac{\cos^2\xi}{2}(\mathrm{tg}\xi+1)(\mathrm{tg}x-1), \text{ agar } 0\le x\le\xi \text{ bo'lsa},\\ -\dfrac{\cos^2\xi}{2}(\mathrm{tg}\xi-1)(\mathrm{tg}x+1), \text{ agar } \xi\le x\le\pi/4 \text{ bo'lsa}.\end{cases}$$

Berilgan masala uchun (6) formulani qo`llash uchun tenglamaning har ikkala tomonini $\cos^2 x$ ga bo`lamiz va (6) ga ko`ra ushbu

$$y(x)=\int_0^{\pi/4}G(x,\xi)\frac{g(\xi)}{\cos^2\xi}d\xi=$$

$$=-\frac{(\mathrm{tg}x+1)}{2}\int_0^x(\mathrm{tg}\xi-1)g(\xi)d\xi-\frac{(\mathrm{tg}x-1)}{2}\int_x^{\pi/4}(\mathrm{tg}\xi+1)g(\xi)d\xi.$$

yechimni topamiz. ☝

Masalalar

Chegaraviy masalalarni yeching (**1 - 5**):

1. $y''-y=2e^x$, $y(0)=1,\ y(1)=0$.

2. $y''-2y'=4x+2$, $y(0)=1,\ y(1)+y'(0)=0$.

3. $y''+y=2\cos x$, $y(\pi/2)=0$, $y'(\pi)+y(\pi)=-1$.

4. $y''+y=x$, $y(\pi/2)=0$, $y(\pi)=0$.

5. $x^2y''+xy'-y=4x$, $y(1)+y'(1)=0,\ y(e)=1$.

Chegaraviy masalalar uchun Grin funksiyasini quring (**6 - 10**).

6. $y''+y=g(x)$, $y(0)=0, y'(1)=0$.

7. $y''-y'=g(x)$, $y(0)=0, y(1)-y'(1)=0$.

8. $y''+y=h(x)$, $y(0)+y'(0)=0, y(1)-y'(1)=0$.

9. $xy''+y'=h(x),\ y(1)=0,\ y(2)=0$.

10. $(x^2+1)y''+2xy'=g(x),\ y(0)=0,\ y'(1)=0$.

Mustaqil ish № 12 topshiriqlari:

I. Berilgan chegaraviy masalani yeching.

1. $y''-y'=1+x;\ y(0)=1,\ y'(1)+y(1)=0.$

2. $y''-y=x;\ y(0)=0,\ y'(1)-y(1)=2.$

3. $y''-y'=0;\ y(0)=2,\ y'(1)+y(1)=2.$

4. $y''+y'=1;\ y(0)+y'(0)=0,\ y'(1)-2y(1)=0.$

5. $y''+y'=e^x;\ y(0)-y'(0)=0,\ y'(1)=0.$

6. $y''+y=1;\ y(0)=1,\ y'(\pi/2)+y(\pi/2)=0.$

7. $xy''+y'=0;\ x\to 0$ da $y(x)-$ chegaralangan, $y(1)=y'(1)$.

8. λ ning qaysi qiymatlarida $y''+\lambda y=0;\ y(0)=y'(\pi)=0$ chegarviy masala noldan farqli yechimga ega bo`ladi?

9. $y''-y=0;\ y(0)=0, y(2\pi)=1$.

10. $y''+y=0;\ y(0)=0, y(2\pi)=1$.

11. λ ning qaysi qiymatlarida $y''+\lambda y=0;\ y(0)=y(\pi), y'(0)=y'(\pi)$ chegarviy masala noldan farqli yechimga ega bo`ladi?

12. λ ning qaysi qiymatlarida $y'' + \lambda y = 0;\ y(0) = y(1) = 0$ chegarviy masala noldan farqli yechimga ega bo`ladi?

13. $y'' - 2y' - 3y = 0;\ y'(0) = 1,\ y(+\infty) = 0$.

14. $y'' - 3y' - 4y = 0;\ y'(1) = 1,\ y(+\infty) = 0$.

15. $x^2 y'' + 2xy' + y = 0;\ y(1) = 0,\ y'(e) = 1$.

16. $y'' - 4y' - 5y = x;\ y'(0) = 1,\ y(1) = 0$.

17. $y'' + 2y' - 3y = 0;\ y(-\infty) = 0,\ y'(0) = 1$.

18. $y'' + 5y' + 6y = x;\ y(0) = 0,\ y'(\ln 2) = y(\ln 2)$.

19. $y'' - 5y' + 6y = 0;\ y(0) = 0,\ y'(1) - y(1) = 1$.

20. $y'' - y' = x;\ y(0) = -1,\ y'(1) = y(1)$.

21. $y'' + y = \cos x;\ y(0) = 0$, $y'(\pi) - y(\pi) = 1$.

22. $y'' - 3y' + 2y = 0; y'(0) - y(0) = 1,\ y'(1) = 0$.

23. $y'' - 2y' + y = e^{-x}$; $y(0) = 0,\ y'(1) = y(1)$.

24. $x^2 y'' + xy' + 4y = 0;\ y(1) = 0,\ y'(e) = 2$.

25. $y'' - y' - 2y = 0;\ y'(0) = 1,\ y(1) = 2$.

26. $y'' + y = \sin x;\ y(\pi/2) = 0,\ y'(\pi) - y(\pi) = 1$.

27. $y'' - 5y' + 6y = 0;\ y(0) = 0,\ y'(1) - y(1) = 1$

28. $y'' + y' - 2y = x;\ y'(0) = y(0),\ y'(1) = 0$.

29. $y'' - y' - 2y = 0;\ y'(0) - y(0) = 1,\ y(1) = -1$.

30. $x^2 y'' + 3xy' - 4y = 0;\ y'(0+) - y(0+) = 0,\ y'(1) = -1$.

II. Berilgan chegaraviy masala uchun Grin funksiyasini quring va yechim formulasini yozing.

1. $x^2 y'' + 5xy' + 3y = g(x);\ y(1) = 0,\ y'(2) = y(2)$

2. $x^2 y'' - 2y = g(x);\ y(1) = 0,\ y'(2) = y(2)$.

3. $x^2 y'' + 5xy' + 3y = g(x);\ y(1) = 0,\ y'(2) = y(2)$.

4. $x^2 y'' + 3xy' + y = f(x);\ y'(1) = y(1),\ y(2) = 0$.

5. $x^2 y'' + 2xy' + y = \psi(x);\ y'(1) - y(1) = 0,\ y'(e) = 0$.

6. $y'' - y = g(x);\ y'(0) + 3y(0) = 0,\ y'(2) = 0$.

7. $y'' - 4y' - 5y = f(x);\ y'(0) - y(0) = 0,\ y(1) = 0$.

8. $y'' - y' = g(x);\ y'(0) + 2y(0) = 0,\ y(1) = 0$.

9. $y'' - 3y' - 4y = f(x);\ y'(0) = 0,\ y(1) = 0$.

10. $y'' + y = g(x);\ y(0) = 0,\ y'(\pi) - y(\pi) = 0$.

11. $x^2 y'' + 2xy' = g(x);\ y(1) = y'(1),\ y'(3) = 0$.

12. $y'' - y' = \psi(x);\ y'(0) - y(0) = 0,\ y'(1) = 0$.

13. $xy'' + y' = f(x);\ y(1) = -y'(1),\ y'(2) = y(2)$.

14. $y'' + y' = g(x);\ y'(0) + 2y(0) = 0,\ y(1) = 0$.

15. $y'' + y = \varphi(x);\ y'(0) + y(0) = 0,\ y'(\pi) = 0$.

16. $y'' + y = g(x);\ y'(0) + y(0) = 0,\ y'(\pi) - y(\pi) = 0$.

17. $xy'' + y' = g(x);\ x \to 0$ da $y(x)$ chegaralangan, $y(1) - 2y'(1) = 0$.

18. $y'' + \omega^2 y = g(x),\ \omega = \text{const} > 0$; $y(0) = y'(0),\ y(\pi/\omega) = 0$.

19. $xy'' + y' = f(x);\ y(1) = 0,\ y'(2) = y(2)$.
20. $x^2 y'' - 2y = f(x);\ y(1) = -y'(1),\ y(3) = 0$.
21. $x^2 y'' + 2xy' = f(x);\ y(1) = y'(1),\ y(2) = 0$.
22. $x^2 y'' + 2xy' = g(x);\ y'(1) = 0,\ y(2) = 0$.
23. $xy'' + y' = g(x);\ y'(1) = 0,\ y'(2) - 2y(2) = 0$.
24. $y'' + y' = g(x);\ y'(0) + 2y(0) = 0,\ y'(1) - y(1) = 0$.
25. $x^2 y'' + 2xy' + y = f(x);\ y'(1) = y(1),\ y(e) = 0$.
26. $xy'' + y' = f(x);\ y'(1) = 0,\ y(2) = 0$.
27. $x^2 y'' - 3xy' = f(x)\ ;\ y'(1) = y(1),\ y'(2) = y(2)$.
28. $x^2 y'' - 2xy' = g(x)\ ;\ y'(1) = y(1),\ y(2) = 0$.
29. $x^2 y'' + 3xy' = f(x)\ ;\ y'(1) = y(1),\ y'(2) = 0$.
30. $y'' - 3y' + 2y = f(x);\ y'(0) = 2y(0),\ y'(1) = 0$.

13. DIFFERENSIAL TENGLAMALARNING NORMAL SISTEMASI

Maqsad – differensial tenglamalarning normal sistemasi uchun yechimning mavjudligi va yagonaligi to`g`risidagi teoremani hamda bunday sistemalarni yechish usullarini o`rganish

Yordamchi ma’lumotlar:

Ushbu

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(t, x_1, x_2, ..., x_n) \\ \dfrac{dx_2}{dt} = f_2(t, x_1, x_2, ..., x_n) \\ .... \\ \dfrac{dx_n}{dt} = f_n(t, x_1, x_2, ..., x_n) \end{cases} \tag{1}$$

ko`rinishdagi differensial tenglamalar sistemasi ***normal sistema*** deb ataladi; bunda $f_i(t, x_1, x_2, ..., x_n),\quad i = \overline{1,n},$ berilgan funksiyalar $D \subset \mathbb{R}^{1+n}$ sohada aniqlangan va uzluksiz deb hisoblanadi, ya’ni $f_i \in C(D, \mathbb{R}),\ i = \overline{1,n}.$

Agar $x_1 = \phi_1(t),\quad x_2 = \phi_2(t), ...,\quad x_n = \phi_n(t)$ funksiyalar $I \subset \mathbb{R}$ oraliqda uzluksiz differensiallanuvchi va (1) sistemani ayniyatga aylantirsa (qanoatlantirsa), u holda bu funksiyalar (1) tenglamalar sistemasining I oraliqda yechimi deyiladi.

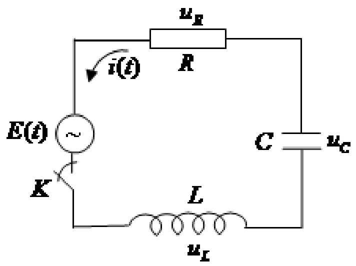
(1) sistemaning ushbu

$$x_1\Big|_{t_0} = x_{10},\quad x_2\Big|_{t_0} = x_{20}\ ,\ ...\ ,\ x_n\Big|_{t_0} = x_{n0} \qquad ((t_0, x_{10}, ..., x_{n0}) \in D - \text{berilgan}) \tag{2}$$

shartlarni qanoatlantiruvchi yechimini biror I, $t_0 \in I$, oraliqda topish Koshi masalasi (boshlang`ich masala) deyiladi. (2) shartlar Koshi shartlari (boshlang`ich shartlar) deb yuritiladi.

Differensial tenglamalar sistemasiga keltiriluvchi ba’zi amaliy masalalarni keltiraylik.

Masala 1. Induktivlik L, kondensator C, qarshilik R va elektr yurituvchi kuch manba'si $E(t)$ dan tuzilgan yopiq elektr zanjirini qaraylik (12.1- rasm).



13.1- rasm. $L, C, R, E(t)$ elektr zanjiri

K kalit qo'shilishidan avval kondensator zaryadlanmagan, zanjirda tok yo'q. Zanjirdagi tok kuchini toping.

⊶ Zanjirdagi tok kuchini $i = i(t)$, R qarshilikdagi, C kondensatordagi va L induktivlikdagi potensiallar ayirmasini mos ravishda u_R, u_C va u_L bilan belgilaylik. Fizikadan ma'lumki,

$$u_R = iR,\ i = C\frac{du_C}{dt},\ u_L = L\frac{di}{dt}\ .$$

To'la zanjir uchun Om qonuniga ko'ra

$$u_L + u_R + u_C = E(t),$$

ya'ni

$$L\frac{di}{dt} + Ri + u_C = E(t).$$

Boshlang'ich shartlar:

$$u_C(0) = 0,\ i(0) = 0.$$

Demak, quyidagi boshlang'ich masalaga egamiz:

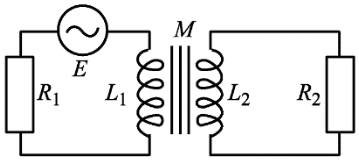
$$\begin{cases} \dfrac{di}{dt} = -\dfrac{R}{L}i - \dfrac{1}{L}u_C + \dfrac{E(t)}{L} \\ \dfrac{du_C}{dt} = \dfrac{1}{C}i \\ i(0) = 0,\ u_C(0) = 0 \end{cases}$$

Bu yerdan $i = i(t)$ ni yo'qotib, ushbu

$$\begin{cases} \dfrac{d^2u_C}{dt^2} + \dfrac{R}{L}\dfrac{du_C}{dt} + \dfrac{1}{LC}u_C = \dfrac{E(t)}{LC} \\ u_C(0) = \dfrac{du_C(0)}{dt} = 0 \end{cases}$$

masalani hosil qilish ham mumkin. 👍

Masala 2. $R_1, L_1, E = E(t)$ va R_1, L_1 elektr zanjirlarining induktivlik g`altaklari umumiy o`zakka ega (13.2- rasm). G`altaklarning o`zaro induktivlik koeffitsienti M. Zanjirlardagi mos $i_1 = i_1(t)$ va $i_2 = i_2(t)$ toklarni toping, $i_1(0) = i_{10}$ va $i_2(0) = i_{20}$ boshlang`ich qiymatlar ma'lum.



13.2- rasm

Har bir zanjir uchun Om qonunini yozaylik:

$$i_1R_1 + L_1\frac{di_1}{dt} + M\frac{di_2}{dt} = E(t)\,,\; i_2R_2 + L_2\frac{di_2}{dt} + M\frac{di_1}{dt} = 0\,.$$

Demak, $i_1 = i_1(t)$ va $i_2 = i_2(t)$ noma'lumlarni topish uchun ushbu

$$\begin{cases} L_1\dfrac{di_1}{dt} + M\dfrac{di_2}{dt} = -i_1R_1 + E(t) \\ L_2\dfrac{di_2}{dt} + M\dfrac{di_1}{dt} = i_2R_2 \\ i_1(0) = i_{10},\; i_2(0) = i_{20} \end{cases}$$

Koshi masalasini hosil qildik.

Masala 3. Volterra-Lotka modeli. Faraz qilaylik, yopiq sistemada (muhitda) o'lja va yirtqichlar (ikki tur individuumlari) yashasin. $N_1 = N_1(t)$ va $N_2 = N_2(t)$ mos ravishda t paytdagi o'lja va yirtqichlar sonini belgilasin. Agar yirtqichlar bo'lmasa, o'ljalarning $N_1'(t)$ o'sish tezligi ularning soniga proporsional, ya'ni $aN_1(t)$ ga ($a > 0$) teng va eksponensial tezlik bilan o'sadi; $N_2(t)$ sondagi yirtqichlar bu o'sish tezligini $bN_2(t)N_1(t)$ ga, ya'ni uchrashishlar soniga proporsianal miqdorga $(b > 0)$ kamaytiradi. Demak, $N_1' = (a - bN_2)N_1$ tenglik o'rinli bo'ladi. Agar o'ljalar bo'lmasa, yirtqichlar o'zgarishining tezligi $-cN_2(t)$ bo'ladi ($c > 0$, ularning soni eksponensial kamayadi); $N_1(t)$ sondagi o'ljalarning mavjudligi natijasida bu tezlik $dN_1(t)N_2(t)$ ga ortadi, ya'ni $N_2' = (-c + dN_1)N_2$ bo'ladi. Shunday qilib, bizning farazlarimizda o'lja va yirtqichlar (populyatsiyasi) soni quyidagi differensial tenglamalar sistemasi bilan boshqariladi:

$$\begin{cases} N_1' = (a - bN_2)N_1 \\ N_2' = (-c + dN_1)N_2 \end{cases}$$

Bu sistema Volterra-Lotka tenglamalari (o'lja-yirtqich modeli) deb yuritiladi.

Masala 4. **Ikki tur orasidagi raqobat modeli.** Ikki tur orasida birining ikkinchisiga yemish bo'lishidan farqli o'zaro ta'sir ham bo'lishi mumkin. Turlar joy, oziq-ovqat yoki boshqa resurslar uchun raqobatlashishi mumkin. Har bir tur individuumlari soni ikkinchi tur mavjud bo'lmaganda o'zining logistik tenglamasi asosida o'zgarardi. Lekin raqobat natijasida tur individuumlari sonining o'sish tezligi kamayadi. Turlar individuumlari sonini N_1 va N_2 bilan belgilab, turlar orasidagi raqobatni quyidagi tenglamalar bilan modellashtirish mumkin:

$$\begin{cases} \dfrac{dN_1}{dt} = k_1N_1(N_{01} - N_1 - r_{12}N_2) \\ \dfrac{dN_2}{dt} = k_2N_2(N_{02} - N_2 - r_{21}N_1) \end{cases}$$

Bu yerda $k_1, N_{01}, r_{12}, k_2, N_{02}, r_{21}$ – musbat sonlar; r_{12} parametr ikkinchi turning birinchi turga ta’sirini, r_{21} parametr esa birinchi turning ikkinchi turga ta’sirini xarakterlaydi. Agar $r_{12} > r_{21}$ bo’lsa, ikkinchi tur raqobatda dominantlik qiladi: ikkinchi turning birinchi turga ta’siri birinchi turning ikkinchi turga ta’siriga nisbatan katta. ✍

Masala 5. Simbiotik turlar o’zaro ta’sirining modeli. Ba’zan ikki turning har biri ikkinchining o’sishini ta’minlashi mumkin. Bunday turlar simbiotik turlar deyiladi. Masalan, yukka guli va yukka kapalagi simbiotik turlardir: yukka gulini yukka kapalagi changlatadi, yukka kapalagi esa yukka gulining nektarini iste’mol qilib yashaydi. Simbiotik turlarnng o’zaro ta’sirini quyidagi sistema bilan modellashtirish mumkin:

$$\begin{cases} \dfrac{dN_1}{dt} = N_1(N_{01} - k_1N_1 + r_{12}N_2) \\ \dfrac{dN_2}{dt} = N_2(N_{02} - k_2N_2 + r_{21}N_1) \end{cases}$$

Bu yerda $N_{01}, k_1, r_{12}, N_{02}, k_2, r_{21}$ – musbat sonlar. ✍

Masala 6. O’lja-yirtqich, ikki tur orasidagi raqobat va ikki simbiotik tur o’zaro ta’sirining modellarini n ta o’zaro ta’sir etuvchi turlar uchun quyidagicha umumlashtirish mumkin:

$$x_j' = x_j\left(a_j + \sum_{k=1}^{n} b_{jk}x_k\right),\ j = \overline{1,n};$$

bu yerda $x_j = x_j(t)$ – j- tur individuumlarining soni, $a_j > 0$ va $b_{jj} < 0$ o’zgarmas sonlar, b_{jk}, $j \neq k$, ixtiyoriy ishorali doimiylar. Bunda, agar $b_{jk}b_{kj} < 0$, $j \neq k$, bo’lsa, o’lja-yirtqich modeli, $b_{jk} < 0$, $b_{kj} < 0$, $j \neq k$, bo’lganda raqobat modeli, $b_{jk} > 0$, $b_{kj} > 0$, $j \neq k$, bo’lganda esa simbiotik model hosil bo’ladi. ✍

Endi mavjudlik va yagonalik reoremasini keltiramiz.

Biror $f(t, x_1, x_2, .., x_n)$, $(t, x_1, x_2, .., x_n) \in E \subset \mathbb{R}^{1+n}$, haqiqiy qiymatli funksiya berilgan bo`lsin. Agar biror $L > 0$ soni mavjud bo`lib, $\forall\{(t, x_1, ..., x_n), (t, \tilde{x}_1, .., \tilde{x}_n)\} \subset E$ uchun

$$|f(t, x_1, ..., x_n) - f(t, \tilde{x}_1, ..., \tilde{x}_n)| \le L\sum_{j=1}^{n}|x_j - \tilde{x}_j|$$

tengsizlik bajarilsa, u holda $f(t, x_1, ..., x_n)$ funksiya E da $(x_1, x_2, ..., x_n)$ o`zgaruvchilarga nisbatan Lipshits shartini qanoatlantiradi deyiladi.

Teorema (**MYaT**). Agar $f_i(t, x_1, ..., x_n)$, $i = \overline{1,n}$, funksiyalarning barchasi $(t_0, x_{10}, ..., x_{n0}) \in \mathbb{R}^{1+n}$ nuqtaning biror atrofida uzluksiz va shu atrofda $(x_1, x_2, ..., x_n)$ o`zgaruvchilarga nisbatan Lipshits shartini qanoatlantirsa, u holda (1), (2) Koshi masalasi biror $|t - t_0| \le h$ $(h > 0)$ oraliqda yechimga ega va bu yechim yagona.

Agar aytilgan shartlar D sohaning har bir nuqtasining biror atrofida o`rinli bo`lsa (buning uchun, masalan, $f_i \in C(D)$; $\dfrac{\partial f_i}{\partial x_j} \in C(D)$, $i = \overline{1,n}$, $j = \overline{1,n}$, bo`lishi yetarli), u holda (1), (2) Koshi masalasi eng katta aniqlanish sohasiga ega bo`lgan yagona yechimga ega (u davomsiz yechim deyiladi). Bu yechim D sohaning chegarasidan chegarasigacha boradi.

Quyida $f_i, i=\overline{1,n},$ funksiyalarni yetarlicha marta differensiallanuvchi deb faraz qilamiz.

(1) sistemani yechimini topishning ikki asosiy usuli mavjud. Ularning biri ***yo`qotish usuli***dir. Bu usulga ko`ra (1) sistemani $n-1$ ta noma'lumni yo`qotish natijasida bitta $n$-tartibli differensial tenglamaga olib kelinadi. (1) sistemadan $x_2, x_3, \ldots, x_n$ noma'lumlarni yo`qotaylik. Buning uchun (1) dagi birinchi tenglamani differensiallaymiz (uchraydigan barcha hosilalar mavjud deb faraz qilinadi) va bunda $\frac{dx_2}{dt}, ..., \frac{dx_n}{dt}$ hosilalar o`rniga ularning qiymatlarini (1) sistemaning tenglamalaridan keltirib qo`yamiz:

$$\frac{d^2x_1}{dt^2} = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f_1}{\partial x_2}\frac{dx_2}{dt} + ... + \frac{\partial f_1}{\partial x_n}\frac{dx_n}{dt} =$$

$$= \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x_1} f_1 + \frac{\partial f_1}{\partial x_2} f_2 + ... + \frac{\partial f_1}{\partial x_n} f_n$$

yoki

$$\frac{d^2x_1}{dt^2} = F_2(t, x_1, x_2, ..., x_n), \tag{3}$$

bunda

$$F_2 = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x_1} f_1 + \frac{\partial f_1}{\partial x_2} f_2 + ... + \frac{\partial f_1}{\partial x_n} f_n .$$

Endi (3) tenglamani differensiallaymiz va yuqoridagiga o`xshash ishlarni bajaramiz. Natijada

$$\frac{d^3x_1}{dt^3} = F_3(t, x_1, x_2, ..., x_n)$$

ko`rinishdagi munosabatga kelamiz. Bunaqa ishlarni takrorlab quyidagi sistemani hosil qilamiz:

$$\begin{cases} \frac{dx_1}{dt} = F_1(t, x_1, x_2, ..., x_n) \quad (F_1 = f_1) \\ \frac{d^2x_1}{dt^2} = F_2(t, x_1, x_2, ..., x_n) \\ .... \\ \frac{d^nx_1}{dt^n} = F_n(t, x_1, x_2, ..., x_n) \end{cases}$$

Bu sistemadagi dastlabki $n-1$ ta tenglamadan $x_2, ..., x_n$ larni topamiz (agar mumkin bo`lsa):

$$x_2 = x_2\left(t, x_1, \frac{dx_1}{dt}, \ldots \frac{d^{n-1}x_1}{dt^{n-1}}\right), \ \ldots, \ x_n = x_n\left(t, x_1, \frac{dx_1}{dt}, \ldots \frac{d^{n-1}x_1}{dt^{n-1}}\right) \tag{4}$$

va ularni sistemadagi oxirgi tenglamaga qo`yamiz. Natijada $x_1 = x_1(t)$ ga nisbatan ushbu

$$\frac{d^nx_1}{dt^n} = F\left(t, \frac{dx_1}{dt}, ..., \frac{d^{n-1}x_1}{dt^{n-1}}\right)$$

$n-$tartibli differensial tenglamani hosil qilamiz. Bu tenglamani yechib, $x_1 = x_1(t)$ ni topamiz, so`ngra (4) formumlalarga ko`ra $x_2 = x_2(t),..., x_n = x_n(t)$ larni topamiz. Umumiy holda shu yo`sinda topilgan $x_1 = x_1(t), x_2 = x_2(t),..., x_n = x_n(t)$ funksiyalar (1) sistemaning yechimini beradi. Ba'zi hollarda $x_1 = x_1(t)$ ga nisbatan bir dona $n$-tartibli differensial tenglama hosil bo`lmasligi mumkin. Masalan, bir-biriga bog`liq bo`lmagan ikki tenglamadan tuzilgan ushbu

$$\begin{cases} x' = x \\ y' = y \end{cases}$$

sistemadan, tushunarliki, $x = x(t)$ ga nisbatan bir dona ikkinchi tartibli differensial tenglama hosil qilib bo`lmaydi. Sistemaning yechimi: $x = c_1 e^t$, $y = c_2 e^t$.

(1) sistemani yechishning ikkinchi usuli — bu integrallanuvchi kombinatsiyalar tuzishdir. Berilgan differensial tenglamalar sistemasidan kelib chiquvchi va osongina yechiladigan (integrallanadigan) natijaviy tenglama ***integrallanuvchi kombinatsiya*** deyiladi. Integrallanuvchi kombinatsiyalar yordamida birinchi integrallar topiladi. Agar $\varphi(t, x_1, x_2, \ldots, x_n)$ funksiya C^1 sinfga tegishli va o`zgarmasdan farqli bo`lib, (1) sistemanining har qanday $x_1 = x_1(t), x_2 = x_2(t),..., x_n = x_n(t)$ yechim bo`ylab (yechimida) o`zgarmasga aylansa, ya'ni $\varphi(t, x_1(t), x_2(t), \ldots, x_n(t)) = \text{const}$ bo`lsa, bunday $\varphi(t, x_1, x_2, \ldots, x_n)$ funksiya (1) ***sistemanining birinchi integrali*** deb ataladi. Bu holda $\varphi(t, x_1, x_2, \ldots, x_n)$ funksiya bilan birgalikda $\varphi(t, x_1, x_2, \ldots, x_n) = c$ $(c-\text{const})$ ham birinchi integral deb yuritiladi. $\varphi(t, x_1, x_2, \ldots, x_n)$ funksiyaning $x_1 = x_1(t), x_2 = x_2(t),..., x_n = x_n(t)$ yechim bo`ylab o`zgarmasligi $\frac{d}{dt}\varphi(t, x_1(t), x_2(t), \ldots, x_n(t)) = 0$ ekanligidan kelib chiqadi.

Agar (1) sistemaning n dona erkli birinchi integrallari

$$\begin{aligned} &\varphi_1(t, x_1, x_2, \ldots, x_n) = c_1 \\ &\varphi_2(t, x_1, x_2, \ldots, x_n) = c_2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\varphi_n(t, x_1, x_2, \ldots, x_n) = c_n \end{aligned} \tag{5}$$

topilgan bo`lsa, u holda (5) tengliklar sistemasi (1) ning umumiy yechimini aniqlaydi.

(5) integrallarning erkliligi ular orasida $\psi(\varphi_1, \varphi_2, ..., \varphi_n) = 0$ ko`rinishdagi funksional bog`lanishning mavjud emasligini anglatadi. $\varphi_1, \varphi_2, ..., \varphi_n$ funksiyalarining erkliligini (lokal) ushbu

$$\begin{vmatrix} \frac{\partial\varphi_1}{\partial x_1} & \frac{\partial\varphi_1}{\partial x_2} & \dots & \frac{\partial\varphi_1}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \frac{\partial\varphi_n}{\partial x_1} & \frac{\partial\varphi_n}{\partial x_2} & \dots & \frac{\partial\varphi_n}{\partial x_n} \end{vmatrix} \neq 0 \tag{6}$$

shart ta'minlaydi.

(1) sistemani integrallash to`g`risidagi yuqorida aytilgan fikrlar ushbu

$$\frac{dx_1}{g_1(x_1,...,x_n)} = \frac{dx_2}{g_2(x_1,...,x_n)} = ... = \frac{dx_n}{g_n(x_1,...,x_n)} \tag{7}$$

simmetrik ko`rinishdagi differensial tenglamalar sistemasi uchun ham o`rinli. Bu sistemada bitta o`zgaruvchi erkli o`zgaruvchi, qolgan $(n-1)$ ta o`zgaruvchi esa – noma'lum funksiyalar. (7) sistemadan integrallanuvchi kombinatsiyalarni tuzishda quyidagi tasdiqdan foydalanish ba'zan qo`l keladi:

agar

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_n}{b_n}$$

bo`lsa, u holda ixtiyoriy $\lambda_1, \lambda_2, ..., \lambda_n$ sonlar uchun

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_n}{b_n} = \frac{\lambda_1 a_1 + \lambda_2 a_2 + ... + \lambda_n a_n}{\lambda_1 b_1 + \lambda_2 b_2 + ... + \lambda_n b_n} \tag{8}$$

proporsiya ham o`rinli bo`ladi.

Misol 1. Ushbu

$$\frac{dx}{dt} = 2x + y^2 + t^2, \quad \frac{dy}{dt} = -\frac{x}{2y} \tag{9}$$

differensial tenglamalar sistemasini yeching.

🗝 Yo`qotish usulini qo`llaymiz. Birinchi tenglamani differensiallaymiz:

$$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} + 2y\frac{dy}{dt} + 2t \tag{10}$$

(9) sistemaning ikkinchi tenglamasidan

$$2y\frac{dy}{dt} = -x$$

kelib chiqadi. Buni (10)ga qo`yib, $x = x(t)$ ga nisbatan ikkinchi tartibli chiziqli tenglamaga kelamiz:

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = 2t\,.$$

Bu tenglamani yechib, uning umumiy yechimini topamiz:

$$x = c_1e^t + c_2te^t + 2t + 4\ . \tag{11}$$

(9) sistemaning birinchi tenglamasidan

$$y^2 = \frac{dx}{dt} - 2x - t^2 = (-c_1 + c_2)e^t - c_2te^t + -t^2 - 4t - 6 \tag{12}$$

(11) va (12) tengliklar (9) sistemaning ushbu

$$\begin{cases} x = c_1e^t + c_2te^t + 2t + 4 \\ y^2 = (-c_1 + c_2)e^t - c_2te^t + -t^2 - 4t - 6 \end{cases}$$

umumiy yechimini aniqlaydi. ✍

Misol 2. $\varphi(t,x,y) = x^2 + ty$ funksiya ushbu

$$\frac{dx}{dt} = -\frac{x}{t},\ \frac{dy}{dt} = \frac{2x^2 - ty}{t^2}$$

sistemaning birinchi integrali ekanligini isbotlang.

⊶ Ixtiyoriy $x = x(t), y = y(t)$ yechim bo`ylab berilgan funksiyaning hosilasi aynan nolga teng:

$$\frac{d}{dt}\varphi(t, x(t), y(t)) = \frac{d}{dt}(x^2(t) + ty(t)) = 2x(t)\frac{dx(t)}{dt} + y(t) + t\frac{dy(t)}{dt} =$$

$$= 2x(t)\left(-\frac{x(t)}{t}\right) + y(t) + t\frac{2x^2(t) - ty(t)}{t^2} = 0\,.$$

Demak, berilgan funksiya har qanday yechim bo`ylab o`zgarmaydi, ya'ni u birinchi integral. ✍

Misol 3. Ushbu

$$\frac{dx}{dt} = y,\ \frac{dy}{dt} = x$$

sistemani yeching.

⊶ Berilgan sistemaning ikkita erkli birinchi integralini sistema tenglamalarini hadma-had qo`shib va ayirib, topamiz:

$$\frac{d(x+y)}{dt} = x + y,\ x + y = c_1e^t;$$

$$\frac{d(x-y)}{dt} = -(x-y),\ x - y = c_2e^{-t}.$$

Birinchi integrallardan yechim osongina topiladi:

$$\begin{cases} x + y = c_1e^t, \\ x - y = c_2e^{-t}. \end{cases} \Rightarrow \begin{cases} x = \dfrac{c_1}{2}e^t + \dfrac{c_2}{2}e^{-t}, \\ y = \dfrac{c_1}{2}e^t - \dfrac{c_2}{2}e^{-t}. \end{cases}$$

yoki qisqaroq ko`rinishda

$$\begin{cases} x = c_1e^t + c_2e^{-t}, \\ y = c_1e^t - c_2e^{-t}. \end{cases}$$ ✍

Misol 4. Ushbu

$$(x^2 + y^2 - t^2)\frac{dx}{dt} = tx,\ (x^2 + y^2 - t^2)\frac{dy}{dt} = ty \qquad (13)$$

sistemani yeching.

⊶ Berilgan sistemaning ikkita erkli birinchi integralini topamiz. (13) dagi tenglamalarni hadma-had bo`lib, quyidagilarni topamiz:

$$\frac{dy}{dx} = \frac{y}{x},\quad x \cdot dy - y \cdot dx = 0,\ d\left(\frac{y}{x}\right) = 0.$$

Demak, $\frac{y}{x} = c_1$ birinchi untegral.

(13) sistemaning yana bir birinchi integralini quyidagicha topish mumkin. (13) sistemaning birinchi tenglamasini $2x$ ga ikkinchisini $2y$ ga ko`paytiramiz. Natijada quyidagi tenglamalarni hosil qilamiz:

$$(x^2+y^2-t^2)\frac{d(x^2)}{dt}=2tx^2,\ (x^2+y^2-t^2)\frac{d(y^2)}{dt}=2ty^2 \qquad (14)$$

$z=x^2+y^2$ deymiz va (14) tenglamalarni hadma-had qo`shamiz:

$$(z-t^2)\frac{dz}{dt}=2tz\,.$$

Bundan $z\frac{dz}{dt}=t^2\frac{dz}{dt}+2tz$, $\frac{d}{dt}(\frac{z^2}{2})=\frac{d}{dt}(t^2z)$ kelib chiqadi. Oxirgi tenglamani integrallab, $\frac{z^2}{2}=t^2z+\frac{c_2}{2}$ $(c_2-const)$ yoki $z^2-2t^2z=c_2$ munosabatni topamiz. Oxirgi tenglikka $z=x^2+y^2$ ni qo`yib berilgan sistemaning quyidagi yana bir birinchi integralini hosil qilamiz:

$$(x^2+y^2)(x^2+y^2-2t^2)=c_2\,.$$

Topilgan birinchi integrallar erkli (nega?). Demak, (13) sistemaning umumiy yechimi ushbu

$$\begin{cases} \frac{y}{x}=c_1 \\ (x^2+y^2)(x^2+y^2-2t^2)=c_2 \end{cases}$$

munosabatlar bilan beriladi. ☝

Misol 5. Ushbu

$$\frac{dx}{z^2-y^2}=\frac{dy}{z}=-\frac{dz}{y}$$

sistemani yeching.

🗝 Proporsiyalarning (8) xossasidan foydalanib, quyidagini yozamiz:

$$\frac{dx}{z^2-y^2}=\frac{zdy+ydz}{z^2-y^2}$$

Bundan

$$dx=d(zy)$$

yoki

$$x-zy=c_1 \quad (c_1=\text{const})\,.$$

Yana bitta birinchi integralni

$$\frac{dy}{z}=-\frac{dz}{y}$$

tenglikdan topamiz:

$$ydy+zdz=0\,,\quad y^2+z^2=c_2 \quad \left(c_2=\text{const}\right).$$

Endi topilgan

$$\varphi_1=x-zy, \qquad \varphi_2=y^2+z^2$$

birinchi integrallarni erklilika tekshiramiz. Buning uchun quyidagi matritsani tuzamiz:

$$\begin{pmatrix} \frac{\partial\varphi_1}{\partial x} & \frac{\partial\varphi_1}{\partial y} & \frac{\partial\varphi_1}{\partial z} \\ \frac{\partial\varphi_2}{\partial x} & \frac{\partial\varphi_2}{\partial y} & \frac{\partial\varphi_2}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & -z & -y \\ 0 & 2y & 2z \end{pmatrix}$$

Bu matritsaning rangi 2 ga teng ($y \neq 0$ yoki $z \neq 0$ bo`lganda). Demak, topilgan φ_1 ва φ_2 birinchi integrallar erkli, ya'ni

$$\begin{cases} x - zy = c_1 \\ y^2 + z^2 = c_2 \end{cases}$$

munosabatlar berilgan sistemaning umumiy yechimini aniqlaydi: ikkita o`zgaruvchini uchinchisi va c_1, c_2 lar orqali (lokal) ifodalash mumkin. ♘

Misol 6. Quyidagi differensial tenglamalar sistemasini yeching:

$$\frac{dx}{dt} = y - z, \quad \frac{dy}{dt} = x^2 + y, \quad \frac{dz}{dt} = x^2 + z. \tag{15}$$

⊶ Ikkinchi va uchinchi tenglamalarga ko`ra

$$\frac{dy}{dt} - \frac{dz}{dt} = y - z.$$

Bundan va berilgan sistemaning birinchi tenglamasidan

$$\frac{d(y-z)}{dt} = \frac{dx}{dt},$$

$$y - z = x + c_1 \text{ — birinchi integral.} \tag{16}$$

Buni (15) sistemaning birinchi tenglamasiga qo`yib x ga nisbatan tenglama hosil qilamiz:

$$\frac{dx}{dt} = x + c_1$$

Bu birinchi tartibli chiziqli tenglamani yechamiz:

$$x = c_2 e^t - c_1. \tag{17}$$

x uchun bu ifodani (15) ning ikkinchi tenglamasiga qo`yamiz va y ga nisbatan tenglama hosil qilamiz:

$$\frac{dy}{dt} = y + c_2^2 e^{2t} - 2c_1 c_2 e^t + c_1^2.$$

Bu birinchi tartibli chiziqli tenglamani yechib $y = y(t)$ ni topamiz:

$$y = -c_1^2 + (c_3 - 2c_1 c_2 t)e^t + c_2^2 e^{2t} \tag{18}$$

Endi (17) va (18) ni (16)ga qo`yib $z(t)$ ni topamiz:

$$z = -c_1^2 + (c_3 - 2c_1 c_2 t)e^t - c_2 e^t + c_2^2 e^{2t}. \tag{19}$$

(17), (18) va (19) formulalar berilgan (15) sistemaning umumiy integralini ifodalaydi. ♘

Misol 7. Sistemani birinchi integrallar yordamida yeching

$$x' = y + z, y' = x + z, \; z' = x + y \; .$$

⊶ Tenglamalarni hadma-had qo`shib topamiz

$$(x + y + z)' = 2(x + y + z) \;\Rightarrow\; (x + y + z)e^{-2t} = 3c_1. \tag{20}$$

Sistemadagi birinchi tenglamadan ikkinchisini va ikkinchisidan uchunchisini hadma-had ayirib, yana ikkita birinchi integral hosil qilamiz

$$x'-y'=y-x \Rightarrow (x-y)e^t=3c_2\,, \tag{21}$$

$$y'-z'=z-y \;\Rightarrow (y-z)e^t=3c_3. \tag{22}$$

Topilgan (20)-(22) birinchi integrallar erkli. Ularni birgalikda yechib, berilgan sistemaning umumiy yechimini topamiz

$$\begin{cases} x+y+z=3c_1e^{2t}\,, \\ x-y=3c_2e^{-t}\,, \\ y-z=3c_3e^{-t}\,. \end{cases} \Rightarrow \begin{cases} x=c_1e^{2t}+(c_3+2c_2)e^{-t}, \\ y=c_1e^{2t}+(c_3-c_2)e^{-t}, \\ z=c_1e^{2t}-(2c_3+c_2)e^{-t}. \end{cases}$$

Masalalar

Differensial tenglamalar sistemasini yo`qotish usuli yordamida yeching $\left(x'=\dfrac{dx}{dt},\,...\right)$ (**1 - 6**):

1. $x'=-4x+2y,\; y'=-2x+y$.

2. $x'=-x-y,\; y'=4x-5y$.

3. $\dfrac{dx}{dt}=y^2+t+\cos t,\; \dfrac{dy}{dt}=\dfrac{x}{2y}$.

4. $\dfrac{dx}{dt}=1-\dfrac{2}{y},\; \dfrac{dy}{dt}=\dfrac{1}{x-t}+y$.

5. $x'=x-y,\; y'=x+z,\; z'=x+z$.

6. $x'=x+y,\; y'=-x-y,\; z'=x+y+z$.

Berilgan φ funksiya berilgan sistemaning birinchi integrali ekanligini isbotlang (**7 - 9**):

7. $\varphi(t,x,y)=3y\cos x-y^3,\; \left\{\dfrac{dx}{dt}=-y^2+\cos x,\; \dfrac{dy}{dt}=y\sin x\right\}$.

8. $\varphi(t,x,y,z)=\dfrac{y}{x},\; \left\{\dfrac{dx}{dt}=2xz,\; \dfrac{dy}{dt}=2yz, \dfrac{dz}{dt}=z^2-x^2-y^2\right\}$.

9. $\varphi=\dfrac{x^2+y^2+z^2}{x},\; \left\{\dfrac{dx}{dt}=2xz,\; \dfrac{dy}{dt}=2yz, \dfrac{dz}{dt}=z^2-x^2-y^2\right\}$.

Birinchi integralni toping (**10 - 11**):

10. $\dfrac{dx}{dt}=y,\; \dfrac{dy}{dt}=\dfrac{t-y^2}{x}$.

11. $\dfrac{dx}{dt}=y-z\,,\; \dfrac{dy}{dt}=x^2+y, \dfrac{dz}{dt}=x^2+z$.

Birinchi integrallarni topib, sistemani yeching (**12 - 18**):

12. $\dfrac{dx}{y^2(y-x^2)}=\dfrac{dy}{y(x^2+xy^2-1)}=\dfrac{dz}{z(x^2+xy^2-1)}$.

13. $x'=x^2y,\; y'=xy^2, (x>0, y>0)$.

14. $x'=x-xy,\; y'=-x+xy,\; (x>0,\; y>1-x)$.

15. $\frac{dx}{dt}=\frac{y^2+1}{2x},\ \frac{dy}{dt}=\frac{x^2+1}{2y},\ (x>0,\ y>0).$

16. $x'=x-y+z^2,\ y'=y-z^2,\ z'=z.$

17. $x'=xy-x^2,\ y'=y^2,\ z'=yz,\ (x>0).$

18. $x'=x^2+z^2,\ y'=y(x-z),\ z'=2xz,\ (x>z>0, y>0).$

Mustaqil ish № 13 topshiriqlari:

I. *Differensial tenglamalar sistemasini yeching.*

1. $\frac{dx}{x-y}=\frac{dy}{x+y}=\frac{dz}{z}.$ **2.** $\frac{dx}{y-u}=\frac{dy}{z-x}=\frac{dz}{u-y}=\frac{du}{x-z}.$

3. $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{xy+z}.$ **4.** $\frac{dx}{x(y+z)}=\frac{dy}{z(z-y)}=\frac{dz}{y(y-z)}.$

5. $\frac{dx}{x(z-y)}=\frac{dy}{y(y-x)}=\frac{dz}{y^2-xz}.$ **6.** $\frac{dx}{dt}=y^2+\cos t+1,\ \frac{dy}{dt}=\frac{x}{2y}.$

7. $\frac{dx}{dt}=\frac{1}{y^2-\alpha x},\ \frac{dy}{dt}=\frac{1}{2xy}\ (\alpha-\text{const}).$ **8.** $\frac{dx}{dt}=\frac{1}{2x^3+x^2-y^2},\ \frac{dy}{dt}=\frac{1}{2xy}.$

9. $\frac{dy}{dx}=\frac{z}{x},\ \frac{dz}{dx}=\frac{z(y+2x-1)}{x(y-1)}.$ **10.** $\frac{dx}{x}=\frac{dy}{2y}=\frac{dz}{xy}.$ **11.** $\frac{dx}{x}=\frac{dy}{-2y}=\frac{dz}{x^2+y^2}.$

12. $\frac{dx}{x}=\frac{dy}{-y}=\frac{dz}{x^2(x-3y)}.$ **13.** $\frac{dx}{xy^3}=\frac{dy}{x^2z^2}=\frac{dz}{y^3z}.$ **14.** $\frac{dx}{1}=\frac{dy}{y+z}=\frac{dz}{-26y-z}.$

15. $\frac{dx}{dt}=y^2+2\cos x,\ \frac{dy}{dt}=\frac{4x}{y}.$ **16.** $\frac{dx}{dt}=\frac{2t}{1+t^2}x,\ \frac{dy}{dt}=-\frac{1}{t}y+x+t.$

17. $\frac{dx}{dt}=y,\ \frac{dy}{dt}=x,\ \frac{dz}{dt}=x+2y+3z.$ **18.** $\frac{dx}{x}=\frac{dy}{-y}=\frac{dz}{z}.$ **19.** $\frac{dx}{dt}=\frac{y-1}{y},\ \frac{dy}{dt}=\frac{1}{x-t}.$

20. $\frac{dx}{dt}=-\frac{y}{t},\ \frac{dy}{dt}=-\frac{x}{t}.$ **21.** $\frac{dx}{dt}=x^2y,\ \frac{dy}{dt}=\frac{y}{t}-xy^2.$

22. $\frac{dx}{dt}=y-z,\ \frac{dy}{dt}=x+y+t,\ \frac{dz}{dt}=x+z+t.$ **23.** $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{2x-y}.$

24. $\frac{dx}{2xy}=\frac{dy}{y^2-x^2-z^2}=\frac{dz}{2yz}.$ **25.** $\frac{dx}{x+y-xy^2}=\frac{dy}{x^2y-x-y}=\frac{dz}{y^2-x^2}.$

26. $\frac{dx}{dt}=\frac{y}{t^2},\ \frac{dy}{dt}=3x+\frac{3}{t}y.$ **27.** $\frac{dx}{dt}=x\cos t,\ \frac{dy}{dt}=xe^{\sin t}.$ **28.** $\frac{dy}{dx}=e^{x-y},\ \frac{dz}{dx}=\frac{2z}{2x-z^2}.$

29. $\frac{dx}{\sqrt{x}}=\frac{dy}{\sqrt{y}}=\frac{dz}{0,5}.$ **30.** $\frac{dx}{\cos y}=\frac{dy}{\cos x}=\frac{dz}{\cos x\cdot\cos y}.$ **31.** $\frac{dx}{dt}=e^{-x}\cos y,\ \frac{dy}{dt}=-e^{-x}\sin y.$

32. $\frac{dx}{dt}=-x^2+y^2,\ \frac{dy}{dt}=-2xy.$ **33.** $\frac{dy}{dx}=e^{x-y},\ \frac{dz}{dx}=\frac{2z}{2x-z^2}.$

34. $x\frac{dy}{dx}=y+\sqrt{y^2-x^2},\ \frac{dz}{dx}=\frac{y+z}{z^2-x}.$ **35.** $\frac{dy}{dx}=2xy^2,\ \frac{dz}{dx}=\frac{z-x}{x}.$ **36.** $\frac{dy}{dx}=z,\ \frac{dz}{dx}=\frac{z^2}{y}.$

37. $\frac{dx}{(z-y)^2}=\frac{dy}{z}=\frac{dz}{y}.$ **38.** $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}=\frac{du}{\alpha u}\ (\alpha\neq 0-\text{const}).$

39. $\frac{dx}{dt}=-y^2+\sin x,\ \frac{dy}{dt}=-y\cos x$ (birinchi integralni toping).

40. $\frac{dt}{x^2-2xy-y^2}=\frac{dx}{t(x+y)}=\frac{dy}{t(x-y)}.$

II. *Matematik modeli differensial tenglamalar sistemasiga keltiriluvchi amaliy masalalarga misollar keltiring.*

14. NORMAL KO'RINISHDAGI CHIZIQLI DIFFERENSIAL TENGLAMALAR SISTEMASI

Maqsad – normal ko'rinishdagi chiziqli bir jinsli va bir jinsli bo'lmagan differensial tenglamalar sistemasini o'rganish

Yordamchi ma'lumotlar:

$x_1 = x_1(t), x_2 = x_2(t), \dots, x_n = x_n(t)$ noma'lum funksiyalarga nisbatan ***normal ko'rinishdagi chiziqli differensial tenglamalar sistemasi*** quyidagi ko'rinishga ega:

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n + f_1(t) \\ \dfrac{dx_2}{dt} = a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n + f_2(t) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n + f_n(t) \end{cases} \tag{1}$$

bunda $a_{kj}(t), k = \overline{1,n}, \ j = \overline{1,n},$ koeffitsientlar va $f_k(t), \ k = \overline{1,n},$ ozod hadlar biror I oraliqda aniqlangan va uzluksiz funksiyalar.

Ushbu

$$\boldsymbol{x} = \boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ . \\ x_n(t) \end{pmatrix}, \quad \frac{d\boldsymbol{x}}{dt} = \boldsymbol{x}' = \begin{pmatrix} x_1' \\ x_2' \\ . \\ . \\ . \\ x_n' \end{pmatrix}, \quad \boldsymbol{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ . \\ . \\ . \\ f_n(t) \end{pmatrix},$$

vektorlarni va

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

matritsani kiritib, (1) sistemani

$$\boldsymbol{x}' = A(t)\,\boldsymbol{x} + \boldsymbol{f}(t) \tag{2}$$

vektor ko‘rinishda yozamiz.

(2) chiziqli tenglamaga mos bir jinsli tenglama

$$\boldsymbol{x}' = A(t)\,\boldsymbol{x} \tag{3}$$

ko‘rinishda bo‘ladi (bu yerda ozod had $\boldsymbol{f}(t) \equiv 0$).

Bir jinsli tenglama (3) ning umumiy yechimini topish uning n dona chiziqli erkli yechimlari bo‘lmish

$$\boldsymbol{x}_1(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ . \\ . \\ x_{n1}(t) \end{pmatrix}, \ \boldsymbol{x}_2(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \\ . \\ . \\ x_{n2}(t) \end{pmatrix}, \dots , \ \boldsymbol{x}_n(t) = \begin{pmatrix} x_{1n}(t) \\ x_{2n}(t) \\ . \\ . \\ x_{nn}(t) \end{pmatrix}$$

vektor-funksiyalarni ((3) ning fundamental (bazis) yechimlarini) qurishga keltiriladi. Bu fundamental yechimlar orqali tuzilgan ushbu

$$\Phi(t) = \Phi[\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \dots, \boldsymbol{x}_n(t)] = \begin{pmatrix} x_{11}(t) & x_{12}(t) \dots\dots x_{1n}(t) \\ x_{21}(t) & x_{22}(t) \dots\dots x_{2n}(t) \\ . \quad\ . & . \qquad\quad . \\ x_{n1}(t) & x_{n2}(t) \dots\dots x_{nn}(t) \end{pmatrix} \tag{4}$$

matritsa (3) ning **fundamental matritsa**si deyiladi. Fundamental matritsa $\Phi(t)$ orqali (3) ning umumiy yechimi

$$\boldsymbol{x} = \Phi(t)\boldsymbol{c}, \quad \boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ . \\ c_n \end{pmatrix}, \tag{5}$$

ko‘rinishida ifodalanadi; bunda $c_1, c_2, \dots, c_n -$ ixtiyoriy o‘zgarmas sonlar, $\boldsymbol{c}$ – o‘zgarmas vektor.

Fundamental matritsa ushbu

$$\Phi'(t) = A(t)\Phi(t)$$

matritsaviy differensial tenglamaning yechimidir.

$\boldsymbol{y}_1(t), \boldsymbol{y}_2(t), ..., \boldsymbol{y}_k(t)$ ($n\times1$ o‘lchamli) vektor-funksiyalar I intervalda aniqlangan bo‘lsin. Agar kamida bittasi noldan farqli $c_1, c_2, ..., c_k$ sonlari mavjud bo‘lib, ular uchun I da

$$c_1\boldsymbol{y}_1(t) + c_2\boldsymbol{y}_2(t) + ... + c_k\boldsymbol{y}_k(t) = \boldsymbol{0} \quad (\boldsymbol{0} - \text{ nol-vektor}) \qquad (6)$$

ayniyat o‘rinli bo‘lsa, u holda qaralayotgan vektor-funksiyalar I intervalda chiziqli bog‘liq deyiladi; aks holda ular I da chiziqli bog‘lanmagan (chiziqli erkli) deb ataladi.

Demak, $\boldsymbol{y}_1(t), \boldsymbol{y}_2(t), ..., \boldsymbol{y}_k(t)$ vektor-funksiyalarni chiziqli bog‘liqlikka tekshirish (6) tenglamaning $c_1, c_2, ..., c_k$ larga nisbatan notrivial $\left(\exists i \in \{1,2,...,k\}\ c_i \neq 0\right)$ yechimga ega bo‘lishini aniqlashga keltiriladi.

Agar $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), ..., \boldsymbol{x}_n(t)$ vektor-funksiyalar (3) differensial tenglamalar sistemasining I da yechimi bo‘lsa, u holda ularning I da chiziqli erkli bo‘lishi osongina tekshiriladi: (3) sistemaning n dona $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), ..., \boldsymbol{x}_n(t)$ yechimlarining chiziqli erkli bo‘lishi uchun ulardan tuzilgan ushbu

$$\det[\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), ..., \boldsymbol{x}_n(t)] = \begin{vmatrix} x_{11}(t) & x_{12}(t) ..... x_{1n}(t) \\ x_{21}(t) & x_{22}(t) ..... x_{2n}(t) \\ . \quad . & . \quad . \quad . \quad . \quad . \\ x_{n1}(t) & x_{n2}(t) ...... x_{nn}(t) \end{vmatrix}$$

determinantning biror $t_0 \in I$ nuqtada noldan farqli bo‘lishi yetarli va zarurdir.

$\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), ..., \boldsymbol{x}_n(t)$ vektor-funksiyalarning Vronskiani deb ushbu

$$W[\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), ..., \boldsymbol{x}_n(t)] = \det[\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), ..., \boldsymbol{x}_n(t)]$$

determinantga aytiladi. (3) differensial tenglamalar sistemasining $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), ..., \boldsymbol{x}_n(t)$ $(t \in I)$ yechimlari uchun tuzilgan ushbu

$$W(t) = W[\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), ..., \boldsymbol{x}_n(t)]$$

Vronskian uchun

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} SpA(s)ds\right); \ t_0 \in I, \ t \in I, \ SpA(s) = \sum_{j=1}^{n} a_{jj}(s), \qquad (7)$$

formula o‘rinli. Bu (7) formula ***Liuvill formulasi*** deb taladi.

Agar bir jinsli differensial tenglamalar sistemasi (3) ning fundamental matritsasi $\Phi(t)$ ma’lum bo‘lsa, unga mos bir jinsli bo‘lmagan (2) tenglamaning xususiy yechimini (5) formuladagi ixtiyoriy o‘zgarmas vektor $\boldsymbol{c}$ ni variatsiyalash usuli (Lagranj usuli) bilan topish mumkin. Bunda (2) ning xususiy yechimi

$$\boldsymbol{x} = \Phi(t)\boldsymbol{c}(t) \tag{8}$$

ko‘rinishda izlanadi. (8) ni (2) ga qo‘yib, noma’lum vektor funksiya $\boldsymbol{c}(t)$ topiladi:

$$\Phi(t)\boldsymbol{c}'(t) = \boldsymbol{f}(t). \tag{9}$$

$\forall t \in I$ uchun $\det\Phi(t) \neq 0$ bo‘lgani uchun (9) dan

$$\boldsymbol{c}(t) = \int \Phi^{-1}(t)\boldsymbol{f}(t)dt \tag{10}$$

ekanligini topamiz. Demak, (3) ning xususiy yechimi (10) va (8) ga ko‘ra

$$\boldsymbol{x} = \Phi(t)\int \Phi^{-1}(t)\boldsymbol{f}(t)dt \tag{11}$$

shaklda bo‘ladi. Bir jinsli bo‘lmagan (2) tenglamaning umumiy yechimi uning xususiy yechimi (11) ga mos bir jinsli tenglama (3) ning umumiy yechimi (5) ni qo‘shishdan hosil bo‘ladi, ya’ni

$$\boldsymbol{x} = \Phi(t)\int \Phi^{-1}(t)\boldsymbol{f}(t)dt + \Phi(t)\boldsymbol{c}. \tag{12}$$

Misol 1. Ushbu

$$\text{a) } \boldsymbol{y}_1(t) = \begin{pmatrix} t \\ 2t \\ 1 \end{pmatrix}, \ \boldsymbol{y}_2(t) = \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}$$

vektor-funksiyalar biror I oraliqda chiziqli bog‘liqmi? Mana bu

$$\text{b) } \boldsymbol{y}_1(t) = \begin{pmatrix} t \\ 2t \\ 1 \end{pmatrix}, \ \boldsymbol{y}_2(t) = \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}, \ \boldsymbol{y}_3(t) = \begin{pmatrix} 1+t \\ 3t \\ 1+t^2 \end{pmatrix}$$

funksiyalarchi?

🗝 a). Faraz qilaylik, ular biror I oraliqda chiziqli bog‘liq bo‘lsin. U holda kamida bittasi noldan farqli bo‘lgan c_1 va c_2 sonlar mavjud bo‘lib, $\forall t \in I$ uchun

$$c_1\boldsymbol{y}_1(t)+c_2\boldsymbol{y}_2(t)=0$$

bo‘ladi. Demak,

$$\forall t\in I \text{ uchun } \quad c_1\begin{pmatrix} t \\ 2t \\ 1 \end{pmatrix}+c_2\begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}=\begin{pmatrix} c_1t+c_2 \\ 2c_1t+c_2t \\ c_1+c_2t^2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

ya’ni

$$\begin{cases} c_1t+c_2=0, \\ 2c_1t+c_2t=0, \qquad (t\in I). \\ c_1+c_2t^2=0. \end{cases}$$

Bu yerdagi birinchi tenglamadan $c_2=-c_1t$ ni topib, uni uchinchi tenglamaga qo‘yaylik:

$$c_1-c_1t^3=0, \quad t\in I.$$

Oxirgi tenglik $t\in I$ ga nisbatan ayniyat bo‘lgani uchun $c_1=0$ bo‘lishi kerak. Demak, $c_2=-c_1t=0$. Shunday qilib, $c_1=c_2=0$. Bu esa c_1 va c_2 larning birortasining noldan farqli ekanligiga zid. Farazimiz noto‘g‘ri: berilgan $\boldsymbol{y}_1(t)$ va $\boldsymbol{y}_2(t)$ vektor-funksiyalar hech qanday oraliqda chiziqli bog‘liq emas.

b). Ushbu

$$c_1\boldsymbol{y}_1(t)+c_2\boldsymbol{y}_2(t)+c_3\boldsymbol{y}_3(t)=0, t\in I,$$

ayniyatni yozaylik. Uning skalyar ko‘rinishi quyidagicha:

$$\begin{cases} c_1t+c_2+c_3(1+t)=0, \\ 2c_1t+c_2t+3c_3t=0, \qquad (t\in I) \\ c_1+c_2t^2+c_3(1+t^2)=0. \end{cases}$$

Oxirgi sistemani

$$\begin{cases} (c_1+c_3)t+c_2+c_3=0, \\ (2c_1+c_2+3c_3)t=0, \qquad (t\in I) \\ (c_2+c_3)t^2+c_1+c_3=0. \end{cases}$$

ayniyatlar ko‘rinishida yozib, ushbu

$$c_1 + c_3 = 0,\ c_2 + c_3 = 0,\ 2c_1 + c_2 + 3c_3 = 0$$

shartlarni hosil qilamiz. Ulardan $c_1 = -c_3$, $c_2 = -c_3$ ekanligini topamiz. Demak, masalan, $c_1 = 1,\ c_2 = 1,\ c_3 = -1$ tanlab, notrivial chiziqli kombinatsiya

$$1\cdot \boldsymbol{y}_1(t) + 1\cdot \boldsymbol{y}_2(t) + (-1)\cdot \boldsymbol{y}_3(t) = 0,\ t \in (-\infty, +\infty),$$

bo‘lishini aniqlaymiz. Demak, berilgan $\boldsymbol{y}_1(t),\ \boldsymbol{y}_2(t),\ \boldsymbol{y}_3(t)$ vektor-funksiyalar ixtiyoriy I oraliqda chiziqli bog‘langan. ☝

Misol 2. Sistemani yeching

$$x' = y + 2e^t,\ y' = x + 2t.$$

⊶ Berilgan sistemaga mos bir jinsli sistema

$$\begin{cases} x' = y \\ y' = x \end{cases} \quad \text{yoki} \quad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

ko‘rinishga ega. Bu sistema

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} \quad \text{va} \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$$

chiziqli erkli yechimlarga ega. Haqiqatdan ham, ularning yechim ekanligi bevosita ko‘rinib turibdi (ularni yo‘qotish usuli bilan ham topsa bo‘ladi), chiziqli erkliligi esa mos Vronskianning noldan farqli ekanligidan kelib chiqadi:

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = \begin{vmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{vmatrix} = 2 \neq 0.$$

Demak, bir jinsli sistemaning fundamental matritsasi

$$\Phi(t) = \begin{pmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{pmatrix},$$

uning umumiy yechimi esa

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Phi(t) \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

yoki skalyar ko‘rinishda

$$x = c_1 e^{-t} + c_2 e^t,\ y = -c_1 e^{-t} + c_2 e^t.$$

Berilgan

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2e^t \\ 2t \end{pmatrix}$$

bir jinsli bo‘lmagan sistemaning yechimini

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Phi(t) \cdot \boldsymbol{c}(t)$$

ko‘rinishda izlaymiz. Noma’lum $\boldsymbol{c}(t)$ vektor-funksiya uchun (9) ga ko‘ra

$$\Phi(t) \cdot \boldsymbol{c}'(t) = \begin{pmatrix} 2e^t \\ 2t \end{pmatrix}$$

tenglamaga kelamiz. Bu tenglamaning har ikkala tomonini

$$\Phi^{-1}(t) = \frac{1}{2} \cdot \begin{pmatrix} e^t & -e^t \\ e^{-t} & e^{-t} \end{pmatrix}$$

teskari matritsaga ko‘paytirib $\boldsymbol{c}'(t)$ ni, so‘ngra esa $\boldsymbol{c}(t)$ ni topamiz.

$$\boldsymbol{c}'(t) = \Phi^{-1}(t) \cdot \begin{pmatrix} 2e^t \\ 2t \end{pmatrix}$$

yoki skalyar ko‘rinishda

$$\begin{cases} c_1'(t) = e^{2t} - te^t, \\ c_2'(t) = 1 + te^{-t}. \end{cases}$$

Bu sistemadan integrallashlarni bajarib,

$$\boldsymbol{c}(t) = \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}, \text{ bunda } \begin{cases} c_1(t) = \frac{1}{2}e^{2t} - te^t + e^t \\ c_2(t) = t - te^{-t} - e^{-t} \end{cases},$$

bo‘lishini topamiz (integrallashlar o‘zgarmaslarini nolga teng deb oldik). Demak, berilgan bir jinsli bo‘lmagan sistemaning xususiy yechimi

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Phi(t) \cdot \boldsymbol{c}(t), \text{ ya'ni } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2}e^{2t} - te^t + e^t \\ t - te^{-t} - e^{-t} \end{pmatrix}$$

ko‘rinishda bo‘ladi. Bu xususiy yechimga bir jinsli tenglamaning

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Phi(t)\cdot\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \text{ ya'ni } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{-t} & e^{t} \\ -e^{-t} & e^{t} \end{pmatrix}\cdot\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

umumiy yechimini qo‘shib, berilgan sistemaning umumiy yechimini hosil qilamiz:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{-t} & e^{t} \\ -e^{-t} & e^{t} \end{pmatrix}\cdot\begin{pmatrix} \frac{1}{2}e^{2t} - te^{t} + e^{t} + c_1 \\ t - te^{-t} - e^{-t} + c_2 \end{pmatrix}$$

yoki skalyar ko‘rinishda

$$\begin{cases} x = c_1 e^{-t} + c_2 e^{t} + te^{t} + \frac{1}{2}e^{t} - 2t, \\ y = -c_1 e^{-t} + c_2 e^{t} .+ te^{t} - \frac{1}{2}e^{t} - 2. \end{cases}$$

Faraz qilaylik, berilgan

$$\boldsymbol{x}_1(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \cdots \\ x_{n1}(t) \end{pmatrix}, \ \boldsymbol{x}_2(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \\ \cdots \\ x_{n2}(t) \end{pmatrix}, \ \ldots, \ \boldsymbol{x}_n(t) = \begin{pmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \cdots \\ x_{nn}(t) \end{pmatrix}$$

vektor-funksiyalar I oraliqda uzluksiz differensiallanuvchi va ularning Vronskiani noldan farqli bo‘lsin, $W(t) \neq 0$. Ushbu

$$\boldsymbol{x} = \boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{pmatrix}$$

vektor-funksiyaning koordinata funksiyalari $x_1(t), x_2(t), \ldots, x_n(t)$ ga nisbatan chiziqli differensial tenglamalar sistemasi

$$\frac{1}{W(t)}\begin{vmatrix} x_j' & x_{j1}'(t) & x_{j2}'(t)........ & x_{jn}'(t) \\ x_1 & x_{11}(t) & x_{12}(t)........ & x_{1n}(t) \\ .... & .... & .... & .... \\ x_j & x_{j1}(t) & x_{j2}(t)........ & x_{jn}(t) \\ .... & .... & .... & .... \\ x_n & x_{n1}(t) & x_{n2}(t)........ & x_{nn}(t) \end{vmatrix} = 0, \quad j=\overline{1,n}\,, \tag{13}$$

uchun berilgan $\boldsymbol{x}=\boldsymbol{x}_1(t),...,\boldsymbol{x}=\boldsymbol{x}_n(t)$ funksiyalar yechimlardir ($\boldsymbol{x}=\boldsymbol{x}_k(t)$, $k=\overline{1,n}$, bo‘lganda bu determinantning 1- va $(k+1)$ - ustunlari bir xil). (13) dagi j- determinantning birinchi satri uning j- satri hosilasidan iborat. (13) dagi determinantlarni birinchi ustun bo‘yicha yoyib (Laplas formulasiga ko‘ra)

$$\frac{1}{W(t)}\Big(W(t)x_j' + b_{j1}(t)x_1 + b_{j2}(t)x_2 + ... + b_{jn}x_n\Big) = 0, \quad j=\overline{1,n},$$

ya’ni

$$x_j' = -\frac{b_{j1}(t)}{W(t)}x_1 - \frac{b_{j2}(t)}{W(t)}x_2 - ... - \frac{b_{jn}}{W(t)}x_n, \quad j=\overline{1,n},$$

sistemaga kelamiz; bunda berilganga ko‘ra $W(t)=W[\boldsymbol{x}_1(t),\boldsymbol{x}_2(t),...,\boldsymbol{x}_n(t)]\neq 0$.

(13) sistemani **boshqacha usulda** ham **tuzish mumkin**. Agar berilgan Vronskiani noldan farqli vektor-funksiyalardan

$$\Phi(t)=\begin{pmatrix} x_{11}(t) & x_{12}(t)..... & x_{1n}(t) \\ x_{21}(t) & x_{22}(t)..... & x_{2n}(t) \\ . & . & . \quad . \\ x_{n1}(t) & x_{n2}(t)...... & x_{nn}(t) \end{pmatrix}$$

matritsani tuzsak, ushbu

$$\boldsymbol{x}' = A(t)\,\boldsymbol{x},\; A(t)=\Phi'(t)\cdot\Phi^{-1}(t), \tag{14}$$

normal sistemaning bazis yechimlari berilgan funksiyalardan, fundamental matritsasi esa tuzilgan $\Phi(t)$ dan iborat bo‘ladi.

Misol 3. Berilgan

$$\boldsymbol{x}_1(t)=\begin{pmatrix}1\\ \cos t\end{pmatrix} \text{ va } \boldsymbol{x}_2(t)=\begin{pmatrix}\sin t\\ 1\end{pmatrix}$$

yechimlarga ega bo‘lgan normal ko‘rinishdagi 2- tartibli chiziqli differensial tenglamalar sistemasini tuzing.

🗝 Berilgan vektor-funksiyalarning Vronskiani

$$W(t)=\begin{vmatrix}1 & \sin t\\ \cos t & 1\end{vmatrix}=(2-\sin 2t)/2$$

ixtiyoriy $t\in(-\infty,+\infty)$ nuqtada noldan farqli va izlangan sistema (13) ga asosan

$$\frac{1}{W(t)}\begin{vmatrix}x_1' & 0 & \cos t\\ x_1 & 1 & \sin t\\ x_2 & \cos t & 1\end{vmatrix}=0,\quad \frac{1}{W(t)}\begin{vmatrix}x_2' & -\sin t & 0\\ x_1 & 1 & \sin t\\ x_2 & \cos t & 1\end{vmatrix}=0$$

ko‘rinishga ega. Mos determinantlarni birinchi ustun bo‘ylab yoyamiz va ixchamlashlarni bajarib, izlangan sistemani hosil qilamiz:

$$\begin{cases}x_1'=-\dfrac{2\cos^2 t}{2-\sin 2t}x_1+\dfrac{2\cos t}{2-\sin 2t}x_2\\ x_2'=-\dfrac{2\sin t}{2-\sin 2t}x_1+\dfrac{2\sin^2 t}{2-\sin 2t}x_2\end{cases}\quad(-\infty<t<+\infty). \qquad (15)$$

Eslatma. Berilgan vektor-funksiyalardan

$$\Phi(t)=\begin{pmatrix}1 & \sin t\\ \cos t & 1\end{pmatrix}$$

matritsani tuzib, (14) formulaga ko‘ra normal sistemani tuzsak ham o‘sha (15) sistemani hosil qilamiz. 👍

Masalalar

Vektor-funksiyalarni chiziqli erklilikka tekshiring (**1 - 5**):

1. $\boldsymbol{y}_1(t)=\begin{pmatrix}1\\ t\end{pmatrix},\ \boldsymbol{y}_2(t)=\begin{pmatrix}1\\ t^2\end{pmatrix}.$ **2.** $\boldsymbol{y}_1(t)=\begin{pmatrix}t\\ 2t\end{pmatrix},\ \boldsymbol{y}_2(t)=\begin{pmatrix}1\\ t\end{pmatrix},\ \boldsymbol{y}_3(t)=\begin{pmatrix}2t+1\\ 5t\end{pmatrix}.$

3. $\boldsymbol{y}_1(t)=\begin{pmatrix}1\\ t\\ 2t\end{pmatrix},\ \boldsymbol{y}_2(t)=\begin{pmatrix}0\\ t\\ t\end{pmatrix}.$ **4.** $\boldsymbol{y}_1(t)=\begin{pmatrix}1\\ 2t\\ 3t\end{pmatrix},\ \boldsymbol{y}_2(t)=\begin{pmatrix}0\\ t\\ t^2\end{pmatrix},\ \boldsymbol{y}_3(t)=\begin{pmatrix}0\\ 2t\\ 2t^2\end{pmatrix}.$

5. $\boldsymbol{y}_1(t)=\begin{pmatrix}1\\ 2t\\ 3t^2\end{pmatrix},\ \boldsymbol{y}_2(t)=\begin{pmatrix}1\\ t\\ t^2\end{pmatrix},\ \boldsymbol{y}_3(t)=\begin{pmatrix}3\\ 4t\\ 5t^2\end{pmatrix}.$

Berilgan vektor-funksiyalar qanoatlantiruvchi $\boldsymbol{x}'=A(t)\boldsymbol{x}$ ko‘rinishdagi chiziqli differensial tenglamalar sistemasini tuzing (**6 - 9**):

6. $\boldsymbol{x}_1(t)=\begin{pmatrix}t\\ 2t\end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix}t\\ 3t\end{pmatrix}.$ **7.** $\boldsymbol{x}_1(t)=\begin{pmatrix}1\\ e^t\end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix}t\\ e^t\end{pmatrix}.$

8. $\boldsymbol{x}_1(t)=\begin{pmatrix}\cos t\\ \sin t\end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix}t\\ 1\end{pmatrix}.$ **9.** $\boldsymbol{x}_1(t)=\begin{pmatrix}1\\ t\\ t^2\end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix}t\\ t^2\\ t^2\end{pmatrix},\ \boldsymbol{x}_3(t)=\begin{pmatrix}1\\ t^{1/2}\\ t^{3/2}\end{pmatrix}.$

Sistemani Lagranj usuli yordamida yeching (**10 - 16**):

10. $x'=2x+y+t,\ y'=-y+t.$ **11.** $x'=x+y+e^t,\ y'=-x+3y-4t.$

12. $x'=x-2y+1/\cos t,\ y'=x-y.$ **13.** $x'=x+y,\ y'=-x-y+4\ln t.$

14. $x'=y+t,\ t^2y'=4ty-6x+6.$ **15.** $\dfrac{dx}{dt}=-\dfrac{x}{t}+\dfrac{y}{t}-\dfrac{2}{t^2},\ \dfrac{dy}{dt}=-2\dfrac{x}{t}+2\dfrac{y}{t}.$

16. $x'=x+2y+3z,\ y'=y+2z+3e^{2t},\ z'=2y+z.$

Mustaqil ish № 14 topshiriqlari:

1. Berilgan $\boldsymbol{x}_1(t),\boldsymbol{x}_2(t)$ vektor-funksiyalarni chiziqli erklilikka tekshiring.

2. Berilgan vektor-funksiyalar qanoatlantiruvchi 2- tartibli chiziqli differensial tenglamalar sistemasi $\boldsymbol{x}'=A(t)\boldsymbol{x}$ ni tuzing. Hosil bo‘lgan tenglamani skalyar differensial tenglamalar sistemasi ko‘rinishida yozing. Bu tenglamalar sistemasi qaysi $t\in I$ oraliqda aniqlangan?

3. Ushbu $\boldsymbol{x}'=A(t)\,\boldsymbol{x}+\boldsymbol{f}(t)$ tenglamani tuzing va uning umumiy yechimini toping.

1. $\boldsymbol{x}_1(t)=\begin{pmatrix} t \\ \cos t \end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix} \sin t \\ 1 \end{pmatrix};\quad \boldsymbol{f}(t)=\begin{pmatrix} 1 \\ t^2 \end{pmatrix}$

2. $\boldsymbol{x}_1(t)=\begin{pmatrix} t \\ \sin t \end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix} \cos t \\ 1+t \end{pmatrix};\quad \boldsymbol{f}(t)=\begin{pmatrix} \cos t \\ 1 \end{pmatrix}$

3. $\boldsymbol{x}_1(t)=\begin{pmatrix} e^t \\ t \end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix} e^{-t} \\ t \end{pmatrix};\quad \boldsymbol{f}(t)=\begin{pmatrix} 2t \\ 1 \end{pmatrix}$

4. $\boldsymbol{x}_1(t)=\begin{pmatrix} e^t \\ e^{2t} \end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix} t \\ t+1 \end{pmatrix};\quad \boldsymbol{f}(t)=\begin{pmatrix} e^t \\ t \end{pmatrix}$

5. $\boldsymbol{x}_1(t)=\begin{pmatrix} e^{2t} \\ t \end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix} 2t \\ e^t \end{pmatrix};\quad \boldsymbol{f}(t)=\begin{pmatrix} t \\ 1+t^2 \end{pmatrix}$

6. $\boldsymbol{x}_1(t)=\begin{pmatrix} \cos t \\ 2\sin t \end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix} t \\ 2t \end{pmatrix};\quad \boldsymbol{f}(t)=\begin{pmatrix} 2\sin t \\ \cos t \end{pmatrix}$

7. $\boldsymbol{x}_1(t)=\begin{pmatrix} 1 \\ t \end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix} 2 \\ 3t \end{pmatrix};\quad \boldsymbol{f}(t)=\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

8. $\boldsymbol{x}_1(t)=\begin{pmatrix} 3 \\ t \end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix} 2 \\ t^2 \end{pmatrix};\quad \boldsymbol{f}(t)=\begin{pmatrix} 3t \\ 1+t \end{pmatrix}$

9. $\boldsymbol{x}_1(t)=\begin{pmatrix} 2\cos t \\ t \end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix} \sin t \\ 3t \end{pmatrix};\quad \boldsymbol{f}(t)=\begin{pmatrix} 2t \\ 3\cos t \end{pmatrix}$

10. $\boldsymbol{x}_1(t)=\begin{pmatrix} t \\ t^2 \end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix} 2t \\ t \end{pmatrix};\quad \boldsymbol{f}(t)=\begin{pmatrix} \cos t \\ t \end{pmatrix}$

11. $\boldsymbol{x}_1(t)=\begin{pmatrix} \ln t \\ t \end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix} 1 \\ t \end{pmatrix};\quad \boldsymbol{f}(t)=\begin{pmatrix} 2t \\ 3 \end{pmatrix}$

12. $\boldsymbol{x}_1(t)=\begin{pmatrix} 1 \\ \ln t \end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix} t \\ 1 \end{pmatrix};\quad \boldsymbol{f}(t)=\begin{pmatrix} t \\ \ln t \end{pmatrix}$

13. $\boldsymbol{x}_1(t)=\begin{pmatrix} 1 \\ t \end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix} 1 \\ t^2 \end{pmatrix};\quad \boldsymbol{f}(t)=\begin{pmatrix} 2 \\ t \end{pmatrix}$

14. $\boldsymbol{x}_1(t)=\begin{pmatrix}\sqrt{t}\\ 1\end{pmatrix}$, $\boldsymbol{x}_2(t)=\begin{pmatrix}t\\ 1+\sqrt{t}\end{pmatrix}$; $\boldsymbol{f}(t)=\begin{pmatrix}1+\cos t\\ t\end{pmatrix}$

15. $\boldsymbol{x}_1(t)=\begin{pmatrix}\cos 2t\\ \sin t\end{pmatrix}$, $\boldsymbol{x}_2(t)=\begin{pmatrix}1\\ t\end{pmatrix}$; $\boldsymbol{f}(t)=\begin{pmatrix}\sin t\\ 1\end{pmatrix}$

16. $\boldsymbol{x}_1(t)=\begin{pmatrix}\sin 2t\\ 2t\end{pmatrix}$, $\boldsymbol{x}_2(t)=\begin{pmatrix}1\\ t\end{pmatrix}$; $\boldsymbol{f}(t)=\begin{pmatrix}\cos t\\ 2\end{pmatrix}$

17. $\boldsymbol{x}_1(t)=\begin{pmatrix}\ln(t+1)\\ 1\end{pmatrix}$, $\boldsymbol{x}_2(t)=\begin{pmatrix}t\\ 2\end{pmatrix}$; $\boldsymbol{f}(t)=\begin{pmatrix}1+t\\ t\end{pmatrix}$

18. $\boldsymbol{x}_1(t)=\begin{pmatrix}\sqrt{t+1}\\ 1\end{pmatrix}$, $\boldsymbol{x}_2(t)=\begin{pmatrix}1\\ \sqrt{t}\end{pmatrix}$; $\boldsymbol{f}(t)=\begin{pmatrix}2t\\ \sqrt{t}\end{pmatrix}$

19. $\boldsymbol{x}_1(t)=\begin{pmatrix}2t\\ 1\end{pmatrix}$, $\boldsymbol{x}_2(t)=\begin{pmatrix}t\\ e^t\end{pmatrix}$; $\boldsymbol{f}(t)=\begin{pmatrix}1+t\\ 2e^t\end{pmatrix}$

20. $\boldsymbol{x}_1(t)=\begin{pmatrix}e^{-t}\\ t\end{pmatrix}$, $\boldsymbol{x}_2(t)=\begin{pmatrix}e^t\\ 1\end{pmatrix}$; $\boldsymbol{f}(t)=\begin{pmatrix}t\\ 2\end{pmatrix}$

21. $\boldsymbol{x}_1(t)=\begin{pmatrix}t\\ t^2\end{pmatrix}$, $\boldsymbol{x}_2(t)=\begin{pmatrix}t^2\\ 1\end{pmatrix}$; $\boldsymbol{f}(t)=\begin{pmatrix}1+t\\ 1-t\end{pmatrix}$

22. $\boldsymbol{x}_1(t)=\begin{pmatrix}1+t\\ t\end{pmatrix}$, $\boldsymbol{x}_2(t)=\begin{pmatrix}2t\\ t^2\end{pmatrix}$; $\boldsymbol{f}(t)=\begin{pmatrix}\cos t\\ 2\sin t\end{pmatrix}$

23. $\boldsymbol{x}_1(t)=\begin{pmatrix}\sqrt{t}\\ 1+t\end{pmatrix}$, $\boldsymbol{x}_2(t)=\begin{pmatrix}1\\ t\end{pmatrix}$; $\boldsymbol{f}(t)=\begin{pmatrix}1+t\\ 2\sqrt{t}\end{pmatrix}$

24. $\boldsymbol{x}_1(t)=\begin{pmatrix}e^{3t}\\ 1\end{pmatrix}$, $\boldsymbol{x}_2(t)=\begin{pmatrix}t\\ e^{2t}\end{pmatrix}$; $\boldsymbol{f}(t)=\begin{pmatrix}e^t\\ 2+t\end{pmatrix}$

25. $\boldsymbol{x}_1(t)=\begin{pmatrix}e^{-t}\\ e^{-2t}\end{pmatrix}$, $\boldsymbol{x}_2(t)=\begin{pmatrix}1\\ t\end{pmatrix}$; $\boldsymbol{f}(t)=\begin{pmatrix}e^t\\ 1+t\end{pmatrix}$

26. $\boldsymbol{x}_1(t)=\begin{pmatrix}\sin t\\ \sin 2t\end{pmatrix}$, $\boldsymbol{x}_2(t)=\begin{pmatrix}\cos t\\ \cos 2t\end{pmatrix}$; $\boldsymbol{f}(t)=\begin{pmatrix}t\\ 1+t\end{pmatrix}$

27. $\boldsymbol{x}_1(t)=\begin{pmatrix}\cos 2t\\ 1\end{pmatrix},\quad \boldsymbol{x}_2(t)=\begin{pmatrix}\sin 2t\\ 2\end{pmatrix};\qquad \boldsymbol{f}(t)=\begin{pmatrix}1+\cos 2t\\ t\end{pmatrix}$

28. $\boldsymbol{x}_1(t)=\begin{pmatrix}e^t\\ t\end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix}e^{2t}\\ t\end{pmatrix};\qquad \boldsymbol{f}(t)=\begin{pmatrix}1+t\\ e^t\end{pmatrix}$

29. $\boldsymbol{x}_1(t)=\begin{pmatrix}1+t\\ t\end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix}t^2\\ 1+t^2\end{pmatrix};\qquad \boldsymbol{f}(t)=\begin{pmatrix}2\\ \sqrt{t}\end{pmatrix}$

30. $\boldsymbol{x}_1(t)=\begin{pmatrix}\ln(t+1)\\ 2\end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix}t\\ 2t\end{pmatrix};\qquad \boldsymbol{f}(t)=\begin{pmatrix}t\\ 1+t\end{pmatrix}$

31. $\boldsymbol{x}_1(t)=\begin{pmatrix}1+t^2\\ 2\end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix}t\\ 3t\end{pmatrix};\qquad \boldsymbol{f}(t)=\begin{pmatrix}\cos t\\ 2+t\end{pmatrix}$

32. $\boldsymbol{x}_1(t)=\begin{pmatrix}\sin t\\ \cos t\end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix}\sin 2t\\ \cos 2t\end{pmatrix};\qquad \boldsymbol{f}(t)=\begin{pmatrix}1+t\\ 2\end{pmatrix}$

33. $\boldsymbol{x}_1(t)=\begin{pmatrix}\mathrm{tg}t\\ t\end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix}t\\ 1\end{pmatrix};\qquad \boldsymbol{f}(t)=\begin{pmatrix}1\\ 2+t\end{pmatrix}$

34. $\boldsymbol{x}_1(t)=\begin{pmatrix}t\\ \mathrm{tg}2t\end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix}1\\ t\end{pmatrix};\qquad \boldsymbol{f}(t)=\begin{pmatrix}t\\ 1+t\end{pmatrix}$

35. $\boldsymbol{x}_1(t)=\begin{pmatrix}\sqrt{t+1}\\ \sqrt{t}\end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix}1\\ 2\end{pmatrix};\qquad \boldsymbol{f}(t)=\begin{pmatrix}1+\sqrt{t}\\ 3\end{pmatrix}$

36. $\boldsymbol{x}_1(t)=\begin{pmatrix}t\\ 2t\end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix}1\\ \sqrt{t}\end{pmatrix};\qquad \boldsymbol{f}(t)=\begin{pmatrix}1-t\\ 2t\end{pmatrix}$

37. $\boldsymbol{x}_1(t)=\begin{pmatrix}\sin t\\ \cos 2t\end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix}1\\ t\end{pmatrix};\qquad \boldsymbol{f}(t)=\begin{pmatrix}1\\ 2-t\end{pmatrix}$

38. $\boldsymbol{x}_1(t)=\begin{pmatrix}\cos t\\ \sin t\end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix}t\\ 2t\end{pmatrix};\qquad \boldsymbol{f}(t)=\begin{pmatrix}\sin t\\ 2\end{pmatrix}$

39. $\boldsymbol{x}_1(t)=\begin{pmatrix}e^t\\ e^{-t}\end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix}e^{2t}\\ e^t\end{pmatrix};\qquad \boldsymbol{f}(t)=\begin{pmatrix}1+t\\ 1-2t\end{pmatrix}$

40. $\boldsymbol{x}_1(t)=\begin{pmatrix}1+t\\ t^2\end{pmatrix},\ \boldsymbol{x}_2(t)=\begin{pmatrix}1\\ t\end{pmatrix};\quad \boldsymbol{f}(t)=\begin{pmatrix}\cos t\\ 1+t\end{pmatrix}$

15. CHIZIQLI O‘ZGARMAS KOEFFITSIENTLI NORMAL DIFFERENSIAL TENGLAMALAR SISTEMASI. EKSPONENSIAL MATRITSA

Maqsad – chiziqli o‘zgarmas koeffitsientli normal differensial tenglamalar sistemasining umumiy yechimini qurishni va matritsaning eksponentasini o‘rganish

Yordamchi ma’lumotlar:

Chiziqli o‘zgarmas koeffitsientli normal differensial tenglamalar sistemasining umumiy ko‘rinishi quyidagicha :

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + f_1(t) \\ x_2' = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + f_2(t) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + f_n(t) \end{cases} \tag{1}$$

bunda $a_{kj}(t)$ $(k=\overline{1,n},\ j=\overline{1,n})$ koeffitsientlar — o‘zgarmas sonlar, $f_1(t)$, $f_2(t)$, ..., $f_n(t)$ ozod hadlar — biror I oraliqda uzluksiz funksiyalar. (1) ning vektor ko‘rinishi

$$\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{f}(t); \tag{2}$$

bunda

$$\boldsymbol{x} = \boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{pmatrix},\ \boldsymbol{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \cdots \\ f_n(t) \end{pmatrix},\ A = \| a_{kj} \|_{(n\times n)}\ .$$

I. (2) – bir jinsli bo‘lmagan tenglama (umumiy holda). Unga mos bir jinsli tenglama

$$\boldsymbol{x}' = A\boldsymbol{x} \tag{3}$$

ko‘rinishda bo‘ladi (ozod had nolga teng).

Agar biz (3) bir jinsli differensial tenglamaning umumiy yechimini qura olsak, (2) bir jinsli bo‘lmagan temglamaning ham yechimini topa olamiz. Buni ixtiyoriy o‘zgarmaslarni variatsiyalash (Lagranj usuli) bilan amalga oshirishimiz mumkin.
(3) tenglamaning umumiy yechimini qurishda Eyler usulini qo‘llaymiz. Bir jinsli tenglama (3) ning xarakteristik tenglamasi deb

$$\det(A-\lambda E)=0 \tag{4}$$

tenglamaga aytiladi; bunda $E-n\times n$ o‘lchamli birlik matritsa. (4) xarakteristik tenglama n-darajali algebraik tenglamadan iborat. Uning ildizlari A matritsaning xarakteristik (xos) sonlari deb ataladi.
(3) differensial tenglamalar sistemasining umumiy yechimi uning n dona chiziqli erkli yechimlarining chiziqli kombinatsiyasidan iborat. Bu chiziqli erkli yechimlar (4) xarakteristik tenglamaning ildizlaridan (xos sonlardan) foydalanib quriladi.

1⁰. *Xarakteristik sonlar oddiy*. Xarakteristik tenglama (4) ning barcha ildizlari $\lambda_1,\lambda_2,...,\lambda_n$ turli bo‘lsin $(\lambda_j\neq\lambda_k,\ j\neq k)$. A matritsaning $\lambda=\lambda_j$ xarakteristik soniga mos keluvchi xos vektori $\boldsymbol{h}_j$ ni ushbu

$$(A-\lambda_j E)\boldsymbol{h}_j=0\ ,\ \ j=\overline{1,n}\ , \tag{5}$$

tenglamadan topamiz va (3) ning

$$e^{\lambda_1 t}\boldsymbol{h}_1,\ e^{\lambda_2 t}\boldsymbol{h}_2,...,\ e^{\lambda_n t}\boldsymbol{h}_n$$

yechimlarini quramiz (ular chiziqli erkli bo‘ladi). (3) ning umumiy yechimi

$$\boldsymbol{x}=\sum_{j=1}^{n}c_j e^{\lambda_j t}\boldsymbol{h}_j\quad (c_j-\text{const}) \tag{6}$$

ko‘rinishda yoziladi.

2⁰. *Karrali xarakteristik son mavjud.* Aytaylik, $\lambda=\lambda_j$ son (4) xarakteristik tenglamaning $r=r_j$ karrali ildizi bo‘lsin. U holda (3) tenglamaning bu xarakteristik songa mos keluvchi yechimi

$$\boldsymbol{h}=\boldsymbol{p}(t)e^{\lambda t} \tag{7}$$

ko‘rinishda bo‘ladi; bunda

$$\boldsymbol{p}(t) = \boldsymbol{p}_{r-1}t^{r-1} + \ldots + \boldsymbol{p}_1 t + \boldsymbol{p}_0 \tag{8}$$

vektor koeffitsientli ko‘phad.

$\boldsymbol{p}(t)$ ko‘phadni topish uchun (7) ni (3) ga qo‘yamiz:

$$\boldsymbol{p}'(t)e^{\lambda t} + \lambda \boldsymbol{p}(t)e^{\lambda t} = A\boldsymbol{p}(t)e^{\lambda t}$$

yoki

$$(A - \lambda E)\boldsymbol{p}(t) = \boldsymbol{p}'(t). \tag{9}$$

(8) ga ko‘ra (9) tenglikdan quyidagini topamiz:

$$(A - \lambda E)(\boldsymbol{p}_{r-1}t^{r-1} + \ldots + \boldsymbol{p}_1 t + \boldsymbol{p}_0) = (r-1)\boldsymbol{p}_{r-1}t^{r-2} + \ldots + \boldsymbol{p}_1. \tag{10}$$

(10) ayniyat bo‘lishi uchun t ning bir xil darajalari oldidagi koeffitsientlar teng bo‘lishi kerak:

$$\begin{cases} (A - \lambda E)\boldsymbol{p}_{r-1} = 0 \\ (A - \lambda E)\boldsymbol{p}_{r-2} = (r-1)\boldsymbol{p}_{r-1} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (A - \lambda E)\boldsymbol{p}_0 = \boldsymbol{p}_1 \end{cases}$$

Oxirgi sistemani yuqoridagi tenglamadan boshlab quyiga qarab yechamiz va $\boldsymbol{p}_{r-1},\ldots,\boldsymbol{p}_1,\boldsymbol{p}_0$ vektor-koeffitsientlarni topamiz. Ular $r = r_j$ dona ixtiyoriy o‘zgarmas son orqali ifodalanadi. (7) yechimda r ta ixtiyoriy o‘zgarmas qatnashadi. Bu o‘zgarmaslarning bittasini birga qolganlarini esa nolga tenglashtirib, r dona chiziqli erkli yechim topishimiz mumkin.

Barcha xarakteristik sonlarga mos kelgan chiziqli erkli yechimlarning (ular n ta bo‘ladi) chiziqli kombinatsiyasi (3) ning umumiy yechimini beradi. Umumiy yechimni ushbu

$$\boldsymbol{x} = \Phi(t)\boldsymbol{c}$$

ko‘rinishda ham yozish mumkin. Bu yerda $\Phi(t)$ – fundamental matritsa, $\boldsymbol{c}$ esa ixtiyoriy o‘zgarmas vektor. Fundamental matritsa n ta chiziqli erkli yechim koordinatalarini ustunlar bo‘ylab yozishdan hosil bo‘ladi.

Agar (3) tenglamaning koeffitsientlari haqiqiy sonlardan iborat bo‘lsa, odatda bunday holda uning haqiqiy yechimlarini topish talab qilinadi. Bu holda xarakteristik kompleks sonlar o‘zaro qo‘shma kompleks sonlar ko‘rinishida uchraydi. (3) ning haqiqiy yechimlarini topish uchun kompleks yechimlarning haqiqiy va mavhum qismlarini ajratish kerak.

II. (3) tenglamaning normalangan $\left(\Phi(0)=E\right)$ fundamental matritsasini eksponensial martitsa orqali

$$\Phi(t)=e^{At}\ \left(=\exp(At)\right)$$

ko‘rinishda ham ifodalash mumkin. Bunda

$$e^{At}=E+At+\frac{A^2}{2!}t^2+\ldots+\frac{A^k}{k!}t^k+\ldots\ ,\ t\in\mathbb{R}, \tag{11}$$

Bu yerdagi matritsaviy qator ixtiyoriy segmentda tekis yaqinlashuvchi va uni hadma-had xohlagancha marta differensiallash mumkin:

$$\frac{d}{dt}e^{At}=A\left(E+At+\frac{A^2}{2!}t^2+\ldots+\frac{A^k}{k!}t^k+\ldots\right)=Ae^{At}\,.$$

A matritsaning eksponentasi e^A ni hisoblashda ta’rifdagi matritsaviy qatordan foydalanish umumiy holda noqulay. Bunda mos (3) tenglamaning biror fundamental matritsasi $\Phi(t)$ ni topib, ushbu

$$e^{At}=\Phi(t)\Phi^{-1}(0)$$

formuladan foydalanish mumkin.

e^A ni hisoblashning boshqa usuli quyidagicha. Dastlab A matritsaning xarakteristik ko‘phadi $\chi(\lambda)\overset{def}{=}\det(A-\lambda E)$ ni tuzamiz. So‘ngra

$$\chi\Big(\frac{d}{dt}\Big)\varphi_j(t)=0\,,\ \frac{d^i\varphi_j(0)}{dt^i}=\delta_{ij}\ \ (i,j=\overline{0,n-1};\,\delta_{ij}-\text{Kroneker simvoli})\,,$$

boshlang‘ich masalalarni yechamiz va, nihoyat,

$$e^{tA}=\varphi_0(t)E+\varphi_1(t)A+\varphi_2(t)A^2+...+\varphi_{n-1}(t)A^{n-1} \tag{12}$$

matritsani hisoblaymiz.

$A\in\mathbb{M}_{n\times n}(\mathbb{C})$ matritsaning eksponentasini hisoblashning yana bir usuli matritsaning Jordan ko‘rinishiga asoslanadi. Shunday teskarilanuvchi S matritsa topamizki, uning uchun

$$J = S^{-1}AS \ \ (A = SJS^{-1}),$$

$$J = \operatorname{diag}(J_{\lambda_1,n_1}, \dots, J_{\lambda_2,n_2}, \dots, J_{\lambda_k,n_k}) = \begin{pmatrix} J_{\lambda_1,n_1} & & & & \\ & \ddots & & & \\ & & J_{\lambda_2,n_2} & & \\ & & & \ddots & \\ & & & & J_{\lambda_k,n_k} \end{pmatrix}, \qquad (13)$$

bo‘ladi, bunda J Jordan (katakli-diagonal) matritsasining diagonali bo‘ylab Jordan kataklari, boshqa o‘rinlarda esa nollar joylashgan bo‘lib, u quyidagicha tuziladi. Faraz qilaylik, A matritsaning turli xos sonlari $\lambda_1, \lambda_2, \dots, \lambda_s$ $(s \le n)$ mos ravishda $k_1, k_2, \dots, k_s$ karrali ($k_1 + k_2 + \dots + k_s = n$) hamda λ_q xos songa mos kelgan chiziqli erkli vektorlar soni p_q, ya’ni $\dim\{\mathbf{x} \,|\, (A\text{-}\lambda_q E)\mathbf{x} = 0\} = p_q$ ($p_q = n - \operatorname{rank}(A - \lambda_q E)$) bo‘lsin. U holda λ_q xos songa p_q dona

$$J_{\lambda_q, d_{qj}} = \begin{pmatrix} \lambda_q & 1 & & & \\ & \lambda_q & 1 & & \\ & & \lambda_q & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_q \end{pmatrix} \in \mathrm{M}_{d_{qj} \times d_{qj}}(\mathbb{C}),$$

$$j = 1, 2, \dots, p_q \ \ (d_{q1} + d_{q2} + \dots + d_{q\,p_q} = k_q),$$

$d_{q1}, d_{q2}, \dots, d_{q\,p_q}$ o‘lchamli Jordan kataklari mos keladi; bu yerda ham bo‘sh o‘rinlarda nollar yozilgan deb tushunish kerak. Ravshanki, λ_q ga mos kelgan Jordan kataklarining eng katta o‘lchami $\tilde{k}_q \overset{def}{=} \max\{d_{q1}, d_{q2}, \dots, d_{q\,p_q}\} \le k_q$ bo‘ladi. (13) formulada Jordan kataklari boshqacha indekslangan. S matritsa umumlashgan xos vektorlardan tuziladi: $S = [\boldsymbol{h}_{1,n_1}, \boldsymbol{h}_{2,n_1}, \dots, \boldsymbol{h}_{n_1,n_1}, \dots]$,

$$(A - \lambda_1 E)\boldsymbol{h}_{1,n_1} = 0, \ \ (A - \lambda_1 E)\boldsymbol{h}_{2,n_1} = \boldsymbol{h}_{1,n_1},$$
$$\dots, \ (A - \lambda_1 E)\boldsymbol{h}_{n_1,n_1} = \boldsymbol{h}_{n_1-1,n_1}, \ \dots .$$

$A = SJS^{-1}$ bo‘lgani uchun $tA = S(tJ)S^{-1}$ va (13) ga ko‘ra

$$e^{tA} = Se^{tJ}S^{-1} = S\mathrm{diag}(e^{tJ_{\lambda_1,n_1}}, \ldots, e^{tJ_{\lambda_2,n_2}}, \ldots, e^{tJ_{\lambda_k,n_k}})S^{-1}. \tag{14}$$

Demak, biz Jordan katagini t ga ko‘paytmasining eksponentasini hisoblashimiz kerak. Ushbu

$$J_{\lambda,\, p} = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix} \in \mathbb{M}_{p\times p}(\mathbb{C})$$

tipik Jordan katagini olaylik. Qulaylik uchun

$$E_p = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in \mathbb{M}_{p\times p}(\mathbb{C}),\; N_p = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \in \mathbb{M}_{p\times p}(\mathbb{C})$$

belgilashlarni kiritamiz. U holda $J_{\lambda,\, p} = \lambda E_p + N_p$ va

$$e^{tJ_{\lambda,p}} = e^{t\lambda}e^{tN_p} = e^{t\lambda}\begin{pmatrix} 1 & \frac{t}{1!} & \frac{t^2}{2!} & \cdots & \frac{t^{p-1}}{(p-1)!} \\ 0 & 1 & \frac{t}{1!} & \cdots & \frac{t^{p-2}}{(p-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{t}{1!} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{15}$$

bo‘ladi. Shunday qilib, agar A matritsani Jordan kanonik ko‘rinishiga keltiruvchi S matritsa ma’lum bo‘lsa, u holda (15) va (14) formulalar orqali e^{tA} ($t \in \mathbb{R}$) matritsani hisoblash mumkin.

III. Bir jinsli bo‘lmagan (2) tenglamaning umumiy yechimi e^{At} eksponensial matritsa yordamida quyidagicha ifodalanadi:

$$\boldsymbol{x} = \int e^{(t-s)A} \boldsymbol{f}(s)\,ds + e^{tA}\boldsymbol{c}\,.$$

Bu yerda $\int e^{(t-s)A} \boldsymbol{f}(s)\,ds$ qo‘shiluvchi (2) tenglamaning xususiy yechimi, $e^{tA}\boldsymbol{c}$ qo‘shiluvchi esa mos bir jinsli (3) tenglamaning umumiy yechimi $\big(\boldsymbol{c}-$ixtiyoriy o‘zgarmas vektor$\big)$. Xususiy yechimni $\boldsymbol{f}(t)$ ozod had kvaziko‘phaddan iborat bo‘lganda integrallash amalini ishlatmasdan topish mumkin.

Kvaziko‘phad (kompleks sohada) deb

$$\boldsymbol{f}(t) = e^{\gamma t}\boldsymbol{p}(t) \tag{16}$$

ko‘rinishdagi vektor-funksiyaga aytiladi; bu yerda $\gamma \in \mathbb{C}$ va $\boldsymbol{p}(t) = \boldsymbol{b}^m t^m + \boldsymbol{b}^{m-1} t^{m-1} + \ldots + \boldsymbol{b}^1 t + \boldsymbol{b}^0$ – (kompleks) vektor koeffitsientli ko‘phad.

Ozod hadi kvaziko‘phaddan iborat bo‘lgan ushbu

$$\boldsymbol{x}' = A\boldsymbol{x} + e^{\gamma t}\boldsymbol{p}(t) \tag{17}$$

sistema kvaziko‘phad ko‘rinishidagi xususiy yechimga ega. Bu yechimni noma’lum koeffitsientlar metodi yordamida topish mumkin. Buning uchun dastlab quyidagi k nomanfiy butun sonni aniqlash kerak:

agar γ son A matritsaning xarakteristik soni bo‘lmasa, $k = 0$ deymiz;

agar γ son A matritsaning xarakteristik soni bo‘lsa, k bilan uning karralilik darajasini belgilaymiz.

Endi xususiy yechim

$$\boldsymbol{x} = e^{\gamma t}\boldsymbol{q}(t),\ \deg \boldsymbol{q}(t) = \deg \boldsymbol{p}(t) + k\ , \tag{18}$$

kvaziko‘phad ko‘rinishida izlanadi. $\boldsymbol{q}(t)$ ko‘phadning koeffitsientlarini hozircha noma’lum deb hisoblab, ularni sistemaning qanoatlanishi shartidan topish kerak.

Agar haqiqiy sohada berilgan (2) sistemada $\boldsymbol{f}(t)$ ozod had

$$\boldsymbol{f}(t) = e^{\alpha t}\left(\boldsymbol{p}(t)\cos\beta t + \boldsymbol{q}(t)\sin\beta t\right)$$

(haqiqiy) kvaziko‘phaddan iborat bo‘lsa $\big(\alpha, \beta \in \mathbb{R}$, $\boldsymbol{p}(t), \boldsymbol{q}(t) -$ haqiqiy vektor-koeffitsientli ko‘phadlar$\big)$, ya’ni

$$\boldsymbol{x}' = A\boldsymbol{x} + e^{\alpha t}\left(\boldsymbol{p}(t)\cos\beta t + \boldsymbol{q}(t)\sin\beta t\right) \tag{19}$$

sistema berilgan bo‘lsa ($A-$ haqiqiy sonlardan tuzilgan matritsa), u holda Eyler formulasidan topilgan

$$e^{\alpha t}\cos\beta t = \mathrm{Re}(e^{(\alpha+i\beta)t}),\ \sin\beta t = \mathrm{Re}(-ie^{(\alpha+i\beta)t})$$

munosabatlarga ko‘ra

$$\boldsymbol{f}(t) = \boldsymbol{p}(t)e^{\alpha t}\cos\beta t + \boldsymbol{q}(t)e^{\alpha t}\sin\beta t = \boldsymbol{p}(t)\mathrm{Re}(e^{(\alpha+i\beta)t}) + \boldsymbol{q}(t)\mathrm{Re}(-ie^{(\alpha+i\beta)t}) =$$
$$= \mathrm{Re}\left(e^{(\alpha+i\beta)t}(\boldsymbol{p}(t) - i\boldsymbol{q}(t))\right)$$

ifodalashdan kelib chiqib, haqiqiy sohada berilgan (19) sistema o‘rniga kompleks sohadagi ushbu

$$\boldsymbol{x}' = A\boldsymbol{x} + e^{\gamma t}\left(\boldsymbol{p}(t) - i\boldsymbol{q}(t)\right),\ \gamma = \alpha + i\beta,$$

sistemani tuzib, uning xususiy yechimini kompleks holda keltirilgan algoritmga ko‘ra topib, topilgan yechimning haqiqiy qismini ajratish kerak.

Bu holda xususiy yechimni kompleks sohaga chiqmasdan to‘g‘ridan-to‘g‘ri qursa ham bo‘ladi. (19) sistemaning xususiy yechimini birdaniga

$$\boldsymbol{x}(t) = e^{\alpha t}\left(\boldsymbol{m}(t)\cos\beta t + \boldsymbol{n}(t)\sin\beta t\right)$$

ko‘rinishda izlash mumkin. Bu yerda $\boldsymbol{m}(t), \boldsymbol{n}(t) -$vektor koeffitsientli ko‘phadlar va $\max\left(\deg\boldsymbol{m}, \deg\boldsymbol{n}\right) = k + \max\left(\deg\boldsymbol{p}, \deg\boldsymbol{q}\right)$, k bilan A matritsaning $\alpha + i\beta$ xarakteristik sonining karralilik darajasi belgilangan; $\alpha + i\beta$ xarakteristik son bo‘lmaganda esa $k = 0$ deb hisoblanadi. Bunday ko‘rinishdagi yechimni topish uchun uni (19) sistemaga qo‘yib, hosil bo‘lgan ayniyatdan noma’lum koeffitsientlarni topish kerak.

Agar berilgan sistemadagi ozod had ikki (yoki undan ortiq) kvaziko‘phad yig‘indisidan iborat, ya’ni

$$\boldsymbol{x}' = A\boldsymbol{x} + e^{\gamma t}\boldsymbol{p}(t) + e^{\omega t}\boldsymbol{r}(t)$$

bo‘lsa, bu sistemadan ushbu

$$\boldsymbol{x}' = A\boldsymbol{x} + e^{\gamma t}\boldsymbol{p}(t),\ \boldsymbol{x}' = A\boldsymbol{x} + e^{\omega t}\boldsymbol{r}(t)$$

sistemalarni tuzib, ularning xususiy yechimlarini qo‘shib, berilgan sistemaning xususiy yechimi quriladi.

Misol 1. Ushbu

$$\begin{cases} x' = y + z \\ y' = x + z \\ z' = x + y \end{cases} \tag{20}$$

tenglamalar sistemasining umumiy yechimini toping.

Eslatma. Biz bu sistemani birinchi integrallar yordamida yechgan edik (§ 13, misol 7). Hozir esa biz uni o‘rganilayotgan metoddan foydalanib yechamiz.

⊶Berilgan sistemaning vektor ko‘rinishi:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{20}$$

bu yerda A matritsa:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Xarakteristik tenglama

$$\det(A - \lambda E) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

yoki

$$\lambda^3 - 3\lambda - 2 = 0.$$

$\lambda^3 - 3\lambda - 2 = (\lambda - 2)(\lambda + 1)^2$ bo‘lgani uchun xarakteristik tenglamaning $\lambda_1 = 2$ bir karrali (ya’ni oddiy) ildizi, $\lambda_2 = -1$ esa uning ikki karrali ildizi bo‘ladi.

Oddiy ildiz $\lambda_1 = 2$ ga mos keluvchi xos vektor $\boldsymbol{h}$

$$(A - \lambda_1 E)\boldsymbol{h} = 0$$

yoki

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = 0$$

tenglamadan aniqlanadi. Oxirgi sistemani yechib, ushbu

$$\boldsymbol{h} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

xos vektorni topamiz. Demak, $\lambda_1 = 2$ xarakteristik songa berilgan (20) sistemaning

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}, \tag{21}$$

ya'ni $x = e^{2t}$, $y = e^{2t}$, $z = e^{2t}$ yechimi mos keladi.

Endi $\lambda_2 = -1$ ikki karrali xarakteristik songa mos keluvchi yechimni topaylik. Yechim

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left(\boldsymbol{p}_1 t + \boldsymbol{p}_0 \right) e^{-t},$$

ya'ni

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left[\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} t + \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right] e^{-t} \tag{22}$$

ko'rinishda bo'ladi. (22) ni (20) ga qo'yib, $\alpha, \beta, \gamma, a, b, c$ noma'lumlarga nisbatan chiziqli algebraik tenglamalar sistemasini hosil qilamiz:

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} e^{-t} - \left[\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} t + \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right] e^{-t} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \left[\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} t + \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right] e^{-t}$$

yoki

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} t + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} - \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix};$$

bundan

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Hosil bo‘lgan tenglamalarning skalyar ko‘rinishi:

$$\alpha+\beta+\gamma=0,\ a+b+c=\alpha,\ a+b+c=\beta,\ a+b+c=\gamma\,.$$

Oxirgi uch tenglamadan $\alpha=\beta=\gamma$ ekanligi ravshan. Birinchi tenglamadan $\alpha=\beta=\gamma=0$ ni topamiz. Nihoyat, hosil bo‘lgan yagona tenglama $a+b+c=0$ ni yechamiz:

$$a=c_1,\ b=c_2,\ c=-c_1-c_2 \quad (c_1,\ c_2-\text{const})\,.$$

Endi (22) ga ko‘ra berilgan (20) sistemaning $\lambda_2=-1$ xarakteristik songa mos keluvchi yechimini yozamiz:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ -c_1-c_2 \end{pmatrix} e^{-t}\,. \tag{23}$$

Bundan $c_1=1, c_2=0$ va $c_1=0,\ c_2=1$ deb, ikkita chiziqli erkli yechimni topishimiz ham mumkin (agar kerak bo‘lsa).

Endi (21) va (23) yechimlarga ko‘ra (20) ning umumiy yechimini yozamiz:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ -c_1-c_2 \end{pmatrix} e^{-t} + \begin{pmatrix} c_3 \\ c_3 \\ c_3 \end{pmatrix} e^{2t}$$

yoki skalyar ko‘rinishda

$$\begin{cases} x=c_1e^{-t}+c_3e^{2t} \\ y=c_2e^{-t}+c_3e^{2t} \\ z=-(c_1+c_2)e^{-t}+c_3e^{2t} \end{cases} \qquad (c_1,\ c_2,\ c_3-\text{const}).$$

Misol 2. Ushbu

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix},\ A = \begin{pmatrix} 2 & -1 & -1 \\ 2 & -1 & -2 \\ -1 & 1 & 2 \end{pmatrix},\ \left(\begin{cases} x' = 2x - y - z \\ y' = 2x - y - 2z \\ z' = -x + y + 2z \end{cases}\right) \qquad (24)$$

sistemaning umumiy yechimini toping va e^A matritsani hisoblang.

⊶Berilgan sistema (A matritsa)ning xarakteristik tenglamasi

$$\det(A - \lambda E) \equiv \begin{vmatrix} 2-\lambda & -1 & -1 \\ 2 & -1-\lambda & -2 \\ -1 & 1 & 2-\lambda \end{vmatrix} \equiv -(\lambda - 1)^3 = 0$$

bitta $\lambda = 1$ uch karrali ildizga ega. (24) sistemaning yechimini

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left[\begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} t^2 + \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} t + \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{pmatrix}\right] e^t \qquad (25)$$

ko‘rinishda izlaymiz. (25) ni berilgan (24) sistemaga qo‘yib topamiz:

$$A\begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix},\ A\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} + 2\begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix},\ A\begin{pmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{pmatrix} + \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix}.$$

Skalyar tenglamalarga o‘tamiz:

$$\begin{cases} 2\alpha_2 - \beta_2 - \gamma_2 = \alpha_2 \\ 2\alpha_2 - \beta_2 - 2\gamma_2 = \beta_2 \\ -\alpha_2 + \beta_2 + 2\gamma_2 = \gamma_2 \end{cases},\ \begin{cases} 2\alpha_1 - \beta_1 - \gamma_1 = \alpha_1 - 2\alpha_2 \\ 2\alpha_1 - \beta_1 - 2\gamma_1 = \beta_1 - 2\beta_2 \\ -\alpha_1 + \beta_1 + 2\gamma_1 = \gamma_1 - 2\gamma_2 \end{cases},\ \begin{cases} 2\alpha_0 - \beta_0 - \gamma_0 = \alpha_0 - \alpha_1 \\ 2\alpha_0 - \beta_0 - 2\gamma_0 = \beta_0 - \beta_1 \\ -\alpha_0 + \beta_0 + 2\gamma_0 = \gamma_0 - \gamma_1 \end{cases}.$$

Bu yerdagi birinchi sistema bitta tenglamaga aylanadi :

$$\alpha_2 - \beta_2 - \gamma_2 = 0. \qquad (26)$$

Ikkinchi sistemadan

$$\begin{cases} \alpha_1 - \beta_1 - \gamma_1 = -2\alpha_2 \\ \alpha_1 - \beta_1 - \gamma_1 = -\beta_2 \\ \alpha_1 - \beta_1 - \gamma_1 = 2\gamma_2 \end{cases} \Rightarrow -2\alpha_2 = -\beta_2 = \gamma_2;$$

(26) ga ko‘ra $\alpha_2 = \beta_2 = \gamma_2 = 0$; demak, yana bitta tenglama hosil bo‘ladi:

$$\alpha_1 - \beta_1 - \gamma_1 = 0. \qquad (27)$$

Uchinchi sistemadan

$$\begin{cases} \alpha_0-\beta_0-\gamma_0=-\alpha_1 \\ \alpha_0-\beta_0-\gamma_0=-\beta_1/2 \\ \alpha_0-\beta_0-\gamma_0=\gamma_1 \end{cases} \Rightarrow \beta_1=2\alpha_1 \text{ va } \gamma_1=-\alpha_1;$$

oxirgi shartlar (27) ni qanoatlantiradi; demak, bu yerda ham bitta tenglama hosil bo‘ladi:

$$\alpha_0-\beta_0-\gamma_0=-\alpha_1.$$

Endi $\alpha_0=c_1, \beta_0=c_2, \gamma_0=c_3$ deb, qolgan noma’lumlarni aniqlaymiz:

$$\alpha_1=-(\alpha_0-\beta_0-\gamma_0)=-c_1+c_2+c_3,$$

$$\beta_1=2\alpha_1=2(-c_1+c_2+c_3),$$

$$\gamma_1=-\alpha_1=c_1-c_2-c_3;\ \alpha_2=\beta_2=\gamma_2=0 \text{ edi.}$$

Topilgan qiymatlarni (25) ga qo‘yib, berilgan sistema (24) ning umumiy yechimini yozamiz:

$$\begin{cases} x=((-c_1+c_2+c_3)t-c_1)e^t \\ y=(2(-c_1+c_2+c_3)t-c_2)e^t \\ z=((c_1-c_2-c_3)t-c_3)e^t \end{cases}$$

yoki vektor ko‘rinishida

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\Phi(t)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},\ \ \Phi(t)=e^t\begin{pmatrix} -t-1 & t & t \\ -2t & 2t-1 & 2t \\ t & -t & -t-1 \end{pmatrix}.$$

e^A ni hisoblash uchun $e^{At}=\Phi(t)\Phi^{-1}(0)$ formuladan foydalanamiz. Quyidagilarni ketma-ket hisoblaymiz:

$$\Phi(0)=\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}=-E\ ,\ \ \Phi^{-1}(0)=-E\ ,$$

$$e^A=\Phi(1)\Phi^{-1}(0)=-e\begin{pmatrix} -2 & 1 & 1 \\ -2 & 1 & 2 \\ 1 & -1 & -2 \end{pmatrix}E=e\begin{pmatrix} 2 & -1 & -1 \\ 2 & -1 & -2 \\ -1 & 1 & 2 \end{pmatrix}.$$

Misol 3. Ushbu

$$\begin{cases} x' = 2x + y \\ y' = x + 3y - z \\ z' = -x + 2y + 3z \end{cases} \tag{28}$$

sistemaning umumiy yechimini toping. Sistemaning fundamental matritsasini yozing

🗝 Berilgan sistema uchun

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix}$$

matritsaning xarakteristik sonlarini ushbu

$$\det(A - \lambda E) = 0$$

algebraik tenglamadan aniqlaymiz. Bu tenglamani yechib, $\lambda_1 = 2,\ \lambda_2 = 3 + i,\ \lambda_3 = 3 - i$ oddiy ildizlarni topamiz.

$\lambda_1 = 2$ xarakteristik songa

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{2t} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

ko‘rinishdagi yechim mos keladi. Buni berilgan sistema (28)ga qo‘yib, s_1, s_2, s_3 larni aniqlaymiz ($s_3 = 1$ tanlangan):

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Demak, $\lambda_1 = 2$ xarakteristik songa mos yechim

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \tag{29}$$

Endi $\lambda_2 = 3 + i$ xos songa mos kelgan

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{(3+i)t} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

yechimlarni topamiz. Buni berilgan sistemaga qo‘yamiz va a,b,c larni topamiz:

$$\begin{pmatrix} -1-i & 1 & 0 \\ 1 & -i & -1 \\ -1 & 2 & -i \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a = 1(\text{tanlangan}), \\ b = 1+i, \\ c = 2-i. \end{cases}$$

Demak, sistema yechimi

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{(3+i)t} \begin{pmatrix} 1 \\ 1+i \\ 2-i \end{pmatrix} = e^{3t} e^{it} \left(\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right)$$

yoki, $e^{it} = (\cos t + i \sin t)$ Eyler formulasiga ko‘ra haqiqiy va mavhum qismlarni ajratsak,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{3t} \begin{pmatrix} \cos t \\ \cos t - \sin t \\ 2\cos t + \sin t \end{pmatrix} + i e^{3t} \begin{pmatrix} \sin t \\ \cos t + \sin t \\ 2\sin t - \cos t \end{pmatrix}.$$

Yechimning haqiqiy va mavhum qismlari ham yechim bo‘lgani uchun bundan $\lambda = 3 \pm i$ xarakteristik sonlarga mos kelgan ikkita haqiqiy yechimni aniqlaymiz

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{3t} \begin{pmatrix} \cos t \\ \cos t - \sin t \\ 2\cos t + \sin t \end{pmatrix}, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{3t} \begin{pmatrix} \sin t \\ \cos t + \sin t \\ 2\sin t - \cos t \end{pmatrix}. \qquad (30)$$

Endi (28) sistemaning umumiy yechimini (29) va (30) bazis yechimlarning ixtiyoriy chiziqli kombinatsiyasi sifatida yozamiz

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} \cos t \\ \cos t - \sin t \\ 2\cos t + \sin t \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} \sin t \\ \cos t + \sin t \\ 2\sin t - \cos t \end{pmatrix}.$$

Fundamental matritsa (25) va (26) chiziqli erkli yechimlarni ustunlar bo‘ylab yozishdan hosil bo‘ladi:

$$\Phi(t)=\begin{pmatrix} e^{2t} & e^{3t}\cos t & e^{3t}\sin t \\ 0 & e^{3t}(\cos t-\sin t) & e^{3t}(\cos t+\sin t) \\ e^{2t} & e^{3t}(2\cos t+\sin t) & e^{3t}(2\sin t-\cos t) \end{pmatrix}.$$

Misol 4. Ushbu

$$A=\begin{pmatrix} 1 & 1 & 0 \\ -4 & -2 & 1 \\ 4 & 1 & -2 \end{pmatrix}$$

matritsa uchun e^{At} ni hisoblang.

Hisoblashni matritsaning Jordan ko‘rinishidan foydalanib bajaramiz. Berilgan matritsaning bitta uch karrali $\lambda=-1$ xarakteristik soni bor. Unga mos kelgan xos vektorlarni aniqlaymiz:

$$(A+E)\boldsymbol{h}=0,\ \begin{pmatrix} 2 & 1 & 0 \\ -4 & -1 & 1 \\ 4 & 1 & -1 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Xos vektor bitta (o‘zgarmas ko‘paytuvchi aniqligida):

$$\boldsymbol{h}=\boldsymbol{h}_1=\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}.$$

Demak, yana ikkita umumlashgan xos vektor bor. Ularni topamiz:

$$(A+E)\boldsymbol{h}_2=\boldsymbol{h}_1,\ \begin{pmatrix} 2 & 1 & 0 \\ -4 & -1 & 1 \\ 4 & 1 & -1 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}=\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix},\ \boldsymbol{h}_2=\begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix};$$

$$(A+E)\boldsymbol{h}_3=\boldsymbol{h}_2,\ \begin{pmatrix} 2 & 1 & 0 \\ -4 & -1 & 1 \\ 4 & 1 & -1 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}=\begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix},\ \boldsymbol{h}_3=\begin{pmatrix} 1/4 \\ 0 \\ 1 \end{pmatrix}.$$

Endi

$$S=[\boldsymbol{h}_1\ \boldsymbol{h}_2\ \boldsymbol{h}_3]=\begin{pmatrix}1 & 1/2 & 1/4\\ -2 & 0 & 0\\ 2 & 0 & 1\end{pmatrix},\ S^{-1}=\begin{pmatrix}0 & -1/2 & 0\\ 2 & 1/2 & -1/2\\ 0 & 1 & 1\end{pmatrix}$$

matritsalarni tuzamiz. Shunday qilib,

$$A=SJS^{-1},\ J=\begin{pmatrix}-1 & 1 & 0\\ 0 & -1 & 1\\ 0 & 0 & -1\end{pmatrix}.$$

Yuqoridagi (15) formulaga ko‘ra

$$e^{Jt}=\begin{pmatrix}e^{-t} & te^{-t} & \dfrac{t^2}{2}e^{-t}\\ 0 & e^{-t} & te^{-t}\\ 0 & 0 & e^{-t}\end{pmatrix}.$$

Demak,

$$e^{At}=Se^{Jt}S^{-1}=\begin{pmatrix}e^{-t}(2t+1) & \dfrac{te^{-t}(t+2)}{2} & \dfrac{t^2e^{-t}}{2}\\ -4te^{-t} & -e^{-t}(t^2+t-1) & -te^{-t}(t-1)\\ 4te^{-t} & te^{-t}(t+1) & e^{-t}(t^2-t+1)\end{pmatrix}.$$

Misol 5. Ushbu

$$A=\begin{pmatrix}-1 & 9 & 0\\ -1 & 5 & 0\\ -1 & 3 & 2\end{pmatrix}$$

matritsa berilgan. e^{At} ni hisoblang.

e^{At} ni A matritsaning Jordan ko‘rinishidan foydalanib hisoblaymiz. A matritsaninig xaraktestik sonlarini topamiz:

$$\det(A-\lambda E)=-(\lambda^3-6\lambda^2+12\lambda-8)=-(\lambda-2)^3=0;$$

$\lambda=2$ uch karrali xarakteristik son. Xos vektorlarni aniqlaymiz:

$$(A-2E)\boldsymbol{h}=0,\ \begin{pmatrix} -3 & 9 & 0 \\ -1 & 3 & 0 \\ -1 & 3 & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

Ikkita chiziqli erkli xos vektor mavjud:

$$\boldsymbol{v}=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ \boldsymbol{h}=\boldsymbol{h}_1=\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

Umumlashgan xos vektor $\boldsymbol{h}_2$ uchun

$$(A-2E)\boldsymbol{h}_2=\boldsymbol{h}_1,\ \begin{pmatrix} -3 & 9 & 0 \\ -1 & 3 & 0 \\ -1 & 3 & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}=\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix},\ \boldsymbol{h}_2=\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}.$$

Endi S, S^{-1} va J matritsalarni tuzamiz:

$$S=[\boldsymbol{v}\ \ \boldsymbol{h}_1\ \ \boldsymbol{h}_2]=\begin{pmatrix} 0 & 3 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix},\ S^{-1}=\begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & 3 & 0 \end{pmatrix};$$

$$S^{-1}AS=J=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}\ \left(A=SJS^{-1}\right).$$

(15) formulaga asosan

$$e^{Jt}=\begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix}.$$

Nihoyat, e^{At} matritsani hisoblaymiz:

$$e^{At}=Se^{Jt}S^{-1}=\begin{pmatrix} -(3t-1)e^{2t} & 9te^{2t} & 0 \\ -te^{2t} & (3t+1)e^{2t} & 0 \\ -te^{2t} & 3te^{2t} & e^{2t} \end{pmatrix}.$$

Misol 6. Ushbu

$$A=\begin{pmatrix}1 & 2\\ -1 & -1\end{pmatrix}$$

matritsa uchun e^{At} ni hisoblang.

⊶Izlanayotgan e^{At} matritsani misol 2 dagidek ushbu

$$\begin{pmatrix}x\\ y\end{pmatrix}'=A\begin{pmatrix}x\\ y\end{pmatrix}$$

sistemaning fundamental matritsasi orqali tuzish mumkin. Hozir sa biz e^{At} matritsani (12) formulaga ko‘ra quramiz. A matritsaning xarakteristik ko‘phadi

$$\chi(\lambda)\equiv\begin{vmatrix}1-\lambda & 2\\ -1 & -1-\lambda\end{vmatrix}=\lambda^2+1.$$

Ushbu $\chi\left(\frac{d}{dt}\right)\varphi_j(t)=0,\ \frac{d^i\varphi_j(0)}{dt^i}=\delta_{ij}\ (i,j=0,1)$ masalalarni tuzamiz:

$$\varphi_0''+\varphi_0=0,\ \varphi_0(0)=1,\ \varphi_0'(0)=0;$$

$$\varphi_1''+\varphi_1=0,\ \varphi_1(0)=0,\ \varphi_1'(0)=1.$$

Bu masalalarning yechimi osongina topiladi:

$$\varphi_0(t)=\cos t,\ \varphi_1(t)=\sin t.$$

Endi (12) formulaga ko‘ra

$$e^{tA}=\varphi_0(t)E+\varphi_1(t)A=$$

$$=\cos t\cdot\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\sin t\cdot\begin{pmatrix}1 & 2\\ -1 & -1\end{pmatrix}=\begin{pmatrix}\cos t+\sin t & 2\sin t\\ -\sin t & \cos t-\sin t\end{pmatrix}.$$ 👍

Misol 7. Ushbu

$$A=\begin{pmatrix}-1 & 2 & 2\\ -4 & 5 & 2\\ 8 & -4 & -1\end{pmatrix}$$

matrisaning eksponentasi e^{A} ni hisoblang.

⊶Hisoblashni (12) formulaga ko‘ra amalga oshiramiz. Berilgan matritsaning xarakteristik ko‘phadi

$$\chi(\lambda) \equiv \begin{vmatrix} -1-\lambda & 2 & 2 \\ -4 & 5-\lambda & 2 \\ 8 & -4 & -1-\lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 + 9\lambda - 27 .$$

Xarakteristik sonlar: $\lambda = 3$ ikki karrali va $\lambda = -3$ oddiy. Endi ushbu

$$\chi(\frac{d}{dt})\varphi_j(t) = 0, \ \frac{d^i\varphi_j(0)}{dt^i} = \delta_{ij} \ \ (i, j = 0,1)$$

masalalarni yechishimiz kerak:

$$-\varphi_0''' + 3\varphi_0'' + 9\varphi_0' - 27\varphi_0 = 0, \ \varphi_0(0) = 1, \ \varphi_0'(0) = 0, \varphi_0''(0) = 0 ;$$

$$-\varphi_1''' + 3\varphi_1'' + 9\varphi_1' - 27\varphi_1 = 0, \ \varphi_1(0) = 0, \ \varphi_1'(0) = 1, \ \varphi_0''(0) = 0 ;$$

$$-\varphi_2''' + 3\varphi_2'' + 9\varphi_2' - 27\varphi_2 = 0, \ \varphi_2(0) = 0, \ \varphi_2'(0) = 0, \ \varphi_2''(0) = 1 .$$

Ularning yechimi:

$$\varphi_0(t) = \frac{3e^{3t}}{4} - \frac{3te^{3t}}{2} + \frac{e^{-3t}}{4} ; \varphi_1(t) = \frac{e^{3t}}{6} - \frac{e^{-3t}}{6} ; \varphi_2(t) = -\frac{e^{3t}}{36} + \frac{te^{3t}}{6} + \frac{e^{-3t}}{36} .$$

A^2 ni hisoblaymiz:

$$A^2 = \begin{pmatrix} -1 & 2 & 2 \\ -4 & 5 & 2 \\ 8 & -4 & -1 \end{pmatrix}^2 = 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

(11) formulaga ko‘ra

$$e^{tA} = \varphi_0(t)E + \varphi_1(t)A + \varphi_2(t)A^2 .$$

Hisoblashlarni bajarib, topamiz:

$$e^{tA} = \frac{1}{3}\begin{pmatrix} e^{3t} + 2e^{-3t} & e^{3t} - e^{-3t} & e^{3t} - e^{-3t} \\ -2e^{3t} + 2e^{-3t} & 4e^{3t} - e^{-3t} & e^{3t} - e^{-3t} \\ 4e^{3t} - 4e^{-3t} & -2e^{3t} + 2e^{-3t} & e^{3t} + 2e^{-3t} \end{pmatrix} .$$

So‘ralgan matritsa bu formuladan $t = 1$ da hosil bo‘ladi.

Misol 8. Quyidagi sistemalarning xususiy yechimlarini toping:

1) $\begin{cases} x' = 2x + y + 8e^t \cos t, \\ y' = x + 3y - z - 8e^t \cos t, \\ z' = -x + 2y + 3z + 8e^t \sin t; \end{cases}$ 2) $\begin{cases} x' = 2x + y + 2e^{3t} \cos t, \\ y' = x + 3y - z, \\ z' = -x + 2y + 3z + 2e^{3t} \sin t. \end{cases}$

🗝 Misol 3 ga qarang. III bo‘lim ma’lumotlaridan foydalanamiz. 1) sistemaning vektorli ko‘rinishi:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix} + e^t\left(\begin{pmatrix} 8 \\ -8 \\ 0 \end{pmatrix}\cos t + \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix}\sin t\right),\ A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix}.$$

$\lambda = \alpha + i\beta = 1 + i$ son A matritsaning xos soni emas. Shuning uchun xususiy yechimni

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^t\left(\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}\cos t + \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}\sin t\right)$$

ko‘rinishda izlaymiz; bu yerda $m_1, m_2, m_3, n_1, n_2, n_3$ – hozircha noma’lumlar. Oxirgi ifodani qaralayotgan sistemaga qo‘yamiz:

$$e^t\left(\begin{pmatrix} m_1 + n_1 \\ m_2 + n_2 \\ m_3 + n_3 \end{pmatrix}\cos t + \begin{pmatrix} n_1 - m_1 \\ n_2 - m_2 \\ n_3 - m_3 \end{pmatrix}\sin t\right) =$$

$$= e^t\left(A\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}\cos t + A\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}\sin t + \begin{pmatrix} 8 \\ -8 \\ 0 \end{pmatrix}\cos t + \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix}\sin t\right).$$

Bu t bo‘yicha ayniyatdan quyidagilarni topamiz:

$$\begin{pmatrix} m_1 + n_1 \\ m_2 + n_2 \\ m_3 + n_3 \end{pmatrix} = A\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} + \begin{pmatrix} 8 \\ -8 \\ 0 \end{pmatrix},\ \begin{pmatrix} n_1 - m_1 \\ n_2 - m_2 \\ n_3 - m_3 \end{pmatrix} = A\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix}.$$

Bu chiziqli tenglamalar sistemasini yechib, $m_1, m_2, m_3, n_1, n_2, n_3$ noma’lumlarni topamiz:

$$m_1 = -4, m_2 = 3, m_3 = -3, n_1 = 7, n_2 = -3, n_3 = -4.$$

Demak, izlangan xususiy yechim

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^t \left(\begin{pmatrix} -4 \\ 3 \\ -3 \end{pmatrix} \cos t + \begin{pmatrix} 7 \\ -3 \\ -4 \end{pmatrix} \sin t \right).$$

2) holda $\lambda = \alpha + i\beta = 3 + i$ son A matritsaning (bir karrali) xos soni. Shuning uchun bu holda xususiy yechimni

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{3t} \left(\left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} t + \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \right) \cos t + \left(\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} t + \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \right) \sin t \right)$$

ko‘rinishda izlaymiz; bu yerdagi noma’lum koeffitsientlarni topish uchun bu ifodani qaralayotgan sistemaga qo‘yib, hosil bo‘lgan tenglikni aynan bajarilishini talab qilamiz. Bunda hosil bo‘lgan chiziqli algebraik sistemani yechib, izlangan noma’lumlarni va, demak, xususiy yechimni topamiz:

$$\left(\begin{pmatrix} 3a_1 + b_1 \\ 3a_2 + b_2 \\ 3a_3 + b_3 \end{pmatrix} t + \begin{pmatrix} 3m_1 + a_1 + n_1 \\ 3m_2 + a_2 + n_2 \\ 3m_3 + a_3 + n_3 \end{pmatrix} \right) \cos t + \left(\begin{pmatrix} 3b_1 - a_1 \\ 3b_2 - a_2 \\ 3b_3 - a_3 \end{pmatrix} t + 3 \begin{pmatrix} 3n_1 + b_1 - m_1 \\ 3n_2 + b_2 - m_2 \\ 3n_3 + b_3 - m_3 \end{pmatrix} \right) \sin t =$$

$$= A \left(\left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} t + \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \right) \cos t + \left(\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} t + \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \right) \sin t \right) + 2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cos t + 2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \sin t;$$

$$\begin{pmatrix} 3a_1 + b_1 \\ 3a_2 + b_2 \\ 3a_3 + b_3 \end{pmatrix} = A \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \begin{pmatrix} 3b_1 - a_1 \\ 3b_2 - a_2 \\ 3b_3 - a_3 \end{pmatrix} = A \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

$$\begin{pmatrix} 3m_1 + a_1 + n_1 \\ 3m_2 + a_2 + n_2 \\ 3m_3 + a_3 + n_3 \end{pmatrix} = A \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 3n_1 + b_1 - m_1 \\ 3n_2 + b_2 - m_2 \\ 3n_3 + b_3 - m_3 \end{pmatrix} = A \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix};$$

$$a_1 = 0, a_2 = 1, a_3 = -1, m_1 = 2, m_2 = 1, m_3 = 1, b_1 = 1, b_2 = 1, b_3 = 2, n_1 = 1, n_2 = 0, n_3 = 1;$$

$$\begin{cases} x = e^{3t}(2\cos t + t\sin t + \sin t), \\ y = e^{3t}(t\cos t + \cos t + t\sin t), \\ z = e^{3t}(-t\cos t + \cos t + 2t\sin t + \sin t). \end{cases}$$

Bu − skalyar ko‘rinishdagi xususiy yechim. ☝

Masalalar

Bir jinsli chiziqli o‘zgarmas koeffitsientli differensial tenglamalar sistemasini yeching. Yechimdan foydalanib e^{tA} matritsani hisoblang (**1 - 2**):

1. $\dfrac{d\boldsymbol{x}}{dt} = A\boldsymbol{x},\ \boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix};\ a).\ A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix},\ b).\ A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}.$

2. $\dfrac{d\boldsymbol{x}}{dt} = A\boldsymbol{x},\ \boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix};\ \ a).\ A = \begin{pmatrix} 1 & -1 & -1 \\ 3 & -3 & -3 \\ -2 & 2 & 2 \end{pmatrix},\ \ b).\ A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{pmatrix},$

$c).\ A = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 1 & -5 \\ -1 & 1 & 6 \end{pmatrix}.$

Berilgan A matritsalarning xarakteristik ko‘phadini toping va e^{tA} matritsani ((12) formulaga ko‘ra) hisoblang (**3 - 8**):

3. $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}.$ **4.** $A = \begin{pmatrix} -2 & -3 \\ 6 & 7 \end{pmatrix}.$ **5.** $A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}.$

6. $A = \begin{pmatrix} 2 & 1 & -3 \\ 3 & -2 & -3 \\ 1 & 1 & -2 \end{pmatrix}.$ **7.** $A = \begin{pmatrix} 2 & -3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$ **8.** $A = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 2 & 2 \\ 3 & -2 & 1 \end{pmatrix}.$

Berilgan A matritsalarni J Jordan ko‘rinishiga keltiruvchi S matritsani toping $\left(A = SJS^{-1}\right)$ va undan foydalanib e^{tA} matritsani hisoblang (**9 - 15**):

9. $A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}.$ **10.** $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}.$ **11.** $A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}.$

12. $A = \begin{pmatrix} 2 & 1 & -3 \\ 3 & -2 & -3 \\ 1 & 1 & -2 \end{pmatrix}.$ **13.** $A = \begin{pmatrix} 2 & -3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}.$

14. $A=\begin{pmatrix} 3 & -2 & 2 \\ 2 & 0 & 1 \\ -2 & 2 & -2 \end{pmatrix}$. **15.** $A=\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$.

Berilgan sistemalarning xususiy yechimini toping (**16 -18**):

16. $\begin{cases} x' = x + y - 4t \\ y' = -x + 3y \end{cases}$. **17.** $\begin{cases} x' = x + y - 6te^{2t} \\ y' = -x + 3y \end{cases}$. **18.** $\begin{cases} x' = x + y - 5\cos t \\ y' = -x + 3y - 5\sin t \end{cases}$.

Mustaqil ish № 15 topshiriqlari:

a) Berilgan bir jinsli differensial tenglamalar sistemasi uchun uning matritsasi A ni yozing.

b) Sistemaning umumiy yechimini toping. Fundamental matritsani quring va undan foydalanib e^{At} matritsani hisoblang.

c) e^{At} matritsani (11) formulaga ko‘ra boshqa usulda hisoblang.

d) Agar sizga $\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{f}(t)$ tenglama berilgan bo‘lib, $\boldsymbol{f}(t)$ ozod hadning kvaziko‘phad ekanligi ma’lum bo‘lsa, bu tenglamaning xususiy yechimini qanday qurgan bo‘lardingiz? Umumiy yechimini-chi?

1. $\begin{cases} x' = 2x + 5y - 3z \\ y' = x - 2y + 3z \\ z' = 3x - 15y + 12z \end{cases}$ **2.** $\begin{cases} x' = 2x + y - z \\ y' = 4y - x - z \\ z' = x - y + 2z \end{cases}$

3. $\begin{cases} x' = 2x - y + 2z \\ y' = x + 2z \\ z' = -2x + y - z \end{cases}$ **4.** $\begin{cases} x' = 4x + y + z \\ y' = 2x + 4y + z \\ z' = y + 4z \end{cases}$

5. $\begin{cases} x' = 6x - 12y - z \\ y' = x - 3y - z \\ z' = -4x + 12y + 3z \end{cases}$ **6.** $\begin{cases} x' = -3x + 2y + 2z \\ y' = -3x - y + z \\ z' = -x + 2y \end{cases}$

7. $\begin{cases} x' = 2x + y \\ y' = 2y + 4z \\ z' = x - z \end{cases}$ **8.** $\begin{cases} x' = -x + y + z \\ y' = x - y + z \\ z' = x + y - z \end{cases}$

9. $\begin{cases} x' = 3x - y + z \\ y' = -x + 5y - z \\ z' = x - y + 3z \end{cases}$

10. $\begin{cases} x' = 4x - y \\ y' = 3x + y - z \\ z' = x + z \end{cases}$

11. $\begin{cases} x' = x - 3y + z \\ y' = x - 2y \\ z' = y - z \end{cases}$

12. $\begin{cases} x' = -x + y + z \\ y' = x - y + z \\ z' = x + y + z \end{cases}$

13. $\begin{cases} x' = y - x \\ y' = x + y \\ z' = x + z \end{cases}$

14. $\begin{cases} x' = -4x + 2y + 5z \\ y' = 6x - y - 6z \\ z' = -8x + 3y + 9z \end{cases}$

15. $\begin{cases} x' = 3x + 12y - 4z \\ y' = -x - 3y + z \\ z' = -x - 12y + 6z \end{cases}$

16. $\begin{cases} x' = 2x - y - z \\ y' = 12x - 4y - 12z \\ z' = -4x + y + 5z \end{cases}$

17. $\begin{cases} x' = 10x - 3y - 9z \\ y' = -18x + 7y + 18z \\ z' = 18x - 6y - 17z \end{cases}$

18. $\begin{cases} x' = -x - 4y \\ y' = x - y + z \\ z' = 3y - z \end{cases}$

19. $\begin{cases} x' = x - z \\ y' = -6x + 2y + 6z \\ z' = 4x - y - 4z \end{cases}$

20. $\begin{cases} x' = 3x - y - 4z \\ y' = -6x + 2y + 6z \\ z' = 6x - 2y - 6z \end{cases}$

21. $\begin{cases} x' = 5x - y - 4z \\ y' = -12x + 5y + 12z \\ z' = 10x - 3y - 9z \end{cases}$

22. $\begin{cases} x' = 3x + 12y - 4z \\ y' = -x - 3y + z \\ z' = -x - 12y + 6z \end{cases}$

23. $\begin{cases} x' = 5x - y - 4z \\ y' = -12x + 5y + 12z \\ z' = 10x - 3y - 9z \end{cases}$

24. $\begin{cases} x' = x - y + z \\ y' = x + y \\ z' = 3x + z \end{cases}$

25. $\begin{cases} x' = x + y - z \\ y' = -x + 2y - z \\ z' = 2x - y + 4z \end{cases}$

26. $\begin{cases} x' = 3x - 2y + 2z \\ y' = 2x + z \\ z' = -2x + 2y - 2z \end{cases}$

27. $\begin{cases} x' = x + 2z \\ y' = y - 4z \\ z' = -x - 2z \end{cases}$

28. $\begin{cases} x' = -2x + 3z + 4z \\ y' = -6x + 7y + 6z \\ z' = x - y + z \end{cases}$

29. $\begin{cases} x' = -3x + 4y - 2z \\ y' = x + z \\ z' = 6x - 6y + 5z \end{cases}$

30 $\begin{cases} x' = 2x + y + z \\ y' = -2x - z \\ z' = 2x + y + 2z \end{cases}$

31. $\begin{cases} x' = 4x - y \\ y' = 3x + y - z \\ z' = x + z \end{cases}$

32. $\begin{cases} x' = x - y - z \\ y' = x + y \\ z' = 3x + z \end{cases}$

33. $\begin{cases} x' = 4x - 4y + 2z \\ y' = 2x - 2y + z \\ z' = -4x + 4y - 2z \end{cases}$

34. $\begin{cases} x' = -3x + z \\ y' = -3y + 2z \\ z' = 3x - 2y - 3z \end{cases}$

35. $\begin{cases} x' = x - y + z \\ y' = x + y - z \\ z' = 2x - y \end{cases}$

36. $\begin{cases} x' = -5x + y - 2z \\ y' = -x - y \\ z' = 6x - 2y + 2z \end{cases}$

37. $\begin{cases} x' = x + z \\ y' = -2x + 2y + 2z \\ z' = 3x - 2y + z \end{cases}$

38. $\begin{cases} x' = 4x - y - 2z \\ y' = x + y - z \\ z' = 5x - 2y - 2z \end{cases}$

39. $\begin{cases} x' = -2y + 2z \\ y' = x - y + z \\ z' = y - z \end{cases}$

40. $\begin{cases} x' = 5x - y - z \\ y' = 2x + 2y - z \\ z' = -4x + 2y + 5z \end{cases}$

16. TEKISLIKDA AVTONOM SISTEMALAR

Maqsad – tekislikda chiziqli avtonom sistemalar muvozanat holatlari tiplari va trayektoriyalar manzarasini hamda nochiziqli avtonom sistemalarni tekshirish (muvozanat holatlarini topish, ularning tiplarini chiziqlilashtirish yordamida aniqlash va trayektoriyalar portretini qurish)ni o‘rganish.

Yordamchi ma’lumotlar:

I. Ikki o‘lchamli avtonom (muxtor) sistema

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \tag{1}$$

ko‘rinishga ega. Bu yerda $f(x,y)$ va $g(x,y)$ funksiyalar tekislikdagi biror G sohada (fazalar fazosida) C^1 sinfga tegishli bo‘lgan berilgan haqiqiy funksiyalar; $x = x(t)$ va $y = y(t)$ noma’lum funksiyalar. (1) avtonom sistema har bir $(x,y) \in G$ nuqtada tezlik vektori deb ataluvchi $\vec{\boldsymbol{v}} = (f(x,y); g(x,y))$ vektorni aniqlaydi. Shunday qilib, (1) avtonom sistema G sohada vektorlar maydonini aniqlaydi.(1) avtonom sistemaning $x = x(t)$, $y = y(t)$ yechimi tekislikdagi chiziqning parametrik tenglamasini ifodalaydi. Bu chiziq avtonom sistemaning trayektoriyasi deb ataladi. t vaqt o‘tishi bilan nuqta trayektoriya bo‘ylab harakat qiladi. Trayektoriya o‘zining har bir nuqtasida shu nuqtadagi tezlik vektoriga urinadi. Berilgan shartlarda G sohaning har bir nuqtasidan yagona trayektoriya o‘tadi.

Agar

$$\begin{cases} f(x_0, y_0) = 0 \\ g(x_0, y_0) = 0 \end{cases} \tag{2}$$

bo‘lsa, $(x_0, y_0) \in G$ nuqta (1) sistemaning muvozanat holati (muvozanat nuqtasi, kritik nuqtasi, maxsus nuqtasi) deyiladi. Muvozanat nuqtasidan boshlangan yechim shu nuqtadaligicha qoladi, ya’ni nuqta harakatlanmaydi, trayektoriya bitta nuqtadan iborat bo‘ladi.

1^0. Teorema. Yuqorida aytilgan $\{f, g\} \subset C^1(G;\mathbb{R})$ shart bajarilganda (1) avtonom sistemaning har qanday trayektoriyasi quyidagi uch turning bittasiga mansub bo‘ladi:

— nuqta, ya’ni muvozanat nuqtasi (yechimning davri ixtiyoriy son);

— o‘z-o‘zini kesmaydigan yopiq chiziq, ya’ni sikl (eng kichik musbat davrli yechim);

— o‘z-o‘zini kesmaydigan yopiqmas chiziq (davrsiz yechim).

G dagi muvozanat nuqtalardan tashqari qismida (1) sistema ushbu

$$g(x,y)dx = f(x,y)dy \tag{3}$$

birinchi tartibli differensial tenglamani aniqlaydi.

2⁰. (1) sistemaning muvozanat holatidan farqli har qanday trayektoriyasi (3) tenglamaning integral chizig'idan iborat va aksincha, ya'ni (3) tenglamaning ixtiyoriy integral chizig'i (1) sistemaning muvozanat holatidan farqli trayektoriyasidan iborat bo'ladi.

II. Tekislikda ushbu

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \quad \text{yoki} \quad \begin{pmatrix} x \\ y \end{pmatrix}' = A\begin{pmatrix} x \\ y \end{pmatrix}, \; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{4}$$

avtonom sistemani qaraylik; A haqiqiy matritsa va $\det A \neq 0$ deb faraz qilinadi; demak, (4) sistema yagona muvozanat nuqta $(0;0)$ ga ega. Har bir (x,y) nuqtada $\boldsymbol{v} = (ax+by; cx+dy)$ tezlik vektori aniqlangan. Muvozanat nuqtasida tezlik vektori nol-vektordan iborat.

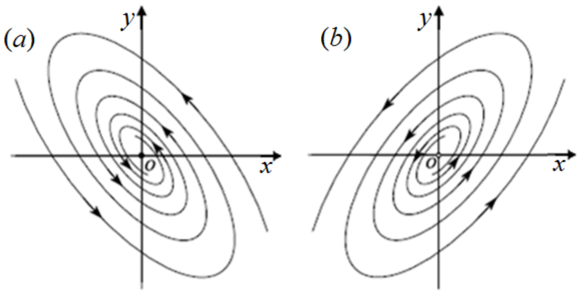
A matritsaning xos (xarakteristik) sonlari

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \text{ yoki } \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

tenglamadan topiladi. Xarakteristik sonlarni λ_1 va λ_2 bilan belgilaylik. $\det A = ad - bc \neq 0$ bo'lgani uchun $\lambda_1 \neq 0$ va $\lambda_2 \neq 0$. λ xos songa mos keluvchi $\vec{\boldsymbol{h}} \neq 0$ xos vektor $A\vec{\boldsymbol{h}} = \lambda\vec{\boldsymbol{h}}$ tenglikdan topiladi. Muvozanat nuqtaning tipi (tabiati, xarakteri) va trayektoriyalar manzarasi A matritsaning xos sonlari va xos vektorlari orqali aniqlanadi.

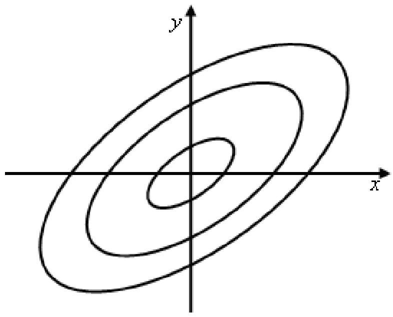
3⁰. Xos sonlar λ_1 va λ_2 kompleks bo'lsin. A matritsa haqiqiy bo'lgani uchun ular o'zaro qo'shma bo'ladi: $\lambda_{1,2} = \alpha \pm i\beta$, $\{\alpha, \beta\} \subset \mathbb{R}$, $\beta \neq 0$; aniqlik uchun $\beta > 0$ deb hisoblanadi. $\alpha < 0$ bo'lganda $(0;0)$ dan farqli ixtiyoriy trayektoriya vaqt o'tishi bilan $(0;0)$ ga buralib intiluvchi spiralsimon chiziqdan iborat bo'ladi. Bu holda ($\mathrm{Re}\,\lambda_{1,2} = \alpha < 0$) muvozanat nuqta **turg'un fokus** (16.1- rasm, (a)) deyiladi.

$\alpha > 0$ bo‘lganda $(0;0)$ dan farqli ar qanday trayektoriya vaqt o‘tishi bilan $(0;0)$ dan buralib uzoqlashuvchi spiralsimon chiziqdan iborat bo‘ladi. Bu holda ($\operatorname{Re}\lambda_{1,2} = \alpha > 0$) muvozanat nuqta ***noturg‘un fokus*** (16.1- rasm, (b)) deyiladi.



16.1 – rasm.

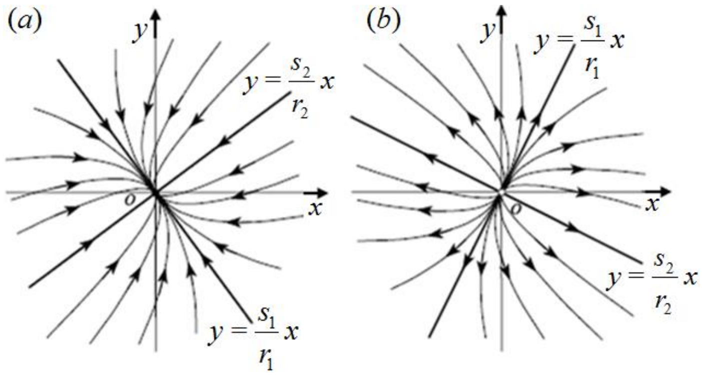
(a) turg‘un fokus: $\operatorname{Re}\lambda_{1,2} = \alpha < 0$; (b) noturg‘un fokus: $\operatorname{Re}\lambda_{1,2} = \alpha > 0$



16.2- rasm. Markaz: $\operatorname{Re}\lambda_{1,2} = 0$.

$\operatorname{Re}\lambda_{1,2} = \alpha = 0$ holida trayektoriyalar $(0;0)$ markazli konsentrik ellipslardan iborat bo‘ladi; bu holda muvozanat nuqta **markaz** (16.2- rasm) deb ataladi.

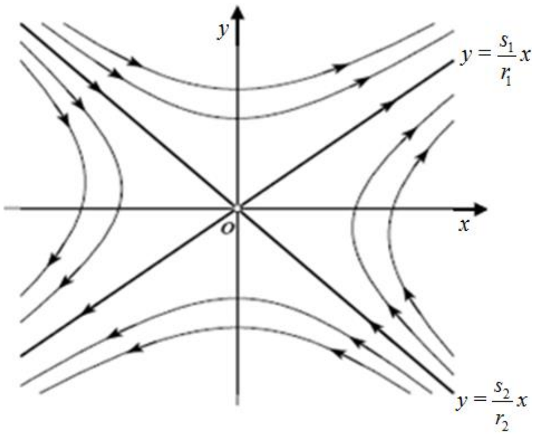
4⁰. λ_1 va λ_2 xos sonlar haqiqiy, turli va bir xil ishorali, ya’ni $\lambda_1 \cdot \lambda_2 > 0$, $\lambda_1 \neq \lambda_2$, bo‘lsin. Bu holda muvozanat nuqta ***tugun*** deb ataladi. $|\lambda_1| < |\lambda_2|$ deylik. Mos xos vektorlarni $\boldsymbol{h}_1 = \begin{pmatrix} r_1 \\ s_1 \end{pmatrix}$ va $\boldsymbol{h}_2 = \begin{pmatrix} r_2 \\ s_2 \end{pmatrix}$ bilan belgilaylik. Trayektoriyalar orasida nurdan iborat bo‘lgan to‘rtta to‘g‘risi (egri bo‘lmagani) bor; ularni $\boldsymbol{h}_1$ va $\boldsymbol{h}_2$ xos vektorlar yo‘nalishi aniqlaydi. Boshqa trayektoriyalar moduli bo‘yicha kichik xos songa mos kelgan $\boldsymbol{h}_1$ xos vektorga urinadi. Ikkala xos son ham manfiy bo‘lganda muvozanat nuqta ***turg‘un tugun*** (16.3- rasm, (a)), ikkala xos son ham musbat bo‘lganda esa u ***noturg‘un tugun*** (16.3- rasm, (b)) deyiladi. Turg‘un tugun holida trayektoriyalar bo‘ylab harakat muvozanat nuqtasiga tomon yo‘nalishida, noturg‘un tugun holida esa bu harakat teskari yo‘nalishda bo‘ladi.



16.3- rasm.

(a) turg‘un tugun: $\lambda_1<0, \lambda_2<0$ $(|\lambda_1|<|\lambda_2|)$;

(b) noturg‘un tugun: $\lambda_1>0, \lambda_2>0$ $(|\lambda_1|<|\lambda_2|)$.

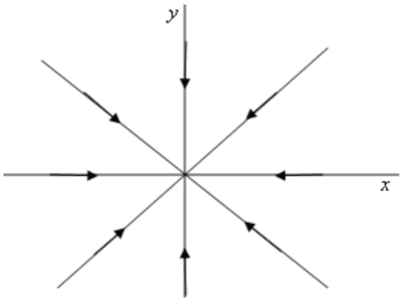


16.4- rasm. Egar: $\lambda_1\lambda_2<0$.

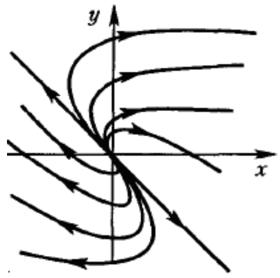
5⁰. λ_1 va λ_2 xos sonlar haqiqiy, turli va har xil ishorali, ya’ni $\lambda_1\cdot\lambda_2<0$, $\lambda_1\neq\lambda_2$, bo‘lsin. Bu holda muvozanat nuqtasi ***egar*** deb ataladi (16.4- rasm). Egardan chiquvchi yoki unga kiruvchi hamda trayektoriyalar oilasini to‘rt qismga ajratuvchi nurlardan iborat bo‘lgan to‘rt dona trayektoriya mavjud. Ularni $\boldsymbol{h}_1$ va $\boldsymbol{h}_2$ xos vektorlar yo‘nalishlari aniqlaydi. Trayektoriyalar bo‘ylab harakat yo‘nalishini tezlik vektori orqali topish mumkin.

6⁰. λ_1 va λ_2 xos sonlar haqiqiy va teng (karrali xos sonlar) bo‘lsin. Agar A matritsaning bu karrali xos soniga ikki dona chiziqli erkli xos vektorlari mos kelsa, ya’ni A matritsa diagonallashtiriluvchi bo‘lsa, muvozanat nuqta ***dikritik tugun***

deyiladi (16.5- rasm). Trayektoriyalar muvozanat nuqtaga kiruvchi (***turg‘un dikritik tugun,*** $\lambda_1 = \lambda_2 < 0$) yoki undan chiquvchi (***noturg‘un dikritik tugun,*** $\lambda_1 = \lambda_2 > 0$) nurlardan iborat bo‘ladi.



16.5 - rasm. Dikritik tugun: $\lambda_1 = \lambda_2 < 0$ ($\lambda_1 = \lambda_2 > 0$ holda yo‘nalishlar teskari)



16.6- rasm. Aynigan tugun: $\lambda_1 = \lambda_2 > 0$ ($\lambda_1 = \lambda_2 < 0$ holda yo‘nalishlar teskari)

Agar A matritsa bir dona (skalyar ko‘paytuvchi aniqligida) chiziqli erkli xos vektorga ega bo‘lsa, ya’ni A matritsa diagonallashtiriluvchi bo‘lmasa, muvozanat nuqta **aynigan tugun** deyiladi (16.6- rasm). Bu holda xos vektor aniqlovchi nurlardan iborat bo‘lgan ikkita to‘g‘ri trayektoriya mavjud bo‘lib, qolgan barcha trayektoriyalar ana shu nurlarga (xos vektorga) urinadi. Bunda muvozanat holati $\lambda_1 = \lambda_2 < 0$ holida ***turg‘un aynigan tugun*** (trayektoriyalar bo‘ylab harakat muvozanat nuqtaga yo‘nalgan), $\lambda_1 = \lambda_2 > 0$ holida esa ***noturg‘un aynigan tugun*** **(**trayektoriyalar bo‘ylab harakatlanuvchi nuqta muvozanat nuqtasidan uzoqlashadi) deb yuritiladi.

Misol 1. Ushbu

$$\begin{cases} x' = x + y \\ y' = -2x + y \end{cases}$$

sistemaning maxsus nuqtasini tekshiring.

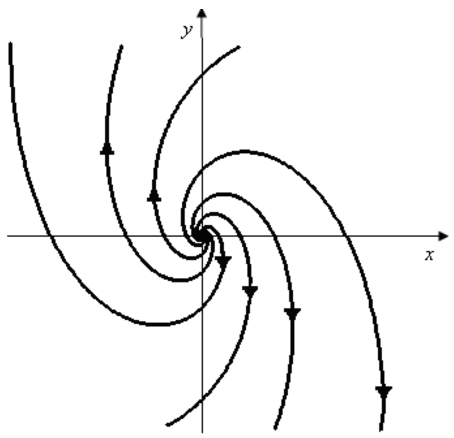
🗝Maxsus nuqta (muvozanat holati) ushbu

$$\begin{cases} x+y=0 \\ -2x+y=0 \end{cases}$$

sistemadan topiladi. Demak, maxsus nuqta $(0;0)$. Qaralayotgan sistema uchun A matritsa va uning xarakteristik tenglamasi

$$A=\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix},\ \det(A-\lambda E)=\begin{vmatrix} 1-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix}=\lambda^2-2\lambda+3=0.$$

Demak, xarakteristik sonlar kompleks $\lambda_{1,2}=1\pm i\sqrt{2}$. Maxsus nuqta − fokus. Lekin $\operatorname{Re}\lambda_{1,2}=1>0$ bo‘lgani uchun maxsus nuqta noturg‘un fokusdan iborat. Trayektoriyalar manzarasi 16.7- rasmda ko‘rsatilgan. 👍



16.7- rasm. Misol 1 uchun trayektoriyalar.

Misol 2. Ushbu

$$\begin{cases} x'=-2x+y \\ y'=3x-4y \end{cases}$$

sistemaning muvozanat nuqtasini tekshiring.

🗝Ravshanki, muvozanat nuqtasi $(0;0)$. Berilgan sistema uchun A matritsa va uning xarakteristik tenglamasini tuzamiz:

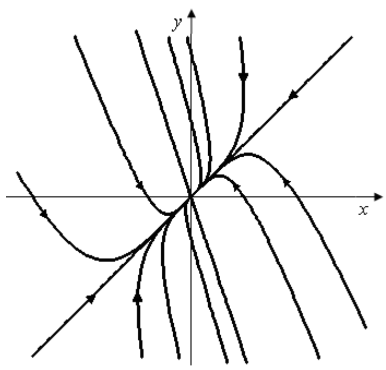
$$A=\begin{pmatrix} -2 & 1 \\ 3 & -4 \end{pmatrix},\ \det(A-\lambda E)=\begin{vmatrix} -2-\lambda & 1 \\ 3 & -4-\lambda \end{vmatrix}=\lambda^2+6\lambda+5=0.$$

Xos sonlar haqiqiy, bir xil manfiy ishorali: $\lambda_1 = -1,\ \lambda_2 = -5$. Demak, muvozanat nuqtasi turg'un tugun. Trayektoriyalar manzarasini chizish uchun A matritsaning λ_1, λ_2 xos sonlariga mos keluvchi xos vektorlarini topaylik:

$$\lambda_1 = -1;\ (A - \lambda_1 E)\boldsymbol{h} = 0;\ \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};\ h_1 = h_2 = 1;\ \boldsymbol{h}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\lambda_2 = -5;\ (A - \lambda_2 E)\boldsymbol{h} = 0;\ h_1 = -1,\ h_2 = 3;\ \boldsymbol{h}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$

Demak, trayektoriyalar moduli bo'yicha kichik xos son $\lambda_1 = -1$ ga mos keluvchi $\boldsymbol{h}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ vektorga (shu vektor orqali o'tuvchi trayektoriyaga) urinadi. Trayektoriyalar bo'ylab harakatlanuvchi nuqta $(0;0)$ nuqtaga yaqinlashadi. Trayektoriyalar manzarasi 16.8- rasmda ko'rsatilgan. ☝



16.8- rasm. Misol 2 uchun trayektoriyalar.

Misol 3. Ushbu

$$\begin{cases} x' = x - 2y \\ y' = -4x - y \end{cases}$$

sistemaning muvozanat nuqtasini tekshiring.

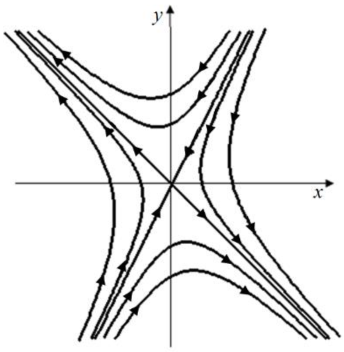
🗝 Muvozanat nuqtasi $(0;0)$. Sistema uchun A matritsa, uning xarakteristik tenglamasi va xos sonlarini hamda xarakteristik vektorlarini topamiz:

$$A = \begin{pmatrix} 1 & -2 \\ -4 & -1 \end{pmatrix};\ \det(A - \lambda E) = \lambda^2 - 9 = 0; \lambda_1 = -3, \lambda_2 = 3.$$

$$\lambda_1 = -3;\ (A-\lambda_1 E)\boldsymbol{h}_1 = 0;\ \boldsymbol{h}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix};$$

$$\lambda_2 = 3;\ (A-\lambda_2 E)\boldsymbol{h}_2 = 0;\ \boldsymbol{h}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Xarakteristik sonlar $\lambda_1 = -3$ va $\lambda_2 = 3$ turli ishorali bo‘lgani uchun muvozanat nuqtasi - egar. $\boldsymbol{h}_1$ va $\boldsymbol{h}_2$ xos vektorlar to‘rt dona nurlardan iborat bo‘lgan (to‘g‘ri) trayektoriyalarni aniqlaydi. Boshqa egri trayektoriyalar uchun ana shu to‘g‘ri trayektoriyalar asimptotalar bo‘lib xizmat qiladi (16.9- rasm.). ☝



16.9- rasm. Misol 3 uchun trayektoriyalar.

III. Nochiziqli avtonom sistema (1) ni tekshirish uchun uni har bir muvozanat nuqtasi atrofida chiziqlilashtirish metodidan foydalanamiz. $f(x,y)$ va $g(x,y)$ funksiyalar berilgan sohada C^2 sinfga tegishli deb faraz qilamiz.

(1) avtonom sistemaning maxsus nuqtalari

$$f(x,y) = 0\ ,\ g(x,y) = 0 \tag{5}$$

sistemadan topiladi. (x_0, y_0) maxsus nuqtaning ($f(x_0, y_0) = 0$, $g(x_0, y_0) = 0$) tabiatini tekshirish uchun bu nuqta atrofida $f(x,y)$ va $g(x,y)$ funksiyalarni Teylor formulasiga ko‘ra tasvirlab

$$f(x,y) = a(x-x_0) + b(y-y_0) + O(r^2),\ a = \frac{\partial f(x_0,y_0)}{\partial x},\ b = \frac{\partial f(x_0,y_0)}{\partial y},$$

$$g(x,y) = c(x-x_0) + d(y-y_0) + O(r^2),\ c = \frac{\partial g(x_0,y_0)}{\partial x},\ d = \frac{\partial g(x_0,y_0)}{\partial y},$$

$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2} \to 0,$$

ularni chiziqli funksiyalar bilan almashtiramiz:

$$f(x,y) \approx a(x-x_0)+b(y-y_0),$$
$$g(x,y) \approx c(x-x_0)+d(y-y_0).$$

Bu almashtirish natijasida (1) sistema o‘rniga

$$\begin{cases} x' = a(x-x_0)+b(y-y_0) \\ y' = c(x-x_0)+d(y-y_0) \end{cases} \tag{6}$$

chiziqli sistemani hosil qilamiz. Ushbu $x = x_0 + u, y = y_0 + v$ almashtirish yordamida (x_0, y_0) maxsus nuqtani koordinatalar boshiga ko‘chiramiz. Natijada

$$\begin{pmatrix} u \\ v \end{pmatrix}' = A\begin{pmatrix} u \\ v \end{pmatrix} \left(\text{bunda } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} f'_x & f'_y \\ g'_x & g'_y \end{pmatrix}\Bigg|_{(x_0,y_0)}\right) \tag{7}$$

tenglamalarga kelamiz. Hosil bo‘lgan **(**7**)** (yoki (6)) chiziqli sistema (1) nochiziqli avtonom sistemaning (x_0, y_0) maxsus nuqta atrofida chiziqlilashtirilishi (yoki birinchi yaqinlashishi) deyiladi. (7) sistemaning $(0,0)$ maxsus nuqtasi, ya’ni (1) sistemaning (x_0, y_0) maxsus nuqtasi tabiatini quyidagi teorema ochadi.

7⁰. Teorema. Agar A matritsaning xos sonlari uchun $\mathrm{Re}\,\lambda_1 \neq 0$ va $\mathrm{Re}\,\lambda_2 \neq 0$ bo‘lsa, (1) nochiziqli sistema $(0,0)$ maxsus nuqtasining tipi (turi) chiziqlilashtirilgan sistema (7) ning maxsus nuqtasi tipi (turi) bilan bir xil. Bunda trayektoriyalarning buralish va maxsus nuqtaga yaqinlashish yoki undan uzoqlashish yo‘nalishlari hamda turg‘unlik tabiatlari saqlanadi.

Berilgan (1) nochiziqli sistemaning barcha maxsus nuqtalari tabiatini maxsus nuqtalar atrofida chiziqlilashtirish yordamida tekshirish mumkin. Bunda ba’zi sohalarda tezliklar maydoni yo‘nalishlarini aniqlab, trayektoriyalar manzarasi quriladi.

Misol 4. Ushbu

$$\begin{cases} x' = 2x - y^2 - 1 \\ y' = x - y^2 \end{cases}$$

sistemaning maxsus nuqtalari tabiatini tekshiring va trayektoriyalar manzarasini (portretini) quring.

🗝 Berilgan misolda $f(x,y)=2x-y^2-1,\ g(x,y)=x-y^2$. Muvozanat (maxsus) nuqtalarni (5) ga ko‘ra topamiz:

$$\begin{cases} 2x-y^2-1=0 \\ x-y^2=0 \end{cases}$$

Bu sistemani yechib, ikkita muvozanat nuqtani hosil qilamiz: $(1;1)$ va $(1;-1)$. Berilgan avtonom sistemani har bir maxsus nuqta atrofida alohoda-alohida chiziqlilashtiramiz. Dastlab xususiy hosilalarni hisoblaymiz:

$$\frac{\partial f(x,y)}{\partial x}=2\ ,\ \frac{\partial f(x,y)}{\partial y}=-2y\ ,\ \frac{\partial g(x,y)}{\partial x}=1,\ \frac{\partial g(x,y)}{\partial y}=-2y\ .$$

$(1;1)$ nuqta atrofida birinchi yaqinlashishni

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{(1;1)}=2\ ,\ \left.\frac{\partial f(x,y)}{\partial y}\right|_{(1;1)}=-2\ ,\ \left.\frac{\partial g(x,y)}{\partial x}\right|_{(1;1)}=1,\ \left.\frac{\partial g(x,y)}{\partial y}\right|_{(1;1)}=-2$$

qiymatlarga ko‘ra (7) formulaga asosan yozamiz:

$$\begin{pmatrix} u \\ v \end{pmatrix}'=A\begin{pmatrix} u \\ v \end{pmatrix},\ \ A=\left.\begin{pmatrix} f'_x & f'_y \\ g'_x & g'_y \end{pmatrix}\right|_{(1;1)}=\begin{pmatrix} 2 & -2 \\ 1 & -2 \end{pmatrix}.$$

Bu yerda shuni e’tirof etish lozimki, birinchi yaqinlashishni to‘g‘ridan- to‘g‘ri Teylor formulasisiz ham hosil qilish mumkin. Buning uchun $(1;1)$ nuqta atrofida $u=x-1$, $v=y-1$ kichik o‘zgaruvchilarni kiritish kerak. Bunda

$$2x-y^2-1=2(u+1)-(v+1)^2-1=2u-2v-v^2\approx 2u-2v$$
$$x-y^2=u+1-(v+1)^2=u-2v-2v^2\approx u-2v$$

sababli berilgan sistema o‘rniga ushbu

$$\begin{cases} u'=2u-2v \\ v'=u-2v \end{cases}$$

chiziqlilashgan sistemaga kelamiz. A matritsaning xos sonlari va mos xos vektorlarini aniqlaymiz:

$$\det(A-\lambda E)=\begin{vmatrix}2-\lambda & -2\\ 1 & -2-\lambda\end{vmatrix}=0\ ;\ \lambda_1=\sqrt{2}\ ,\ \lambda_2=-\sqrt{2}\ ;$$

$$A\boldsymbol{h}_1=\lambda_1\boldsymbol{h}_1,\ \boldsymbol{h}_1=\begin{pmatrix}2+\sqrt{2}\\ 1\end{pmatrix};\quad A\boldsymbol{h}_2=\lambda_2\boldsymbol{h}_2,\ \boldsymbol{h}_2=\begin{pmatrix}2-\sqrt{2}\\ 1\end{pmatrix}.$$

Xos sonlar turli ishorali bo‘lgani uchun $(1;1)$ maxsus nuqta egar. Bu nuqtadan separatissalar $\begin{pmatrix}2+\sqrt{2}\\ 1\end{pmatrix}$ va $\begin{pmatrix}2-\sqrt{2}\\ 1\end{pmatrix}$ xos vektorlar yo‘nalishida chiqadi.

$(1;-1)$ muvozanat nuqta atrofida chiziqlilashtirishni ham yuqoridagiga o‘xshash amalga oshiramiz:

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{(1;-1)}=2\ ,\ \left.\frac{\partial f(x,y)}{\partial y}\right|_{(1;-1)}=2\ ,\ \left.\frac{\partial g(x,y)}{\partial x}\right|_{(1;-1)}=1,\ \left.\frac{\partial g(x,y)}{\partial y}\right|_{(1;-1)}=2\ ;$$

$$\begin{pmatrix}u\\ v\end{pmatrix}'=A\begin{pmatrix}u\\ v\end{pmatrix},\ A=\left.\begin{pmatrix}f'_x & f'_y\\ g'_x & g'_y\end{pmatrix}\right|_{(1;-1)}=\begin{pmatrix}2 & 2\\ 1 & 2\end{pmatrix};$$

Xos son va xos vektorlar:

$$\det(A-\lambda E)=\begin{vmatrix}2-\lambda & 2\\ 1 & 2-\lambda\end{vmatrix}=0\ ;\ \lambda_1=2+\sqrt{2}\ ,\ \lambda_2=2-\sqrt{2}\ ;$$

$$A\boldsymbol{h}_1=\lambda_1\boldsymbol{h}_1\ ,\ \boldsymbol{h}_1=\begin{pmatrix}\sqrt{2}\\ 1\end{pmatrix};$$

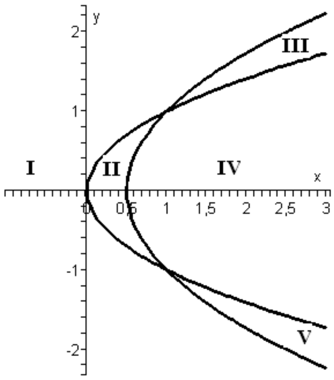
$$A\boldsymbol{h}_2=\lambda_2\boldsymbol{h}_2\ ,\ \boldsymbol{h}_2=\begin{pmatrix}-\sqrt{2}\\ 1\end{pmatrix}\ \ (|\lambda_2|<|\lambda_1|)\ ;$$

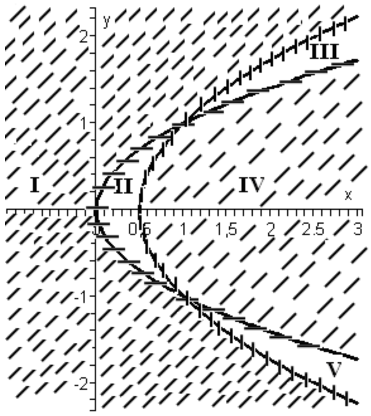
Xos sonlarning ikkalasi ham musbat bo‘lgani uchun $(1;-1)$ muvozanat nuqta noturg‘un tugun. Trayektoriyalar $(1;-1)$ nuqtada $\begin{pmatrix}-\sqrt{2}\\ 1\end{pmatrix}$ xos vektorga urinadi; $(1;-1)$ muvozanat nuqtaning kichik atrofidan chiqib trayektoriyalar bo‘ylab harakat qiluvchi nuqta vaqt o‘tishi bilan $(1;-1)$ nuqtadan uzoqlashadi.

Trayektoriyalar ushbu

$$\frac{dy}{dx}=\frac{x-y^2}{2x-y^2-1}$$

tenglamaning yechimlari grafiklaridan (integral chiziqlaridan) iborat.





16.10- rasm. 16.11- rasm.

Ravshanki, $x-y^2=0$ va $2x-y^2-1=0$ parabolalar butun tekislikni besh sohaga ajratadi (16.10- rasm); bu sohalarning har birida $\frac{dy}{dx}$ hosila o‘z ishorasini saqlaydi (16.11- rasm):

$$\mathbf{I}=\{(x,y)|\ x<y^2,\ 2x<y^2+1\} \text{ sohada } \frac{dy}{dx}>0,$$

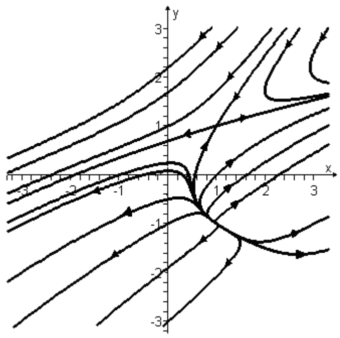
$$\mathbf{II}=\{(x,y)|\ x>y^2,\ 2x<y^2+1\} \text{ sohada } \frac{dy}{dx}<0,$$

$$\mathbf{III}=\{(x,y)|\ x<y^2,\ 2x>y^2+1,\ y>0\} \text{ sohada } \frac{dy}{dx}<0,$$

$$\mathbf{IV}=\{(x,y)|\ x>y^2,\ 2x>y^2+1\} \text{ sohada } \frac{dy}{dx}>0,$$

$$\mathbf{V}=\{(x,y)|\ x<y^2,\ 2x>y^2+1,\ y<0\} \text{ sohada } \frac{dy}{dx}<0.$$

Bu sohalarda trayektoriyalar monoton funksiyalar grafiklaridan iborat bo‘ladi. Bu mulohazalarni hisobga olib, trayektoriyalar manzarasini quramiz (16.12- rasm.). ☝



16.12- rasm. Misol 4 uchun trayektoriyalar.

Misol 5. Ushbu

$$\begin{cases} x' = x - ye^x + xy \\ y' = x(1-x^2)(1-y) \end{cases}$$

sistemaning maxsus nuqtalari tabiatini tekshiring va trayektoriyalar portretini quring.

🗝 Maxsus nuqtalarni

$$f(x,y) \overset{def}{=} x - ye^x + xy = 0\ ,\ \ g(x,y) \overset{def}{=} x(1-x^2)(1-y) = 0$$

sistemani yechib topamiz. Ular uchta:

$(0;0)$, $(1;1/(e-1)) = (1;\alpha)$ $(\alpha = 1/(e-1) \approx 0,58)$ va

$(-1;-e/(e+1)) = (-1;-\beta)$ $(\beta = e/(e+1) \approx 0,73)$.

Endi muvozanat nuqtalarining tabiatini chiziqlilashtirish yordamida aniqlaymiz. Hosilalar

$$\frac{\partial f(x,y)}{\partial x} = 1 - ye^x + y\ ,\ \frac{\partial f(x,y)}{\partial y} = -e^x + x\ ,$$

$$\frac{\partial g(x,y)}{\partial x} = (1-3x^2)(1-y), \frac{\partial g(x,y)}{\partial y} = -x(1-x^2)$$

$(0;0)$ maxsus nuqta atrofida chiziqlilashtirilgan sistema

$$\begin{pmatrix} u \\ v \end{pmatrix}' = A\begin{pmatrix} u \\ v \end{pmatrix},\ \ A = \left.\begin{pmatrix} f'_x & f'_y \\ g'_x & g'_y \end{pmatrix}\right|_{(0;0)} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

A matritsaning xos sonlari $\lambda_{1,2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$; $\mathrm{Re}\lambda_{1,2} = \frac{1}{2} > 0$ bo‘lganligi uchun $(0;0)$ maxsus nuqta noturg‘un fokus.

$(1;\alpha)$ muvozanat nuqta atrofida chiziqlilashtirilgan sistema

$$\begin{pmatrix} u \\ v \end{pmatrix}' = A\begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \left.\begin{pmatrix} f'_x & f'_y \\ g'_x & g'_y \end{pmatrix}\right|_{(1;\alpha)} = \begin{pmatrix} 0 & -e+1 \\ -2(1-\alpha) & 0 \end{pmatrix}.$$

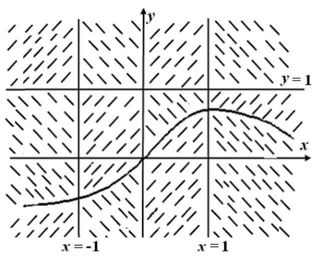
Bu A matritsaning xos sonlari $\lambda_{1,2} = \pm\sqrt{2(e-2)}$ turli ishorali. Demak, $(1;\alpha)$ muvozanat nuqta - egar.

Shunga o‘xshash $(-1;-\beta)$ maxsus nuqta ham egar ekanligi aniqlanadi.

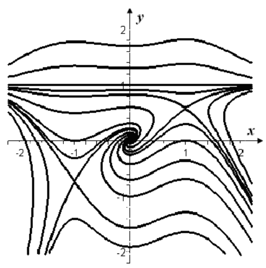
Endi trayektoriyalarning yana ba`zi xususiyatlarini ushbu

$$\frac{dy}{dx} = \frac{x(1-x^2)(1-y)}{x - ye^x + xy} \qquad (*)$$

tenglamadan aniqlashtiramiz. Ravshanki, $x - ye^x + xy = 0$, ya’ni $y = \frac{x}{e^x - x}$ chiziq nuqtalarida trayektoriyaga urinmalar ordinatalar o‘qiga parallel bo‘ladi $\frac{dy}{dx} = \infty$. (*) formuladan $\frac{dy}{dx}$ hosilaning ishorasi saqlanuvchi sohalarni osongina topish va har bir sohada trayektoriya yo‘nalishlarini aniqlash mumkin (16.13- rasm). Ko‘rinib turibdiki, $y=1$ to‘g‘ri chiziq trayektoriyadan iborat. Bu trayektoriya bo‘ylab harakat x ning kamayish yo‘nalishida bo‘ladi, chunki trayektoriyada $x' = 2x - e^x < 0$. Yuqoridagilardan kelib chiqib trayektoriyalar manzarasini quramiz (16.14- rasm). 👍



16.13- rasm



16.14- rasm

Masalalar

Chiziqli avtonom sistemalar muvozanat holatining turini (tabiatini) aniqlang (**1 - 10**):

1. $x' = x - 2y,\ y' = 2x + 2y$.

2. $x' = x + 2y,\ y' = 2x - 2y$.

3. $x' = x - 2y,\ y' = x - y$.

4. $x' = y,\ y' = -2x - 3y$.

5. $x' = -2x - y,\ y' = 2x - y$.

6. $x' = x - y,\ y' = -x + 3y$.

7. $x' = -x + 2y,\ y' = 2x + y$.

8. $x' = 2x,\ y' = 2y$.

9. $x' = -x - y,\ y' = x - 3y$.

10. $x' = x + 2y,\ y' = 3x + 4y$.

Nochiziqli avtonom sistemalarning maxsus (muvozanat) nuqtalarini toping, ularning tabiatini tekshiring (**11 - 20**):

11. $x' = xy - x - 2y + 2,\ y' = xy - 2x - y + 2$.

12. $x' = 2 + x - x^2 - 2y^2,\ y' = -x + y^2$.

13. $x' = y^2 + y - 2,\ y' = -xy + 3y^2 + 2$.

14. $x' = \ln(2 - x + y),\ y' = x - y - \exp(x^2 - 1)$.

15. $x' = xy,\ y' = e^{xy} - x$.

16. $x' = \operatorname{arctg}(x^2 - x + 2y),\ y' = \ln(x^2 + x - y + 1)$.

17. $x' = \operatorname{arctg}(x^2 + x + 3y),\ y' = \ln(x^2 + 2x - y + 1)$.

18. $x' = x - y - 4,\ y' = 2x - 2y - \sqrt[3]{x^2 - 1}$.

19. $x' = \arcsin(2xy - 4y + 8),\ y' = \ln(1 - x^2 + 4y^2)$.

20. $x' = \sqrt[3]{x - y} - y - 2,\ y' = \ln(1 + 2x)$.

Mustaqil ish № 16 topshiriqlari:

Berilgan avtonom sistemaning maxsus (muvozanat) nuqtalarini toping, ularning tabiatini tekshiring va trayektoriyalar portretini quring.

1. $\begin{cases} x' = x - y \\ y' = x + y - xy \end{cases}$

2. $\begin{cases} x' = x^2 + y^2 - 2 \\ y' = y - x^2 \end{cases}$

3. $\begin{cases} x' = x^2 - y^2 + 1 \\ y' = y - x^2 + 5 \end{cases}$

4. $\begin{cases} x' = -y\sqrt[3]{1 - x^2} \\ y' = x\sqrt[3]{1 - x^2} \end{cases}$

5. $\begin{cases} x' = x - y^2 \\ y' = x - y \end{cases}$

6. $\begin{cases} x' = y - x \\ y' = (x^2 - 1)y - x \end{cases}$

7. $\begin{cases} x' = -y\sqrt[3]{1 - x} \\ y' = x\sqrt[3]{1 - x} \end{cases}$

8. $\begin{cases} x' = (x^2 + y^2 - 1)x - y \\ y' = (x^2 + y^2 - 1)y + x \end{cases}$

9. $\begin{cases} x' = (1 + y^2)x + y \\ y' = (1 + x^2)(1 - y^2) \end{cases}$

10. $\begin{cases} x' = (1 + y^2)x + y \\ y' = x + x^2 y \end{cases}$

11. $\begin{cases} x' = y + x^3 \\ y' = x + y + y^3 \end{cases}$

12. $\begin{cases} x' = 2xy + y \\ y' = x + x^2 - y^2 \end{cases}$

13. $\begin{cases} x' = (x - 1)^3 + xy^3 \\ y' = y + y^3 \end{cases}$

14. $\begin{cases} x' = 2xy + y^3 \\ y' = x - x^2 - y^2 + x^3 \end{cases}$

15. $\begin{cases} x' = 1 - y^3 + x^2 \\ y' = 2xy \end{cases}$

16. $\begin{cases} x' = (1 - y^2)x \\ y' = x - (1 + e^x)y \end{cases}$

17. $\begin{cases} x' = (x^2 + y^2)y - 6xy \\ y' = 3y^2 - 3x^2 - x(x^2 + y^2) \end{cases}$

18. $\begin{cases} x' = 4 - y^2 \\ y' = y - (1 - x^2)(1 - y^2) \end{cases}$

19. $\begin{cases} x' = (1 - x^2)(x + 2y) \\ y' = (1 - y^2)(2x - y) \end{cases}$

20. $\begin{cases} x' = 5x - (x^2 - y)(y - 1) \\ y' = y(2x^2 + 2y^2 - x - 3) \end{cases}$

21. $\begin{cases} x' = x - 2y + x^2 + y^2 \\ y' = x / 2 - y + xy \end{cases}$

22. $\begin{cases} x' = 2 + x - y^2 \\ y' = -(x - y)y \end{cases}$

23. $\begin{cases} x' = x^2 + y^2 - 2 \\ y' = (x - 1)(x + y - 1) \end{cases}$

24. $\begin{cases} x' = 3 - \sqrt{8x + y^2} \\ y' = \ln(1 - x + x^2) \end{cases}$

25. $\begin{cases} x' = x + 2y - xy - 1 \\ y' = e^{2x} + e^{x} - 2 \end{cases}$

26. $\begin{cases} x' = (e^{y} - 1)(x - 1) \\ y' = (y - 2)\ln(1 + x) \end{cases}$

27. $\begin{cases} x' = (2x - y + y^4)(x - 2) \\ y' = (x + y - xy)(y - 1) \end{cases}$

28. $\begin{cases} x' = 2x - y + xy \\ y' = x - 2y + x^2 + y^4 \end{cases}$

29. $\begin{cases} x' = (1 + x - 3y)x \\ y' = (x - 2)y \end{cases}$

30. $\begin{cases} x' = -6x + 2xy - 8 \\ y' = y^2 - x^2 \end{cases}$

31. $\begin{cases} x' = 4 - x^2 - 4y^2 \\ y' = xy \end{cases}$

32. $\begin{cases} x' = x - y \\ y' = 4x^2 / (1 + 3x^2) - y \end{cases}$

33. $\begin{cases} x' = x + y + x^2 - 2 \\ y' = 2xy \end{cases}$

34. $\begin{cases} x' = x + y + 1 \\ y' = x + \sqrt[3]{1 + 20y^2} \end{cases}$

35. $\begin{cases} x' = \operatorname{arctg}(y^2 + xy) \\ y' = \sqrt{1 - x + y^2} - 1 \end{cases}$

36. $\begin{cases} x' = 2xy \\ y' = x + y^2 \end{cases}$

37. $\begin{cases} x' = 1 - x - y^2 \\ y' = (x + y)^2 - 1 \end{cases}$

38. $\begin{cases} x' = x + y + 1 \\ y' = x + \sqrt[3]{1 + 20y^2} \end{cases}$

39. $\begin{cases} x' = (1 - x^2 - y^2)x \\ y' = \left(0{,}5 - x - y\right)y \end{cases}$

40. $\begin{cases} x' = x - xy/2 \\ y' = -xy - 3x^2/2 \end{cases}$

17. DIFFERENSIAL TENGLAMALAR YECHIMLARINING TURG‘UNLIGI

Maqsad - differensial tenglamalar yechimlarini turg‘unlikka tekshirishni o‘rganish

Yordamchi ma’lumotlar:

Differensial tenglamalarning quyidagi normal sistemasini qaraylik:

$$\boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}); \tag{1}$$

bu yerda $\boldsymbol{f} \in \boldsymbol{C}(\mathbb{R}_+ \times D; \mathbb{R}^n)$ $(\mathbb{R}_+ = [0, +\infty)$, $D \subset \mathbb{R}^n -$soha) va $\boldsymbol{f}(t, \boldsymbol{x})$ vektor-funksiya $\boldsymbol{x}$ bo‘yicha lokal Lipshits shartini qanoatlantiradi deb hisoblanadi. Bu shartlarda $\forall (t_0, \boldsymbol{x}^0) \in \mathbb{R}_+ \times D$ uchun ushbu

$$\begin{cases} \boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}) \\ \boldsymbol{x}(t_0) = \boldsymbol{x}^0 \end{cases}$$

masala $t \in [t_0, T)$ oraliqda aniqlangan (o‘ngga davomsiz) yagona $\boldsymbol{x} = \boldsymbol{x}(t, t_0, \boldsymbol{x}^0)$ yechimga ega. Yechimlarni turg‘unlikka tekshirish masalalarida berilgan yechimlar o‘ngga cheksiz davom etadi, ya’ni ular $t \in [t_0, +\infty)$ oraliqda aniqlangan deb hisoblanadi.

Bizga (1) tenglamaning $\mathbb{R}_+$ da aniqlangan $\boldsymbol{x} = \boldsymbol{\varphi}(t)$, $\boldsymbol{\varphi} : \mathbb{R}_+ \to D$, yechimi berilgan bo‘lsin. Agar ixtiyoriy $t_0 \in \mathbb{R}_+$ va ixtiyoriy $\varepsilon > 0$ sonlarga ko‘ra shunday $\delta > 0$ son topilsaki, (1) tenglamaning $\boldsymbol{x}(t_0) = \boldsymbol{x}^0$ shartni qanoatlantiruvchi $\boldsymbol{x} = \boldsymbol{x}(t, t_0, \boldsymbol{x}^0)$ yechimlari $\left\| \boldsymbol{x}^0 - \boldsymbol{\varphi}(t_0) \right\| < \delta$ bo‘lganda mavjud va o‘ngga $+\infty$ gacha davom ettirilib, barcha $t \in [t_0, +\infty)$ paytlarda $\| \boldsymbol{x}(t, t_0, \boldsymbol{x}^0) - \boldsymbol{\varphi}(t) \| < \varepsilon$ bo‘lsa, u holda $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ yechim Lyapunov ma’nosida turg‘un yechim deb ataladi (17.1-rasm).

Keltirilgan shart boshlang‘ich qiymatlarning yaqinligidan $\left(\left\| \boldsymbol{x}^0 - \boldsymbol{\varphi}(t_0) \right\| < \delta \right)$ barcha keyingi paytlarda ham yechimlarning yaqinligi $(\| \boldsymbol{x}(t, t_0, \boldsymbol{x}^0) - \boldsymbol{\varphi}(t) \| < \varepsilon,\ t \in [t_0, +\infty))$ kelib chiqishini anglatadi.

Umumiy holda topiladigan $\delta > 0$ soni tayinlangan $t_0 \in \mathbb{R}_+$ va berilgan $\varepsilon > 0$ sonlarga bog‘liq bo‘ladi, ya’ni $\delta = \delta(t_0, \varepsilon)$. Agar keltirilgan ta’rifdagi $\delta > 0$ sonni $t_0 \in [0, +\infty)$ ga bog‘liqsiz holda tanlash mumkin, ya’ni $\delta = \delta(\varepsilon)$ bo‘lsa, bu holdagi turg‘unlik ($\mathbb{R}_+$ da yoki $t_0 \in [0, +\infty)$ ga nisbatan) tekis turg‘unlik deb ataladi.

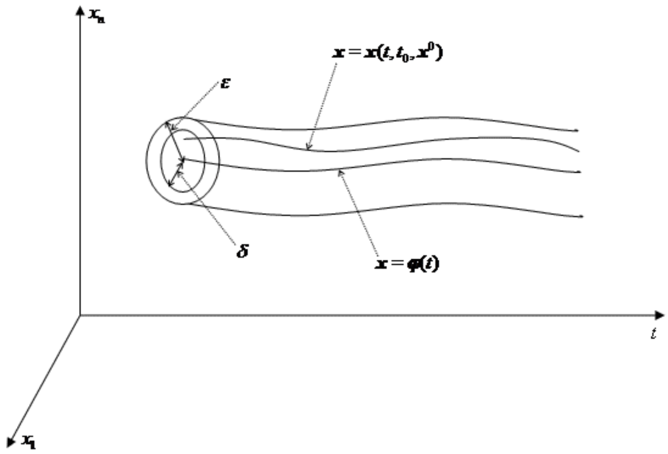
Agar

1) $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ yechim turg‘un va

2) shunday $\delta_0 > 0$ mavjud bo‘lib, $\left\| \boldsymbol{x}^0 - \boldsymbol{\varphi}(t_0) \right\| < \delta_0$ ekanligidan

$\lim\limits_{t\to+\infty} \| \boldsymbol{x}(t, t_0, \boldsymbol{x}^0) - \boldsymbol{\varphi}(t) \| = 0$ bo‘lishi kelib chiqsa,

u holda $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ yechim asimptotik turg‘un yechim deyiladi.



17.1-rasm. Turg‘un yechim atrofidagi yechimlar.

Misol 1. Ushbu

$$\frac{dx}{dt} = 1 - x^2 \tag{2}$$

skalyar differensial tenglamaning $x = 1$ va $x = -1$ yechimlarini turg‘unlikka tekshiring.

$t_0 = 0$ deb hisoblaymiz. Tenglamada t o‘zgaruvchi oshkor ko‘rinishda qatnashmaganligi sababli hosil qilingan natijalar har qanday $t_0 \geq 0$ uchun ham o‘rinli bo‘ladi. Qaralayotgan tenglamada o‘zgaruvchilarni ajratib integrallashlarni bajaramiz va uning $x(0) = x_0$ boshlang‘ich shartli yechimini topamiz:

$$x(t) = x(t, 0, x_0) = \frac{e^{2t} + x_0(1 + e^{2t}) - 1}{e^{2t} + x_0(e^{2t} - 1) + 1}\,.$$

Dastlab $x = 1$ yechimni qaraylik. Uning turg‘unligi ta’rifga ko‘ra quyidagini anglatadi:

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \left| x_0 - 1 \right| < \delta \ \Rightarrow \ \forall t \geq 0 \ \left| x(t) - 1 \right| < \varepsilon \tag{3}$$

Quyidagilarga egamiz: $t \geq 0$ va $\left| x_0 - 1 \right| < 1$ bo‘lganda $e^{2t} \geq 1$ va $x_0 > 0$ hamda

$$|x(t)-1| = \left|\frac{e^{2t}+x_0(e^{2t}+1)-1}{e^{2t}+x_0(e^{2t}-1)+1}-1\right| = \frac{2|x_0-1|}{|e^{2t}+x_0(e^{2t}-1)+1|} \le \frac{2|x_0-1|}{1+1} = |x_0-1|.$$

Demak, (3) shart bajarilishi uchun $\delta > 0$ ni

$$|x_0-1|<1 \text{ va } |x_0-1|<\varepsilon$$

shartlardan tanlash lozim. Buning uchun $\delta = \min(1,\varepsilon)$ deyish kifoya. Shunday qilib, $x=1$ yechim turg‘un. U asimptotik turg‘un hamdir, chunki $x_0 \ne 1$ bo‘lganda

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} x(t,0,x_0) = 1.$$

Endi $x=-1$ yechimni turg‘unlikka tekshiraylik. $x(t)$ yechim formulasidan ravshanki, $x = x_0 < -1$nuqtadan boshlangan yechimlar chekli vaqtda $-\infty$ka ketib qoladi, ya’ni ular o‘ngga cheksiz davom etmaydi. Bundan tashqari, -1ga xohlagancha yaqin, lekin $x = x_0 > -1$ nuqtadan boshlangan yechimlar $t\to+\infty$da 1 ga intiladi, ya’ni ular $x=-1$yechimdan uzoqlashadi. Aniqrog‘i, o‘sha x_0 lar uchun

$$|x(t)-(-1)| = \frac{2e^{2t}|1+x_0|}{|e^{2t}+x_0(e^{2t}-1)+1|}$$

ifodani barcha $t \ge 0$ lar uchun oldindan berilgan ixtiyoriy $\varepsilon \in (0;2)$sondan kichik qilib bo‘lmaydi, chunki $\lim\limits_{t\to+\infty}|x(t)+1| = 2.$ Demak, $x=-1$ yechim turg‘un emas. Yuqoridagi fikrlar turli boshlang‘ich qiymatli yechimlar grafiklarining joylashishidan osongina kelib chiqadi (grafiklarni quring).

Umumiy holda turg‘unlikdan tekis turg‘unlik kelib chiqmaydi; xuddi shuningdek, turg‘unlikdan asimptotik turg‘unlik ham kelib chiqmaydi.

(1) ning o‘ng tomonidan talab qilingan shartlarda $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ yechim boshlang‘ich ma’lumotlarga uzluksiz bog‘liq, ya’ni $\forall[\alpha,\beta]\subset\mathbb{R}_+$ segment va $\forall\varepsilon>0$ soni uchun shunday $\delta>0$ topiladiki, $\gamma\in[\alpha,\beta]$ paytda $\boldsymbol{z}^0$ qiymat qabul qiluvchi $\boldsymbol{x} = \boldsymbol{x}(t,\gamma,\boldsymbol{z}^0)$ yechim, agar $\|\boldsymbol{x}(\gamma,\gamma,\boldsymbol{z}^0)-\boldsymbol{\varphi}(\gamma)\| = \|\boldsymbol{z}^0-\boldsymbol{\varphi}(\gamma)\| < \delta$ bo‘lsa,

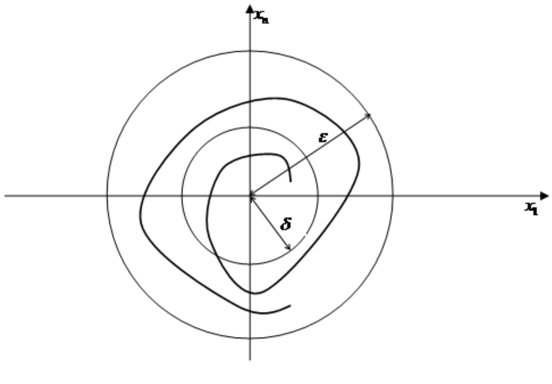
barcha $t\in[\alpha,\beta]$ larda aniqlangan va $\forall t\in[\alpha,\beta]$ uchun $\left\|\boldsymbol{x}(t,\gamma,\boldsymbol{z}^0)-\boldsymbol{\varphi}(t)\right\|<\varepsilon$ bo‘ladi. Bu xossadan ravshanki, $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ yechimning turg‘unligi (asimptotik turg‘unligi) boshlang‘ich payt $t_0\in[0,+\infty)$ ning tanlanishiga bog‘liq emas, ya’ni agar biror $t_0\in[0,+\infty)$ da $\boldsymbol{x}=\boldsymbol{x}(t,t_0,\boldsymbol{\varphi}(t_0))$ yechim turg‘un (asimptotik turg‘un) bo‘lsa, u holda ixtiyoriy $\tilde{t}_0\in[0,+\infty)$ uchun $\boldsymbol{x}=\boldsymbol{x}(t,\tilde{t}_0,\boldsymbol{\varphi}(\tilde{t}_0))$ yechim ham turg‘un (asimptotik turg‘un) bo‘ladi.

Berilgan tenglamaning ixtiyoriy tayinlangan $\boldsymbol{x}=\boldsymbol{\varphi}(t)$, $t\ge t_0$ $(t_0\ge 0)$, yechimini turg‘unlikka tekshirishni boshqa bir tenglamaning trivial yechimini (nol yechimini) turg‘unlikka tekshirishga keltirish mumkin. Buning uchun (1) tenglamada $\boldsymbol{y}=\boldsymbol{x}-\boldsymbol{\varphi}(t)$ almashtirishni bajarish kerak. Yangi noma’lum $\boldsymbol{y}$ quyidagi tenglamani qanoatlantiradi:

$$\boldsymbol{y}'=\boldsymbol{f}(t,\boldsymbol{y}+\boldsymbol{\varphi}(t))-\boldsymbol{f}(t,\boldsymbol{\varphi}(t))\equiv\boldsymbol{g}(t,\boldsymbol{y}),\ \ \boldsymbol{g}(t,0)=0.$$

Oxirgi differensial tenglama $\boldsymbol{y}=0$ trivial yechimga ega. Bu yechimning turg‘unligi (asimptotik turg‘unligi) (1) tenglamaning $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ yechimini turg‘unligiga (asimptotik turg‘unligiga) teng kuchli.

Fazalar fazosida nol-yechimning (ya’ni muvozanat nuqtaning) turg‘unligi quyidagini anglatadi: Fazalar fazosida ixtiyoriy $B(0,\varepsilon)$ sharni olaylik; shunday yetarli kichik $\delta>0$ radiusli $B(0,\delta)$ shar topiladiki, $t=t_0$ da bu shar ichidan boshlangan trayektoriyalar $t\ge t_0$ paytlarda to‘laligicha $B(0,\varepsilon)$ shar ichida joylashadi (17.2- rasm).



17.2- rasm.

Misol 2. Ushbu $x' = y, y' = 2x^3(1+y^2)$ sistemaning fazaviy trayektoriyalarini $(0;0)$ nuqta atrofida chizib, uning muvozanat holatini turg‘unlikka tekshiring.

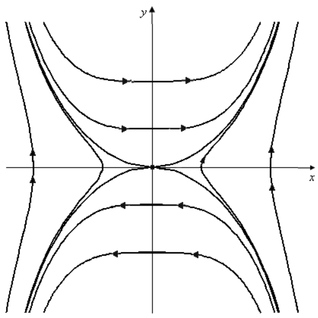
🗝 Fazaviy trayektoriyalarni chizish uchun

$$\frac{dy}{dx} = \frac{2x^3(1+y^2)}{y}$$

tenglamani yechish kerak. Tenglamada o‘zgaruvchilarni ajratib integrallashlarni bajaramz va

$$1 + y^2 = c\exp(x^4)$$

yechimni topamiz. Fazaviy trayektoriyalarning bir nechtasi va ulardagi harakat yo‘nalishi 17.3- rasmda tasvirlangan.



17.3- rasm. Misol 2 uchun trayektoriyalar.

Ravshanki, $(0;0)$ nuqtaga yaqin nuqtalardan boshlangan yechimlar vaqt o‘tishi bilan $(0;0)$ muvozanat nuqtadan uzoqlashadi. Demak, berilgan sistemaning muvozanat holati noturg‘un. ✍

Chiziqli sistemaning turg‘unligi

Ushbu

$$\boldsymbol{x}' = A(t)\boldsymbol{x} + \boldsymbol{b}(t) \tag{4}$$

chiziqli sistemani qaraylik; bunda $A(t) \in C\big([0;+\infty); \mathbb{M}_{n\times n}(\mathbb{R})\big)$ va $\boldsymbol{b}(t) \in C\big([0;+\infty); \mathbb{R}^n\big)$ deb hisoblanadi. Demak, ixtiyoriy $\boldsymbol{x}|_{t_0} = \boldsymbol{x}^0 \in \mathbb{R}^n$, $t_0 \in [0;+\infty)$, boshlang‘ich shart uchun bira-to‘la $[0;+\infty)$ oraliqda aniqlangan $\boldsymbol{x} = \boldsymbol{x}(t) = \boldsymbol{x}(t; t_0, \boldsymbol{x}^0)$ yagona yechim mavjud. (4) ga mos bir jinsli sistema

$$\boldsymbol{y}' = A(t)\boldsymbol{y} \tag{5}$$

ko‘rinishda bo‘ladi.

Quyidagi alternativa o‘rinli:

yo (4) chiziqli sistemaning barcha yechimlari turg‘un (asimptotik turg‘un); bu holda (4) sistema ***turg‘un sistema*** (mos ravishda ***asimptotik turg‘un sistema***) deb ataladi,

yoki uning barcha yechimlari noturg‘un (asimptotik noturg‘un); bu holda esa (4) sistema ***noturg‘un sistema*** (mos ravishda ***asimptotik noturg‘un sistema***) deb ataladi.

Bir jinsli (5) sistema bir dona trivial yechiminining turg‘unli (4) va (5) sistemalarning turg‘unligiga teng kuchli.

Ushbu

$$\boldsymbol{x}' = A\boldsymbol{x} \quad (A-\text{haqiqiy sonlardan tuzilgan } n \times n \text{ matritsa}, A \in \mathbb{M}_{n\times n}(\mathbb{R})) \tag{6}$$

chiziqli o‘zgarmas koeffitsientli sistemaning turg‘unligi (noturg‘unligi) A maritsaning xos sonlari bilan aniqlanadi.

Teorema.

1^0. Agar A maritsa xos sonlarining barchasi manfiy haqiqiy qismga ega bo‘lsa, (6) sistema asimptotik turg‘un bo‘ladi.

2^0. Agar A maritsa xos sonlarining hammasi nomusbat haqiqiy qismga ega bo‘lib, haqiqiy qismi nol bo‘lgan xos sonlarga faqat 1- tartibli Jordan kataklari mos kelsa, (6) sistema turg‘un bo‘ladi.

3^0. Agar A maritsa xos sonlarining birortasi musbat haqiqiy qismga ega bo‘lsa, yoki haqiqiy qismi nol bo‘lgan xos sonlarning birortasiga tartibi ikkidan kichik bo‘lmagan Jordan katagi mos kelsa, u holda (6) sistema noturg‘un bo‘ladi.

Ushbu

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda^1 + a_0 = 0,\ a_n > 0, \tag{7}$$

haqiqiy koeffitsientli algebraik tenglama ildizlarining haqiqiy qismi manfiy bo‘lishini aniqlash uchun foydalaniladigan mezonni keltiraylik. Bu mezonni birinchi bo‘lib E. Raus (1987 y.) va A. Gurvits (1895 y.) topishgan. Keltirilagan mezon L’enar- Shipar mezoni deb ataladi va u tatbiq etish uchun qulayroq.

Dastlab ushbu

$$\begin{vmatrix} a_{n-1} & a_n & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n & 0 & 0 & 0 & \cdots & 0 \\ a_{n-5} & a_{n-4} & a_{n-3} & a_{n-2} & a_{n-1} & a_n & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_0 \end{vmatrix}$$

determinantni tuzaylik. Uning bosh diagonalida (7) ko‘phadning $a_{n-1}, a_{n-2}, a_{n-3}, \ldots, a_0$ koeffitsientlari, satrlarida esa a_j lar indeksnig o‘sish tartibida joylashgan bo‘lib, bunda $j < 0$ yoki $j > n$ indekslar uchun $a_j = 0$ deb hisobanadi. Bu detrminantning bosh diagonal minorlarini

$$\Delta_1 = a_{n-1},\ \Delta_2 = \begin{vmatrix} a_{n-1} & a_n \\ a_{n-3} & a_{n-2} \end{vmatrix},\ \Delta_3 = \begin{vmatrix} a_{n-1} & a_n & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} \\ a_{n-5} & a_{n-4} & a_{n-3} \end{vmatrix},\ \ldots$$

bilan belgilaylik.

Teorema (L’enar–Shipar mezoni). (7) tenglama barcha ildizlarining haqiqiy qismi manfiy bo‘lishi uchun ushbu

1) barcha a_j lar musbat, ya’ni $a_n > 0,\ a_{n-1} > 0,\ a_{n-2} > 0,\ \ldots,\ a_0 > 0$;

2) $\Delta_{n-1} > 0,\ \Delta_{n-3} > 0,\ \Delta_{n-5} > 0,\ \ldots$

shartlarning bir vaqtda bajarilishi yetarli va zarurdir.

Misol 3. Ushbu

$$\begin{cases} x' = -3x + 3y - 3z \\ y' = \ 6x - y - 7z \\ z' = \ 3x + 4y - 8z \end{cases}$$

sistemani turg‘unlikka tekshiring.

🗝 Berilgan sistemani

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = A\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad A = \begin{pmatrix} -3 & 3 & -3 \\ 6 & -1 & -7 \\ 3 & 4 & -8 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

ko‘rinishda yozib olamiz. A matritsaning xarakteristik tenglamasini tuzamiz:

$$\det(\lambda E - A) = 0, \quad \begin{vmatrix} \lambda+3 & -3 & 3 \\ -6 & \lambda+1 & 7 \\ -3 & -4 & \lambda+8 \end{vmatrix} = 0, \quad \lambda^3 + 12\lambda^2 + 54\lambda + 108 = 0.$$

L’enar–Shipar mezonini qo‘llaymiz:

1). $a_3 = 1 > 0,\ a_2 = 12 > 0,\ a_1 = 54 > 0,\ a_0 = 108 > 0;$

2). $\Delta_2 = \begin{vmatrix} a_2 & a_3 \\ a_0 & a_1 \end{vmatrix} = \begin{vmatrix} 12 & 1 \\ 108 & 54 \end{vmatrix} = 12 \cdot 54 - 108 = 540 > 0.$

Demak, A maritsa xos sonlarining barchasi manfiy haqiqiy qismga ega va berilgan sistema asimptotik turg‘un. 👍

Birinchi yaqinlashishga ko‘ra turg‘unlik

Endi (1) sistema ushbu

$$\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{g}(t, \boldsymbol{x}) \tag{8}$$

ko‘rinishda bo‘lgan holni qaraylik. Bunda $A \in \mathbb{M}_{n\times n}(\mathbb{R})$, $\boldsymbol{g} \in C(\mathbb{R}_+ \times B_\rho; \mathbb{R}^n)$, $\boldsymbol{g}(t, \boldsymbol{x})$ vektor-funksiya $\boldsymbol{x}$ bo‘yicha lokal Lipshits shartini qanoatlantiradi va $\|\boldsymbol{g}(t, \boldsymbol{x})\| \le \alpha(\boldsymbol{x})\|\boldsymbol{x}\|$, $\alpha(\boldsymbol{x}) \xrightarrow[x\to 0]{} 0$, deb hisoblanadi. Xususan, $\boldsymbol{g}(t, 0) \equiv 0$ va (8) sistema $\boldsymbol{x}(t) \equiv 0$ yechimga ega. Agar (8) sistemada $\mathbf{x} \to 0$ da yuqori tartibli cheksiz kichik miqdor $\boldsymbol{g}(t, \boldsymbol{x})$ ni tashlab yuborsak, birinchi yaqinlashish sistemasi deb ataluvchi ushbu

$$\boldsymbol{x}' = A\boldsymbol{x} \tag{9}$$

sistemani hosil qilamiz. Oxirgi chiziqli sistema (8) sistemaning chiziqlilashtirilishi deb ham yuritiladi.

Teorema (Birinchi yaqinlashishga ko‘ra asimptotik turg‘unlik to‘g‘risidagi). Agar A matritsaning barcha λ_j xarakteristik sonlari uchun $\operatorname{Re}\lambda_j < 0$ bo‘lsa, (8) sistemaning $\boldsymbol{x}(t) \equiv 0$ yechimi asimptotik turg‘un bo‘ladi.

Teorema (Birinchi yaqinlashishga ko‘ra noturg‘unlik to‘g‘risidagi). Agar A matritsaning biror λ_j xarakteristik soni uchun $\operatorname{Re}\lambda_j > 0$ bo‘lsa, (8) sistemaning $\boldsymbol{x}(t) \equiv 0$ yechimi noturg‘un bo‘ladi.

Misol 4. Ushbu

$$\begin{cases} x' = \sin x + \cos y - e^{2z} + x^2 y \\ y' = 2x + \ln(1 + y + 2z) - 3yz \\ z' = x - y - 4 + 4\sqrt{1-z} + 3z^2 \end{cases}$$

sistemaning nol-yechimini turg‘unlikka tekshiring.

🗝 Turg‘unlikka tekshirishni birinchi yaqinlashishga ko‘ra amalga oshiramiz. Buning uchun berilgan sistemani chiziqlilashtiramiz. Analizdan ma’lum bo‘lgan ushbu

$$\sin x = x + O(x^3)\ , \ \ x \to 0;$$

$$\cos y = 1 + O(y^2)\ , \ \ y \to 0;$$

$$e^t = 1 + t + O(t^2)\ , \ \ t \to 0;$$

$$\ln(1+t) = t + O(t^2)\ , \ \ t \to 0;$$

$$\sqrt{1+t} = 1 + \frac{t}{2} + O(t^2)\ , \ \ t \to 0;$$

asimptotik yoyilmalarga asosan $x \to 0, y \to 0, z \to 0$ bo‘lganda

$$\sin x + \cos y - e^{2z} + x^2 y = x + O(x^2) + 1 + O(y^2) - 1 - 2z + O(z^2) + x^2 y =$$
$$= x - 2z + O(x^2 + y^2 + z^2),$$

$$2x + \ln(1 + y + 2z) - 3yz = 2x + y + 2z + O((y+2z)^2) - 3yz =$$
$$= 2x + y + 2z + O(x^2 + y^2 + z^2),$$

$$x - y - 4 + 4\sqrt{1-z} + 3z^2 = x - y - 4 + 4 - 2z + O(z^2) + 3z^2 =$$
$$= x - y - 2z + O(x^2 + y^2 + z^2),$$

asimptotik tengliklarni hosil qilamiz (Ularni Teylor formulasiga ko‘ra ham topish mumkin edi). Demak, berilgan sistema uchun birinchi yaqinlashish quyidagi ko‘rinishga ega:

$$\begin{cases} x' = x - 2z \\ y' = 2x + y + 2z \\ z' = x - y - 2z \end{cases}$$

Bu chiziqli sistemaning matritsasi

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 1 & 2 \\ 1 & -1 & -2 \end{pmatrix}.$$

Uning xarakteristik tenglamasi

$$\det(\lambda E - A) \equiv \lambda^3 + \lambda - 6 = 0.$$

$(\lambda^3 + \lambda - 6)\big|_{\lambda=0} = -6$, $(\lambda^3 + \lambda - 6)\big|_{\lambda=+\infty} = +\infty$ bo‘lganligi uchun musbat xarakteristik son mavjud. Shuning uchun birinchi yaqinlashishga ko‘ra noturg‘unlik to‘g‘risidagi teoremaga asosan berilgan sistemaning nol-yechimi noturg‘un. ✍

Birinchi yaqinlashishga ko‘ra turg‘unlik va noturg‘unlik to‘g‘risidagi teoremalar ishlamaganda Lyapunov funksiyalaridan foydalanish mumkin.

Lyapunov funksiyalari yordamida turg‘unlikka tekshirish

Agar $v(\mathbf{x})$ (t ga bog‘liq bo‘lmagan) funksiya $\mathbf{x} = 0$ nuqtaning biror $B_{\tilde{\rho}}$ $(\tilde{\rho} > 0)$ atrofida C^1 sinfga tegishli, $v(0) = 0$ va shu atrofdagi barcha $\mathbf{x} \neq 0$ nuqtalarda $v(\mathbf{x}) > 0$ bo‘lsa, u holda bu $v(\mathbf{x})$ funksiyani *aniq musbat funksiya* deymiz va buni $v(\mathbf{x}) \succ 0$ kabi ifodalaymiz. Masalan, $v(\mathbf{x}) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2$ funksiya (kvadratik forma) $\lambda_1 > 0, \lambda_2 > 0, \ldots, \lambda_n > 0$ holida aniq musbat. Lekin $v(x, y) = x^2 - 2xy + y^2 = (x - y)^2$ funksiya, nomanfiy bo‘lsada, aniq musbat emas.

Ushbu

$$\boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}), \;\; \boldsymbol{f}(t, 0) \equiv 0, \; t \geq 0, \tag{10}$$

sistemani qaraylik. U, ravshanki, $x=0$ yechimga ega. $v=v(\mathbf{x})$ funksiyaning (10) sistemaga ko‘ra (to‘la) hosilasi deb

$$\left.\frac{dv}{dt}\right|_{(10)} = \frac{\partial v}{\partial x_1}\cdot f_1 + \frac{\partial v}{\partial x_2}\cdot f_2 + \cdots + \frac{\partial v}{\partial x_n}\cdot f_n \qquad (11)$$

miqdorga aytiladi.

Teorema (Lyapunovning turg‘unlik to‘g‘risidagi teoremasi). Agar (10) sistema uchun ushbu

$$v(\mathbf{x}) \succ 0 \text{ va } \left.\frac{dv}{dt}\right|_{(10)} \le 0$$

shartlarni qanoatlantiruvchi $v(\mathbf{x})$ (aniq musbat va (10) sistemaga ko‘ra hosilasi noldan kichik yoki unga teng) funksiya mavjud bo‘lsa, (10) sistemaning $\mathbf{x}(t)\equiv 0$ yechimi turg‘un bo‘ladi.

Teorema (Lyapunovning asimptotik turg‘unlik to‘g‘risidagi teoremasi). Aytaylik, (10) sistema uchun nol nuqtaning biror atrofida ushbu

$$v(\mathbf{x}) \succ 0 \text{ va } \left.\frac{dv}{dt}\right|_{(1)} \le -w(\mathbf{x}) < 0,\ \mathbf{x}\ne 0,$$

shartlarni qanoatlantiruvchi $v(\mathbf{x})$ va uzluksiz $w(\mathbf{x})$ funksiyalar mavjud bo‘lsin.U holda (10) sistemaning $\mathbf{x}(t)\equiv 0$ yechimi asimptotik turg‘un bo‘ladi.

Quyidagi teorema Lyapunovning noturg‘unlik to‘g‘risidagi teoremasining umumlashishidir.

Teorema (Chetayevning noturg‘unlik to‘g‘risidagi teoremasi). Aytaylik, (10) sistema uchun quyidagi shartlarni qanoatlantiruvchi U soha va $v=v(\mathbf{x})$ funksiya (Lyapunov funksiyasi) mavjud bo‘lsin:

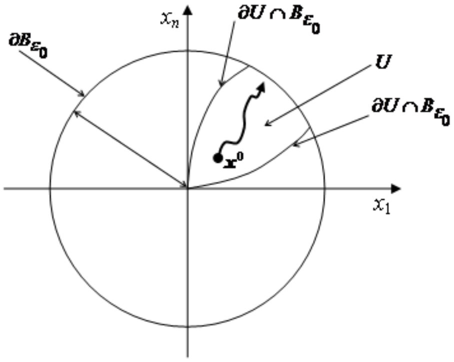
U soha $0\in\mathbb{R}^n$ nuqtaning biror B_{ε_0} atrofida yotadi, ya’ni $U\subset B_{\varepsilon_0}$ va $0\in\partial U$;

$v\in C(U\cup\partial U)$, U sohada $v>0$, lekin ∂U ning B_{ε_0} dagi qismida $v(\mathbf{x})|_{\partial U\cap B_{\varepsilon_0}}=0$;

$v\in C^1(U)$ va biror $w\in C(U\cup\partial U)$ funksiya uchun $t\in[0;+\infty)$, $\mathbf{x}\in U$ bo‘lganda

$$\left.\frac{dv}{dt}\right|_{(1)} \geq w(\mathbf{x}) > 0.$$

U holda (10) sistemaning $\mathbf{x}(t) \equiv 0$ yechimi noturg‘un bo‘ladi (17.3- rasm).



17.3- rasm.

Lyapunov funksiyalarini ba’zan kvadratik forma, $v(\mathbf{x}) = v(x_1) + v(x_2) + \ldots + v(x_n)$ (o‘zgaruvchilar ajralgan) ko‘rinishida izlash mumkin.

Misol 5. Ushbu

$$\begin{cases} x' = y - x \\ y' = x - y - x^3 \end{cases}$$

sistemaning nol-yechimini turg‘unlikka tekshiring.

⊶Berilgan sistemaning nol-yechim ($x = 0, y = 0$) atrofida chiziqli-lashtirilishi

$$\begin{cases} x' = y - x \\ y' = x - y \end{cases} \quad \text{yoki} \quad \begin{pmatrix} x \\ y \end{pmatrix}' = A\begin{pmatrix} x \\ y \end{pmatrix}, \ A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

ko‘rinishga ega. Bu yerdagi A matritsaning xos sonlari $\lambda_1 = 0$, $\lambda_2 = -2$. $\lambda_1 = 0$ bo‘lganligi uchun birinchi yaqinlashishga ko‘ra turg‘unlik va noturg‘unlik to‘g‘risidagi teoremalar ishlamaydi.

Nol-yechimini turg‘unlikka tekshirish uchun Lyapunov funksiyasidan foydalanamiz. $v = v(x, y) = (x + y)^2 + \frac{1}{2}x^4$ funksiyani qaraylik. Bu funksiya aniq musbat, chunki $v(0,0)=0$ va $(x,y)\neq(0,0)$ bo‘lganda $v(x,y)>0$. Bundan tashqari, uning berilgan sistemaga ko‘ra hosilasi noldan kichik yoki unga teng:

$$\frac{dv}{dt}=\frac{\partial v}{\partial x}(y-x)+\frac{\partial v}{\partial y}(x-y-x^3)=$$

$$=(2(x+y)+2x^3)(y-x)+2(x+y)(x-y-x^3)=-4x^4\leq 0$$

Demak, Lyapunovning turg‘unlik to‘g‘risidagi teoremasiga ko‘ra berilgan sistemaning nol-yechimi turg‘un. ✍

Misol 6. Ushbu

$$\begin{cases} x'=-y+x^3 \\ y'=x+y^3 \end{cases}$$

sistemaning nol-yechimini turg‘unlikka tekshiraylik.

🗝 U soha sifatida $U=\{(x,y)\,|\,0<x^2+y^2<1\}=B_1\setminus\{(0;0)\}$ to‘plamni olaylik. Tushunarliki, $(0;0)\in\partial U$. $v=x^2+y^2$ bo‘lsin. U sohada $v>0$ va ∂U ning B_1 dagi qismida $v|_{\partial U\cap B_1}=v|_{(0;0)}=0$; $(x,y)\in U$ bo‘lganda v funksiyaning berilgan sistemaga ko‘ra hosilasi

$$\frac{dv}{dt}=2x(-y+x^3)+2y(x+y^3)=w(x,y)>0\,,\;\; w(x,y)\equiv 2(x^4+y^4)\,.$$

Demak, Chetayevning noturg‘unlik to‘g‘risidagi teoremasiga ko‘ra berilgan sistemaning nol-yechimi noturg‘un.

Chiziqlashtirilgan sistema:

$$\begin{cases} x'=-y \\ y'=x \end{cases},\;\; A=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\; \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix}=0,\lambda^2+1=0,\lambda_{1,2}=\pm i\,.$$

Demak, berilgan sistema nol-yechimining turg‘unligini birinchi yaqinlashishga ko‘ra aniqlab bo‘lmaydi. ✍

Misol 7. Ushbu

$$\begin{cases} x'=x^5+y^3 \\ y'=x^3+y^5 \end{cases}$$

sistemaning nol-yechimini turg‘unlikka tekshiring.

⊷ $U=\{(x,y)\mid |x|>y>0\}$ bo‘lsin. Tushunarliki, $(0;0)\in\partial U$. $v=x^4-y^4$ deylik. U sohada $v>0$ va ∂U da $v|_{\partial U}=0$; v funksiyaning berilgan sistemaga ko‘ra hosilasi

$$\frac{dv}{dt}=4x^3(x^5+y^3)-4y^3(x^3+y^5)=w(x,y),$$
$$w(x,y)\equiv 4(x^4-y^4)(x^4+y^4)>0,\ (x,y)\in U.$$

Chetayevning noturg‘unlik to‘g‘risidagi teoremasi qralayotgan sistemaning nol-yechimi noturg‘un ekanligini asoslaydi. ☝

Masalalar

Boshlang‘ich masala yechimini turg‘unlikka ta’rifga ko‘ra tekshiring (**1 - 4**):

1. $x'=(1-t)x,\ x(0)=0,\ t\ge 0.$ **2.** $x'=(1+t^2)x,\ x(0)=0,\ t\ge 0.$

3. $2tx'=x(1-x^2),\ x(t)\equiv 1,\ t>0.$ **4.** $3tx'=x-x^4,\ x(t)\equiv 0,\ t>0.$

Chiziqli sistemani turg‘unlikka tekshiring (**5 - 11**):

5. $\begin{cases}x'=x+2y,\\ y'=3x+4y.\end{cases}$ **6.** $\begin{cases}x'=2x-5y,\\ y'=3x-y.\end{cases}$ **7.** $\begin{cases}x'=x-4y,\\ y'=3x-y.\end{cases}$

8. $\begin{cases}x'=-3x+2y+z,\\ y'=x-2z,\\ z'=2x+y-2z.\end{cases}$ **9.** $\begin{cases}x'=y-z,\\ y'=-y+z+t,\\ z'=x-z-\sin t.\end{cases}$

10. $\begin{cases}x'=x+5y-z+e^t,\\ y'=x-2y+z,\\ z'=x+y-z-1.\end{cases}$ **11.** $\begin{cases}x'=-3x+y-z,\\ y'=x+y-2z,\\ z'=2x+5y-2z.\end{cases}$

Sistemani chiziqlilashtirish yordamida nol-yechimni turg‘unlikka tekshiring (**12 -14**):

12. $\begin{cases}x'=\ln(1-2x+z)-(1+y)^2+1,\\ y'=\sqrt{1-x+y}+z-1,\\ z'=\sin(x+x^2+2y-z).\end{cases}$ **13.** $\begin{cases}x'=\exp(x-y+z^2)-2z-1,\\ y'=(1+2x-y)^2-\cos z,\\ z'=\arcsin(2x-y-z).\end{cases}$

14. $\begin{cases} x' = (1-2x+y+z)^2 - 1, \\ y' = \operatorname{arctg}(x+y) - z, \\ z' = \sin(x+y-2z). \end{cases}$

Sistemalarning barcha muvozanat nuqtalarini toping va ularni turg‘unlikka tekshiring (**15 - 17**)

15. $\begin{cases} x' = 1-2x-y+z^2, \\ y' = e^{-x} - y - 2, \\ z' = x+y. \end{cases}$

16. $\begin{cases} x' = \operatorname{arctg}(2x+y+z^2-1), \\ y' = 1 - e^{x+y+z-1}, \\ z' = x^2 - y^2. \end{cases}$

17. $\begin{cases} x' = \ln(1-x^2+y^2), \\ y' = x+3y+z+1, \\ z' = 2x+y-z^2+1. \end{cases}$

Mustaqil ish № 17 topshiriqlari:

I. Yechimni turg‘unlikka ta’rifga ko‘ra tekshiring.

1. $x' = t - 2x;\ x(0) = 1,\ t \geq 0$.

2. $x' = -x^2 + \frac{4}{t}x - \frac{4}{t^2};\ x = \frac{1}{t},\ t \geq 1$.

3. $x' = 9x - t^2 x;\ x(0) = 0,\ t \geq 0$.

4. $(t-1)x' = 2x;\ x(2) = 1,\ t \geq 2$.

5. $x' + t(x+1) = 0;\ x = -1,\ t \geq 0$.

6. $xx' = 3tx^2 - t;\ x(1) = 1,\ t \geq 1$.

7. $tx' = x - te^{x/t};\ x(1) = 0,\ t \geq 1$.

8. $x' + x\cos t = e^{-\sin t};\ x(0) = 1,\ t \geq 0$.

9. $x' = -x^2 + \frac{2}{t^2};\ x = \frac{2}{t},\ t \geq 1$.

10. $\begin{cases} x' = -y \\ y' = \sin x \end{cases}$ $;x(t) \equiv 0,\ y(t) \equiv 0,\ t \geq 0$.

11. $\begin{cases} x' = -y \\ y' = x \end{cases}$ $;x(t) \equiv 0,\ y(t) \equiv 0,\ t \geq 0$

12. $\begin{cases} x' = y \\ y' = x \end{cases}$ $;x(t) \equiv 0,\ y(t) \equiv 0,\ t \geq 0$.

13. $tx' = -2x - te^{x/t};\ x(1) = 0,\ t \geq 1$.

14. $x' = 1 + t - x;\ x(0) = 1,\ t \geq 0$.

15. $x' - 2tx = 2te^{t^2},\ x(0) = 1,\ t \geq 0$.

16. $x' - 2tx = t;\ x(0) = 0,\ t \geq 0$.

17. $x' + tx = t^2;\ x(1) = 1, t \geq 0$.

18. $x' - 3tx = 3t^2;\ x(0) = 1, t \geq 0$.

19. $x' - x + x^3 = 0;\ x(1) = 0, t \geq 1$.

20. $2x' = x - x^4;\ x(1) = 1,\ t \geq 1$.

21. $x' = -x - 2tx^3;\ x(1) = 1, t \geq 1$.

22. $x' + 2(x-t)x = 1;\ x = t,\ t \geq 1$.

23. $\begin{cases} x' = 20y \\ y' = -x^3 \end{cases}$ **;** $x(t) \equiv 0, y(t) \equiv 0,\ t \geq 0$**.**

24. $\begin{cases} x' = y \\ y' = -x^2 - x \end{cases}$ **;** $x(t) \equiv 0, y(t) \equiv 0,\ t \geq 0$**.**

25. $\begin{cases} x' = y \\ y' = x^2 - x \end{cases}$ **;** $x(t) \equiv 0, y(t) \equiv 0,\ t \geq 0$**.**

26. $\begin{cases} x' = y \\ y' = -4x^3 - 2x \end{cases}$ **;** $x(t) \equiv 0, y(t) \equiv 0,\ t \geq 0$**.**

27. $\begin{cases} x' = y \\ y' = 4x^3 - 2x \end{cases}$ **;** $x(t) \equiv 0, y(t) \equiv 0,\ t \geq 0$.

28. $\begin{cases} x' = -y\cos x \\ y' = x \end{cases}$ **;** $x(t) \equiv 0, y(t) \equiv 0, t \geq 0$**.**

29. $x' + x\cos t = \sin t \cdot \cos t; x(t) \equiv 2e^{-\sin t} + \sin t - 1, t \geq 0$**.**

30. $x' - x\cos t = \sin t \cdot \cos t,\ x(0) = 1,\ t \geq 0$**.**

II. Sistemaning nol-yechimini birinchi yaqinlashishga ko‘ra turg‘unlikka tekshiring**.**

1. $\begin{cases} x' = x - \cos y + e^{-z} + xy^2 \\ y' = 3x + \sin(y + 2z) \\ z' = 2x - y + 2(1 - z)^3 - 2 \end{cases}$

2. $\begin{cases} x' = x - \ln(1 - 3y) + e^{-2z} - 1 \\ y' = 3\sin x + y + (y + 2z)^3 \\ z' = 2x - 3y + 2(1 - z)^4 - 2 \end{cases}$

3. $\begin{cases} x' = 2\sin x - e^{-z} + 1 \\ y' = x/2 + \ln(1 + y - 3z) + z^2 \\ z' = x - y + 2(1 - z)^{1/3} - 2 \end{cases}$

4. $\begin{cases} x' = x/3 - e^{x+y} + e^{-z} + xy^2 \\ y' = -2x + \cos(y + 2z) - z - 1 \\ z' = 0{,}25x - 3y + 2(1 - z)^3 - 2 \end{cases}$

5. $\begin{cases} x' = \ln(1 + 2x) - \cos^2 y - z + xy^2 + 1 \\ y' = 0{,}3x + \sin(x + y - 2z) - y^3 \\ z' = 2x + 2(1 - y + z)^3 - 2 \end{cases}$

6. $\begin{cases} x' = x + \cos y - e^{-z} + xy^2 \\ y' = 3x + \sin(y + 2z) \\ z' = 0{,}5x - 2(1 - y)^{1/3} + z + 2 \end{cases}$

7. $\begin{cases} x' = 1{,}5x - e^{-z} + 1 + xy^2 \\ y' = 3x + 0{,}5\sin(y+2z) \\ z' = x - y + \ln(1-z)^{2/3} \end{cases}$

8. $\begin{cases} x' = x + \cos y - e^{x-2z} + xy^2 \\ y' = 1{,}3x + \sin(y+z) \\ z' = -2x + y + 2(1-z)^{2/3} - 2 \end{cases}$

9. $\begin{cases} x' = 3y - e^{-x-2z} + xy^2 + 1 \\ y' = 0{,}5x + \sin(y+2z) \\ z' = 2{,}5x - y + \ln(1-y+z)^3 \end{cases}$

10. $\begin{cases} x' = x/4 - y - \sin(y+2z) \\ y' = 3x + e^{-z} - 1 + z^3 \\ z' = 0{,}5x + (1-y)^2 + z - 1 \end{cases}$

11. $\begin{cases} x' = -3y - \sin(y+2z) \\ y' = x + 2e^{-z} - 2 + xz^3 \\ z' = 1{,}5x + y + \ln(1-z)^2 \end{cases}$

12. $\begin{cases} x' = x - (1-y+2z)^{3/4} + 1 \\ y' = x + e^{-z} - 1 + xz^4 \\ z' = 2{,}5x + 2y + \ln(1-z)^2 \end{cases}$

13. $\begin{cases} x' = -3y - \sin(y+2z) \\ y' = x + 2e^{y-z} - 2 + xz \\ z' = 1{,}5x + y + \ln(1-z)^4 \end{cases}$

14. $\begin{cases} x' = -3(1-x)^{1/3} - \sin(y+2z) + 3 \\ y' = \ln(1-x-z)^2 + e^{0{,}5y-z} - 1 \\ z' = 1{,}5x + y + \ln^2(1-z)^4 \end{cases}$

15. $\begin{cases} x' = x - \cos(y+2z) + z + 1 + y^3 \\ y' = \sin x + e^{y-z} - 1 + xz \\ z' = 2{,}5x - y + \ln(1-y+z)^4 \end{cases}$

16. $\begin{cases} x' = -y - \sin(x-y+2z) \\ y' = 1{,}5x - e^{y-z} + 1 + xz \\ z' = -x + 2y + (1-y-z)^4 - 1 \end{cases}$

17. $\begin{cases} x' = -x - y - \sin(y+3z) \\ y' = x - e^{2x+y-z} + 1 + x^3 z \\ z' = 1{,}5x - 2y + \ln(1-z)^4 \end{cases}$

18. $\begin{cases} x' = y + \sin(x+y+2z) \\ y' = 0{,}5x + y - (e^{y-z} - 1)^2 + xz \\ z' = x + y + \ln(1+z)^3 - x^4 \end{cases}$

19. $\begin{cases} x' = x - 3y - y\cos y \\ y' = x + \sin(y+2z) - xy \\ z' = x + 1{,}5y + \ln(1-2z)^4 \end{cases}$

20. $\begin{cases} x' = x - y - y\cos y + 2{,}5z \\ y' = x + 0{,}5\sin(y+2z) - xy \\ z' = x + y + \sqrt{1-3z} + 1 \end{cases}$

21. $\begin{cases} x' = \sqrt[3]{1+x-3y} - \cos y \\ y' = \sin(x-y+2z) - xy \\ z' = x + 1{,}5y + e^{(1-2z)^4} - 1 \end{cases}$

22. $\begin{cases} x' = z - 2(1-x+y+2z)^{2/3} + 2 \\ y' = 0{,}5x + y - \sin(e^{y-z} - 1) \\ z' = y + \ln(1+z)^2 - x^4 \end{cases}$

23. $\begin{cases} x' = x + \sin(x - y + 2z) \\ y' = 0,5x + y - (\cos z - 1)^2 + xz \\ z' = x + y + \ln(1+z)^{2/3} - x^4 \end{cases}$ **24.** $\begin{cases} x' = z + \sin\ln(1 - x + 2z)^{1/3} \\ y' = 0,5x + y + 2(e^{y-z} - 1) \\ z' = y + (1-z)^2 - 1 \end{cases}$

25. $\begin{cases} x' = z - (1 - x + y + 2z)^{2/3} + 1 \\ y' = 0,5x + y - \sin(e^{y-z} - 1) \\ z' = y + \ln(1+z)^2 - x^4 \end{cases}$ **26.** $\begin{cases} x' = z + \sqrt[3]{1 - x + 2y + z} - 1 \\ y' = 0,5x + y - \sin(e^{y-z} - 1) \\ z' = y + \ln(1+z)^2 - x^4 \end{cases}$

27. $\begin{cases} x' = 3z + y - (e^{y-z} - 1)^3 \\ y' = x - (1 - x - 2y + 2z)^{2/3} + 1 \\ z' = y\cos y + \ln(1+z)^2 - z^4 \end{cases}$ **28.** $\begin{cases} x' = 3z + 2y - (e^{y-z} - 1)^3 \\ y' = x - (x - y + z)^{1/3} \\ z' = x\cos y + \ln(1+z)^3 - z^2 \end{cases}$

29. $\begin{cases} x' = z + \sqrt[4]{1 - x + 2y + z} - 1 \\ y' = x + 0,5y - \sin(y - z) \\ z' = y + \ln(1 - x - z)^2 \end{cases}$ **30.** $\begin{cases} x' = -x + 2\sin(x - 2y + 3z) \\ y' = 0,5x + y - (\cos z - 1)^3 \\ z' = x - 3y + \ln(1+z)^{2/3} - x^3 \end{cases}$

III. Ushbu $\boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x})$ differensial tenglamalar sistemasi va uning $\boldsymbol{x} = \boldsymbol{\varphi}(t)$, $t \geq 0$, yechimi berilgan bo‘lsin. Bu yechimni turg‘unlikka tekshirish algoritmini (yo‘lini) keltiring.

18. DIFFERENSIAL TENGLAMALAR YECHIMLARINI QATORLAR YORDAMIDA QURISH

Maqsad – differensial tenglamalar yechimlarini darajali va umumlashgan darajali qatorlar yordamida qurishni o‘rganish

Yordamchi ma’lumotlar:

I. Analitik koeffitsientli differensial tenglamani uning regular nuqtasi atrofida yechish.

Differensial tenglamada qatnashgan funksiyalarning barchasi analitik bo‘lganda, uning yechimlarni darajali qator yig‘indisi sifatida topish mumkin bo‘ladi. Shu munosabat bilan dastlab matematik analiz kursidan ma’lum bo‘lgan analitik funksiya ta’rifi va uning ba’zi xossalarini eslaylik.

Agar bir o‘zgaruvchining $y=f(x)$ funksiyasi x_0 nuqtaning biror atrofida biror $\sum_{n=0}^{+\infty} a_n(x-x_0)^n$ (x_0 markazli) darajali qatorning yig‘indisi sifatida tasvirlansa, ya’ni

$$f(x)=\sum_{n=0}^{+\infty} a_n(x-x_0)^n,\ |x-x_0|<\delta\ \ (\delta>0), \tag{1}$$

yoyilma (tenglik) o‘rinli bo‘lsa, u holda $y=f(x)$ funksiya x_0 nuqtada analitik funksiya deyiladi. Analizdan ma’lumki, masalan, $\sin x, \cos x, e^x$ funksiyalari ixtiyoriy $x_0\in(-\infty;+\infty)$, $\ln x$ funksiya esa ixtioriy $x_0>0$ nuqtada analitik.

Ushbu $\sum_{n=0}^{+\infty} a_n(x-x_0)^n$ darajali qatorning R yaqinlashish radiusi uchun, ma’lumki, $\frac{1}{R}=\overline{\lim_{n\to+\infty}}\sqrt[n]{a_n}$ Koshi formulasi o‘rinli. Yaqinlashish radiusi qator yaqinlashadigan eng katta $|x-x_0|<R\ (R>0)$ itervalni aniqlaydi, $R=+\infty$ bo‘lganda darajali qator $(-\infty,+\infty)$ oraliqda yaqinlashuvchi bo‘ladi. (1) formula aslida $|x-x_0|<R$ yaqinlashish intervalida o‘rinli bo‘ladi. (1) dagi darajali

qatorni xohlagancha marta hadma-had differensiallash mumkin; bunda qatorning R yaqinlashish radiusi o‘zgarmaydi va

$$f^{(k)}(x)=\sum_{n=k}^{+\infty} n(n-1)...(n-k+1)a_n(x-x_0)^{n-k},\ |x-x_0|<R,\ k\in\mathbb{N},$$

formulalar o‘rinli bo‘ladi. x_0 nuqtada analitik funksiya (1) darajali qatorning yaqinlashish intervalida, ya’ni bu intervalning har bir nuqtasida analitik bo‘ladi. (1) formuladagi darajali qator koeffitsientlari uchun

$$a_n=\frac{f^{(n)}(x_0)}{n!},\ n=0,1,2,...\ (0!=1;\ 1!=1;\ n!=1\cdot 2\cdot ...\cdot n,\ n\in\mathbb{N}),$$

formulalar o‘rinli, ya’ni analitik funksiya o‘zining Teylor qatori yig‘indisidan iborat.

Agar

$$f(x)=\sum_{n=0}^{+\infty} a_n(x-x_0)^n=\sum_{n=0}^{+\infty}\tilde{a}_n(x-x_0)^n,\ |x-x_0|<\delta\ \ (\delta>0),$$

bo‘lsa, bu darajali qatorlarning mos koeffitsientlari bir xil, ya’ni $a_n=\tilde{a}_n\ (n=0,1,2,...)$ (yagonalik xossasi).

Analitik

$$f(x)=\sum_{n=0}^{+\infty} a_n(x-x_0)^n, |x-x_0|<R_1, \text{ va } g(x)=\sum_{n=0}^{+\infty} b_n(x-x_0)^n, |x-x_0|<R_2,$$

funksiyalar berilgan bo‘lsin. Ularning yig‘indisi va ayirmasi ham analitik funksiya va

$$f(x)\pm g(x)=\sum_{n=0}^{+\infty}(a_n\pm b_n)(x-x_0)^n,\ |x-x_0|<\min(R_1,R_2),$$

yoyilma o‘rinli; ko‘paytma funksiya ham analitik va uning darajali qatorga yoyilmasi berilgan qatorlarni formal ravishda ko‘paytirishdan hosil bo‘ladi:

$$f(x)g(x)=\left(\sum_{n=0}^{+\infty} a_n(x-x_0)^n\right)\cdot\left(\sum_{n=0}^{+\infty} b_n(x-x_0)^n\right)=\sum_{n=0}^{+\infty} c_n(x-x_0)^n,$$

$$c_n=\sum_{k=0}^{n} a_k b_{n-k}=a_0b_n+a_1b_{n-1}+...+a_nb_0,\ |x-x_0|<\min(R_1,R_2);$$

bundan tashqari, qo‘shimcha ravishda $g(x_0)\neq 0$ ham bo‘lsa, u holda $f(x)/g(x)$ nisbat x_0 nuqtada analitik funksiya bo‘ladi va

$$\frac{f(x)}{g(x)}=\sum_{n=0}^{+\infty}d_n(x-x_0)^n\,,\ |x-x_0|<\delta,$$

($\delta<\min(R_1,R_2)$ bo‘lishi mumkin) yoyilmaning d_n koeffitsiyentlari

$$\sum_{n=0}^{+\infty}a_n(x-x_0)^n=\left(\sum_{n=0}^{+\infty}b_n(x-x_0)^n\right)\cdot\left(\sum_{n=0}^{+\infty}d_n(x-x_0)^n\right)=\sum_{n=0}^{+\infty}\Big(\sum_{k=0}^{n}b_kd_{n-k}\Big)(x-x_0)^n,$$

tenglikdagi $(x-x_0)$ ning bir xil darajalari oldidagi koeffitsientlarni tenglashtirish yordamida topilishi mumkin. Analitik funksiyalar kompozitsiyasi ham analitik bo‘ladi. Differensial tenglama yechimini darajali qator yig‘indisi sifatida topish, ya’ni analitik yechimni qurish algoritmi (jarayoni) bilan ikkinchi tartibli chiziqli differensial tenglama misolida tanishamiz. Birinchi yoki yuqori tartibli chiziqli tenglama yoki chiziqli tenglamalar sistemasi ychun ham analitik yechimni topish jarayoni shunga o‘xshash amalga oshiriladi. Analitik yechimni topish uchun tenglama(lar)dagi barcha koeffitsientlar analitik bo‘lishi kerak.

Aniqrog‘i, ushbu

$$y''+p_1(x)y'+p_0(x)y=q(x) \tag{2}$$

tenglamani qaraylik. Quyidagi teorema o‘rinli.

Teorema. Faraz qilaylik, $p_1(x), p_0(x)$ va $q(x)$ funksiyalar (x_0-R, x_0+R) intervalda analitik bo‘lsin. U holda har qanday a_0 va a_1 sonlar uchun ushbu

$$y''+p_1(x)y'+p_0(x)y=q(x),\ y(x_0)=a_0\,,\ y'(x_0)=a_1,$$

Koshi masalasi (x_0-R, x_0+R) intervalda aniqlangan yagona analitik yechimga ega.

Analitik yechimni qurish uchun uni x_0 markazli hozircha noma’lum koeffitsientli darajali qator yig‘indisi sifatida yozamiz:

$$y=\sum_{n=0}^{+\infty}a_n(x-x_0)^n,\ |x-x_0|<R_1\ \ (R_1>0), \tag{3}$$

bu yerda $a_n\,(n=0,1,2,...)$ – noma'lum koeffitsientlar. Bu koeffitsientlarni topishning ikki usuli mavjud.

Noaniq koeffitsientlar usuli. (3) yoyilmani va tenglama koeffitsiyentlarining yoyilmalarini (2) tenglamaga qo'yib, kerakli ixchamlashtirishlarni bajarib, $(x-x_0)$ning bir xil darajalari oldidagi koeffitsiyentlarini tenglashtirib, hosil bo'lgan cheksiz sistemadan $a_n\,(n=0,1,2,...)$ koeffitsiyentlarni ketma-ket aniqlaymiz. Aniqrog'i, quyidagicha ish tutamiz.

(3) qatorni hadma-had ikki marta differensiallaymiz:

$$y'=\sum_{n=1}^{+\infty}na_n(x-x_0)^{n-1}=\sum_{n=0}^{+\infty}(n+1)a_{n+1}(x-x_0)^n,\ |x-x_0|<R_1, \tag{4}$$

$$y''=\sum_{n=1}^{+\infty}n(n+1)a_{n+1}(x-x_0)^{n-1}=\sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}(x-x_0)^n,\ |x-x_0|<R_1. \tag{5}$$

Tenglamada berilgan koeffitsientlarni ham x_0 markazli darajali qatorlarga yoyamiz:

$$p_1(x)=\sum_{n=0}^{+\infty}b_n(x-x_0)^n,\ |x-x_0|<R, \tag{6}$$

$$p_0(x)=\sum_{n=0}^{+\infty}c_n(x-x_0)^n,\ |x-x_0|<R, \tag{7}$$

$$q(x)=\sum_{n=0}^{+\infty}d_n(x-x_0)^n,\ |x-x_0|<R, \tag{8}$$

(bu yerda $p_1(x),p_0(x),q(x)$ uchun darajali qatorlar yaqinlashish radiuslarining eng kichigini R bilan belgiladik). Bu qatorlarni qaralayotgan differensial tenglamaga qo'yib, zarur soddalashtirishlarni bajaramiz:

$$\sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}(x-x_0)^n+\sum_{n=0}^{+\infty}b_n(x-x_0)^n\cdot\sum_{n=0}^{+\infty}(n+1)a_{n+1}(x-x_0)^n+$$

$$+\sum_{n=0}^{+\infty} c_n (x-x_0)^n \cdot \sum_{n=0}^{+\infty} a_n (x-x_0)^n = \sum_{n=0}^{+\infty} d_n (x-x_0)^n;$$

$$\sum_{n=0}^{+\infty}\Big((n+1)(n+2)a_{n+2} + \sum_{k=0}^{n}\big((k+1)a_{k+1}b_{n-k} + a_k c_{n-k}\big)\Big)(x-x_0)^n = \sum_{n=0}^{+\infty} d_n (x-x_0)^n .$$

Bu tenglikning chap va o‘ng tomonidagi $(x-x_0)$ ning bir xil darajalari oldidagi koeffitsientlarni tenglashtirib, $a_n\,(n=0,1,2,...)$ noma’lumlarni topish uchun cheksiz chiziqli algebraik sistemani hosil qilamiz:

$$(x-x_0)^0:\ 1\cdot 2\cdot a_2 + a_1 b_0 + a_0 c_0 = d_0 ,$$

$$(x-x_0)^1:\ 2\cdot 3\cdot a_3 + a_1 b_1 + a_0 c_1 + 2a_2 b_0 + a_1 c_0 = d_1 ,$$

....

$$(x-x_0)^n:\ (n+1)(n+2)a_{n+2} + \sum_{k=0}^{n}\big((k+1)a_{k+1}b_{n-k} + a_k c_{n-k}\big) = d_n ,$$

....

Berilgan ixtiyoriy $a_0 = y(x_0)$ va $a_1 = y'(x_0)$ qiymatlarga ko‘ra bu yerdagi birinchi tenglamadan a_2, ikkinchisidan a_3 va h.k. barcha qolgan a_n lar bir qiymatli aniqlanadi.

Ba’zi hollarda a_n ni a_0, a_1 va n ning bevosita oshkor funksiyasi sifatida topish mumkin bo‘ladi.Topilgan a_n larga ko‘ra (3) darajali qatorning yaqinlashish radiusini hisoblaymiz va qurilgan analitik funksiya yaqinlashish intervalida (2) differensial tenglamaning yechimi ekanligini bevosita asoslaymiz. Umumiy nazariyada isbotlanganiga ko‘ra, $R_1 = R$ bo‘lishi, ya’ni qurilgan (3) funksiya (2) tenglamada qatnashgan analitik funksiyalarining umumiy analitiklik intervalida shu tenglamaning (analitik) yechimi bo‘lishi kerak. Shunday qilib, (2) tenglamaning a_0, a_1 ikki parametrli analitik yechimlar oilasini hosil qilamiz.

Ketma-ket differensiallash usuli. Yuqoridagi (2) tenglamaning (3) yechimidagi $a_n\,(n=0,1,2,...)$ koeffitsientlarni

$$a_n = \frac{y^{(n)}(x_0)}{n!} \quad (n = 0,1,2,...)$$

formulalarga ko‘ra topish ham mumkin. Bunda $a_0 = y(x_0)$, $a_1 = y'(x_0)$; qolgan koeffitsientlar esa (2) tenglamadan aniqlanadi. $y = y(x)$ yechim bo‘lgani uchun, u x_0 nuqtaning biror atrofida (3) tenglamani ayniyatga aylantiradi:

$$y''(x) + p_1(x)y'(x) + p_0(x)y(x) = q(x)\,. \tag{13}$$

Bu ayniyatda $x = x_0$ deb $y''(x_0)$ni va, demak, $a_2 = \frac{y''(x_0)}{2!}$ ni topamiz. (13) ayniyatni ketma-ket differensiallab va $x = x_0$ deb, $y'''(x_0), y^{IV}(x_0),...$ hamda $a_3, a_4,...$larni hisoblaymiz:

$$y'''(x) = q'(x) - p_1'(x)y'(x) - p_1(x)y''(x) - p_0'(x)y(x) - p_0(x)y'(x)\,,$$

bundan $y'''(x_0)$ va $a_3 = \frac{y'''(x_0)}{3!}$ qiymat;

$$y^{IV}(x) = \left(q'(x) - p_1'(x)y'(x) - p_1(x)y''(x) - p_0'(x)y(x) - p_0(x)y'(x)\right)' = q''(x) - ...\ ,$$

bundan $y^{IV}(x_0)$ va $a_4 = \frac{y^{IV}(x_0)}{4!}$ qiymat ;

....

topiladi.

Misol 1. Ushbu

$$\begin{cases} y'' + \sin x \cdot y' + y = 0 \\ y(0) = a_0, y'(0) = a_1 \end{cases} \tag{14}$$

boshlang‘ich masalasining $x_0 = 0$ markazli darajali qator ko‘rinishidagi yechimini quring.

⊶ Tenglamaning $p_1(x) = \sin x$, $p_0(x) = 1$ va $q(x) = 0$ koeffitsientlari $(-\infty, +\infty)$ oralig‘ida analitik $(R = +\infty)$ va

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...\ , -\infty < x < +\infty, \tag{15}$$

yoyilma o‘rinli. Berilgan masalaning

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots, \ |x| < R_1, \tag{16}$$

yechimini noma’lum koeffitsientlar usuli yordamida topamiz. Yuqorida keltirilgan teoremaga ko‘ra $R_1 = R = +\infty$ bo‘ladi. (16) yoyilmani hadma-had ikki marta differensiallaymiz:

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots, \tag{17}$$

$$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \dots, \tag{18}$$

Endi (15) va (18) yoyilmalarni berilgan tenglamaga qo‘yamiz:

$$2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \dots + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) \times$$

$$\times\left(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots\right) +$$

$$+ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots = 0\,.$$

Bu yerda qavslarni ochib, o‘xshash hadlarni ixchamlaymiz:

$$(2a_2 + a_0) + (6a_3 + 2a_1)x + (12a_4 + 3a_2)x^2 + (20a_5 + 4a_3 - \frac{a_1}{6})x^3 + \dots = 0.$$

Koeffitsientlarni nolga tenglashtirib topamiz:

$$a_2 = -\frac{a_0}{2}, \ a_3 = -\frac{a_1}{3}, \ a_4 = -\frac{a_2}{4} = \frac{a_0}{8},$$

$$a_5 = -\frac{a_3}{5} + \frac{a_1}{120} = \frac{a_1}{15} + \frac{a_1}{120} = \frac{3a_1}{40}, \dots .$$

Bu qiymatlarni (16) ga qo‘yib, berilgan tenglama yechimining beshinchi darajali hadlarigacha yoyilmasini hosil qilamiz:

$$y = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \dots\right) +$$

$$+ a_1\left(x - \frac{1}{3}x^3 + \frac{3}{40}x^5 + \dots\right), \ -\infty < x < +\infty\ . ☝$$

Misol 2. Ushbu

$$y'' + xy' - \frac{2}{x}y = e^x, y(1) = 1, \ y'(1) = 1, \tag{19}$$

masala yechimi uchun $(x-1)$ ning darajalari bo‘ylab yoyilma qaysi intervalda yaqinlashuvchi? Bu yoyilmani $(x-1)$ ning to‘rtinchi darajasiga qadar hadlarini toping.

🗝 Ravshanki, berilgan tenglama koeffitsientlari uchun

$$p_1(x) = x = 1 + 1\cdot(x-1),\ -\infty < x < +\infty,$$

$$p_0(x) = -\frac{2}{x} = -2\cdot\frac{1}{1+(x-1)} = -2(1-(x-1)+(x-1)^2 - \ldots),\ |x-1|<1,$$

$$q(x) = e^x = e\cdot e^{x-1} = e\left(1+\frac{(x-1)}{1!}+\frac{(x-1)^2}{2!}+\ldots\right),\ -\infty < x < +\infty,$$

Demak, bu darajali qatorlar uchun ymumiy yaqinlashish radiusi $R=1$. Yuqorida keltirilgan teoremaga ko‘ra berilgan masalaning

$$y = \sum_{n=0}^{+\infty} a_n (x-1)^n,\ |x-1| < R_1, \tag{20}$$

yechimi uchun ham yaqinlashish radiusi $R_1 = R = 1$, yaqinlashish intervali esa $(0;2)$ bo‘ladi. Endi bu yoyilma koeffitsientlarinining beshtasini

$$a_n = \frac{y^{(n)}(x_0)}{n!}\ \ (n=0,1,2,\ldots),\ x_0 = 1,$$

formulaga ko‘ra ketma-ket differensiallash usuli yordamida topamiz. Berilgan boshlang‘ich shartlarga ko‘ra

$$a_0 = y(1) = 1,\ a_1 = y'(1) = 1. \tag{21}$$

Berilgan tenglamadan ushbu

$$y'' = e^x - xy' + \frac{2}{x}y. \tag{22}$$

tenglikni topamiz. Bu tenglikda $x=1$ deb va $y(1)=1$, $y'(1)=1$ berilgan qiymat-larni hisobga olib,

$$y''(1) = (e^x - xy' + \frac{2}{x}y)\big|_{x=1} = e-1+2 = e+1,\ a_2 = \frac{y''(1)}{2!} = \frac{e+1}{2} \tag{23}$$

ekanligini topamiz. (22) tenglikni ketma-ket ikki marta differensiallaymiz:

$$y''' = e^x - xy'' - y' - \frac{2}{x^2}y + \frac{2}{x}y' \quad ; \qquad . \tag{24}$$

$$y^{IV} = xe^x + (x^2 - 1 + \frac{2}{x})y'' + (2x - \frac{2}{x^2})y' - 2(1 + \frac{1}{x^2})y' + \frac{4}{x^3}y \ . \tag{25}$$

Endi (24), (25) va (23) formulalardan foydalanib a_3 va a_4 koeffitsientlarni topamiz

$$y'''(1) = \left(e^x - xy'' - y' - \frac{2}{x^2}y + \frac{2}{x}y'\right)\Big|_{x=1} = -2\,,\ a_3 = \frac{y'''(1)}{3!} = -\frac{1}{3}\ ; \tag{26}$$

$$y^{IV}(1) = e + 2,\ a_4 = \frac{y^{IV}(1)}{4!} = \frac{e+2}{24}\ . \tag{27}$$

Koeffitsientlar qiymatlarini (27), (26), (23) va (21) tengliklardan olib (20) ga qo‘yamiz va berilgan masala yechimi uchun yoyilmaning izlangan qismini hosil qilamiz:

$$y = 1 + 1\cdot(x-1) + \frac{e+1}{2}(x-1)^2 - \frac{1}{3}(x-1)^3 + \frac{e+2}{24}(x-1)^4 + \ldots\ ,\ 0 < x < 2\ .$$

Endi nochiziqli differensial tenglama holida yechimning analitikligi haqi-dagi teoremada to‘xtalaylik. Ushbu

$$\begin{cases} y'' = f(x, y, y'), \\ y(x_0) = a_0\,,\ y'(x_0) = a_1 \end{cases}$$

Koshi masalasi berilgan bo‘lsin.

Teorema. Faraz qilaylik, $f(x, y, y')$ funksiya

$$\Pi = \{(x, y, y') \,\big|\, |x - x_0| < r\,, |y - a_0| < r, |y' - y_1| < r\}$$

to‘plam (parallelepiped) da analitik va shu to‘plamda $\left|f(x, y, y')\right| \le M$ $(M = \text{const} > 0)$ bo‘lsin. U holda berilgan Koshi masalasi x_0 nuqtada analitik bo‘lgan yagona

$$y = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!}(x - x_0)^n$$

yechimga ega. Bu qatorning yaqinlashish radiusi $R \ge \min\left\{r, \dfrac{r}{4M}\right\}$ bo‘ladi.

II. Differensial tenglamani uning regular maxsus nuqtasi atrofida yechish. Frobenius metodi.

Endi (2) tenglamaga qaraganda umumiyroq

$$p_2(x)y'' + p_1(x)y' + p_0(x)y = q(x) \tag{28}$$

differensial tenglamani qaraylik. Bu yerdagi $p_2(x),\ p_1(x),\ p_0(x),\ q(x)$ funksiyalar x_0 nuqtaning biror atrofida analitik deb hisoblanadi. Agar $p_2(x_0)\neq 0,$ ya'ni x_0 – tenglamaning regular nuqtasi bo'lsa, u holda (28) tenglama x_0 nuqtaning yetarlicha kichik atrofida ushbu

$$y'' + \frac{p_1(x)}{p_2(x)}y' + \frac{p_0(x)}{p_2(x)}y = \frac{q(x)}{p_2(x)}$$

analitik koeffitsientli tenglamaga ekvivalent. Oxirgi tenglamaning x_0 nuqtada analitik yechimini topish bilan yuqorida tanishdik.

Endi faraz qilaylik, $p_2(x_0)=0$, ya'ni x_0 – (28) tenglamaning maxsus nuqtasi bo'lsin. Bu holda (28) tenglama, umumiy holda, x_0 nuqtada analitik yechimga ega bo'lmasligi mumkin. Lekin ba'zi hollarda yechimni umumlashgan darajali qator yig'ndisi ko'rinishida ifodalasa bo'ladi.

Ushbu

$$(x-x_0)^2y'' + (x-x_0)\tilde{p}_1(x)y' + \tilde{p}_0(x)y = 0$$

yoki

$$y'' + \frac{\tilde{p}_1(x)}{x-x_0}y' + \frac{\tilde{p}_0(x)}{(x-x_0)^2}y = 0 \tag{29}$$

tenglamani qaraylik; bu yerda $\tilde{p}_1(x), \tilde{p}_0(x)$ – x_0 nuqtada analitik funksiyalar. Bu holda x_0 nuqta (29) tenglama uchun ***regular maxsus nuqta*** deyiladi. Bundan keyin qisqalik uchun $x_0=0$ deb hisoblaymiz. Har doim $s=x-x_0$ almashtirish yordamida x_0 nuqtani $s=0$ nuqtaga o'tkazish mumkin.

Teorema. Agar $x_0=0$ nuqta (29) tenglama uchun regular maxsus nuqta bo'lsa, u holda (29) tenglama

$$y = x^{\mu}\sum_{n=0}^{+\infty} a_n x^n \quad (y = y(x) = y(x,\mu)) \tag{30}$$

($\mu, a_n\,(n = 0,1,2,...)$ – o‘zgarmas sonlar) ko‘rinishdagi kamida bitta yechimga ega; bu umumlashgan darajali qator biror $x \in (0,\rho)\,(\rho > 0)$ intervalda yaqinlashuvchi bo‘ladi.

Teoremada aytilgan $\mu, a_n\,(n = 0,1,2,...)$ sonlarni topish uchun, Frobenius metodiga ko‘ra, ushbu ($x > 0$)

$$\tilde{p}_1(x) = \sum_{n=0}^{+\infty} b_n x^n\,,\quad \tilde{p}_0(x) = \sum_{n=0}^{+\infty} c_n x^n\;;\quad y = x^{\mu}\sum_{n=0}^{+\infty} a_n x^n = \sum_{n=0}^{+\infty} a_n x^{n+\mu}\,,$$

$$y' = \sum_{n=0}^{+\infty}(n+\mu)a_n x^{n+\mu-1}\,,\quad y'' = \sum_{n=0}^{+\infty}(n+\mu)(n+\mu-1)a_n x^{n+\mu-2}\;;$$

$$\tilde{p}_0(x)y = \sum_{n=0}^{+\infty} c_n x^n \cdot \sum_{n=0}^{+\infty} a_n x^{n+\mu} = \sum_{n=0}^{+\infty}\sum_{k=0}^{n} a_k c_{n-k} x^{n+\mu};$$

$$x\tilde{p}_1(x)y' = \sum_{n=0}^{+\infty} b_n x^n \cdot \sum_{n=0}^{+\infty}(n+\mu)a_n x^{n+\mu} = \sum_{n=0}^{+\infty}\sum_{k=0}^{n}(k+\mu)a_k b_{n-k} x^{n+\mu}\;;$$

$$x^2 y'' = \sum_{n=0}^{+\infty}(n+\mu)(n+\mu-1)a_n x^{n+\mu}\;;$$

yoyilmalarni (29) tenglamaga qo‘yib, uni quyidagi ko‘rinishga keltiramiz:

$$\sum_{n=0}^{+\infty}\Big((n+\mu)(n+\mu-1)a_n + \sum_{k=0}^{n} a_k\big((k+\mu)b_{n-k} + c_{n-k}\big)\Big)x^{n+\mu} = 0$$

yoki

$$a_0\big(\mu(\mu-1) + \mu b_0 + c_0\big)x^{\mu} + \sum_{n=1}^{+\infty}\Big((a_n(n+\mu)(n+\mu-1) + (n+\mu)b_0 + c_0) +$$

$$+\sum_{k=0}^{n-1} a_k\big((k+\mu)b_{n-k} + c_{n-k}\big)\Big)x^{n+\mu} = 0$$

yoki yana qisqaroq

$$a_0 A(\mu)x^{\mu} + \sum_{n=1}^{+\infty}\Big(a_n A(n+\mu) + \sum_{k=0}^{n-1} a_k\big((k+\mu)b_{n-k} + c_{n-k}\big)\Big)x^{n+\mu} = 0\,; \tag{31}$$

bu yerda

$$A(\mu)=\mu(\mu-1)+\mu b_0+c_0\,. \qquad (32)$$

(31) tenglik ayniyat bo‘lishi uchun x ning darajalari oldidagi koeffitsientlar nolga teng bo‘lishi kerak:

$$a_0A(\mu)=0\ ,$$

$$a_nA(n+\mu)+\sum_{k=0}^{n-1}a_k\left((k+\mu)b_{n-k}+c_{n-k}\right)=0\ \ (n\geq 1)\,. \qquad (33)$$

Bu yerdagi birinchi tenglikdan μ ga nisbatan kvadrat tenglama hosil qilamiz (a_0 - ixtiyoriy noldan farqli son):

$$A(\mu)=0$$

yoki (32) ga ko‘ra

$$\mu(\mu-1)+\mu b_0+c_0=0\,. \qquad (34)$$

Bu kvadrat tenglama ***aniqlovchi tenglama*** deyiladi. Agar berilgan μ uchun $A(1+\mu)\neq 0,\ A(2+\mu)\neq 0,...,A(n+\mu)\neq 0,...$ bo‘lsa, u holda (33) tenglamadan tayinlangan a_0 ga ko‘ra rekurrent usulda barcha $a_1=a_1(\mu),\ \ a_2=a_2(\mu),...,$ $a_n=a_n(\mu),...$koeffitsientlarni bir qiymatli topamiz.

Aniqlovchi tenglamaning μ_1,μ_2 ***ildizlari haqiqiy bo‘lsin.*** Aniqlik uchun $\mu_1\geq\mu_2$ deylik. Demak, $A(\mu_1)=0$ va $\mu>\mu_1$ bo‘lganda $A(\mu)\neq 0$. Shuning uchun (33) rekurrent munosabatdan $a_0=1$ deb, barcha $a_n=a_n(\mu_1)$ $(n\geq 1)$ koeffitsientlarni bir qiymatli aniqlaymiz. Shunday qilib, bu holda (29) tenglamaning bir dona

$$y_1(x)=x^{\mu_1}\sum_{n=0}^{+\infty}a_n(\mu_1)x^n=x^{\mu_1}(1+\sum_{n=1}^{+\infty}a_n(\mu_1)x^n) \qquad (35)$$

yechimini hosil qilamiz. (29) tenglamaning umumiy yechimini topish uchun uning $y_1(x)$ (35) yechimga chiziqli bog‘liq bo‘lmagan yana bir $y_2(x)$ yechimini topishimiz kerak. Bu $y_2(x)$ yechimni qurish usuli aniqlovchi tenglamaning μ_1,μ_2 ildizlariga bog‘liq. Quyidagi hollar bo‘lishi mumkin.

1^0. $\mu_1-\mu_2$ *ayirma butun son bo'lmasin*. U holda, ravshanki, $A(1+\mu_2)\neq 0,\ A(2+\mu_2)\neq 0, ..., A(n+\mu_2)\neq 0,...$. Endi $a_0=1$ deb, (33) rekurrent munosabatdan $\mu=\mu_2$ uchun barcha $a_n=a_n(\mu_2)$ $(n\geq 1)$ koeffitsientlarni bir qiymatli aniqlaymiz. Demak, $y_1(x)$ ga chiziqli bog'liq bo'lmagan $y_2(x)$ yechim sifatida quyidagi funksiyani olish mumkin:

$$y_2(x)=x^{\mu_2}\sum_{n=0}^{+\infty}a_n(\mu_2)x^n=x^{\mu_2}\left(1+\sum_{n=1}^{+\infty}a_n(\mu_2)x^n\right). \tag{36}$$

2^0. $\mu_1=\mu_2$ *bo'lsin*. Birinchi $y_1(x)$ (35) yechim bizga ma'lum. Ikkinchi, undan chiziqli erkli $y_2(x)$ yechim

$$y_2(x)=y_1(x)\ln x+x^{\mu_1}\sum_{n=0}^{+\infty}a_n'(\mu_1)x^n \tag{37}$$

ko'rinishga ega bo'ladi. Bu yerdagi $a_n'(\mu_1)$ larni noma'lum koefftsientlar metodi yordamida, ya'ni (37) ni (29) tenglamaga qo'yib, tenglamaning qanoatlanishi shartidan aniqlash mumkin.

3^0. $\mu_1-\mu_2$ *ayirma natural sondan iborat bo'lsin*. Bu holda ikkinchi $y_2(x)$ chiziqli erkli yechim

$$y_2(x)=\tilde{a}_{-1}y_1(x)\ln x+x^{\mu_2}\sum_{n=0}^{+\infty}\tilde{a}_n(\mu_2)x^n \tag{38}$$

ko'rinishda bo'ladi. Bu $y_2(x)$ yechimni qurish uchun dastlab Frobenius metodidan foydalanib, μ_2 ga ko'ra (29) tenglamaning $y_2(x)$ yechimini qurish kerak. Agar qurilgan $y_2(x)$ yechim $y_1(x)$ (35) yechimga chiziqli bog'liq bo'lmasa, u (38) ko'rinishda bo'ladi (bunda $\tilde{a}_{-1}=0$). Aks holda, ya'ni bu $y_2(x)$ va $y_1(x)$ yechimlar chiziqli bog'liq bo'lsa, $y_1(x)$ yechimga chiziqli bog'liq bo'lmagan yechim sifatida

$$y_2(x)=\frac{\partial}{\partial\mu}\big((\mu-\mu_2)y(x,\mu)\big)\Big|_{\mu=\mu_2} \tag{39}$$

funksiyani olish kerak. $y_2(x)$ (38) yechimni to‘g‘ridan to‘g‘ri noma’lum koefftsientlar metodi yordamida ham qurish mumkin.

Agar aniqlovchi tenglamaning μ_1, μ_2 ***ildizlari kompleks sonlardan iborat bo‘lsa***, u holda $\mu_2 = \bar{\mu}_1$ boladi (tenglamadagi koeffitsienlarning haqiqiyligi uchun) va $e^{\alpha \pm i\beta} = e^{\alpha}(\cos\beta \pm i\sin\beta)$ Eyler formlalaridan foydalanib chiziqli erkli haqiqiy $y_1(x), y_2(x)$ yechimlarni qurish mumkin.

Misol 3. Ushbu

$$9x^2 y'' + 2(1+x)y = 0$$

differensial tenglamaning umumiy yechimini qatorlar yordamida quring.

Tushunarliki, $x=0$ nuqta bu differensial tenlamaning regular maxsus nuqtasi:

$$y'' + \frac{2(1+x)}{9x^2} y = 0; \quad \tilde{p}_1(x) = 0, \ \tilde{p}_0(x) = \frac{2(1+x)}{9} -$$ koeffitsientlar $x=0$ nuqtada analitik. Qaralayotgan holda $b_n = 0$, $c_0 = c_1 = 2/9$, $c_2 = c_3 = ... = 0$. Yechimni $y = x^{\mu}\sum_{n=0}^{+\infty} a_n x^n$ (30) umumlashgan darajali qator ko‘rinishida izlaymiz. Buni berilgan tenglamaga qo‘yib, noma’lum μ va a_n lar uchun (34) va (33) tenglamalarning quyidagi ko‘rinishini va yechimlarini hosil qilamiz:

$$9x^2 y'' + 2(1+x)y = 9x^2 \sum_{n=0}^{+\infty}(n+\mu)(n+\mu-1)a_n x^{n+\mu-2} + 2(1+x)\sum_{n=0}^{+\infty} a_n x^{n+\mu} =$$

$$= \sum_{n=0}^{+\infty}\big(9(n+\mu)(n+\mu-1)+2\big)a_n x^{n+\mu} + \sum_{m=1}^{+\infty} 2a_{m-1}x^{m+\mu} =$$

$$= \big(9\mu(\mu-1)+2\big)a_0 x^{\mu} + \sum_{n=1}^{+\infty}\Big(\big(9(n+\mu)(n+\mu-1)+2\big)a_n + 2a_{n-1}\Big)x^{n+\mu} = 0$$

Bu yerdan aniqlovchi tenglama: $9\mu(\mu-1)+2=0$; yechimlari: $\mu_1 = \dfrac{2}{3}$, $\mu_2 = \dfrac{1}{3}$.

Ushbu $9(n+\mu)(n+\mu-1)+2\big)a_n + 2a_{n-1} = 0$ tenglikni $9\mu(\mu-1)+2=0$ bo‘lganligi uchun

$$9n(n+2\mu-1)a_n+2a_{n-1}=0$$

ko‘rinishda yozish mumkin. Oxirgi tenglikdan

$$a_n=-\frac{2}{9n(n+2\mu-1)}a_{n-1}\ ,\ n\geq 1\ ,$$

rekurrent formulani topamiz. Endi ikki holni qaraymiz.

1) $\mu=\mu_1=\dfrac{2}{3}$ bo‘lsin. U holda

$$a_n=-\frac{2}{9n(n+2\mu-1)}a_{n-1}=-\frac{2}{3n(3n+1)}a_{n-1}\ ,\ n\geq 1\ .$$

Demak ($a_0=1$ deb qabul qildik),

$$a_n=\prod_{k=1}^{n}\left(-\frac{2}{3k(3k+1)}\right)=\left(-\frac{2}{3}\right)^n\frac{1}{n!}\prod_{k=1}^{n}\frac{1}{3k+1}\ ,\ n\geq 1\ ,$$

va $$y_1=x^{\mu}\sum_{n=0}^{+\infty}a_nx^n=x^{2/3}\sum_{n=0}^{+\infty}\left(-\frac{2}{3}\right)^n\frac{1}{n!}\prod_{k=1}^{n}\frac{1}{3k+1}x^n$$

ko‘rinishdagi birinchi yechimni topamiz. Bu yerdagi qator $|x|<+\infty$ bo‘lganda yaqinlashuvchi.

2) $\mu=\mu_2=\dfrac{1}{3}$ bo‘lsin. Bu holda

$$a_n=-\frac{2}{9n(n+2\mu-1)}a_{n-1}=-\frac{2}{3n(3n-1)}a_{n-1}\ ,\ n\geq 1\ .$$

Bundan

$$a_n=\prod_{k=1}^{n}\left(-\frac{2}{3k(3k-1)}\right)=\left(-\frac{2}{3}\right)^n\frac{1}{n!}\prod_{k=1}^{n}\frac{1}{3k-1}\ ,\ n\geq 1\ ,$$

va

$$y_2=x^{\mu}\sum_{n=0}^{+\infty}a_nx^n=x^{1/3}\sum_{n=0}^{+\infty}\left(-\frac{2}{3}\right)^n\frac{1}{n!}\prod_{k=1}^{n}\frac{1}{3k-1}x^n$$

ko‘rinishdagi birinchi yechimga chiziqli bog‘liq bo‘lmagan ikkinchi yechimni topamiz. Demak, berilgan tenglamaning umumiy yechimi

$$y = c_1 y_1 + c_2 y_2 \quad (c_1, c_2 - \text{ const}) ,$$

ya’ni

$$y = c_1 x^{2/3} \sum_{n=0}^{+\infty} \left(-\frac{2}{3}\right)^n \frac{1}{n!} \prod_{k=1}^{n} \frac{1}{3k+1} x^n + c_2 x^{1/3} \sum_{n=0}^{+\infty} \left(-\frac{2}{3}\right)^n \frac{1}{n!} \prod_{k=1}^{n} \frac{1}{3k-1} x^n$$

formula bilan beriladi. 👍

Izoh. Bu yerda shuni e’tirof etaylikki, berilgan tenglamada $y(x) = \sqrt{x} z(x)$ almashtirishni bajarib, uni Bessel tenglamasiga keltirish va umumiy yechimni Bessel funksiyalari orqali

$$y(x) = c_1 \sqrt{x} J_{1/3}(2\sqrt{2}\sqrt{x} / \sqrt{3}) + c_2 \sqrt{x} Y_{1/3}(2\sqrt{2}\sqrt{x} / \sqrt{3})$$

ko‘rinishda yozish mumkin.

Misol 4. Ushbu

$$x^2 y'' + (x^2 - 2x) y' + 2y = 0$$

differensial tenglamanig chiziqli erkli yechimlarini qatorlar yordamida quring.

🗝 Ravshanki, $x = 0$ berilgan tenglama uchun regular maxsus nuqta. Tenglamani $x > 0$ oraliqda qaraymiz. Yechimni $y = x^{\mu} \sum_{n=0}^{+\infty} a_n x^n$ ko‘rinishidagi umumlashgan darajali qator sifatida izlaymiz. Buni berilgan tenglamaga qo‘yib, quyidagi shakl almashtirishlarni bajaramiz:

$$x^2 y'' + (x^2 - 2x) y' + 2y = x^2 \sum_{n=0}^{+\infty} (n+\mu)(n+\mu-1) a_n x^{n+\mu-2} +$$

$$+ (x^2 - 2x) \sum_{n=0}^{+\infty} (n+\mu) a_n x^{n+\mu-1} + 2 \sum_{n=0}^{+\infty} a_n x^{n+\mu} =$$

$$= \sum_{n=0}^{+\infty} ((n+\mu)(n+\mu-1) + 2 - 2(n+\mu)) a_n x^{n+\mu} + \sum_{n=1}^{+\infty} (n+\mu-1) a_{n-1} x^{n+\mu} =$$

$$= (\mu(\mu-3) + 2) a_0 x^{\mu} + \sum_{n=1}^{+\infty} ((n+\mu)(n+\mu-3) + 2) a_n x^{n+\mu} +$$

$$+ \sum_{n=1}^{+\infty} (n+\mu-1) a_{n-1} x^{n+\mu} = 0$$

Demak, aniqlovchi tenglama: $\mu(\mu-3)+2=0$, ildizlari: $\mu_1=2,\ \mu_2=1$ (3^0 bandga qarang). Noma'lum koeffitsientlar uchun tenglama

$$((n+\mu)(n+\mu-3)+2)a_n+(n+\mu-1)a_{n-1}=0.$$

$\mu=\mu_1=2$ uchun a_n koeffitsientlar

$$((n+2)(n-1)+2)a_n+(n+1)a_{n-1}=0,$$

ya'ni

$$na_n+a_{n-1}=0,\ \text{yoki}\ \ a_n=-\frac{a_{n-1}}{n}\ ,\ n=1,2,3,\dots\ ,$$

tenglamadan aniqlanadi. $a_0=1$ deb, oxirgi tenglamadan izlangan koeffitsientlarni topamiz:

$$a_n=\frac{(-1)^n}{n!}\ ,\ n=0,1,2,3,\dots\ .$$

Demak, bitta yechim

$$y_1(x)=x^2\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}x^n,\ \text{ya'ni}\quad y_1=x^2e^{-x}.$$

Ikkinchi chiziqli erkli yechim

$$y_2(x)=\tilde{a}_{-1}y_1(x)\ln x+x^{\mu_2}\sum_{n=0}^{+\infty}\tilde{a}_nx^n,\ \text{ya'ni}\ \ y_2=\tilde{a}_{-1}y_1(x)\ln x+\sum_{n=0}^{+\infty}\tilde{a}_nx^{n+1}$$

ko'rinishga ega. Buni berilgan tenglamaga qo'yamiz, $y_1(x)$ ning tenglama yechimi ekanligini hisobga olamiz hamda zarur shakl almashtirishlarni va ixchamlashtirishlarni bajaramiz:

$$x^2y_2''+(x^2-2x)y_2'+2y_2=\tilde{a}_{-1}\underbrace{x^2y_1''}\ln x+2\tilde{a}_{-1}xy_1'-\tilde{a}_{-1}y_1+\sum_{n=0}^{+\infty}(n+1)n\tilde{a}_nx^{n+1}+$$

$$+\tilde{a}_{-1}\underbrace{(x^2-2x)y_1'}\ln x+(x-2)\tilde{a}_{-1}y_1+(x-2)\sum_{n=0}^{+\infty}(n+1)\tilde{a}_nx^{n+1}+$$

$$+\tilde{a}_{-1}\underbrace{2y_1}\ln x+2\sum_{n=0}^{+\infty}\tilde{a}_nx^{n+1}=$$

$$=2\tilde{a}_{-1}xy_1'-3\tilde{a}_{-1}y_1+x\tilde{a}_{-1}y_1+\sum_{n=0}^{+\infty}(n+1)\tilde{a}_n x^{n+2}+\sum_{n=2}^{+\infty}n(n-1)\tilde{a}_n x^{n+1}=$$

$$=\sum_{n=0}^{+\infty}\left(\frac{(-1)^n(2n+1)}{n!}\tilde{a}_{-1}+(n+1)\tilde{a}_n\right)x^{n+2}+$$

$$+\tilde{a}_{-1}\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}x^{n+3}+\sum_{n=0}^{+\infty}(n+2)(n+1)\tilde{a}_{n+2}x^{n+3}=$$

$$=(\tilde{a}_{-1}+\tilde{a}_0)x^2+\sum_{n=1}^{+\infty}\left(\frac{(-1)^n(2n+1)}{n!}\tilde{a}_{-1}+(n+1)\tilde{a}_n\right)x^{n+2}+$$

$$+\sum_{n=0}^{+\infty}\left(\frac{(-1)^n}{n!}\tilde{a}_{-1}+(n+2)(n+1)\tilde{a}_{n+2}\right)x^{n+3}=$$

$$=(\tilde{a}_{-1}+\tilde{a}_0)x^2+$$

$$+\sum_{n=0}^{+\infty}\left(-\frac{(-1)^n(2n+3)}{(n+1)!}\tilde{a}_{-1}+(n+2)\tilde{a}_{n+1}+(n+2)(n+1)\tilde{a}_{n+2}\right)x^{n+3}=0.$$

Demak, noma’lum koeffitsientlalar uchun quyidagi tenglamalar hosil bo‘ladi:

$$\tilde{a}_{-1}+\tilde{a}_0=0,$$

$$-\frac{(-1)^n(2n+3)}{(n+1)!}\tilde{a}_{-1}+(n+2)\tilde{a}_{n+1}+(n+2)(n+1)\tilde{a}_{n+2}=0,\ n=0,1,2,3,\ldots .$$

Oxirgi rekurrent tenglamadan

$$\tilde{a}_{n+2}=\frac{(-1)^n(2n+3)}{(n+2)!(n+1)}\tilde{a}_{-1}-\frac{1}{n+1}\tilde{a}_{n+1},\quad n=0,1,2,3,\ldots .$$

Soddalik uchun $\tilde{a}_0=1$ deb qabul qilamiz. U holda $\tilde{a}_{-1}=-\tilde{a}_0=-1$ bo‘ladi. Qolgan koeffitsientlarni aniqlash uchun

$$\tilde{a}_n=\frac{(-1)^{n-1}}{(n-1)!}u_n,\ n=1,2,3,\ldots,$$

belgilashni kiritamiz. U holda yuqoridagi tenglamadan

$$\frac{(-1)^{n+1}}{(n+1)!}u_{n+2}=-\frac{(-1)^n(2n+3)}{(n+1)!(n+2)(n+1)}-\frac{(-1)^n}{n!(n+1)}u_{n+1},n=0,1,2,3,\ldots,$$

ya’ni

$$u_{n+2} = \frac{2n+3}{(n+2)(n+1)} + u_{n+1},\ n = 0,1,2,3,\ldots,$$

yoki

$$u_{k+1} = \frac{1}{k+1} + \frac{1}{k} + u_k\,,\ k = 1,2,3,\ldots,$$

munosabatni hosil qilamiz. Qulaylik uchun $u_1 = 1$ deb qabul qilamiz va oxirgi tenglikni $k = 1,2,\ldots,n-1$ uchun yozib, hosil bo‘lgan tengliklarrni hadma-had qo‘shamiz:

$$u_n = 2\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{n}\ ,\ n = 1,2,3,\ldots\ .$$

Demak,

$$\tilde{a}_n = \frac{(-1)^{n-1}}{(n-1)!}\left(2\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{n}\right),\ n = 1,2,3,\ldots\ .$$

Shunday qilib, $y_1 = x^2 e^{-x}$ $\left(y_1 = x^2 \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} x^n \right)$ yechimga chiziqli bog‘liq bo‘lmagan ikkinchi yechim topildi:

$$y_2 = -x^2 e^{-x} \ln x + x\left(1 + \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{(n-1)!}\left(2\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{n}\right)x^n\right).$$

Izoh. Qaralgan tenglamaning chiziqli erkli yechimlari sifatida ushbu

$$x^2 e^{-x} \ \text{ va } \ x(1 + xe^{-x}\mathrm{Ei}(1,-x))$$

funksiyalarni olish mumkin; bu yerda $\mathrm{Ei}(1,-x) = \int_1^{+\infty}\frac{e^{tx}}{t}dt$ – xossalari yaxshi o‘rganilgan eksponensial integral.

Misol 5. Ushbu

$$4x^2 y'' + (1+x)y = 0$$

tenglamaning umumiy yechimini qatorlar yordamida quring.

🗝 Ravshanki, $x=0$ nuqta bu differensial tenlamaning regular maxsus nuqtasi. Yechimni $y=x^{\mu}\sum_{n=0}^{+\infty}a_n x^n$ umumlashgan darajali qator ko‘rinishida izlaymiz ($x>0$). Buni berilgan tenglamaga qo‘yib, noma’lum μ va a_n lar uchun joiz tenglamalarni tuzamiz va ularning yechimlarini topamiz:

$$4x^2y''+(1+x)y=4x^2\sum_{n=0}^{+\infty}(n+\mu)(n+\mu-1)a_n x^{n+\mu-2}x^n+(1+x)\sum_{n=0}^{+\infty}a_n x^{n+\mu}=$$

$$=\sum_{n=0}^{+\infty}\big(4(n+\mu)(n+\mu-1)+1\big)a_n x^{n+\mu}+\sum_{m=1}^{+\infty}a_{m-1}x^{m+\mu}=$$

$$=\big(4\mu(\mu-1)+1\big)a_0x^{\mu}+\sum_{n=1}^{+\infty}\Big(\big(4(n+\mu)(n+\mu-1)+1\big)a_n+a_{n-1}\Big)x^{n+\mu}=0\,;$$

aniqlovchi tenglama $4\mu(\mu-1)+1=(2\mu-1)^2=0$ ning ildizi (ikki karrali) $\mu_1=\mu_2=1/2$ (2^0 bandga qarang);

$\big(4(n+\mu)(n+\mu-1)+1\big)a_n+a_{n-1}=0$ rekurrent tenglamadan $a_0=1$ deb va $\mu=1/2$ ekanligini hisobga olib, a_n $(n\geq 1)$ koeffitsientlarni topamiz:

$$4n^2a_n=-a_{n-1}\,,\ a_n=-\frac{1}{4n^2}a_{n-1}\,,\ a_n=\prod_{k=1}^{n}\Big(-\frac{1}{4k^2}\Big)=\Big(-\frac{1}{4}\Big)^n\frac{1}{(n!)^2}\,,\ n\geq 1.$$

Shunday qilib, berilgan differensial tenglamaning

$$y_1=x^{\mu}\sum_{n=0}^{+\infty}a_nx^n=x^{1/2}\sum_{n=0}^{+\infty}\Big(-\frac{1}{4}\Big)^n\frac{1}{(n!)^2}x^n\quad(0<x<+\infty)$$

yechimini hosil qildik.

Ikkinchi (undan) chiziqli erkli yechim

$$y_2=y_1(x)\ln x+x^{1/2}\sum_{n=0}^{\infty}\tilde{a}_nx^n$$

ko‘rinishga ega bo‘lishi kerak. Ikkinchi tartibli hosilani hisoblaymiz:

$$y_2''=y_1''(x)\ln x+\frac{y_1'(x)}{x}-\frac{y_1(x)}{x^2}+\frac{1}{x^2}\sum_{n=0}^{\infty}(n^2-\tfrac{1}{4})\tilde{a}_nx^{n+1/2}\ .$$

Endi y_2 va y_2'' larni berilgan tenglamaga qo‘yib, $y_1(x)$ ning tenglama yechimi ekanligini hisobga olib, $y_1(x)$ ning o‘rniga uning yoyilmasini qo‘yib va zarur shakl almashtirishlar va ixchamlashtirishlarni bajarib, topamiz

$$4x^2y_2''+(1+x)y_2=8\sum_{n=1}^{+\infty}\left(-\frac{1}{4}\right)^n\frac{n}{(n!)^2}x^{n+1/2}+4\sum_{n=1}^{\infty}n^2\tilde{a}_nx^{n+1/2}+\sum_{n=1}^{\infty}\tilde{a}_{n-1}x^{n+1/2}=$$

$$=\sum_{n=1}^{+\infty}\left(-2\left(-\frac{1}{4}\right)^{n-1}\frac{n}{(n!)^2}+4n^2\tilde{a}_n+\tilde{a}_{n-1}\right)x^{n+1/2}=0$$

Oxirgi tenglikdan $\tilde{a}_n$ noma’lum koeffitsientlarni topish uchun ushbu

$$-2\left(-\frac{1}{4}\right)^{n-1}\frac{n}{(n!)^2}+4n^2\tilde{a}_n+\tilde{a}_{n-1}=0\ ,\ n=1,2,3,...$$

yoki

$$\tilde{a}_n=\frac{1}{2n(n!)^2}\left(-\frac{1}{4}\right)^{n-1}-\frac{\tilde{a}_{n-1}}{4n^2},\ n=1,2,3,...$$

tenglamani hosil qilamiz. Bu tenglamadan $\tilde{a}_n$ lar ketma-ket rekurrent usulda topilishi mumkin. Lekin biz ular uchun bevosita formula hosil qilamiz. Buning uchun $\tilde{a}_n$ larni

$$\tilde{a}_n=\left(-\frac{1}{4}\right)^n\frac{1}{(n!)^2}z_n$$

ko‘rinishda izlaymiz. Buni yuqoridagi tenglamaga qo‘yib, quyidagilarni topamiz:

$$\left(-\frac{1}{4}\right)^n\frac{1}{(n!)^2}z_n=\frac{1}{2n(n!)^2}\left(-\frac{1}{4}\right)^{n-1}-\frac{1}{4n^2}\left(-\frac{1}{4}\right)^{n-1}\frac{1}{((n-1)!)^2}z_{n-1},\ n=1,2,3,...\ ;$$

$$z_n=-\frac{2}{n}+z_{n-1},\ n=1,2,3,...\ ;\quad z_n=z_0-2\sum_{k=1}^{n}\frac{1}{k}\ ,\ n=1,2,3,...\ .$$

Qulaylik uchun $z_0=0$ deb hisoblab,

$$\tilde{a}_n=-2\left(-\frac{1}{4}\right)^n\frac{1}{(n!)^2}\sum_{k=1}^{n}\frac{1}{k}\ ,\ n=1,2,3,...$$

ekanligini topamiz. Shunday qilib, berilgan tenglamaning ikkinchi chiziqli erkli yechimini qurdik:

$$y_2 = \sum_{n=0}^{+\infty}\left(-\frac{1}{4}\right)^n \frac{1}{(n!)^2} x^{n+1/2}\ln x - 2\sum_{n=1}^{\infty}\left(-\frac{1}{4}\right)^n \frac{1}{(n!)^2}\sum_{k=1}^{n}\frac{1}{k} x^{n+1/2}\,.$$

Tenglamaning umumiy yechimi $y = c_1 y_1 + c_2 y_2$ formula bilan ifodalanadi. 👍

Izoh. Ko‘rsatish mumkinki, berilgan tenglama Bessel funksiyalari orqali ifodalangan ushbu

$$\sqrt{x}J_0(\sqrt{x}) \text{ va } \sqrt{x}Y_0(\sqrt{x})$$

chiziqli erkli yechimlarga ega.

Masalalar

Darajali qatorlar yordamida tenglamalarni va masalalarni yeching. Qator-larning yaqinlashish radiusini toping ($x_0 = 0$). (**1 - 7**):

1. $y'' + xy' + y = 0$. **2.** $y'' + e^x y' + y = \sin x$.

3. $y'' + \dfrac{1}{1-x} y' + y = 0$, $y(0) = 0,\ y'(0) = 1.$

4. $y'' + \dfrac{x}{2+x} y' + \sqrt{1-x}\cdot y = 0$, $y(0) = 1,\ y'(0) = 1.$ **5.** $y'' - x^2 y = 0$.

6. $y'' + y'^2 = y^2 + x^2,\ \ y(0) = 1, y'(0) = -1$.

7. $y'' = y'e^y + xy,\ \ y(0) = 0, y'(0) = 1$.

Differensial tenglamalarning ikki dona chiziqli erkli yechimlari uchun umumlashgan darajali ($x_0 = 0$ markazli) qatorning dastlabki 3ta noldan farqli hadini toping (**8 - 10**):

8. $2xy'' + (x+3)y' + 2y = 0$. **9.** $x^2 y'' + (x^3+1)xy' - y = 0$.

10. $x^2 y'' - 3xy' + 4(x+1)y = 0$.

Mustaqil ish № 18 topshiriqlari:

I. Berilgan differensial tenglamaning berilgan boshlang‘ich shartlarni qanoatlantiruvchi yechimi uchun markazi boshlang‘ich nuqtada bo‘lgan darajali qatorning dastlabki m ta hadini toping. Har ikkala metoddan foydalaning.

1. $xy'' + y\sin x = 2 + x^2$, $y(1) = -1, y'(1) = 0, m = 4$.

2. $y'' + xy' + e^x y = 1 + x^2$, $y(0) = -1, y'(0) = 0, m = 4$.

3. $xy'' + (2 + x)y = \dfrac{2 + x^2}{1 + x}$, $y(1) = 1, y'(1) = 1, m = 4$.

4. $y'' + (1 - x)y = \cos 3x$, $y(0) = -1, y'(0) = 2, m = 5$.

5. $y'' + y\ln x = 3 + x - x^2$, $y(1) = 1, y'(0) = 2, m = 4$.

6. $y'' + y' + (1 - x^2)y = \sin 3x$, $y(0) = -1, y'(0) = 1, m = 4$.

7. $x^2 y'' + (1 + x)y = \mathrm{tg}\, x$, $y(0) = 1, y'(0) = 1, m = 4$.

8. $y'' + y\,\mathrm{arctg}\, x = 4 - x^3$, $y(0) = -1, y'(0) = 1, m = 4$.

9. $y'' + y' + \dfrac{y}{3 + x} = \mathrm{tg}\, x$, $y(-1) = 1, y'(-1) = 0, m = 5$.

10. $y'' + 2y' + y\arccos x = 1 - 2x$, $y(0) = -1, y'(0) = 2, m = 4$.

11. $y'' + y\arcsin x = x^3 + x$, $y(0) = 1, y'(0) = 2, m = 5$.

12. $y'' + \dfrac{y}{\sqrt{1 + x^2}} = 1 - 2x$, $y(0) = -1, y'(0) = 1, m = 4$.

13. $\sqrt{1 - x}\,y'' + y\,\mathrm{tg}\, x = x^2 + x$, $y(0) = 1, y'(0) = 2, m = 4$.

14. $y'' + xy = 0$, $y(-1) = a, y'(-1) = b, m = 5$.

15. $y'' + y\arcsin x = 1 + 3x$, $y(0) = 1, y'(0) = 1, m = 4$.

16. $\sqrt{1 + x^2}\,y'' + y = \arcsin x$, $y(0) = 1, y'(0) = 1, m = 4$.

17. $y'' + y' + \dfrac{y}{\sqrt{1 + x^3}} = 2 + 3x$, $y(0) = 1, y'(0) = -1, m = 4$.

18. $y'' + (2 - x)y' + \dfrac{y}{\sqrt{1 + x}} = 1 + x$, $y(0) = -1, y'(0) = 1, m = 4$.

19. $y'' + y'\sin x + \dfrac{y}{\sqrt{1 - x}} = 0$, $y(0) = 1, y'(0) = -1, m = 4$.

20. $y'' + y'\arcsin x + y = 1,\ y(0) = 1, y'(0) = 1, m = 4$.

21. $xy'' + y'\sin x + y = 1,\ y(\pi/2) = 1, y'(\pi/2) = 0, m = 4$.

22. $\sqrt{1-x}\,y'' + (1+x)y' + y = \sin x,\ y(0) = 1, y'(0) = 0, m = 5$.

23. $y'' + \dfrac{y'}{1+2x} + y = \ln(1-x),\ y(0) = 1, y'(0) = 1, m = 4$.

24. $(1+x^3)y'' + y' + y\sin x = x + x^2,\ y(0) = 1, y'(0) = -1, m = 5$.

25. $y'' + (1-x)y' + \dfrac{y}{\sqrt{1+x}} = \sin x,\ y(0) = -1, y'(0) = 1, m = 4$.

26. $y'' + \dfrac{y'}{1+2x} + y = \sqrt[3]{1-x},\ y(0) = 1, y'(0) = 0, m = 5$.

27. $y'' + (1-x)y' + \dfrac{y}{\sqrt{1+x}} = \sin x,\ y(0) = -1, y'(0) = 1, m = 4$.

28. $(1+x^2)y'' + (1-x)y' + y = \cos x,\ y(0) = -1, y'(0) = 1, m = 5$.

29. $(2+x)y'' + \sqrt{1-x}\,y' + y = 1,\ y(0) = 1, y'(0) = 1, m = 4$.

30. $x^2y'' + (1-x)\dfrac{y'}{1-x} + y = \cos x,\ y(2) = -1, y'(2) = 1, m = 4$.

II. Berilgan differensial tenglamaning ikki dona chiziqli erkli yechimlari uchun umumlashgan darajali ($x_0 = 0$ markazli) qatorning dastlabki 4 ta hadini toping. Biror yechimning to‘la yoyilmasini topa olasizmi?

1. $9x^2y'' + xy' + (x+1)y = 0$.

2. $x^2y'' - 3\ln(1-x)\cdot y' + 5(1-x^2)y = 0$.

3. $4x^2y'' + (3+x)y = 0$.

4. $2x^2y'' - xy' + (1+x^2)y = 0$.

5. $4xy'' + y\cos x = 0$.

6. $x^2y'' + 6(x-2x^2)y' - (1+2x^3)y = 0$.

7. $2xy'' + 2(1-x)y' + y = 0$.

8. $x^2(1+x)y'' - (1-x)y' + 2\sin x\cdot y = 0$.

9. $xy'' + 2xy' + 6\cos x\cdot y = 0$.

10. $4x^2y'' + (3+\sin x)y = 0$.

11. $(1-x^2)y'' + xy' + (3+x)y = 0$.

12. $(1-x^2)y'' + \sin x\cdot y' + 3y = 0$.

13.. $3x^2y'' + \ln(1-x)y' + y = 0$.

14. $\sin x\cdot y'' - 6x^3y' + 3y = 0$.

15. $xy'' + (1-\ln x)y' - y = 0$.

16. $\cos x\cdot y'' + 4xy' + 6y = 0$.

17. $2x(1+\cos x)y''+4xy'+6y=0$. **18.** $xy''+2\sqrt[3]{1+x}\cdot y=0$.

19. $2(1-\cos x)y''-xy'+y=0$. **20.** $\sin^2 x\cdot y''-5xy'+9(1-x)y=0$

21. $x^2y''+(x-2)\sin x\cdot y'+2y=0$. **22.** $x^2(1+x)y''-3\operatorname{arctg} x\cdot y'+4y=0$.

23. $xy''+y'-(1+x)y=0$ **24.** $x^2y''+3\arcsin x\cdot y'-8(1-x)y=0$.

25. $3xy''+3(1+x)y'-y=0$ **26.** $2x^2\sqrt{1+x}\cdot y''-xy'+\cos x\cdot y=0$.

27. $x^2y''+2\left(1-\sqrt{1-x}\right)y'-8(1+x^2)y=0$.

28. $x^2y''-3\ln(1-x)\cdot y'+5(1-x^2)y=0$.

29. $2x^2y''+5(x-2x^3)y'-2\sqrt{1+x^2}\cdot y=0$.

30. $\sin^2 x\cdot y''-4\left(1-\sqrt{1+x}\right)y'-6(1+x)y=0$.

19. KICHIK PARAMETR METODI

Maqsad – normal sistema yechimlarining parametr va boshlang‘ich ma’lumotlarga uzluksiz va silliq bog‘liqligini hamda kichik parametr metodini o‘rganish

Yordamchi ma’lumotlar:

Ushbu

$$\begin{cases} \boldsymbol{x}'=\boldsymbol{f}(t,\boldsymbol{x},\boldsymbol{\mu}) \\ \boldsymbol{x}|_{t_0}=\boldsymbol{x}^0 \end{cases} \tag{1}$$

$\boldsymbol{\mu}=(\mu_1,\mu_2,\ldots,\mu_m)\in M$ $(M\subset\mathbb{R}^m-\text{soha})$ $\boldsymbol{\mu}$ parametr(lar)ga bog‘liq bo‘lgan Koshi masalasi berilgan bo‘lsin. Faraz qilaylik, $\boldsymbol{f}(t,\boldsymbol{x},\boldsymbol{\mu})$ vektor-funksiya $(t,\boldsymbol{x})\in D^{1+n}$, $\boldsymbol{\mu}\in M$ bo‘lganda aniqlangan va (barcha argumentlari bo‘yicha) uzluksiz ($\boldsymbol{f}\in C(D\times M,\mathbb{R}^n)$) hamda $D\times M$ da $\mathbf{x}$ vektor o‘zgaruvchi bo‘yicha lokal Lipshits shartini qanoatlantirsin, ya’ni har qanday $(t,\boldsymbol{x},\boldsymbol{\mu})\in D\times M$

nuqtaning yetarlicha kichik atrofi uchun shunday $L>0$ son mavjudki, shu atrofdagi barcha $(t,\boldsymbol{x}^1,\boldsymbol{\mu})$ va $(t,\boldsymbol{x}^2,\boldsymbol{\mu})$ nuqtalar uchun

$$\|\boldsymbol{f}(t,\boldsymbol{x}^2,\boldsymbol{\mu})-\boldsymbol{f}(t,\boldsymbol{x}^1,\boldsymbol{\mu})\|\le L\|\boldsymbol{x}^2-\boldsymbol{x}^1\|$$

tengsizlik o‘rinli. Oxirgi shart bajarilishi uchun, masalan, ixtiyoriy $(t,\boldsymbol{x},\boldsymbol{\mu})\in D\times M$ nuqtaning biror atrofida $\left|\frac{\partial f_i}{\partial x_j}\right|\le \text{const}$ bo‘lishi yetarli.

Qo‘yilgan shartlarda har qanday $(t_0,\boldsymbol{x}^0,\boldsymbol{\mu})\in D\times M$ $\left((t_0,\boldsymbol{x}^0)\in D,\ \boldsymbol{\mu}\in M\right)$ uchun (1) masala yagona davomsiz $\boldsymbol{x}=\boldsymbol{\varphi}(t;t_0,\boldsymbol{x}^0,\boldsymbol{\mu})$, $t\in I$, yechimga ega. Bu davomsiz yechimning aniqlanish intervali, tushunarliki, tayinlangan $(t_0,\boldsymbol{x}^0,\boldsymbol{\mu})$ qiymatlarga bog‘liq bo‘ladi, $I=I(t_0,\boldsymbol{x}^0,\boldsymbol{\mu})$. Demak, $\boldsymbol{x}=\boldsymbol{\varphi}(t;t_0,\boldsymbol{x}^0,\boldsymbol{\mu})$ yechim $(t;t_0,\boldsymbol{x}^0,\boldsymbol{\mu})\in I\times D\times M\subset\mathbb{R}^{2+n+m}$ sohada aniqlangan. Agar $(t_0,\boldsymbol{x}^0)$ tayinlangan bo‘lsa, u holda $\boldsymbol{x}=\boldsymbol{\varphi}(t;t_0,\boldsymbol{x}^0,\boldsymbol{\mu})$ yozuv o‘rniga $\boldsymbol{x}=\boldsymbol{\varphi}(t;\boldsymbol{\mu})$ yozuvni ishlatamiz.

***Teorema 1.* (*yechimning parametrlarga uzluksiz bog‘liqligi*).** Faraz qilaylik, $\boldsymbol{f}\in C(D\times M,\mathbb{R}^n)$ bo‘lsin va u $D\times M$ sohada $\mathbf{x}$ vektor o‘zgaruvchi bo‘yicha lokal Lipshits shartini qanoatlantirsin hamda $\boldsymbol{\mu}=\boldsymbol{\mu}^0$ bo‘lganda (1) masala $t\in[t_1,t_2]$ $(t_0\in[t_1,t_2],(t_0,\boldsymbol{x}^0,\boldsymbol{\mu}^0)\in D\times M)$ segmentda aniqlangan $\boldsymbol{x}=\boldsymbol{\varphi}(t;\boldsymbol{\mu}^0)$ yechimga ega bo‘lsin. U holda shunday yetarlicha kichik $\delta>0$ son mavjudki, $\|\boldsymbol{\mu}-\boldsymbol{\mu}^0\|<\delta$ bo‘lganda $\boldsymbol{x}=\boldsymbol{\varphi}(t;\boldsymbol{\mu})\ (=\boldsymbol{\varphi}(t;t_0,\boldsymbol{x}^0,\boldsymbol{\mu}))$ yechim barcha $t\in[t_1,t_2]$ larda aniqlangan va $(t;\boldsymbol{\mu})$ o‘zgaruvchilar bo‘yicha uzluksiz vektor-funksiyadan iborat bo‘ladi, $\boldsymbol{\varphi}(t;\boldsymbol{\mu})\in C([t_1,t_2]\times B_\delta(\boldsymbol{\mu}^0);\mathbb{R}^n)$.

***Teorema 1′* (*yechimning boshlang‘ich ma’lumotlar va parametrlarga uzluksiz bog‘liqligi*).** Faraz qilaylik, $\boldsymbol{f}\in C(D\times M,\mathbb{R}^n)$ bo‘lsin va u $D\times M$ sohada $\mathbf{x}$ vektor o‘zgaruvchi bo‘yicha lokal Lipshits shartini qanoatlantirsin hamda $\boldsymbol{\mu}=\boldsymbol{\mu}^0$ bo‘lganda (1) masala $t\in[t_1,t_2]$ $(t_0\in[t_1,t_2],(t_0,\boldsymbol{x}^0,\boldsymbol{\mu}^0)\in D\times M)$ segmentda aniqlangan $\boldsymbol{x}=\boldsymbol{\varphi}(t;t_0,\boldsymbol{x}^0,\boldsymbol{\mu}^0)$ yechimga ega bo‘lsin. U holda shunday yetarlicha kichik $\delta>0$ soni topiladiki, ushbu $|\tau-t_0|<\delta$, $\|\boldsymbol{\xi}-\boldsymbol{x}^0\|<\delta$ va $\|\boldsymbol{\mu}-\boldsymbol{\mu}^0\|<\delta$ shartlar bajarilganda $\boldsymbol{x}=\boldsymbol{\varphi}(t;\tau,\boldsymbol{\xi},\boldsymbol{\mu})$ yechim barcha $t\in[t_1,t_2]$ larda aniqlangan va $(t;\tau,\boldsymbol{\xi},\boldsymbol{\mu})$ (t yechimning argumenti, $(\tau,\boldsymbol{\xi})$ boshlang'ich ma'lumotlar, $\boldsymbol{\mu}$ parametrlar) o'zgaruvchilar bo'yicha uzluksiz vektor-funksiyadan iborat bo'ladi, ya'ni . $\boldsymbol{\varphi}(t;\tau,\boldsymbol{\xi},\boldsymbol{\mu})\in C([t_1,t_2]\times(t_0-\delta,t_0+\delta)\times B_\delta(\boldsymbol{x}^0)\times B_\delta(\boldsymbol{\mu}^0);\mathbb{R}^n)$

Quyidagi teoremada soddalik uchun parametrlar soni birga teng deb hisoblanadi. Bu holda $m=1$, $M-$sonli interval, $\boldsymbol{\mu}=\mu\in M$.

***Teorema 2* (*yechimning parametr bo'yicha differensiallanuvchiligi*).** Aytaylik, $\boldsymbol{f}(t,\boldsymbol{x},\mu)$, $\dfrac{\partial\boldsymbol{f}(t,\boldsymbol{x},\mu)}{\partial x_j}$ $(j=1,...,n)$, $\dfrac{\partial\boldsymbol{f}(t,\boldsymbol{x},\mu)}{\partial\mu}$ funksiyalar $(t,\boldsymbol{x},\mu)\in D\times M$ sohada uzluksiz, (1) masalaning $\boldsymbol{x}=\boldsymbol{\varphi}(t;\mu)$ yechimi esa har bir $\mu\in M$ uchun $t\in[t_1,t_2]$ $(t_0\in[t_1,t_2])$ segmentda aniqlangan bo'lsin. U holda bu yechimning $\boldsymbol{u}\equiv\dfrac{\partial\boldsymbol{\varphi}(t;\mu)}{\partial\mu}$ $(\boldsymbol{u}=\boldsymbol{u}(t;\mu))$ hosilasi $(t,\mu)\in[t_1,t_2]\times M$ bo'lganda uzluksiz va u variatsiya uchun tenglama deb ataluvchi ushbu

$$\frac{d\boldsymbol{u}}{dt}=\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{x}}\boldsymbol{u}+\frac{\partial\boldsymbol{f}}{\partial\mu},\ \boldsymbol{u}\big|_{t=t_0}=0, \tag{2}$$

chiziqli tenglamani qanoatlantiradi, bunda xususiy hosilalar $\boldsymbol{x}=\boldsymbol{\varphi}(t;\mu)$ bo'lganda hisoblangan, ya'ni

$$\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{x}}=\left.\frac{\partial\boldsymbol{f}(t,\boldsymbol{x},\mu)}{\partial\boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{\varphi}(t;\mu)},\quad \frac{\partial\boldsymbol{f}}{\partial\mu}=\left.\frac{\partial\boldsymbol{f}(t,\boldsymbol{x},\mu)}{\partial\mu}\right|_{\boldsymbol{x}=\boldsymbol{\varphi}(t;\mu)}.$$

Variatsiya uchun (2) vektorli tenglamaning skalyar ko'rinishi quyidagi variatsiyalar uchun tenglamalar sistemasidan iborat:

$$\frac{du_i}{dt}=\sum_{j=1}^{n}\frac{\partial f_i}{\partial x_j}u_j+\frac{\partial f_i}{\partial\mu},\ u_i\big|_{t=t_0}=0\ \ (i=1,...,n). \tag{3}$$

Keltirilgan teorema parametrlar soni bittadan ko‘p bo‘lganda ham o‘rinli. Bu holda $\boldsymbol{x}=\boldsymbol{\varphi}(t;\mu_1,\mu_2,...,\mu_m)$ yechimning har bir $\boldsymbol{u}^j\equiv\dfrac{\boldsymbol{\varphi}(t;\mu)}{\partial\mu_j}$ $(j=1,...,m)$ xususiy hosilasi variatsiya uchun mos (2) chiziqli tenglamani qanoatlantiradi:

$$\frac{d\boldsymbol{u}^j}{dt}=\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{x}}\boldsymbol{u}^j+\frac{\partial\boldsymbol{f}}{\partial\mu_j},\ \ \boldsymbol{u}^j\big|_{t=t_0}=0\ (j=1,...,m).$$

Keltirilgan teoremani quyidagicha qisqaroq (lekin noaniqroq) ifodalash mumkin:

agar $\boldsymbol{x}'=\boldsymbol{f}(t,\boldsymbol{x},\boldsymbol{\mu})$ sistemaning o‘ng tomoni $\boldsymbol{f}(t,\boldsymbol{x},\boldsymbol{\mu})\in C^1$ bo‘lsa, uning $\boldsymbol{x}=\boldsymbol{\varphi}(t;\boldsymbol{\mu})$ yechimi ham C^1 sinfga tegishli bo‘ladi.

Variatsiyalar uchun (3) tenglamalar sistemasini (yoki uning (2) vektor ko‘rinishini) hosil qilish uchun ushbu

$$\frac{d\varphi_i(t,\mu)}{dt}=f_i(t,\varphi_1(t,\mu),...,\varphi_n(t,\mu),\mu),\ \ (i=1,...,n)$$

ayniyatlarni μ bo‘yicha differensiallash va aralash hosilalarda differensiallash tartibini almashtirish kerak. Agar $\boldsymbol{x}=\boldsymbol{\varphi}(t;\mu)$ yechim biror μ da ma’lum bo‘lsa, μ ning shu qiymatida yechimning μ bo‘yicha hosilasi $\boldsymbol{u}=\dfrac{\boldsymbol{\varphi}(t;\mu)}{\partial\mu}$ ni (2) (yoki(3)) masalani yechib aniqlash mumkin.

(1) masala $\boldsymbol{x}=\boldsymbol{\varphi}(t;t_0,\boldsymbol{x}^0,\boldsymbol{\mu})$ yechimining $(t_0,\boldsymbol{x}^0,\boldsymbol{\mu})$ larga bog‘liqligini o‘rganishni boshqa sistema yechimining faqat parametrlarga bog‘liqligini o‘rganishga keltirish mumkin. Buning uchun (1) da $\boldsymbol{x}=\boldsymbol{y}-\boldsymbol{x}^0$ formula bilan $\boldsymbol{y}$ ga va t o‘rniga $t-t_0$ erkli o‘zgaruvchiga o‘tish kifoya.

Endi yechimni boshlang‘ich qiymatlar bo‘yicha differensiallash masalasini qaraylik. Ushbu

$$\begin{cases} \boldsymbol{x}' = \boldsymbol{f}(t,\boldsymbol{x}) \\ \boldsymbol{x}|_{t_0} = \boldsymbol{\xi} \end{cases} \tag{4}$$

Koshi masalasi berilgan bo‘lsin; bunda $(t,\boldsymbol{x}) \in D\ ((t_0,\boldsymbol{\xi}) \in D)$, $D - \mathbb{R}^{1+n}$ fazodagi soha. Bu masalaning yechimini $\boldsymbol{x} = \boldsymbol{\varphi}(t;\boldsymbol{\xi})\ \big(\boldsymbol{\varphi}(t_0;\boldsymbol{\xi}) = \boldsymbol{\xi}\big)$ ko‘rinishda belgilaymiz (boshlang‘ich payt (vaqt) t_0 tayinlangan).

***Teorema 2′* (*yechimning boshlang‘ich qiymatlar bo‘yicha differensial-lanuvchanligi*).** Aytaylik, $\boldsymbol{f}(t,\boldsymbol{x})$ vektor-funksiya va uning $\dfrac{\partial \boldsymbol{f}(t,\boldsymbol{x})}{\partial \boldsymbol{x}}$ xususiy hosilasi D sohada uzluksiz hamda (4) masalaning $\boldsymbol{\xi} = \boldsymbol{\xi}^0$ dagi $\boldsymbol{x} = \boldsymbol{\varphi}(t;\boldsymbol{\xi}^0)$ yechimi $t \in [t_1,t_2]$ oraliqda aniqlangan bo‘lsin. U holda $\boldsymbol{\xi}^0$ nuqtaning biror $B_{\delta_0}(\boldsymbol{\xi}^0)$ atrofiga tegishli bo‘lgan barcha $\boldsymbol{\xi}$ lar uchun $\boldsymbol{x} = \boldsymbol{\varphi}(t;\boldsymbol{\xi})$ yechimning boshlang‘ich qiymatlar bo‘yicha $\boldsymbol{w}^j = \dfrac{\partial \boldsymbol{\varphi}(t;\boldsymbol{\xi})}{\partial \xi_j}\ (j = 1,\ldots,n)$ hosilalari $(t,\boldsymbol{\xi}) \in [t_1,t_2] \times B_{\delta_0}(\boldsymbol{\xi}^0)$ to‘plamda uzluksiz va ular quyidagi masalalar yechimlaridir:

$$\frac{d\boldsymbol{w}^j}{dt} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\boldsymbol{w}^j,\ \boldsymbol{w}^j = \boldsymbol{e}^j\ \Big(\boldsymbol{e}^j = (0,\ldots,0,\underset{j-\text{o'rin}}{1},0,\ldots 0)^T\Big),\ j = \overline{1,n}\ ;$$

bunda $\dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \left.\dfrac{\partial \boldsymbol{f}(t,\boldsymbol{x})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{\varphi}(t;\boldsymbol{\xi})}$.

Bu masalani hosil qilish uchun ushbu

$$\frac{\partial}{\partial t}\boldsymbol{\varphi}(t;\boldsymbol{\xi}) = \boldsymbol{f}(t,\boldsymbol{\varphi}(t;\boldsymbol{\xi})),\ \boldsymbol{\varphi}(t_0;\boldsymbol{\xi}) = \boldsymbol{\xi}$$

ayniyatlarni ξ_j bo‘yicha differensiallash va $\dfrac{\partial^2 \boldsymbol{\varphi}(t;\boldsymbol{\xi})}{\partial \xi_j \partial t} = \dfrac{\partial^2 \boldsymbol{\varphi}(t;\boldsymbol{\xi})}{\partial t \partial \xi_j} = \dfrac{\partial \boldsymbol{w}^j}{\partial t}$ deyish kifoya.

Nihoyat, yechimning boshlang‘ich payt bo‘yicha differensiallanuvchiligini qarab chiqamiz.

Ushbu

$$\begin{cases} \boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}) \\ \boldsymbol{x}|_{\tau} = \boldsymbol{x}^0 \end{cases} \tag{5}$$

boshlang‘ich masalani qaraylik; bunda $(t, \boldsymbol{x}) \in D$ $((\tau, \boldsymbol{x}^0) \in D)$. Bu masalaning yechimini $\boldsymbol{x} = \boldsymbol{\varphi}(t;\tau)$ $(\boldsymbol{\varphi}(\tau;\tau) = \boldsymbol{x}^0)$ ko‘rinishda belgilaymiz (boshlang‘ich qiymat $\boldsymbol{x}^0$ tayinlangan).

***Teorema 2”* (*yechimning boshlang‘ich payt bo‘yicha differensiallanuvchiligi*).** Aytaylik, $\boldsymbol{f}(t, \boldsymbol{x})$ vektor-funksiya va uning $\dfrac{\partial \boldsymbol{f}(t, \boldsymbol{x})}{\partial \boldsymbol{x}}$ xususiy hosilasi D sohada uzluksiz hamda (5) masalaning $\tau = t_0$ bo‘lgandagi $\boldsymbol{x} = \boldsymbol{\varphi}(t;t_0)$ yechimi $t \in [t_1, t_2]$ oraliqda aniqlangan bo‘lsin. U holda t_0 nuqtaning biror yetarlicha kichik $(t_0 - \delta_0, t_0 + \delta_0)$ atrofiga tegishli bo‘lgan barcha τ lar uchun $\boldsymbol{x} = \boldsymbol{\varphi}(t;\tau)$ yechimning boshlang‘ich payt bo‘yicha $\boldsymbol{w} = \dfrac{\partial \boldsymbol{\varphi}(t;\tau)}{\partial \tau}$ hosilalasi $(t,\tau) \in [t_1, t_2] \times (t_0 - \delta_0, t_0 + \delta_0)$ to‘plamda uzluksiz va u ushbu

$$\begin{cases} \dfrac{d\boldsymbol{w}}{dt} = \boldsymbol{f}'_x(t, \boldsymbol{\varphi}(t;\tau))\boldsymbol{w} \\ \boldsymbol{w}|_{t=\tau} = -\boldsymbol{f}(\tau, \boldsymbol{x}^0) \end{cases} \tag{6}$$

masalaning yechimidan iborat bo‘ladi.

Agar $\boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{\mu})$ tenglamaning o‘ng tomoni $\boldsymbol{x}$ va μ bo‘yicha m marta uzluksiz differensiallanuvchi bo‘lsa, uning $\boldsymbol{x} = \boldsymbol{\varphi}(t;\mu)$ yechimi ham μ bo‘yicha m marta uzluksiz differensiallanuvchi bo‘ladi. Bu tasdiqning aniq ifodalanishi quyidagi teoremada keltirilgan.

Teorema 3. Yechimning parametr bo‘yicha differensiallanuvchanligi to‘g‘risidagi teorema 1 shartlariga qo‘shimcha holda $\boldsymbol{f}(t, \boldsymbol{x}, \mu)$ funksiya $x_1, ..., x_n, \mu$ lar bo‘yicha C^m sinfga tegishli bo‘lsin. U holda $\boldsymbol{x} = \boldsymbol{\varphi}(t;\mu)$

yechimning t,μ bo‘yicha birinchi tartibli, μ bo‘yicha esa $m-$ tartibli hosilalari uzluksiz bo‘ladi.

Differensial tenglamalarning taqribiy yechimlarini topishda **kichik parametr metodi** muhim o‘rin tutadi. (1) nochiziqli masalaning μ parametr qiymati $\mu=0$ bo‘lgandagi yechimi $\boldsymbol{x}=\boldsymbol{\varphi}^0(t)$ ma’lum bo‘lsin. U holda μ parametrning 0ga yaqin (kichik) qiymatlarida bu masalaning taqribiy yechimini kichik parametr metodi yordamida qurish mumkin.

Teorema 4. Aytaylik, teorema 3 ning shartlari $(t,\boldsymbol{x})\in D,\ |\mu|<\varepsilon\ (\varepsilon>0)$ sohada o‘rinli, $\mu=0$ bo‘lgandagi $(t_0\in[t_1,t_2])$ masalaning $\boldsymbol{x}=\boldsymbol{\varphi}^0(t)$ yechimi $t\in[t_1,t_2]$ oraliqda aniqlangan bo‘lsin. U holda (1) masalaning $\boldsymbol{x}=\boldsymbol{\varphi}(t;\mu)$ ($t\in[t_1,t_2]$) yechimi uchun

$$\boldsymbol{\varphi}(t;\mu)=\boldsymbol{\varphi}^0(t)+\boldsymbol{\varphi}^1(t)\mu+\boldsymbol{\varphi}^2(t)\mu^2+\ldots+\boldsymbol{\varphi}^m(t)\mu^m+o(\mu^m),\ \mu\to 0\,, \qquad (7)$$

asimptotik yoyilma o‘rinli; bundan tashqari, bu yerdagi kichik o $t\in[t_1,t_2]$ga nisbatan tekis ham bo‘ladi.

Konkret masalalar yechilganda (7) yoyilmani, ya’ni $\boldsymbol{\varphi}^0(t),\ \boldsymbol{\varphi}^1(t),\ \boldsymbol{\varphi}^2(t),\ \ldots,\ \boldsymbol{\varphi}^m(t)$ vektor-funksiyalarni aniqlash uchun yoyilmani qaralayotgan tenglamaga qo‘yib,

$$\frac{d\boldsymbol{\varphi}^0(t)}{dt}+\frac{d\boldsymbol{\varphi}^1(t)}{dt}\mu+\frac{d\boldsymbol{\varphi}^2(t)}{dt}\mu^2+\ldots+\frac{d\boldsymbol{\varphi}^m(t)}{dt}\mu^m+o(\mu^m)=\boldsymbol{f}(t,\boldsymbol{\varphi}(t;\mu),\mu),\mu\to 0,$$

o‘ng tomonni μning darajalari bo‘yicha yoyib,

$$\frac{d\boldsymbol{\varphi}^0(t)}{dt}+\frac{d\boldsymbol{\varphi}^1(t)}{dt}\mu+\frac{d\boldsymbol{\varphi}^2(t)}{dt}\mu^2+\ldots+\frac{d\boldsymbol{\varphi}^m(t)}{dt}\mu^m+o(\mu^m)=$$

$$=\boldsymbol{f}(t,\boldsymbol{\varphi}(t;0),0)+\left(\frac{\partial \boldsymbol{f}(t,\boldsymbol{\varphi}^0(t;0),0)}{\partial \boldsymbol{x}}\boldsymbol{\varphi}^1(t)+\frac{\partial \boldsymbol{f}(t,\boldsymbol{\varphi}^0(t;0),0)}{\partial \mu}\right)\mu+$$

$$+\ldots+\left(\frac{\partial \boldsymbol{f}(t,\boldsymbol{\varphi}^0(t;0),0)}{\partial \boldsymbol{x}}m!\boldsymbol{\varphi}^m(t)+\ldots\right)\mu^m+o(\mu^m),\ \mu\to 0\,,$$

hosil bo‘lgan tenglikning chap va o‘ng tomonlaridagi μning bir xil darajalari oldidagi koeffitsientlarni tenglashtirish kerak:

$$\mu^0: \ \frac{d\boldsymbol{\varphi}^0(t)}{dt} = \boldsymbol{f}(t, \boldsymbol{\varphi}^0(t;0), 0)$$

$$\mu^1: \ \frac{d\boldsymbol{\varphi}^1(t)}{dt} = \frac{\partial \boldsymbol{f}(t, \boldsymbol{\varphi}^0(t;0), 0)}{\partial \boldsymbol{x}} \boldsymbol{\varphi}^1(t) + \frac{\partial \boldsymbol{f}(t, \boldsymbol{\varphi}^0(t;0), 0)}{\partial \mu}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\mu^m: \frac{d\boldsymbol{\varphi}^m(t)}{dt} = \frac{\partial \boldsymbol{f}(t, \boldsymbol{\varphi}^0(t;0), 0)}{\partial \boldsymbol{x}} m! \boldsymbol{\varphi}^m(t) + \dots \ .$$

Bunda $\boldsymbol{\varphi}^1(t), \boldsymbol{\varphi}^2(t), \dots, \boldsymbol{\varphi}^m(t)$ funksiyalar uchun chiziqli tenglamalar hosil bo‘ladi. $\boldsymbol{x}|_{t_0} = \boldsymbol{x}^0$ boshlang‘ich shartdan

$$\boldsymbol{\varphi}(t;\mu)|_{t=t_0} = \boldsymbol{x}^0 = \boldsymbol{\varphi}^0(t)|_{t=t_0} + \boldsymbol{\varphi}^1(t)|_{t=t_0} \mu + \boldsymbol{\varphi}^2(t)|_{t=t_0} \mu^2 +$$
$$+ \dots + \boldsymbol{\varphi}^m(t)|_{t=t_0} \mu^m + o(\mu^m)|_{t=t_0}, \ \mu \to 0,$$

ya’ni

$$\boldsymbol{\varphi}^0(t)|_{t=t_0} = \boldsymbol{x}^0, \ \boldsymbol{\varphi}^1(t)|_{t=t_0} = 0, \ \boldsymbol{\varphi}^2(t)|_{t=t_0} = 0, \ \boldsymbol{\varphi}^m(t)|_{t=t_0} = 0$$

boshlang‘ich shartlar hosil bo‘ladi. Hosil qilingan tenglamalardan $\boldsymbol{\varphi}^0(t)$ dan boshlab ketma-ket $\boldsymbol{\varphi}^1(t), \boldsymbol{\varphi}^2(t), \dots, \boldsymbol{\varphi}^m(t)$ yechimlarni mos boshlang‘ich shartlarga ko‘ra topish kerak.

Misol 1. Ushbu

$$\begin{cases} \dfrac{dx}{dt} = \mu t - x^2, \\ x|_{t=1} = 1, \end{cases}$$

masala yechimining kichik μ parametr darajalari bo‘ylab yoyilmasidagi dastlabki uchta hadni toping.

🗝 Berilgan sistemaning o‘ng tomoni $(t, x) \in D = \mathbb{R}^2$, $|\mu| < +\infty$ sohada xohlagancha marta uzluksiz differensiallanuvchi. Demak, teorema 4 ning shartlari ixtiyoriy m uchun o‘rinli. Biz

$$x=\varphi_0(t)+\varphi_1(t)\mu+\varphi_2(t)\mu^2+o(\mu^2)\ ,\ \mu\to 0,$$

yoyilmalardagi koeffitsientlarni topishimiz kerak. Bu yoyilmalarni berilgan sistema va boshlang‘ich shartlarga qo‘yamiz ($o(\mu^2)$ miqdorlar $\mu\to 0$ da tushuniladi) :

$$\varphi_0'(t)+\varphi_1'(t)\mu+\varphi_2'(t)\mu^2+o(\mu^2)=\mu t-\left(\varphi_0(t)+\varphi_1(t)\mu+\varphi_2(t)\mu^2+o(\mu^2)\right)^2,$$
$$\left(\varphi_0(t)+\varphi_1(t)\mu+\varphi_2(t)\mu^2+o(\mu^2)\right)\big|_{t=1}=1.$$

Bu yerdagi birinchi tenglamaning o‘ng tomonini μ ning darajalari bo‘ylab yoyamiz (qavslarni ochib, tartiblari μ^2 gacha bo‘lgan hadlarni saqlaymiz va ixchamlashlarni bajaramiz):

$$\varphi_0'(t)+\varphi_1'(t)\mu+\varphi_2'(t)\mu^2+o(\mu^2)=-\varphi_0^2(t)+\left(t-2\varphi_0(t)\cdot\varphi_1(t)\right)\mu-$$
$$-(\varphi_1^2(t)+2\varphi_0(t)\cdot\varphi_2(t))\mu^2+o(\mu^2),$$

$$\varphi_0(1)+\varphi_1(1)\mu+\varphi_2(1)\mu^2+o(\mu^2)\big|_{t=1}=1.$$

Endi μ ning bir xil darajalari oldidagi koeffitsientlarni tenglashtirib, quyidagi masalalarni tuzamiz:

$$\mu^0:\ \begin{cases}\varphi_0'(t)=-\varphi_0^2(t)\\ \varphi_0(1)=1\end{cases},$$

$$\mu^1:\ \begin{cases}\varphi_1'(t)=t-2\varphi_0(t)\cdot\varphi_1(t)\\ \varphi_1(1)=0\end{cases},$$

$$\mu^2:\ \begin{cases}\varphi_2'(t)=-\varphi_1^2(t)-2\varphi_0(t)\cdot\varphi_2(t)\\ \varphi_2(1)=0\end{cases}.$$

Bu masalalarni birinchisidan boshlab ketma-ket yechamiz va quyidagilarni topamiz:

$$\varphi_0(t)=\frac{1}{t}\ ,\ \varphi_1(t)=\frac{t^2}{4}-\frac{1}{4t^2}\ ,\ \varphi_2(t)=-\frac{t^5}{112}+\frac{t}{24}-\frac{2}{21t^2}+\frac{1}{16t^3}$$

Demak, berilgan masala yechimi uchun ushbu

$$x=\frac{1}{t}+\left(\frac{t^2}{4}-\frac{1}{4t^2}\right)\mu+\left(-\frac{t^5}{112}+\frac{t}{24}-\frac{2}{21t^2}+\frac{1}{16t^3}\right)\mu^2+o(\mu^2),\ \mu\to 0,$$

asimptotik yoyilma o‘rinli. Bu yerda shuni e’tirof etaylikki, qaralgan masalaning $x = x(t,\mu)$ yechimi uchun $\left.\frac{\partial x}{\partial \mu}\right|_{\mu=0} = \frac{t^2}{4} - \frac{1}{4t^2}$. ☝

Misol 2. Ushbu

$$\begin{cases} \frac{dx}{dt} = x + \mu(x^2 + y), \\ \frac{dy}{dt} = y - \mu(x - y^2), \\ x|_{t=0} = 1, y|_{t=0} = \mu, \end{cases}$$

masala yechimining kichik parametr μ darajalari bo‘ylab yoyilmasidagi dastlabki uchta hadni toping.

🗝 Kichik parametr boshlang‘ich shartda ham uchragan. y ning o‘rniga $y - \mu$ kiritib, uni boshlang‘ich shrartdan tenglamaning o‘ng tomoniga o‘tkazish mumkin. Lekin bu ishni bajarmasdan ham kerakli yoyilmani topsa bo‘ladi. Yechim yoyilmasini yozamiz:

$$\begin{cases} x = \varphi_0(t) + \varphi_1(t)\mu + \varphi_2(t)\mu^2 + o(\mu^2) \\ y = \psi_0(t) + \psi_1(t)\mu + \psi_2(t)\mu^2 + o(\mu^2) \end{cases}, \ \mu \to 0.$$

Bu yoyilmalarni berilgan tenglamalar va boshlang‘ich shartlarga qo‘yamiz:

$$\varphi_0'(t) + \varphi_1'(t)\mu + \varphi_2'(t)\mu^2 + o(\mu^2) = \varphi_0(t) + \varphi_1(t)\mu + \varphi_2(t)\mu^2 + o(\mu^2) +$$
$$+\mu\big((\varphi_0(t) + \varphi_1(t)\mu + \varphi_2(t)\mu^2 + o(\mu^2))^2 + \psi_0(t) + \psi_1(t)\mu + \psi_2(t)\mu^2 + o(\mu^2)\big),$$
$$\psi_0'(t) + \psi_1'(t)\mu + \psi_2'(t)\mu^2 + o(\mu^2) = \psi_0(t) + \psi_1(t)\mu + \psi_2(t)\mu^2 + o(\mu^2) -$$
$$-\mu\big(\varphi_0(t) + \varphi_1(t)\mu + \varphi_2(t)\mu^2 + o(\mu^2) - (\psi_0(t) + \psi_1(t)\mu + \psi_2(t)\mu^2 + o(\mu^2))^2\big),$$
$$\big(\varphi_0(t) + \varphi_1(t)\mu + \varphi_2(t)\mu^2 + o(\mu^2)\big)\Big|_{t=0} = 1,$$
$$\big(\psi_0(t) + \psi_1(t)\mu + \psi_2(t)\mu^2 + o(\mu^2)\big)\Big|_{t=0} = \mu,$$

Bu yerdagi birinchi va ikkinchi tenglamalarning o‘ng tomonini μ ning darajalari bo‘ylab yoyamiz va barcha tengliklarda μ ning bir xil darajalari oldidagi koeffitsientlarni tenglashtirib, quyidagi boshlang‘ich masalalarni hosil qilamiz:

$$\mu^0 : \begin{cases} \varphi_0'(t) = \varphi_0(t), \\ \psi_0'(t) = \psi_0(t), \\ \varphi_0(0) = 1\,,\ \psi_0(0) = 0. \end{cases}$$

$$\mu^1 : \begin{cases} \varphi_1'(t) = \varphi_1(t) + \varphi_0^2(t) + \psi_0(t), \\ \psi_1'(t) = \psi_1(t) - \varphi_0(t) + \psi_0^2(t), \\ \varphi_1(0) = 0,\ \psi_1(0) = 1. \end{cases}$$

$$\mu^2 : \begin{cases} \varphi_2'(t) = \varphi_2(t) + 2\varphi_0(t)\varphi_1(t) + \psi_1(t), \\ \psi_2'(t) = \psi_2(t) - \varphi_1(t) + 2\psi_0(t)\psi_1(t), \\ \varphi_2(0) = 0\,,\ \psi_2(0) = 0. \end{cases}$$

Bu masalalarni yechib quyidagilarni topamiz:

$$\begin{cases} \varphi_0(t) = e^t \\ \psi_0(t) = 0 \end{cases},\ \begin{cases} \varphi_1(t) = e^{2t} - e^t \\ \psi_1(t) = (1-t)e^t \end{cases},\ \begin{cases} \varphi_2(t) = e^{3t} - 2e^{2t} + (-\frac{1}{2}t^2 + t + 1)e^t \\ \psi_2(t) = -e^{2t} + (t+1)e^t \end{cases}.$$

Demak, izlangan yoyilma quyidagi ko‘rinishda bo‘ladi:

$$\begin{cases} x = e^t + (e^{2t} - e^t)\mu + \left(e^{3t} - 2e^{2t} + (-\frac{1}{2}t^2 + t + 1)e^t\right)\mu^2 + o(\mu^2) \\ y = (1-t)e^t\mu + (-e^{2t} + (t+1)e^t)\mu^2 + o(\mu^2) \end{cases},\ \mu \to 0.$$ ✍

Misol 3. Ushbu

$$x' = x + \frac{x^2}{t^2} + t - 2,\ x(1) = a$$

masalaning yechimi uchun $\left.\frac{\partial x}{\partial a}\right|_{a=-1}$ ni hisoblang.

🗝 $x = x(t;a)$ berilgan masalaning yechimi bo‘lsin. U holda quyidagi ayniyatlar o‘rinli

$$\frac{\partial x(t;a)}{\partial t} = x(t;a) + \frac{x^2(t;a)}{t^2} + t - 2,\ x(1;a) = a\,.$$

Bu ayniyatlarni a parametr bo‘yicha differensiallaymiz va $w = w(t) = \left.\dfrac{\partial x(t;a)}{\partial a}\right|_{a=-1}$ belgilash kiritib topamiz:

$$w' = w + \frac{2}{t^2}\tilde{x} \cdot w,\ w(1) = 1,$$

bu yerda $\tilde{x}$ funksiya berilgan masalaning $a = -1$ bo‘lgandagi yechimi, ya’ni

$$\tilde{x}' = \tilde{x} + \frac{\tilde{x}^2}{t^2} + t - 2,\ \tilde{x}(1) = -1\,.$$

Oxirgi masalaning yechimi, ravshanki, $\tilde{x} = -t$. Demak, noma’lum w uchun quyidagi masala hosil bo‘ldi:

$$w' = w - \frac{2}{t}w,\ w(1) = 1.$$

Bu masalani yechib, izlangan hosilani topamiz:

$$\left.\frac{\partial x(t;a)}{\partial a}\right|_{a=-1} = w = \frac{e^t}{et^2}\,.$$

Masalalar

Masalalar yechimlarining kichik parametr μ darajalari bo‘ylab yoyilmasidagi dastlabki ikkita hadni toping (**1** - **5**):

1. $x' = t\mu - x^3,\ x(0) = 1 + \mu.$

2. $x' = x - t + \mu(x-1)^2,\ x(1) = 1.$

3. $x' = \mu e^{3t} + \mathrm{tg}x,\ x(0) = \dfrac{3\pi - 4\mu}{12}.$

4. $x'' + x + \mu x'^2 = 0,\ x(0) = a, x'(0) = b\,.$

5. $x'' - x' = \mu x^2, x(0) = 2, x'(0) = 1 - \mu\,.$

6. $\begin{cases} x' = x + 2y,\ x(0) = 0, \\ y' = x + \mu x^2,\ y(0) = 3. \end{cases}$

Masalalar yechimlarining parametr yoki boshlang‘ich ma’lumotlar bo‘yicha hosilalarini toping (**6** - **12**):

7. $tx' = \mu t^3 + \ln x,\ x(1) = 1;\ \left.\dfrac{\partial x}{\partial \mu}\right|_{\mu=0} - ?$

8. $x' = e^{x-t} + \mu e^x,\ x(0) = -1 + \mu;\ \left.\dfrac{\partial x}{\partial \mu}\right|_{\mu=0} - ?$

9. $x' = x + t^2x^4$, $x(1) = a$; $\left.\dfrac{\partial x}{\partial a}\right|_{a=0}$ $-$?

10. $x' = x + \mu y, y' = x^2 + 1, x(0) = 2, y(0) = 2$; $\left.\dfrac{\partial x}{\partial \mu}\right|_{\mu=0}$ $-$?

11. $\begin{cases} x' = x + 2y + \mu x^2, \ x(0) = 1, \\ y' = x + \mu y^2, \ y(0) = -1; \end{cases}$ $\left.\dfrac{\partial y}{\partial \mu}\right|_{\mu=0}$ $-$?

12. $\begin{cases} x' = \dfrac{x}{y} + t, \ x(1) = a, \\ y' = \dfrac{1}{t^2} - \dfrac{1}{y^2} + 1, y(1) = b; \end{cases}$ $\left.\dfrac{\partial x}{\partial b}\right|_{\substack{a=1\\b=1}}$ $-$?

Mustaqil ish № 19 topshiriqlari:

Berilgan masala yechimining kichik μ parametr darajalari bo‘ylab yoyilmasidagi dastlabki uchta hadni toping.

1. $\begin{cases} x' = y, \\ y' = x + \mu \sin y, \\ x|_{t=0} = 0, y|_{t=0} = 1 + \mu. \end{cases}$

2. $\begin{cases} x' = y, \\ y' = -x + \mu \sin t + \mu y^2, \\ x|_{t=0} = 0, \ \ y|_{t=0} = 0. \end{cases}$

3. $\begin{cases} x' = y, \\ y' = x + \mu e^y + 1, \\ x|_{t=0} = 0, \ \ y|_{t=0} = 0. \end{cases}$

4. $\begin{cases} x' = y, \\ y' = -x + \mu \ln(1 - y), \\ x|_{t=0} = 0, \ \ y|_{t=0} = \mu. \end{cases}$

5. $\begin{cases} x' = y, \\ y' = -y + \mu(e^x - 1), \\ x|_{t=0} = \mu, \ \ y|_{t=0} = 0. \end{cases}$

6. $\begin{cases} x' = 1 + \mu(x^2 + xy), \\ y' = -y + \mu(x + x^2), \\ x|_{t=0} = 1, \ \ y|_{t=0} = 0. \end{cases}$

7. $\begin{cases} x' = y + \mu x^2, \\ y' = x + \mu(e^y - 1), \\ x|_{t=0} = 1, \ y|_{t=0} = 0. \end{cases}$

8. $\begin{cases} x' = \mu(y + xy), \\ y' = x^2 - 1 + \mu y^2, \\ x|_{t=0} = 0, y|_{t=0} = 1. \end{cases}$

9. $\begin{cases} x' = 1 + \mu(x^2 + xy), \\ y' = (\mu - 1)y + x^2, \\ x|_{t=0} = 1, \ y|_{t=0} = 0. \end{cases}$

10. $\begin{cases} x' = 1 - x + \mu x^2 y, \\ y' = x - \mu x\sqrt{1 + y}, \\ x|_{t=0} = 0, \ y|_{t=0} = 0. \end{cases}$

11. $\begin{cases} x' = x/t + \mu xy, \\ y' = -y + \mu y^2, \\ x\big|_{t=1} = 1, y\big|_{t=1} = 0. \end{cases}$

12. $\begin{cases} x' = x - xy, \\ y' = 1 + \mu(x + y^2), \\ x\big|_{t=0} = 0,\ y\big|_{t=0} = \mu. \end{cases}$

13. $\begin{cases} x' = 1 - 4\mu y^2, \\ y' = y - xy + y^2, \\ x\big|_{t=0} = \mu^2,\ y\big|_{t=0} = 1 - \mu. \end{cases}$

14. $\begin{cases} x' = 2/x - \mu ty \\ y' = y + x^2 \\ x\big|_{t=0} = 1,\ \ y\big|_{t=0} = 0 \end{cases}$

15. $\begin{cases} x' = x^2 y + \mu / x, \\ y' = y + \mu x^2, \\ x\big|_{t=0} = 1,\ y\big|_{t=0} = \mu. \end{cases}$

16. $\begin{cases} x' = x + \mu xy, \\ y' = y + x^2 y, \\ x\big|_{t=0} = 0,\ y\big|_{t=0} = 1 - \mu. \end{cases}$

17. $\begin{cases} x' = x + \mu xy, \\ y' = y + \mu(x^2 + y^2), \\ x\big|_{t=0} = 1 + \mu,\ \ y\big|_{t=0} = 1. \end{cases}$

18. $\begin{cases} x' = x - y, \\ y' = 2x + \mu xy, \\ x\big|_{t=0} = 1, y\big|_{t=0} = -2. \end{cases}$

19. $\begin{cases} x' = y, \\ y' = y + 1 - \mu y^2, \\ x\big|_{t=0} = 1/2, y\big|_{t=0} = -1 + \mu. \end{cases}$

20. $\begin{cases} x' = x - \mu y^2, \\ y' = -y + \mu\sqrt[3]{2 + x}, \\ x\big|_{t=0} = -1,\ y\big|_{t=0} = 1. \end{cases}$

21. $\begin{cases} x' = x + y + \mu xy, \\ y' = 2x + \mu y^2, \\ x\big|_{t=0} = 1 - \mu,\ y\big|_{t=0} = -2. \end{cases}$

22. $\begin{cases} x' = 2t + \mu xy, \\ y' = y + x^2, \\ x\big|_{t=0} = 0, y\big|_{t=0} = 1. \end{cases}$

23. $\begin{cases} x' = x - y - \mu y^2, \\ y' = y + \mu x^3, \\ x\big|_{t=0} = 0,\ y\big|_{t=0} = \mu. \end{cases}$

24. $\begin{cases} x' = x + \mu \ln y, \\ y' = y + \mu x^2, \\ x\big|_{t=0} = -1, y\big|_{t=0} = 1. \end{cases}$

25. $\begin{cases} x' = x - \mu y^2, \\ y' = y - x^2, \\ x\big|_{t=0} = 1, y\big|_{t=0} = \mu. \end{cases}$

26. $\begin{cases} x' = x - \mu x^2 y, \\ y' = y - \ln x, \\ x\big|_{t=0} = 1,\ y\big|_{t=0} = -\mu. \end{cases}$

27. $\begin{cases} x' = 2/x - \mu ty, \\ y' = x + \mu xy, \\ x\big|_{t=0} = 2,\ y\big|_{t=0} = 1. \end{cases}$

28. $\begin{cases} x' = x + \mu xy, \\ y' = y - \mu x^2, \\ x\big|_{t=0} = 1, y\big|_{t=0} = 1 + \mu. \end{cases}$

29. $\begin{cases} x' = y - t + \mu y^2, \\ y' = y - \mu x^2, \\ x\big|_{t=0} = 1, y\big|_{t=0} = 1. \end{cases}$ **30.** $\begin{cases} x' = x + \mu\sqrt{1+y}, \\ y' = y + x^2, \\ x\big|_{t=0} = 0,\ y\big|_{t=0} = 1. \end{cases}$

20. BIRINCHI TARTIBLI XUSUSIY HOSILALI DIFFERENSIAL TENGLAMALAR

Maqsad – chiziqli va kvazichiziqli birinchi tartibli xususiy hosilali differensial tenglamalar yechimlarini qurishni o‘rganish

Yordamchi ma’lumotlar:

I. Ushbu $u = u(x) = u(x_1, x_2, ..., x_n)$ noma’lum funksiyaga nisbatan **birinchi tartibli xususiy hosilali differensial tenglama** deb

$$F\left(x_1, x_2, ..., x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n}\right) = 0 \qquad (1)$$

ko‘rinishdagi tenglamaga aytiladi; bu yerda $F(x_1, x_2, ..., x_n, u, p_1, p_2, ..., p_n)$ - $2n+1$ haqiqiy o‘zgaruvchining berilgan haqiqiy funksiyasi, u o‘zgaruvchilarning biror $G \subset R^{2n+1}$ sohasida C^1 sinfga tegishli va $\left|\frac{\partial F}{\partial p_1}\right| + \left|\frac{\partial F}{\partial p_2}\right| + ... + \left|\frac{\partial F}{\partial p_n}\right| \neq 0$ deb hisoblanadi.

(1) tenglamaning D sohada aniqlangan yechimi deb shunday $u = \varphi(x)$ funksiyaga aytiladiki, uning barcha birinchi tartibli xususiy hosilalari D sohada mavjud va uzluksiz (ya’ni $\varphi(x) \in C^1(D, \mathbb{R})$) hamda u D sohada (1) tenglamani ayniyatga aylantiradi:

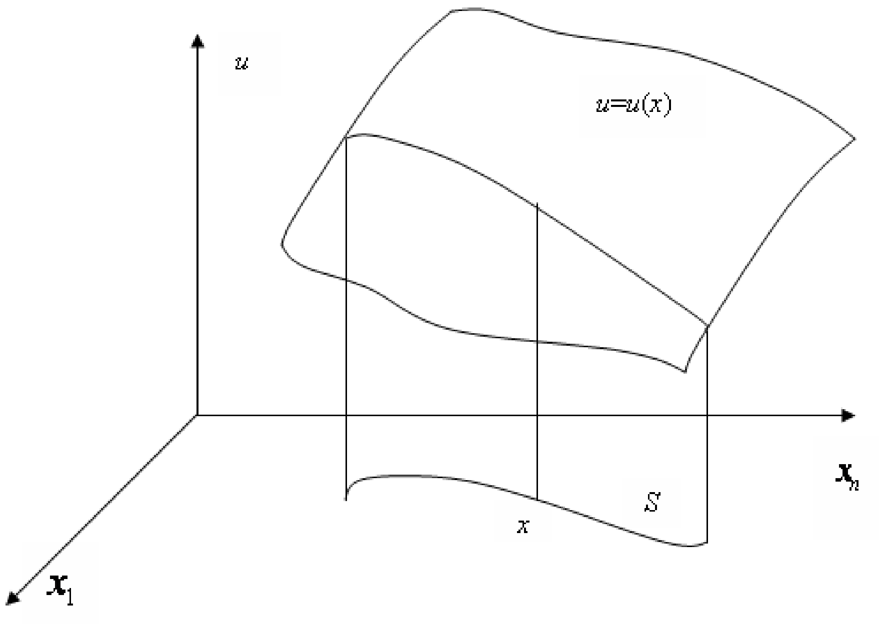
$$F\left(x_1, x_2, ..., x_n, \varphi(x), \frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2}, ..., \frac{\partial \varphi}{\partial x_n}\right) \equiv 0\ ,\ x = (x_1, x_2, ..., x_n) \in D.$$

(1) tenglama uchun Koshi masalasi (yoki boshlang‘ich masala) quyidagicha qo‘yiladi. $\mathbb{R}^n$ fazoda joylashgan S (20.1- rasm) silliq gipersirt $(\dim S = n-1)$ hamda S da aniqlangan $\varphi_0(x)$ funksiya beriladi va (1)

tenglamaning biror D, $S \subset D$, sohada aniqlangan va S da φ_0 ga aylanuvchi yechimini topish talab etiladi:

$$\begin{cases} F(x_1, x_2, ..., x_n, u, \dfrac{\partial u}{\partial x_1}, \dfrac{\partial u}{\partial x_2}, ..., \dfrac{\partial u}{\partial x_n}) = 0, \ x \in D \\ u(x) = \varphi_0(x), \ x \in S \end{cases}$$

Bu yerdagi $u(x) = \varphi_0(x), x \in S$, shart boshlang‘ich shart yoki Koshi sharti deyiladi.



20.1- rasm. Koshi masalasining qo‘yilishi.

Misol 1. *a).* $n = 3$ va soddalik uchun $x_1 = x, x_2 = y, x_3 = z$ deb, ushbu

$$yzu'_x + xzu'_y - 2xyu'_z = 0, (x, y, z) \in \mathbb{R}^3,$$

tenglamani qaraylik.

🗝 Ixtiyoriy $h \in C^1(\mathbb{R})$ funksiyaga ko‘ra tuzilgan $\varphi(x, y, z) = h(x^2 + y^2 + z^2)$ funksiya berilgan tenglamaning $D = \mathbb{R}^3$ sohada aniqlangan yechimini beradi. Haqiqatan ham, $\varphi \in C^1(\mathbb{R})$ ekanligi ravshan va

$$yz\varphi'_x + xz\varphi'_y - 2xy\varphi'_z = yzh' \cdot 2x + xzh' \cdot 2y - 2xyh' \cdot 2z = 0, (x, y, z) \in \mathbb{R}^3.$$

b). Ushbu

$$u = \frac{x^2}{y^2} + \frac{y^2}{z^2}$$

funksiya $xu'_x + yu'_y + zu'_z = 0$ tenglamaning $y > 0,\ z > 0$ sohada aniqlangan yechimi ekanligini tekshiraylik.

Berilgan funksiya $y > 0,\ z > 0$ sohada uzluksiz differensiallanuvchi, chunki uning birinchi tartibli xususiy hosilalari usluksiz:

$$u'_x = \frac{2x}{y^2} \in C(y > 0, z > 0)\ ,\ u'_y = -\frac{2x^2}{y^3} + \frac{2y}{z^2} \in C(y > 0, z > 0)\ ,$$

$$u'_z = -\frac{2y^2}{z^3} \in C(y > 0, z > 0).$$

Berilgan funksiya $y > 0,\ z > 0$ sohada berilgan tenglamani ayniyatga ham aylantiradi:

$$xu'_x + yu'_y + zu'_z = x \cdot \frac{2x}{y^2} + y\left(-\frac{2x^2}{y^3} + \frac{2y}{z^2}\right) + z \cdot \left(-\frac{2y^2}{z^3}\right) = 0\ (y > 0, z > 0)\ .$$

Birinchi tartibli xususiy hosilali chiziqli tenglamaning ko‘rinishi quyidagicha:

$$f_1(x)\frac{\partial u}{\partial x_1} + f_2(x)\frac{\partial u}{\partial x_2} + ... + f_n(x)\frac{\partial u}{\partial x_n} = g(x)u + f_0(x);$$

bu yerda $f_1, f_2, ..., f_n, g, f_0$ – berilgan funksiyalar. **Kvazichiziqli tenglama** deb esa

$$f_1(x,u)\frac{\partial u}{\partial x_1} + f_2(x,u)\frac{\partial u}{\partial x_2} + ... + f_n(x,u)\frac{\partial u}{\partial x_n} = g(x,u)$$

ko‘rinishdagi tenglamaga aytiladi. Bu tenglama $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n}$ hosilalarga nisbatan chiziqli, $f_1, f_2, ..., f_n$ koeffitsientlar va o‘ng tomondagi g funksiya $u = u(x)$ noma’lum funksiyaga bog‘liq bo‘lishi mumkin, lekin uning hosilalariga bog‘liq emas.

II. Ushbu

$$f_1(x)\frac{\partial u}{\partial x_1}+f_2(x)\frac{\partial u}{\partial x_2}+...+f_n(x)\frac{\partial u}{\partial x_n}=0,\ x\in D\subset\mathbb{R}^n, \tag{2}$$

$$\left(f_j\in C^1(D),\ j=\overline{1,n};\sum_{j=1}^{n}\left|f_j(x)\right|>0, x\in D\right)$$

ko'rinishdagi chiziqli tenglamani qaraylik. Ushbu

$$\frac{dx_1}{f_1(x)}=\frac{dx_2}{f_2(x)}=...=\frac{dx_n}{f_n(x)} \tag{3}$$

oddiy differensial tenglamalar sistemasi (1) tenglamaning xarakteristik sistemasi deyiladi. (3) ning yechimlari D da egri chiziqlarni aniqlaydi. Bu egri chiziqlar (2) tenglamaning xarakteristikalari deb ataladi. Bizning (2) farazimizga ko'ra D ning har bir nuqtasidan yagona xarakteristika o'tadi.

Xarakteristik sistema (3) ning ixtiyoriy $u=u(x)$ birinchi integrali (2) tenglamaning yechimi bo'ladi va aksincha, ya'ni (2) ning ixtiyoriy $u=u(x)$ yechimi (3) sistemaning birinchi integralini beradi.

(2) tenglama umumiy yechimining tuzilishini quyidagi teorema ochadi.

Teorema 1. Agar $u_1(x), u_2(x), ..., u_{n-1}(x)$ funksiyalar (3) sistemaning erkli birinchi integrallari ((2) tenglamaning erkli yechimlari) bo'lsa, u holda ixtiyoriy

$$u=\Phi(u_1(x),u_2(x),...,u_{n-1}(x)),\ \Phi\in C^1, \tag{4}$$

ko'rinishdagi funksiya (2) tenglamaning yechimidir. Ixtiyoriy $x^0\in D$ nuqtaning biror atrofida (2) tenglamaning shunday erkli $u_1(x), u_2(x), ..., u_{n-1}(x)$ yechimlari mavjudki, (2) tenglamaning shu atrofda aniqlangan har qanday yechimi (4) ko'rinishda yoziladi. Demak, (4) formula $x^0\in D$ nuqtaning biror atrofida (2) tenglamaning umumiy yechimini ifodalaydi.

Misol 2. Ushbu

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}-xy\frac{\partial u}{\partial z}=0 \tag{5}$$

tenglamaning umumiy yechimini topaylik.

🗝 Berilgan (5) tenglama uchun xarakteristik sistemani tuzamiz ((3) ga qarang):

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{-xy}.$$

Bu sistemaning 2 ta erkli birinchi integralini topishimiz kerak. Buning uchun quyidagilarni bajaramiz:

$$\frac{dx}{x}=\frac{dy}{y}\Rightarrow ydx-xdy=0\Rightarrow d\left(\frac{x}{y}\right)=0\ \Rightarrow u_1=\frac{x}{y}\ ;$$

$$\left.\begin{aligned} dx&=-\frac{dz}{y}\\ dy&=-\frac{dz}{x}\end{aligned}\right\}\Rightarrow \left.\begin{aligned} ydx+dz=0\\ xdy+dz=0\end{aligned}\right\}\Rightarrow d(xy+2z)=0\Rightarrow u_2=xy+2z\ .$$

Ravshanki, topilgan u_1 va u_2 birinchi integrallar erkli va, demak, qaralayotgan (5) tenglamaning umumiy yechimi

$$u=\Phi\left(\frac{x}{y},xy+2z\right),\ \Phi\in C^1,$$

ko‘rinishda ifodalanadi. ✍

Chiziqli tenglamaning yechimini qurishda ba’zan superpozitsiya prinsipidan foydalanish qo‘l keladi.

Misol 3. Ushbu

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}-xy\frac{\partial u}{\partial z}=x+y$$

tenglamaning umumiy yechimini quraylik.

🗝 Ravshanki, $u = x$ va $u = y$ funksiyalari - bu tenglamaning o‘ng tomonini mos ravishda x va y bilan almashtirishdan hosil bo‘lgan tenglamalarning yechimi; demak, ularning yig‘indisi $u = x + y$ berilgan tenglamaning yechimi. Endi berilgan tenglamada

$$u = x + y + v \tag{*}$$

deb almashtirish bajarsak, v noma’lum funksiyaga nisbatan ushbu

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} - xy\frac{\partial v}{\partial z} = 0$$

bir jinsli tenglamani hosil qilamiz. Oxirgi tenglamaning umumiy yechimi misol 2 da topilgan edi:

$$v = \Phi(\frac{x}{y}, xy + 2z)\ ,\ \Phi \in C^1.$$

Endi (*) almashtirishga ko‘ra berilgan tenglamaning umumiy yechimini yozamiz:

$$v = x + y + \Phi(\frac{x}{y}, xy + 2z)\ ,\ \Phi \in C^1.$$ 👍

(2) tenglamaning berilgan boshlang‘ich shartni qanoatlantiruvchi yechimini topish uchun umumiy yechim (4) ni topib, boshlang‘ich shart yordamida bu yerdagi Φ funksiyani aniqlash kerak bo‘ladi.

Misol 4. Ushbu

$$\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = 0 \tag{6}$$

tenglamaning $u|_{x=1} = y + z$ shartni qanoatlantiruvchi yechimini topaylik.

🗝 Dastlab

$$\frac{dx}{1} = \frac{dy}{2} = \frac{dz}{3}$$

xarakteristik sistemaning 2 ta erkli birinchi integralini topamiz:

$$\frac{dx}{1}=\frac{dy}{2}\Rightarrow\ 2x-y=c_1.$$

$$\frac{dx}{1}=\frac{dz}{3}\ \Rightarrow\ 3x-z=c_2.$$

Demak, berilgan (6) tenglamaning umumiy yechimi

$$u=\Phi(2x-y,3x-z),\ \Phi\in C^1, \tag{7}$$

ko‘rinishda bo‘ladi. Endi Φ funksiyani boshlang‘ich shartning qanoatlanishidan topamiz:

$$u|_{x=1}=y+z\ \Rightarrow\ y+z=\Phi(2-y,3-z). \tag{8}$$

Oxirgi tenglikda $2-y=t$, $3-z=s$ deymiz. Bunda $y=2-t$, $z=3-s$ va (8) ga ko‘ra $2-t+3-s=\Phi(t,s)$ bo‘ladi. Demak,

$$\Phi(t,s)=5-t-s\,.$$

Izlangan yechim (7) ga ko‘ra topiladi: $u=5-(2x-y)-(3x-z)$, ya’ni $u=5-5x+y+z$. 👍

III. Endi $O\subset\mathbb{R}^{n+1}$ sohada aniqlangan ushbu

$$f_1(x,u)\frac{\partial u}{\partial x_1}+f_2(x,u)\frac{\partial u}{\partial x_2}+...+f_n(x,u)\frac{\partial u}{\partial x_n}=g(x,u)\ ,\ (x,u)\in O, \tag{9}$$

$$\left(\text{bunda } g,\ f_j\in C^1(O),\ j=\overline{1,n};\ \sum_{j=1}^{n}\left|f_j(x,u)\right|>0,\ (x,u)\in O\right)$$

kvazichiziqli tenglamani qaraylik. (9) tenglamaning xarakteristik sistemasi quyidagicha:

$$\frac{dx_1}{f_1(x,u)}=\frac{dx_2}{f_2(x,u)}=...=\frac{dx_n}{f_n(x,u)}=\frac{du}{g(x,u)}\,. \tag{10}$$

Bu sistemaning O sohadagi yechimlari (9) tenglamaning xarakteristikalari deb ataladi. (9) tenglamaning yechimi $u=u(x)$ gipersirtni ($n+1$o‘lchamli fazodagi n o‘lchamli sirtni) ifodalaydi. $u=u(x)$ gipersirt (9) ning yechimi bo‘lishi uchun uning xarakteristikalardan tuzilgan bo‘lishi yetarli va zarurdir.

Agar $v(x,u)$ ($v(x_1, x_2,..., x_n, u)$) funksiya (10) sistemaning birinchi integrali va $(x^0, u_0) \in O$ nuqtada $v(x^0, u_0) = v_0$, $\dfrac{\partial v(x^0, u_0)}{\partial u} \neq 0$ bo‘lsa, u holda $v(x,u) = v_0$ tenglik (x^0, u_0) nuqtaning biror atrofida (9) tenglamaning oshkormas ko‘rinishdagi $u = u(x)$ yechimini aniqlaydi.

(9) kvazichiziqli tenglama umumiy yechimining tuzilishi quyidagi teoremada ochiladi.

Teorema 2. Aytaylik, $v_1(x,u), v_2(x,u),..., v_n(x,u)$ funksiyalar (10) xarakteristik sistemaning n dona erkli birinchi integrallari hamda $(x^0, u_0) \in O$, $v_1(x^0, u_0) = v^0$, $v_2(x^0, u_0) = v^0, ..., v_n(x^0, u_0) = v^0$ bo‘lsin. Φ funksiya $(v_1^0, v_2^0, ..., v_n^0) \in \mathbb{R}^n$ nuqtaning biror atrofida $\in C^1$, $\Phi(v_1(x,u), v_2(x,u),..., v_n(x,u))$ murakkab funksiyaning u bo‘yicha hosilasi (x^0, u_0) nuqtada noldan farqli ($\Phi'_u|_{(x^0, u_0)} \neq 0$) hamda $\Phi(v_1^0, v_2^0, ..., v_n^0) = 0$ shartlar bajrilsin. U holda

$$\Phi(v_1(x,u), v_2(x,u),..., v_n(x,u)) = 0 \tag{11}$$

tenglik (9) tenglamaning $x^0 \in \mathbb{R}^n$ nuqta atrofida aniqlangan $u = u(x)$, $u(x^0) = u^0$, oshkormas yechimini beradi; aksincha, ixtiyoriy bunday $u = u(x)$ yechim yuqoridagi xususiyatlarga ega bo‘lgan Φ funksiya orqali (11) tenglik bilan oshkormas ko‘rinishda beriladi. Demak, (11) munosabat (9) tenglamaning umumiy yechimini ifodalaydi (lokal).

(9) tenglama uchun Koshi masalasini yechish uchun tenglamaning umumiy yechimini (11) ko‘rinishda yozib, boshlang‘ich shartdan foydalanib noma’lum Φ funksiyani aniqlash lozim.

Boshqacha ish tutish ham mumkin. Aytaylik, (9) tenglamaning

$$u|_{x_k = x_k^0} = \varphi_0(x_1, x_2, ..., x_{k-1}, x_{k+1}, ..., x_n) \tag{12}$$

boshlang‘ich shartni qanoatlantiruvchi yechimini topish kerak bo‘lsin. Faraz qilaylik, (10) sistemaning n dona erkli birinchi integrallari topilgan bo‘lsin:

$$\begin{cases} v_1(x_1, x_2, ..., x_{k-1}, x_k, x_{k+1}, ..., x_n, u) = c_1, \\ v_2(x_1, x_2, ..., x_{k-1}, x_k, x_{k+1}, ..., x_n, u) = c_2, \\ .... \\ v_n(x_1, x_2, ..., x_{k-1}, x_k, x_{k+1}, ..., x_n, u) = c_n. \end{cases} \tag{13}$$

Bu yerda boshlang‘ich shartga ko‘ra $x_k = x_k^0$ va $u = \varphi_0$ deymiz:

$$\begin{cases} v_1(x_1, x_2, ..., x_{k-1}, x_k^0, x_{k+1}, ..., x_n, \varphi_0) = c_1, \\ v_2(x_1, x_2, ..., x_{k-1}, x_k^0, x_{k+1}, ..., x_n, \varphi_0) = c_2, \\ .... \\ v_n(x_1, x_2, ..., x_{k-1}, x_k^0, x_{k+1}, ..., x_n, \varphi_0) = c_n. \end{cases} \tag{14}$$

Hosil bo‘lgan bu (14) sistemadan $x_1, x_2, ..., x_{k-1}, x_{k+1}, ..., x_n$ o‘zgaruvchilarni yo‘qotib, quyidagi bog‘lanishni hosil qilamiz:

$$\Psi(x_k^0, c_1, c_2, ..., c_n) = 0.$$

Bu yerdagi $c_1, c_2, ..., c_n$ larni (13) birinchi integrallardagi mos qiymatlari $v_1(x,u), v_2(x,u), ..., v_n(x,u)$ bilan almashtirib, qaralayotgan (9),(12) Koshi masalasining oshkormas yechimini topamiz:

$$\Psi(x_k^0, v_1(x,u), v_2(x,u), ..., v_n(x,u)) = 0. \tag{15}$$

Agar boshlang‘ich shart ushbu

$$\varphi_1(x_1, x_2, ..., x_n, u) = 0,\ \varphi_2(x_1, x_2, ..., x_n, u) = 0 \tag{16}$$

oshkormas ko‘rinishda berilgan bo‘lsa, u holda

$$\begin{cases} \varphi_1(x_1, x_2, ..., x_n, u) = 0, \\ \varphi_2(x_1, x_2, ..., x_n, u) = 0, \\ v_1(x_1, x_2, ..., x_n, u) = c_1, \\ v_2(x_1, x_2, ..., x_n, u) = c_2, \\ .... \\ v_n(x_1, x_2, ..., x_n, u) = c_n. \end{cases} \tag{17}$$

sistemadan $x_1, x_2, ..., x_n, u$ o‘zgaruvchilarni yo‘qotib

$$\mathrm{H}(c_1, c_2, ..., c_n) = 0$$

bog‘lanishni hosil qilamiz va $c_1, c_2, ..., c_n$ larni birinchi integrallardagi mos qiymatlari $v_1(x,u), v_2(x,u), ..., v_n(x,u)$ bilan almashtirib qaralayotgan (9),(16) Koshi masalasining oshkormas yechimini topamiz:

$$\mathrm{H}(v_1(x,u), v_2(x,u), ..., v_n(x,u)) = 0. \tag{18}$$

Misol 5. Koshi masalasini yeching

$$\begin{cases} xuu'_x + yuu'_y = -(x^2 + y^2) \\ u|_{x=1} = y \end{cases} \quad \left(x > 0, y > 0\right). \tag{19}$$

🗝 Dastlab berilgan tenglamaning umumiy yechimini quramiz. Buning uchun xarakteristik sistemani yozamiz:

$$\frac{dx}{xu} = \frac{dy}{yu} = \frac{du}{-(x^2 + y^2)}.$$

Birinchi integrallarni topamiz:

$$\frac{dx}{xu} = \frac{dy}{yu} \Rightarrow \frac{dx}{x} = \frac{dy}{y} \Rightarrow \frac{y}{x} = c_1;$$

$$\frac{dy}{yu} = \frac{du}{-(x^2 + y^2)} \Rightarrow \frac{du}{dy} = -\frac{x^2 + y^2}{yu} \left[y = c_1 x\right] \Rightarrow udu = -x(1 + c_1^2)dx \Rightarrow$$

$$\Rightarrow u^2 + x^2(1 + c_1^2) = c_2 \left[c_1 = \frac{y}{x}\right] \Rightarrow x^2 + y^2 + u^2 = c_2.$$

Topilgan $\frac{y}{x} = c_1$ va $x^2 + y^2 + u^2 = c_2$ birinchi integrallar erkli. Demak, berilgan tenglamaning $u = u(x,y)$ yechimi uchun

$$\Phi\left(\frac{y}{x}, x^2 + y^2 + u^2\right) = 0$$

munosabatni hosil qilamiz; bu yerda $\Phi -$ ikki o‘zgaruvchining ($\in C^1$) funksiyasi. Oxirgi tenglikni $x^2 + y^2 + u^2$ ga nisbatan yechib, qaralayotgan tenglamaning

$$x^2 + y^2 + u^2 = \psi\left(\frac{y}{x}\right) \tag{20}$$

ko‘rinishdagi yechimini hosil qilamiz. Endi $u|_{x=1}=y$ Koshi shartiga ko‘ra (20) dan topamiz:

$$1+y^2+y^2=\psi(y)\;\Rightarrow\;\psi(\tfrac{y}{x})=1+2\left(\frac{y}{x}\right)^2.$$

Oxirgi formulani (20) ga qo‘yib, berilgan (19) Koshi masalasining $u=u(x,y)$ yechimi uchun

$$x^2+y^2+u^2=1+2\left(\frac{y}{x}\right)^2$$

tenglamani hosil qilamiz. Bundan

$$u=\frac{\sqrt{2y^2+x^2-x^4-x^2y^2}}{x}\quad\left(x>0,y>0\right).$$ ♘

Misol 6. Quyidagi Koshi masalasini yechaylik:

$$\begin{cases} xyu'_x+(x+u)u'_y=yu, \\ u|_{x=1}=y^2. \end{cases}$$

⊶ Xarakteristik sistemani tuzamiz:

$$\frac{dx}{xy}=\frac{dy}{x+u}=\frac{du}{yu}.$$

Ikkita erkli birinchi integrallarni topamiz:

$$\frac{dx}{xy}=\frac{du}{yu}\;\Rightarrow\;\frac{u}{x}=c_1,$$

$$\frac{dx}{xy}=\frac{dy}{x+u}\;\Rightarrow\;\frac{dx}{xy}=\frac{dy}{x+c_1x}\;\Rightarrow\;\frac{dx}{y}=\frac{dy}{1+c_1}\Rightarrow$$

$$\Rightarrow y^2-2(1+c_1)x=c_2\Rightarrow y^2-2(1+\tfrac{u}{x})x=c_2\Rightarrow$$

$$\Rightarrow y^2-2x-2u=c_2.$$

Endi quyidagi sistemadan x,y,u o‘zgaruvchilarni yo‘qotamiz:

$$\begin{cases} x = 1, \\ u = y^2, \\ \dfrac{u}{x} = c_1, \\ y^2 - 2x - 2u = c_2. \end{cases}$$

Quyidagiga ega bo‘lamiz:

$$-2 - c_1 = c_2.$$

Bu yerda $c_1 = \dfrac{u}{x}$, $c_2 = y^2 - 2x - 2u$ deb qo‘yilgan Koshi masalasining $u = u(x, y)$ yechimini oshkormas ko‘rinishda hosil qilamiz:

$$-2 - \frac{u}{x} = y^2 - 2x - 2u.$$

Bu tenglikdan u ni osongina topamiz:

$$u = \frac{x(y^2 - 2x + 2)}{2x - 1}.$$

Izoh. Berilgan Koshi masalasining yechimini misol 5 dagiga o‘xshash fikr yuritib ham topish mumkin:

$$\Phi\left(\frac{u}{x}, y^2 - 2x - 2u\right) = 0 \Rightarrow y^2 - 2x - 2u = \psi\left(\frac{u}{x}\right) \Rightarrow ,$$

$$\Rightarrow y^2 - 2 - 2y^2 = \psi\left(y^2\right) \Rightarrow \psi\left(t\right) = -2 - t;$$

Demak, $y^2 - 2x - 2u = -2 - \dfrac{u}{x}$ va $u = \dfrac{x(y^2 - 2x + 2)}{2x - 1}$.

Masalalar

Differensial tenglamalarni yeching (**1 - 6**):

1. $(xu + y)\dfrac{\partial u}{\partial x} + (x + yu)\dfrac{\partial u}{\partial y} = u^2$. **2.** $(x + u)\dfrac{\partial u}{\partial x} + (y + u)\dfrac{\partial u}{\partial y} + (z + u)\dfrac{\partial u}{\partial z} = u$.

3. $(z + u)\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + (x + u)\dfrac{\partial u}{\partial z} = \dfrac{u^3}{2}$. **4.** $xy\dfrac{\partial u}{\partial x} - 3x^4\dfrac{\partial u}{\partial y} = y\sqrt{u^2 + 1}$.

5. $2xy\dfrac{\partial u}{\partial x} + (x^2 + y^2)\dfrac{\partial u}{\partial y} = u^2$. **6.** $xy\dfrac{\partial u}{\partial x} + x^2u^2\dfrac{\partial u}{\partial y} = yu$.

Differensial tenglamalarning umumiy yechimini toping. Undan foydalanib ko‘rsatilgan shartni qanoatlantiruvchi yechimni aniqlang (**7** - **20**):

7. $x\frac{\partial u}{\partial y}-y\frac{\partial u}{\partial x}=0,\ u|_{x=1}=y^2$.

8. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x-y,\ u|_{y=x^2}=x^2\ (x>0)$.

9. $(x^2+y^2)\frac{\partial u}{\partial x}+2xy\frac{\partial u}{\partial y}+u=0,\ u|_{x=2y}=y\ (y>0)$.

10. $(x+y^2+z)\frac{\partial u}{\partial x}+2y\frac{\partial u}{\partial y}+y^2\frac{\partial u}{\partial z}=0,\ u|_{y=1}=x+3z$.

11. $x\frac{\partial u}{\partial x}-3(xz^2+z)\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=1,\ u|_{z=1}=\ln x+\frac{y}{x}$.

12. $z(x+\cos y)\frac{\partial u}{\partial x}-yz\frac{\partial u}{\partial y}+y\cos y\frac{\partial u}{\partial z}=0,\ u|_{x=z^2/y}=z^2\ (0<y<\pi/2)$.

13. $(1+xe^z)u'_x-ye^zu'_y-yu'_z=0,\ u|_{x=1}=z+e^z$.

14. $2x^2zu'_x+(1-2xyz)u'_y+xu'_z=0,\ u|_{x=1}=y+z\ (z>0)$.

15. $xu'_x+yu'_y=2xyu,\ u|_{x=1}=y^2$.

16. $x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=(x-y)u^2,\ u|_{x=1}=\frac{1}{y}$.

17. $(2x-u)\frac{\partial u}{\partial x}+(2y-u)\frac{\partial u}{\partial y}=-2u,\ u|_{x=y+1}=y$.

18. $u^2\frac{\partial u}{\partial x}-xy\frac{\partial u}{\partial y}=xu,\ u|_{x=y}=1$.

19. $2y\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=u^2,\ u|_{x=y^2}=-\frac{1}{y}$.

20. $2y\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=u^2,\ u|_{x=y^2}=y$.

Mustaqil ish № 20 topshiriqlari:

1) Dastlab berilgan differensial tenglamaning umumiy yechimini toping.

2) Tekshirishni bajaring, ya’ni topilgan funksiyaning haqiqatan ham yechim ekanligiga ishonch hosil qiling.

3) So‘ngra tenglamaning ko‘rsatilgan shartni qanoatlantiruvchi yechimini aniqlang.

1. $xu'_x + 2yu'_y + 3zu'_z = u^3,\ u|_{z=1} = \dfrac{1}{\sqrt{x+y}}.$

2. $xu'_x + 2yu'_y + 3zu'_z = u + u^2,\ u|_{x=1} = \dfrac{1}{y+z}.$

3. $xu'_x + 2yu'_y + 3zu'_z = u^2,\ u|_{z=1} = \dfrac{1}{x+y}.$

4. $xu'_x + yu'_y + (x+u)u'_z = u,\ u|_{x=1} = y + z.$

5. $yuu'_x + xuu'_y = e^u,\ u|_{x=0} = \ln y.$

6. $xu'_x + yu'_y + zu'_z = x + y + z + u,\ u|_{x=e} = 2y + 2z.$

7. $u'_x + 2u'_y + 3u'_z = xyz,\ u|_{x=0} = y^2 z^2.$

8. $xu'_x - yu'_y = (x+y)u^2,\ u|_{y=2x} = x + 1.$

9. $xu'_x + xyu'_y = u,\ u|_{y=1} = e^x.$

10. $\sqrt{x}u'_x + \sqrt{y}u'_y + u'_z = u^2,\ \ u|_{x=4y} = \dfrac{1}{z - \sqrt{y}}.$

11. $u'_x - xyu'_y = u,\ u|_{y=1} = e^{x^2+x}.$

12. $xu'_x + 2yu'_y + 3zu'_z = u,\ u|_{y=x} = x + z.$

13. $xu'_x + yu'_y + zu'_z = xyz,\ u|_{z=2} = xy.$

14. $xu'_x + yu'_y + zu'_z = x + y + z,\ u|_{z=1} = 2x + y.$

15. $yzu'_x + xzu'_y + xyu'_z = xyz,\ u|_{y=1} = z.$

16. $uu'_x + u'_y + u'_z = 0,\ u|_{y=1} = x + 2z.$

17. $yuu'_x - xuu'_y = e^u,\ u|_{y=1} = \ln x.$

18. $\cos y \cdot \cos z \cdot u'_x + u'_y + u'_z = u - 1,\ u|_{y=0} = 1 + ze^y.$

19. $yzu'_x + xzu'_y + xyu'_z = 0,\ u|_{x=1} = y + z.$

20. $xu'_x + yu'_y + (z+u)u'_z = u,\ u|_{x=e} = y + z.$

21. $xu'_x + yu'_y + (x+z)u'_z = u + x + y,\ u|_{x=e} = y.$

22. $xu'_x + yu'_y + (x+y+z)u'_z = u + x,\ u|_{x=e} = y + z.$

23. $\sqrt{x}u'_x + \sqrt{y}u'_y + u'_z = u,\ u|_{z=0} = xe^{3\cdot\sqrt{y}}.$

24. $\sin y \cdot u'_x + u'_y + u'_z = 0,\ u|_{y=\pi/2} = x + z.$

25. $xu'_x + yu'_y + zu'_z = y,\ u|_{y=1} = x + z.$

26. $xu'_x + yu'_y + zu'_z = 1,\ u|_{x=2} = yz.$

27. $(y+1)u'_x + xu'_y = u,\ u|_{y=1} = x^2.$

28. $(u+x)u'_x + (u-y)u'_y - zu'_z = 0,\ u|_{y=0} = \sqrt{1+x+z}.$

29. $u'_x + 2u'_y + 3u'_z = u,\ u|_{y=0} = (x+2y)e^x.$

30. $u'_x + 2u'_y + 3u'_z = u^2,\ u|_{z=0} = \dfrac{1}{1+2x+y}.$

31. $u'_x + (2e^x - y)u'_y = 0,\ u|_{y=1} = e^x.$

32. $u'_x + (2e^x - y)u'_y = u,\ u|_{y=1} = (1+x)e^x.$

33. $2u'_x + (2e^x - y)u'_y = 1,\ u|_{y=0} = 1 - \dfrac{x}{2}.$

34. $xu'_x - yu'_y = x + y,\ u|_{x=4y} = 1 + \ln y.$

35. $xu'_x + yu'_y + zu'_z = \ln(x+y+z),\ u|_{z=1} = 1 + x + y.$

36. $xu'_x + yu'_y + zu'_z = \sqrt{u},\ u|_{y=1} = 1 + \dfrac{1}{2}\ln x.$

37. $(x+u)u'_x + (y+u)u'_y + (z+u)u'_z = u,\ u|_{x=0} = y + z.$

38. $xu'_x - yu'_y = (x-2y)u^2,\ u|_{y=x} = \dfrac{1}{1+x}.$

39. $xu'_x + yu'_y = (x+y)u^2,\ u|_{y=2x} = \dfrac{1}{1+x}.$

40. $(x^2 - y^2 - z^2)u'_x + 2xyu'_y + 2xzu'_z = 1,\ u|_{z=1} = 2 + x^2 + y.$

21. DIFFERENSIAL TENGLAMALARNI MAPLE YORDAMIDA TEKSHIRISH

Maqsad – MAPLE (bu inglizcha so‘z meypl deb o‘qiladi) kompyuter matematikasi haqida tasavvurga ega bo‘lish, MAPLE yordamida differensial tenglamalarni, ular uchun qo‘yilgan masalalarni yechishni va yechimlar tabiatini o‘rganish

Yordamchi ma’lumotlar:

Hozirgi kunda MAPLE, MATEMATICA, MATLAB va MATHCAD kabi kompyuter dasturlarlari mavjudki, ular yordamida sof matematikada va amaliy fanlarda uchraydigan ko‘pdan-ko‘p matematik masalalarni analitik (simvolli) va sonli yechish mumkin. Bu dasturlar (sistemalar) foydalanuvchi uchun qulay bo‘lgan interfeyslar(muloqot oynalari)ga ega.

Biz Maple sistemasi yordamida oddiy differensial tenglamalarni tekshirish (ularni yechish, yechim tabiatini o‘rganish va hk.) buyruqlari bilan tanishamiz. Maplening turli versiyalari mavjud, masalan: Maple 8, Maple 9.5, Maple 13, Maple 14. Quyida keltirilgan misollar Maple 13 da bajarilgan.

Maple arifmetik hisoblashlarni quvvatli kalkulyator kabi bajaradi, harfli ifodalarni ixchamlaydi, tenglamalarni analitik usullar bilan birgalikda sonli usullar yordamida ham yechadi, tekislikda va fazoda grafiklar quradi.

Maple kompyuterga yuklangach, displeyda ishchi varaq (work sheet) paydo bo‘ladi. Ishchi varaqda buyruqlar kiritiladi va natijalar chiqariladi. Maple buyruqlar majmuasi bilan to‘laligicha uning Help (yordam) imkoniyati orqali tanishish mumkin. Buyruq kiritish sohasi (satri) ushbu “>” belgi bilan boshlanadi. Shu belgidan keyin kerakli buyruqni va “;” belgini (buyruqning tugallanganlik belgisi) terib, “Enter” tugmasi bosilgach, natija displeyning chiqarish sohasida paydo bo‘ladi.

Mapleda tenglamalardan noma’lum son(lar)ni topish uchun `solve` buyrug‘i ishlatiladi. Masalan, ushbu $2x^2 - x - 3 = 0$ tenglamani $x$ga nisbatan yechishni quyidagicha bajarish mumkin

> solve(2*x^2-x-3=0,x);

$$\frac{3}{2},\ -1$$

> **solve (2*x^3 -x^2-2*a^2*x+a^2=0,x);**

$$\frac{1}{2},\ a,\ -a$$

> solve(eq,a);

$$x,\ -x$$

Funksiyani aniqlash operaratori yordamida funksiyalarni aniqlash (kiritish) mumkin. Masalan, $f(x) = 2\ln x + \exp(3x) + x^2$ funksiyani aniqlash quyidagicha amalga oshiriladi:

> f:=x->2*ln(x)+exp(3*x)+x^2;

$$f := x \to 2\ln(x) + e^{3x} + x^2$$

$$x \to 2\ln(x) + e^{3x} + x^2$$

Aniqlangan bu funksiyaning $f(1)$ va $f(0,5)$ qiymatlarini endi hisoblash oson:

> f(1);

$$1 + e^3$$

> f(0.5);

3.345394709

Maple $f(1)$ ni aniq hisobladi. Agar 1 butun sonning o‘rniga 1.0 taqribiy son ko‘rsatsak, taqribiy qiymat hisoblanadi:

> f(1.0);

21.08553692

Aniqlik ko‘rsatilmaganda hisoblashlar 10 ta qiymatli raqam aniqligida bajariladi. Kerak bo‘lsa, qiymatli raqamlar sonini **Digits** buyrug‘i bilan qo‘shimcha ko‘rsatish mumkin (20 ta qiymatli raqam hisoblash):

> Digits:=20;f(1.0);

$$Digits := 20$$

21.085536923187667741

Bu buyruqdan keyingi hisoblashlar endi 20 ta qiymatli raqam bilan bajariladi. Masalan,

> 1/3.0;

$$0.33333333333333333333$$

Berilgan koʻrsatmalar, aniqlangan funksiyalar va oʻzgaruvchilarning qiymatlarini bekor qilish uchun **restart** buyrugʻini berish kerak:

> restart;

> 1/3.0;

$$0.3333333333$$

> f(1.0);

$$f(1.0)$$

Bu yerda $f(x)$ funksiya bekor qilingani (yoʻqotilgani) uchun $f(1.0)$ yozuvi chiqarilgan (funksiya yoʻq – qiymat ham yoʻq).

Differensiallash buyrugʻi (amali) **diff** bilan belgilanadi. U mavjud, aniqlangan, koʻrsatilgan funksiyaning hosilasini hisoblaydi:

> g:=x->exp(3*x)+2*x^2;

$$g := x \mapsto e^{3\,x} + 2\,x^2$$

> diff(g(x),x);

$$3\,e^{3\,x} + 4\,x$$

> diff(f(x),x);

$$f'(x)$$

Ikkinchi tartibli hosilani hisoblash quyidagicha bajariladi:

> diff(g(x),x,x);

$$9\,e^{3\,x} + 4$$

> diff(f(x),x,x);

$$f''(x)$$

Qisqaroq yozuv ham ishlatish mumkin:

> diff(g(x),x$2);

$$9\,e^{3\,x} + 4$$

> diff(f(x),x$5);

$$f^{(5)}(x)$$

Yana misollar keltiraylik (ikki o‘zgaruvchining $h(x, y)$ funksiyasi aniqlangan va uning y bo‘yicha uchinchi tartibli hosilasi hisoblangan):

> h:=(x,y)->3*x^2*y^3-x*y-x;

$$h := (x, y) \rightarrow 3\, x^2\, y^3 - x\, y - x$$

> diff(h(x,y),y$3);

$$18\, x^2$$

> diff(f(x,y),x$2,y);

$$\frac{\partial^3}{\partial x^2 \partial y} f(x, y)$$

Mapleda **D** differensiallash operatori ham mavjud. **D(f)** yozuv **f** funksiyaning hosilasini anglatadi:

> D(g); diff(g(x),x);

$$x \mapsto 3\, \mathrm{e}^{3\, x} + 4\, x$$

$$3\, \mathrm{e}^{3\, x} + 4\, x$$

> D(f);

$$\mathrm{D}(f)$$

Ko‘p o‘zgaruvchining funksiyadan hosila olish shunga o‘xshash bajariladi:

> h(x,y);

$$3\, x^2\, y^3 - x\, y - x$$

>D[1,2](h);

$$(x, y) \mapsto 18\, x\, y^2 - 1$$

> diff(h(x,y),x,y);

$$18\, x\, y^2 - 1$$

> D[1,2](f);

$$\mathrm{D}_{1,\, 2}(f)$$

Quyidagi misollar **D** operatorining mohiyatini ochadi:

> restart;

> D(cos);

$$-\sin$$

> D(ln);

$$z \to \frac{1}{z}$$

> D(exp+sin^3+tan);

$$3\cos\sin^2 + \tan^2 + \exp + 1$$

> D(ln)(x) = diff(ln(x),x);

$$\frac{1}{x} = \frac{1}{x}$$

> D(D(f));

$$\mathrm{D}^{(2)}(f)$$

> (D@@2)(g);

$$\mathrm{D}^{(2)}(g)$$

> f := (x,y) -> y^2*sin(x) + x^3*y;

$$f := (x, y) \to y^2 \sin(x) + x^3 y$$

> D[1](f);

$$(x, y) \to y^2 \cos(x) + 3x^2 y$$

> diff(f(x,y), x);

$$y^2 \cos(x) + 3x^2 y$$

> g:=D[2](f);

$$g := (x, y) \to 2y\sin(x) + x^3$$

> g(a,b);

$$2b\sin(a) + a^3$$

> f21:=D[2,1](f);

$$f21 := (x, y) \to 2y\cos(x) + 3x^2$$

> f12:=D[1,2](f);

$$f12 := (x, y) \to 2y\cos(x) + 3x^2$$

Maple aniqmas va aniq integrallarni hisoblay oladi. **int** buyrug‘i integral hisoblaydi:

> int(x*ln(x)+x,x);

$$\frac{1}{2}x^2\ln(x)+\frac{1}{4}x^2$$

> int(exp(-x^2),x);

$$\frac{1}{2}\sqrt{\pi}\,\mathrm{erf}(x)$$

evalf buyrug‘i yordamida aniq integralni sonli hisoblash mumkin. Masalan, $\int_0^1 \frac{\sin x}{x}dx$ ning qiymati quyidagicha hisoblanadi:

> evalf(int(sin(x)/x,x=0..1));

$$0.9460830704$$

Int buyrug‘i inert integralni anglatadi. Bu buyruq integralni hisoblamaydi, yozadi xolos. Masalan,

> Int(x*ln(x)+x,x);

$$\int (x\ln(x)+x)\,\mathrm{d}x$$

Oddiy differensial tenglama (ODT) larni Maple yordamida yechishda asosiy qurol, bu **dsolve** buyrug‘idir.

dsolve buyrug‘ining ko‘rinishlari:

dsolve(ODE)

dsolve(ODE, y(x), options)

dsolve({ODE, ICs}, y(x), options)

dsolve({sysODE, ICs}, {funcs}, options)

Bu yerdagi parametrlar

ODE – (ordinary differential equation) oddiy differensial tenglama,

y(x) – x erkli o‘zgaruvchi (argument) ning noma’lum funksiyasi,

ICs – (initial conditions) boshlang‘ich shartlar,

{sysODE} – oddiy differensial tenglamalar sistemasi (to‘plami),

{funcs} – noma’lum funksiyalar to‘plami (sistemasi),

options – yozilishi shartmas (optional) parametrlar (shartmas argumentlar); ular yechiladigan masalaning tipiga bog‘liq.

Buyruq tavsifi.

Erksiz o‘zgaruvchi (noma’lum funksiya)ni ko‘rsatuvchi **options** berilgan ODTda bir necha funksiyalarning hosilalari qatnashgan holda yoziladi, chunki u yozilmaganda Maple tenglamani qaysi noma’lum funksiyaga nisbatan yechish kerakligini tushunmaydi.

Chekli ko‘rinishdagi umumiy yechim oshkor, ya’ni $y(x) = \varphi(x, \_C_1, \_C_2, \ldots, \_C_n)$ yoki oshkormas, ya’ni $\Phi(y(x), x, \_C_1, \_C_2, \ldots, \_C_n) = 0$ ko‘rinishda berilishi mumkin; bu yerda $\_C_1, \_C_2, \ldots, \_C_n -$ ixtiyoriy o‘zgarmaslar ($n-$natural son). Agar mumkin bo‘lsa, yechim oshkor ko‘rinishda chiqariladi.

Birinchi tartibli ODTlarning yechimlari, ayniqsa ular dy/dx ga nisbatan yuqori darajali bo‘lsa, parametrik ko‘rinishda ham berilishi mumkin: [x(_T)=f(_T), y(_T)=g(_T)], bunda _T parametr.

Eslatma: Agar buyruqda ko‘rsatilgan ODE to‘plam yoki ro‘yxat ko‘rinishida yozilgan bo‘lsa, u ODTlar sistemasi deb tushuniladi, bu sistema bir dona tenglamadan iborat bo‘lsa-da!

Shartmas argumentlar **(options).**

dsolve buyrug‘iga shartmas argumentlar orqali qo‘shimcha ko‘rsatmalar berish mumkin. Shartmas argumentlarning to‘la tavsifini Help orqali bilib olsa bo‘ladi.

dsolve buyrug‘ida bir dona ODT ko‘rsatilgan holida ko‘p uchraydigan shartmas argunentlar quyida keltirilgan:

‘**implicit'** yechimni oshkormas ko‘rinishda chiqarish.

‘explicit' yechimni oshkor ko‘rinishda chiqarish.

‘parametric' parametrik yechimni topish (faqat birinchi tartibli tenglamalar uchun)

‘**useInt'** inert integralni ishlatish; yechish jarayoni tezlashadi. Yechim topilgach, value buyrug‘i bilan barcha integrallarni hisoblab ko‘rish mumkin.

‘**useint'** yechish jarayonida barcha integrallarni hisolab yurish (agar mumkin bo‘lsa)

‘_mu = int_factor_hint' integrallovchi ko‘paytuvchini ko‘rsatilgan ko‘rinishda izlash va berilgan ODTni yechish

Optional parametrlar (shartmas argumentlar)ning boshqa xususiyatlarini Help dan **dsolve, setup** orqali bilib olish mumkin.

Quyida birinchi tartibli chiziqli differensial tenglama yechilgan:

> **deq1:=diff(y(x),x)+y(x)=x; dsolve(deq1,y(x));**

$$deq1 := \frac{d}{dx}\, y(x) + y(x) = x$$

$$y(x) = x - 1 + e^{-x}\, \_C1$$

Bu yerda ***_C1*** ixtiyoriy o‘zgarmas. Demak, $y' + y = x$ tenglamaning umumiy yechimi $y = -1 + x + ce^{x}$ ($c = \text{const}$).

Ushbu

$$\begin{cases} y' + y = x \\ y(0) = 1 \end{cases}$$

Koshi masalasining yechimi:

> **dsolve({deq1,y(0)=1});**

$$y(x) = x - 1 + 2\, e^{-x}$$

Quyida differensial tenglamani oshkormas (**implicit)** ko‘rinishda yechishga buyruq berilgan. Yechim parametrik ko‘rinishda chiqarilgan.

> **deq2 := y(x)=2*x*diff(y(x),x)+diff(y(x),x)^3;**

$$deq2 := y(x) = 2\,x \left(\frac{d}{dx}\, y(x) \right) + \left(\frac{d}{dx}\, y(x) \right)^3$$

> **dsolve(deq2, implicit);**

$$\left[x(\_T) = \frac{-\frac{3}{4}\, \_T^4 + \_C1}{\_T^2},\, y(\_T) = \_T^3 + \frac{2\left(-\frac{3}{4}\, \_T^4 + \_C1 \right)}{\_T} \right]$$

Bu yerda ***_T*** parametr, ***_C1*** ixtiyoriy o‘zgarmas. Demak, $y = 2xy' + y'^3$ Lagranj tenglamasining yechimi (odatdagi belgilashlarda):

$$\begin{cases} x = \dfrac{-\dfrac{3p^4}{4} + c}{p^2} \\ y = \dfrac{2\left(-\dfrac{3p^4}{4} + c\right)}{p} + p^3 \end{cases} \qquad \left(\begin{array}{l} p - \text{yechimdagi parametr,} \\ c = \text{const yechimlarni belgilaydi} \end{array} \right).$$

Endi bir Klero tenglamasini yechaylik:

> deq3 := y(x)=x*diff(y(x),x)+diff(y(x),x)^2;

$$deq3 := y(x) = x\left(\frac{d}{dx}\,y(x)\right) + \left(\frac{d}{dx}\,y(x)\right)^2$$

> dsolve(deq3);

$$y(x) = -\frac{1}{4}\,x^2,\, y(x) = \_C1^2 + \_C1\,x$$

Bu yerda $y = xy' + y'^2$ Klero tenglamasining $y = cx + c^2$ umumiy yechimi bilan birgalikda $y = -\dfrac{x^2}{4}$ maxsus yechimi ham topilgan.

Bir ikkinchi tartibli nochiziqli tenglamaning yechimi:

> deq4:=x^2*y(x)*diff(y(x),x,x)+3*(y(x)-x*diff(y(x),x))^2=0;

$$deq4 := x^2\,y(x)\left(\frac{d^2}{dx^2}\,y(x)\right) + 3\left(-x\left(\frac{d}{dx}\,y(x)\right) + y(x)\right)^2 = 0$$

> dsolve(deq4,implicit);

$$-\frac{\_C1}{x} + \_C2 + \frac{1}{4}\,\frac{y(x)^4}{x^4} = 0$$

Ushbu

$$\begin{cases} y'' - 5y' + 6y = x \\ y(0) = a, y'(0) = b \end{cases}$$

Koshi masalasining yechimi:

> deq5:=diff(y(x),x,x)-5*diff(y(x),x)+6*y(x)=x;

$$deq5 := \frac{d^2}{dx^2}\,y(x) - 5\left(\frac{d}{dx}\,y(x)\right) + 6\,y(x) = x$$

> Init_con:=y(0)=a, D(y)(0)=b;

$$Init\_con := y(0) = a, \mathrm{D}(y)(0) = b$$

> dsolve({deq5, Init_con});

$$y(x) = \mathrm{e}^{3x}\left(-2a + \frac{1}{9} + b\right) + \mathrm{e}^{2x}\left(-b + 3a - \frac{1}{4}\right) + \frac{1}{6}x + \frac{5}{36}$$

Endi ushbu

$$y''' - 3y' + 2y = 0,\ y(0) = 1,\ y'(1) = 1,\ y(1) + y''(0) = 0$$

chegaraviy masalani yechaylik:

> deq6:=diff(y(x),x$3)-3*diff(y(x),x)+2*y(x)=0;

$$q6 := \frac{\mathrm{d}^3}{\mathrm{d}x^3}y(x) - 3\left(\frac{\mathrm{d}}{\mathrm{d}x}y(x)\right) + 2y(x) = 0$$

> Bound_con:=y(0)=1, D(y)(1)=1,y(1)+D(D(y))(0)=0;

$$Bound\_con := y(0) = 1, \mathrm{D}(y)(1) = 1, y(1) + \mathrm{D}^{(2)}(y)(0) = 0$$

> dsolve({deq6,Bound_con});

$$y(x) = -\frac{\left(\mathrm{e} + 2 + (\mathrm{e})^2\right)\mathrm{e}^{-2x}}{-(\mathrm{e})^2 + 4\,\mathrm{e}\,\mathrm{e}^{-2} + 8\,\mathrm{e} + 4\,\mathrm{e}^{-2}} - \frac{\left(3\,\mathrm{e}\,\mathrm{e}^{-2} + 5\,\mathrm{e} + \mathrm{e}^{-2} - 3\right)x\,\mathrm{e}^{x}}{-(\mathrm{e})^2 + 4\,\mathrm{e}\,\mathrm{e}^{-2} + 8\,\mathrm{e} + 4\,\mathrm{e}^{-2}} + \frac{\left(4\,\mathrm{e}\,\mathrm{e}^{-2} + 9\,\mathrm{e} + 4\,\mathrm{e}^{-2} + 2\right)\mathrm{e}^{x}}{-(\mathrm{e})^2 + 4\,\mathrm{e}\,\mathrm{e}^{-2} + 8\,\mathrm{e} + 4\,\mathrm{e}^{-2}}$$

Quyida bir ODTlar sistemasi yechilgan

> sysdeq := [(D@@2)(x)(t)+2*(D@@2)(y)(t)-x(t)-2*y(t)=0, D(x)(t)+2*D(y)(t)-x(t)+y(t)=0];

$$sysdeq := \left[\mathrm{D}^{(2)}(x)(t) + 2\mathrm{D}^{(2)}(y)(t) - x(t) - 2y(t) = 0,\ \mathrm{D}(x)(t) + 2\mathrm{D}(y)(t) - x(t) + y(t) = 0\right]$$

> dsolve(sysdeq);

$$\left\{x(t) = \_C1\,\mathrm{e}^{t} + \_C2\,\mathrm{e}^{-t}, y(t) = -2\,\_C2\,\mathrm{e}^{-t}\right\}$$

Endi ODTlar sistemasi uchun Koshi va chegaraviy masalalarni yechaylik:

> sysdeq2:= diff(y(t),t) = x(t)+2*y(t), diff(x(t),t) = 3*x(t)+2*y(t);

$$sysdeq2 := \frac{d}{dt}\, y(t) = x(t) + 2\, y(t),\ \frac{d}{dt}\, x(t) = 3\, x(t) + 2\, y(t)$$

> Boshl_shart:= x(0)=1, y(1)=0;

$$Boshl\_shart := x(0) = 1, y(1) = 0$$

> dsolve({sysdeq2, Boshl_shart});

$$\left\{ x(t) = \frac{e^4\, e^t}{2\, e + e^4} + \frac{2\, e\, e^{4\, t}}{2\, e + e^4}, y(t) = -\frac{e^4\, e^t}{2\, e + e^4} + \frac{e\, e^{4\, t}}{2\, e + e^4} \right\}$$

> Cheg_shart:=x(0)+y(1)=1,2*x(0)-y(0)=1;

$$Cheg\_shart := x(0) + y(1) = 1, 2\, x(0) - y(0) = 1$$

> dsolve({sysdeq2, Cheg_shart});

$$\left\{ x(t) = \frac{1}{3}\, \frac{\left(-1 + e^4\right) e^t}{e + 1 + e^4} + \frac{2}{3}\, \frac{(2 + e)\, e^{4\, t}}{e + 1 + e^4}, \right.$$
$$\left. y(t) = -\frac{1}{3}\, \frac{\left(-1 + e^4\right) e^t}{e + 1 + e^4} + \frac{1}{3}\, \frac{(2 + e)\, e^{4\, t}}{e + 1 + e^4} \right\}$$

Mapledagi DEtools nomli paket ODTlarni tekshirishda katta imkoniyatlar yaratadi. Bu paket va Maplening boshqa imkonoyatlari bilan mavjud o‘quv resurslaridan (masalan, [5] adabiyotdan) va internetdagi www.exponenta.ru saytdan tanishish mumkin.

Mavzuga doir tavsiyalar

Mavjud o‘quv resurslaridan (masalan, [5] adabiyot) foydalanib differensial tenglamalarni Maple yordamida tekshirishga oid bilimingizni chuqurlashtiring. Kompyuterda mashqlar bajaring.

Yordamchi nazariy ma’lumotlar va masalalarning yechilishi keltirilgan mustaqil ish yozing;

Mustaqil ish №21 topshiriqlari:

1. Biror birinchi tartibli ODTni tuzing (o‘zingiz tanlang) va uni Maple yordamida yeching.

2. Biror uchinchi tartibli (skalyar) chiziqli oo‘garmas koeffitsientli bir jinsli bo‘lmagan ODTni tuzing. Bu tenglama uchun Koshi va chegaraviy masalalar qo‘ying. Masalalarni Maple yordamida yeching.

3. Ikkinchi tartibli chiziqli differensial tenglamalarning normal sistemasini tuzing. Bu sistema uchun Koshi va chegaraviy masalalar qo‘ying. Masalalarni Maple yordamida yeching.

JAVOBLAR, KO‘RSATMALAR

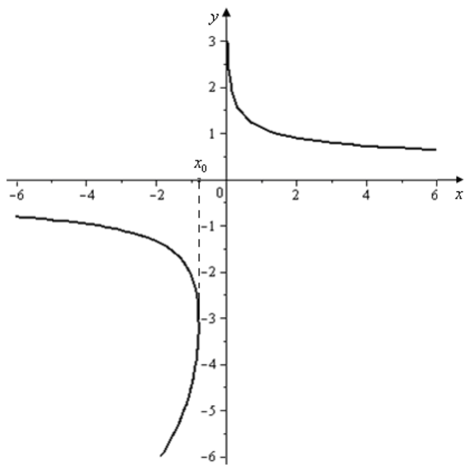
1. Differensial tenglama va uning yechimi

1. Hech qanday oraliqda yechimi emas. **2.** Hech qanday oraliqda yechimi emas.
3. $(-\infty,+\infty)$ oraliqda yechimi bo‘ladi. **4.** Hech qanday oraliqda yechimi emas.

5. Berilgan tenglama biror I oraliqda $y = y(x) \in C^1(I)$ funksiyani aniqlasin: $x^2 + y^2(x) - 2y(x) = 0,\ x \in I$. Bu ayniyatni differensiallab, $y = y(x)$ funksiyaning I da yechim ekanligini isbotlaymiz: $2x + 2y(x)y'(x) - 2y'(x) = 0,\ x \in I,$ $\left(1 - y(x)\right)y'(x) = x,\ x \in I$. Berilgan tenglama ushbu $y = 1 + \sqrt{1 - x^2} \in C^1\left((-1;1)\right)$ va $y = 1 - \sqrt{1 - x^2} \in C^1\left((-1;1)\right)$ funksiyalarni oshkormas ko‘rinishda aniqlaydi.

6. Oshkormas ko‘rinishda berilgan $y = y(x)$ funksiya qaysi oraliqda C^1 sinfga tegishli bo‘lsa, shu oraliqda u berilgan tenglamaning yechimi bo‘lishini tekshiring. Grafik yordamida bu funksiya $(-\infty,+\infty)$ oraliqda aniqlanganligini ko‘rsating. Uning $C^1\left((-\infty;+\infty)\right)$ sinfga tegishli ekanligini asoslang.

7. $I = (0;1)$ intervalda. **8.** Berilgan tenglama $C^1\left((0;+\infty)\right)$ sinfga tegishli bir dona, $C^1\left((-\infty;x_0)\right)$ ($x_0 \approx -0,79$) sinfga tegishli ikki dona funksiyani oshkormas ko‘rinishda aniqlaydi (* rasm). Ular yechimlardir.



* rasm. $xy^3 - e^{-y} - 1 = 0$ tenglama grafigi

9. $y'^2+y^2=1$; **10.** $x^2y''-2xy'+2y=0$; **11.** $x^2y''+xy'-y=0$**; 12.** $(e^y+x^3)y'=3x^2$.

2. O‘zgaruvchilari ajraladigan differensial tenglamalar

1. $y=\dfrac{ce^x}{1-ce^x}$, $y=-1$. **2.** $y=\dfrac{1}{c\sin x-1}$, $y=0$. **3.** $\mathrm{arctg}\,x^2+\sqrt{(1+2y)^3}+c=0$.

4. $\sqrt{1+x^2}-\sqrt{2+y^2}=c$. **5.** $4\ln(1+e^x)-y^2=c$. **6.** $y=\exp\left(\frac{c}{x}-1\right)$, $y=0$.

7. $y=\dfrac{1}{1+ce^{\mathrm{Si}(x)}}$, $\mathrm{Si}(x)=\int\limits_0^x\dfrac{\sin t}{t}dt$. **8.** $3x+2y+2\ln(x+y-1)=c$; $u=x+y$ almashtirish bajaring. **9.** $(3xy+c)^2=16x^3$; $u=xy$ almashtirish bajaring.

10. $x+2\sqrt{x+y}+2\ln\left|\sqrt{x+y}-1\right|=c$, $y=1-x$; $u=x+y$ almashtirish bajaring.

11. $y=\mathrm{tg}\left(\frac{\pi}{4}+\ln x\right)$. **12.** $y=-1+\sqrt{4+\sin 2x}$. **13.** $y=-1$. **14.** $y=\dfrac{4e^x}{(e^x+1)^2}$.

15. $\int\limits_0^x y(s)ds=a^2\ln\dfrac{y(x)}{a}\Rightarrow\ y=a^2\dfrac{1}{y}y',y(0)=a\ \Rightarrow y=\dfrac{a^2}{a-x}$, $x\in[0,a)$.

16. $v=v(t)-$ oniy tezlik bo‘lsin. Ta’sir etuvchi kuch $F=-kv^2t$, $k=const>0-$ proprsionallik koeffitsienti. Masala shartiga ko‘ra $0{,}5=k\cdot 2^2\cdot 10\Rightarrow k=1/80$. Demak, $F=-\dfrac{1}{80}v^2t$. Nyutonning ikkinchi qonuniga ko‘ra

$$\frac{dv}{dt}=-\frac{1}{80}v^2t\ \Rightarrow\frac{1}{v}=\frac{1}{160}t^2+c.$$

Boshlang‘ich shart: $v(10)=2$. Bundan $c=-\dfrac{1}{8}$ kelib chiqadi. Demak,

$$\frac{1}{v}=\frac{1}{160}t^2-\frac{1}{8}\Rightarrow v=\frac{160}{t^2-20}.$$

$t=20(s)$ paytdagi tezlik $v(20)=\dfrac{160}{20^2-20}\approx 0{,}42\ (m/s)$.

17. Nyutonning ikkinchi qonuniga ko‘ra

$$m\frac{dv}{dt}=-kv,\ k=const>0-\text{proprsionallik koeffitsienti.}$$

Bundan $v = c\exp\left(-\frac{k}{m}t\right)$. Lekin, $v(0) = 10$. Demak, $v = 10\cdot\exp\left(-\frac{k}{m}t\right)$. $v(5) = 8$

ekanligidan $\exp\left(-\frac{k}{m}\right)$ miqdorni topamiz: $8 = 10\cdot\exp\left(-\frac{k}{m}\cdot 5\right) \Rightarrow \exp\left(-\frac{k}{m}\right) = \sqrt[5]{4/5}$.

Demak, tezlikning o‘zgarish qonuni $v = 10\cdot\exp\left(-\frac{k}{m}t\right) = 10\cdot\left(\exp\left(-\frac{k}{m}\right)\right)^t = 10\cdot\left(\frac{4}{5}\right)^{t/5}$.

Kemaning tezligi 1 (m/s) gacha tushgan paytni topamiz:

$$1 = 10\cdot\left(\frac{4}{5}\right)^{t/5} \Rightarrow t = \frac{5\cdot\ln 10}{\ln(5/4)} \approx 51{,}59(s).$$

18. $x = x(t) - t$ (soat) paytdagi tuz miqdorini (kg) anglatsin. Shartga ko‘ra $x(0) = x_0 = 10$. Agar $t = 0$ paytda v (kg) miqdordagi suv quyilgan bo‘lsa, aralashmaning miqdori $v + x_0 = const$ bo‘ladi. Yana masalaning shartiga ko‘ra

$$\frac{dx}{dt} = k\cdot x\cdot\left(\frac{x}{v+x_0} - \frac{1}{3}\right),$$

$k = const > 0 -$ proporsionallik koeffitsienti. Bu o‘zgaruchilari ajraladigan differensial tenglamani $x(0) = x_0$ boshlang‘ich shartda yechaylik:

$$\frac{dx}{dt} = -k\cdot x\cdot\frac{(v+x_0)-3x}{3(v+x_0)},\quad \frac{3(v+x_0)}{((v+x_0)-3x)x}dx = -kdt,\quad x = \frac{x_0(v+x_0)}{3x_0 + (v-2x_0)\exp(kt/3)}.$$

Masalaning shartiga ko‘ra $v = 90$ bo‘lganda $x(1) = 5$ (1 soatda 10 kg tuzning yarmi qolgan). Demak,

$$5 = \frac{10(90+10)}{3\cdot 10 + (90 - 2\cdot 10)\exp(k/3)} \Rightarrow \exp(k/3) = \frac{17}{7}.$$

Shuning uchun

$$x = \frac{x_0(v+x_0)}{3x_0 + (v-2x_0)(17/7)^t}.$$

Masalada $v = 180$ suv quyilgandagi $x(1)$ ning qiymati so‘ralgan. Bu $m = x(1)$ ni hisoblaymiz:

$$m = \frac{10(180+10)}{3\cdot 10 + (180 - 2\cdot 10)(17/7)} = \frac{190\cdot 7}{293} \approx 4{,}5.$$

Demak, agar dastlab $v = 180$ (kg) suv quyilganda edi, 1 soatdan so‘ng 10 (kg) tuzdan $m \approx 4{,}5$ (kg) i qolgan bo‘lardi.

19. Shartda aytilgan egri chiziqli trapetsiya yuzi $S=\int\limits_{x_0}^{x} f(x)dx$ ga teng. Masala shartiga ko‘ra $S=k\left(f(x)-f(x_0)\right)^3$, $0<k=\text{const}-$proporsionallik koeffitsienti. Demak,

$$\int\limits_{x_0}^{x} f(x)dx=k\left(f(x)-f(x_0)\right)^3.$$

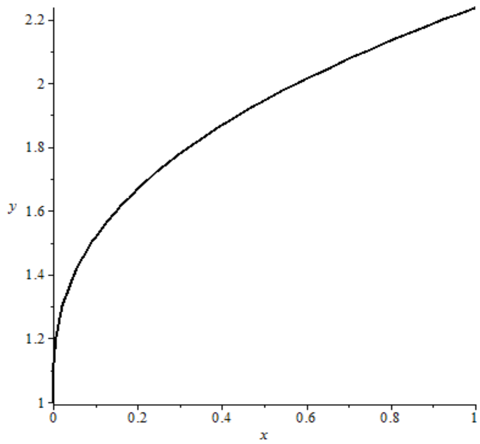
Bu tenglikni hadma-had differensiallab, $y=f(x)$ noma’lum funksiyaga nisbatan differensial tenglama hosil qilamiz:

$$f(x)=3k\left(f(x)-f(x_0)\right)^2 f'(x) \text{ yoki } 3k\left(y-y_0\right)^2 y'=y \quad \left(y_0=f(x_0)>0\right).$$

Bu o‘zgaruvchilari ajraladigan differensial tenglamani $y(x_0)=y_0$ shartda yechib, yechimni oshkormas ko‘rinishda topamiz:

$$x-x_0-\frac{3}{2}ky^2+6kyy_0-3ky_0^2\ln y-\frac{9}{2}ky_0^2+3ky_0^2\ln y_0=0.$$

Bu munosabat $x=x(y)$ funksiyani bir qiymatli aniqlaydi. Teskari funksiya $y=f(x)$ izlangan yechimni beradi. Bu $y=f(x)$, $x\ge x_0$, funksiya aniqlovchi egri chiziq $k=1$, $x_0=0$, $y_0=1$ holda quyidagi **- rasmda ko‘rsatilgan.



**- rasm. $x-\frac{3}{2}y^2+6y-3\ln y-\frac{9}{2}=0,\ x\ge 0,$ tenglama grafigi.

20. $m\frac{dv}{dt}=-kv^2$, $k=const>0-$ proporsionallik koeffitsienti.

$\frac{dv}{v^2}=-\frac{k}{m}dt$, $v=\frac{1}{\eta t+c}$, $\eta \overset{def}{=} \frac{k}{m}$. $t=0$ paytda $v=v_0$ bo‘lgan. Bu boshlang‘ich shartdan $c=\frac{1}{v_0}$. Demak, $v=\frac{1}{\eta t+\frac{1}{v_0}}$. Masala shartiga ko‘ra $v(t_1)=v_1$. Bundan η koeffitsient topiladi:

$$\frac{1}{\eta t_1+\frac{1}{v_0}}=v_1 \Rightarrow \eta=\frac{1}{t_1}\frac{v_0-v_1}{v_0 v_1}.$$

Yechishni davom ettiramiz. Tushunarliki,

$$\frac{dx}{dt}=v,\quad \frac{dx}{dt}=\frac{1}{\eta t+\frac{1}{v_0}},\quad x=\frac{1}{\eta}\ln\left(\eta t+\frac{1}{v_0}\right)+c.$$

Yana masala shartiga ko‘ra $x(0)=0$, $d=x(t_1)$. Demak,

$$d=\frac{1}{\eta}\ln\left(\eta t_1+\frac{1}{v_0}\right)-\frac{1}{\eta}\ln\left(\frac{1}{v_0}\right)=\frac{1}{\eta}\ln\left(\eta t_1 v_0+1\right)=t_1\frac{v_0 v_1}{v_0-v_1}\ln\frac{v_0}{v_1}$$

Javob. $d=t_1\frac{v_0 v_1}{v_0-v_1}\ln\frac{v_0}{v_1}$.

21. Izlangan egri chiziq $y=f(x)$ tenglama bilan berilsin. Uning $(x,f(x))$ nuqtasidan o‘tkazilgan urinma Ox va Oy o‘qlarni mos ravishda $\left(x-f(x)/f'(x),0\right)$ va $\left(0,f(x)-xf'(x)\right)$ nuqtalarda kesadi. Masala shartiga ko‘ra bu nuqtalarning o‘rtasi $(x,f(x))$ nuqtadan iborat bo‘lishi kerak. Bundan ushbu
$xf'(x)=-f(x)$ tenglama hosil bo‘ladi. Oxirgi tenlamaning umumiy yechimi $xf(x)=c$. $f(2)=1$ boshlang‘ich shartga ko‘ra $c=2$. Demak, izlnagan egri chiziq $xf(x)=2$, ya'ni $y=f(x)=2/x$, $0<x<+\infty$ tenglama bilan beriladi. Bu – giperbolaning bir pallasi.

22. Masalaning simmetriyasiga ko‘ra nuqtadagi T temperatura shu nuqtadan faqat sterjen o‘qigacha bo‘lgan masofa r ga bog‘liq xolos, $T=T(r)$, $r_1\le r\le r_2$; $\frac{\partial T}{\partial \boldsymbol{n}}=\frac{dT}{dr}$. Issiqlik holati statsionar bo‘lgani uchun asosi r radiusli uzunligi 1 ga teng bo‘lgan silindrning yon sirti orqali oqib o‘tadigan issiqlik miqori o‘zgarmas:

$$-k\cdot 2\pi r\cdot 1\cdot\frac{dT}{dr}=q=const.$$

Oxirgi tenglamadan T ni topamiz:

$$T = -\frac{q}{2\pi k}\ln r + c.$$

Bu yerdagi $\frac{q}{2\pi k}$ va c larni berilgan $T(r_1) = T_1$ va $T(r_2) = T_2$ shartlardan foydalanib aniqlaymiz:

$$T_1 = -\frac{q}{2\pi k}\ln r_1 + c\,,\ T_2 = -\frac{q}{2\pi k}\ln r_2 + c\ \Rightarrow\ \frac{q}{2\pi k} = \frac{T_1 - T_2}{\ln r_2 - \ln r_1}\,,\ c = \frac{T_1\ln r_2 - T_2\ln r_1}{\ln r_2 - \ln r_1}.$$

Demak, temperatura taqsimoti quyidagi formulaga ko‘ra topiladi:

$$T = \frac{T_2 - T_1}{\ln r_2 - \ln r_1}\ln r + \frac{T_1\ln r_2 - T_2\ln r_1}{\ln r_2 - \ln r_1}.$$

23. Suyuqlik sathining t paytdagi balandligini $z = z(t)$ bilan belgilaymiz. $[t, t+\Delta t]$ kichik vaqt oralig‘ida teshikdan $\Delta V = \sigma v\Delta t + o(\Delta t)$ miqdordagi suyuqlik oqib ketadi. Buning natijasida suyuqlikning sath balandligi Δz ga kamayadi. Ravshanki, $\Delta V = -S(z)\Delta z + o(\Delta z)$ bo‘ladi. Demak,

$\sigma v\Delta t + o(\Delta t) = -S(z)\Delta z + o(\Delta z)$.

Bu tenglikni Δt ga bo‘lib, $\Delta t \to 0$ da limitga o‘tsak, ushbu

$$S(z)\frac{dz}{dt} = -\sigma v$$

munosabatga kelamiz. Bu tenglikdan Torichelli formulasiga ko‘ra noma’lum $z = z(t)$ uchun quyidagi o‘zgaruvchilari ajraladigan differensial tenglama hosil bo‘ladi:

$$S(z)\frac{dz}{dt} = -\sigma k\sqrt{2gz}\ .$$

Bu tenglamani $z(0) = h$ boshlang‘ich shartda yechib, topamiz

$$t = \frac{1}{\sigma k\sqrt{2g}}\int_z^h \frac{S(x)}{\sqrt{x}}dx\,.$$

Izlangan $z = z(t)$ funksiya bu formula bilan oshkormas ko‘rinishda (teskari funksiya kabi) beriladi. Idishning bo‘shash vaqti t_b ni topish uchun oxirgi tenglikda $z = 0$ qo‘yish kerak:

$$t_b = \frac{1}{\sigma k\sqrt{2g}}\int_0^h \frac{S(x)}{\sqrt{x}}dx\,.$$

24. $v = v(t)$ jismning t paytdagi tezligi bo‘lsin. Nyuton qonuniga ko‘ra

$$m\frac{dv}{dt}=-G\frac{Mm}{(r+R)^2},$$

bu yerda $m,M-$ mos ravishda jism va Yerning massalari, $R-$ Yerning radiusi, $G-$ gravitatsion doimiy, $r-$ Yer sathidan uning radiusi yo‘nalishi bo‘yicha hisoblangan masofa. Tenglamada shakl almashtiramiz va o‘zgaruvchilarni ajratib integrallashlarni bajaramiz:

$$\frac{dv}{dr}\frac{dr}{dt}=-G\frac{M}{(r+R)^2},\ \frac{dv}{dr}v=-G\frac{M}{(r+R)^2},\ .vdv=-G\frac{Mdr}{(r+R)^2}.,$$

$$\int vdv+GM\int\frac{dr}{(r+R)^2}+c,\ \frac{v^2}{2}-GM\frac{1}{r+R}=c.$$

Oxirgi tenglik energiyaning saqlanish qonunini anglatadi ($m\frac{v^2}{2}-G\frac{mM}{r+R}=const$).

Masalaning shartiga ko‘ra $v\big|_{r=0}=v_0$. Demak,

$$\frac{v^2}{2}-\frac{GM}{r+R}=\frac{v_0^2}{2}-\frac{GM}{R}.$$

Bundan

$$v=\left(v_0^2-\frac{2GM}{R}+\frac{2GM}{r+R}\right)^{1/2}.$$

Barcha $r\to+\infty$ larda ham v mavjud bo‘lishi uchun boshlng‘ich tezlik uchun $v_0^2-\frac{2GM}{R}\geq 0$ ya’ni $v_0\geq\sqrt{\frac{2GM}{R}}$ shart bajarilishi kerak. $\sqrt{\frac{2GM}{R}}\approx 11{,}2\frac{km}{s}-$ 2- kosmik tezlik deb ataladi. Agar jismning boshlang‘ich tezligi 2- kosmik tezlikdan katta bo‘lsa, u Yerning tortish maydonidan chiqib ketadi.

25. Berilgan boshlang‘ich masalaning yechimi osongina topiladi:

$$Q=\frac{w}{p}+\left(Q_0-\frac{w}{p}\right)e^{pt}.$$

Bu formuladan ravshanki, agar $Q_0-\frac{w}{p}>0$, ya’ni $w<pQ_0$ (pulni o‘sish tezligi kichik) bo‘lsa, vaqt o‘tishi bilan omonat pul miqdori cheksiz ortadi. Agar $w=pQ_0$ bo‘lsa, pul miqdori o‘zgarmay turadi (qo‘shilgan protsentning hamasi olinadi): $Q=\frac{w}{p}=Q_0=const$.

Agar $w > pQ_0$ bo‘lsa, omonat pul miqdori kamaya boshlaydi va uning nolga tenglashgan payti T ushbu

$$0 = \frac{w}{p} - \left(\frac{w}{p} - Q_0\right)e^{pT}$$

tenglamadan topiladi. Natijada

$$T = \frac{1}{p}\ln\frac{w}{w - pQ_0}$$

formula hosil bo‘ladi.

3. O‘zgaruvchilariga nisbatan bir jinsli differensial tenglamalar

1. $(y-x)^4(3x+2y)=c$. **2.** $y^3(2x^3+y^3)=c$. **3.** $y=c(x-1)^2-3x$.

4. $2x^2+2y^2-3x-2y+5=c\cdot\exp\left(-2\operatorname{arctg}\frac{2y-1}{2x-3}\right)$.

5. $(y-x^2)^2(2y+x^2)=c$. $y=z^m$ almashtirish bajaring. **6.** $\ln|x|-\operatorname{arctg}(x^2y+1)=c$. $y=z^m$ almashtirish bajaring. **7.** $y=-2x-\sqrt{5x^2-1}$. **8.** $y=\dfrac{1}{\sqrt{x^4+3}-x^2}$.

9. $y=x\cdot\arcsin(ae^{-1/x})$, $a=e^{2/\pi}\sin(2/\pi)$. **10.** $y=x^3$.

4. Birinchi tartibli chiziqli differensial tenglamalar

1. $y=(x^2+c)e^x$. **2.** $y=x(\ln|x|-\dfrac{1}{x}+c)$. **3.** $y=\dfrac{1}{4x^2}+\dfrac{1}{3x}+cx^2$.

4. $x+y\cos y=cy$. $x=x(y)$ ni qarang. **5.** $y=\dfrac{\ln x-1}{x^2}+\dfrac{c}{x^3}$.

6. $y=c\cos x+\displaystyle\int\frac{e^x\sin x}{\cos x}dx$. **7.** $x+e^y=ce^{4y}$. $x=x(y)$ ni qarang.

8. $y=\dfrac{ce^{2x}-x-1}{e^x(e^x+1)}$. **9.** $\dfrac{1}{y}+\dfrac{x^2\ln x}{2}+cx^2=0$, $y=0$. Bernulli tenglamasi.

10. $y=\dfrac{\cos x}{3\sin x+c}$, $y=0$. Bernulli tenglamasi. **11.** $y=\dfrac{x^6-7}{6x^3}$. **12.** $y=\dfrac{3x^2}{2x^3-1}$.

13. $y^3-3x^3y\ln y+3x^3y-2x^3=0$. $x=x(y)$, Bernulli tenglamasi.

14. $y=x\sqrt{3-2x}$. $u=y^2$ almashtirish bajaring.

15. $y=y=\dfrac{x^4+3c}{x(x^4-c)}$. $y_1=\dfrac{1}{x}$ – xususiy yechim.

16. $y=\dfrac{10e^x(e^{7x}+c)}{5e^{7x}-2c}$. $y_1=2e^x-$ xususiy yechim.

17. $y=\sin x-\dfrac{e^{-3\cos x}}{\int e^{-3\cos x}dx+c}$. $y_1=\sin x-$ xususiy yechim.

5. To‘la differensialli va unga keltiriluvchi tenglamalar

1. $y^4-xy-x^4+2x=c$.

2. $\sin y+xy-x=c$. To‘la differensialli tenglama, $x=x(y)$ ga nisbatan chiziqli. Quyidagicha yechish qulay: $\underline{ydx}-dx+\underline{xdy}+\cos y\cdot dy=0,\ d(xy)-dx+d(\sin y)=0,$ $d(xy-x+\sin y)=0$. **3.** $e^y-xy^2-e^x=c$. **4.** $y^3+x^2\ln y+x=c$.

5. $y\ln y-y+xy-x^2=c$. To‘la differensialli tenglama, $x=x(y)$ ga nisbatan chiziqli. **6.** $xy^2+\cos y+x^2=c$. To‘la differensialli tenglama, $x=x(y)$ ga nisbatan chiziqli. **7.** $e^y+y\ln(xy)+x^2=c$. **8.** $xe^y+\sin y=c$. To‘la differensialli tenglama, $x=x(y)$ ga nisbatan chiziqli. **9.** $y^4+8y\ln x+4x=c$. **10.** $x+y+\sqrt{x^2+y^2}=c$. **11.** $\cos(2x)+4\sin(x+y)=c$. **12.** $x\ln y=c+x^4, x=0;\ \mu=y^{-1}$.

13. $x^2y+x=cy^2, y=0;\ \mu=y^{-3}$. **14.** $ye^{-x}+\ln|y|=c, y=0;\ \mu=y^{-1}e^{-x}$.

15. $y=e^{x^2+cx};\ \mu=x^{-2}y^{-1}$. **16.** $x+\ln(1+e^{-x}y)+\ln(e^{-x}y)=c$; $\mu=e^{-x}$.

17. $xe^{2x}=c\left(y+\sqrt{x^2+y^2}\right);\ \mu=\left(x\sqrt{x^2+y^2}\right)^{-1}$.

18. $x^2-\dfrac{x}{y}+(y-1)\ln y-y=c;\ \mu=\dfrac{1}{y^2}$. **19.** $y^2+\dfrac{y}{x}-\dfrac{x}{y}=c;\ \mu=\dfrac{1}{x^2y^2}$.

20. $(x^2+1)^2+2(x^2+1)\sin y=c;\ t=x^2+1$. **21.** $\ln y+\ln x+2=cy;\ \mu=x^{-1}y^{-2}$.

22. $xy+\sqrt{x^2+y^2}=c;\ \mu=(x^2+y^2)^{-1/2}$. **23.** $x+y^2+x^3\ln y=c;\ \mu=y^{-1}$.

24. $x^3y^3(3x-y^3)=c$; $\mu=(xy)^2$. **25.** Qismlarga ajrating, $\mu=(x^3+y^3)^{-4/3}$.

26. $\mu=(2-x+y)^{-2}$.

6. Birinchi tartibli normal ko‘rinishdagi differensial tenglama

1. T_1da – yo‘q, T_2da – ha. **4.** Misol 4 ning yechilishiga qarang. **10.** Misol 5 ning yechilishiga qarang.

7. Hosilaga nisbatan yechilmagan birinchi tartibli differensial tenglamalar. Maxsus yechimlar

1. $y = cx - \dfrac{c^2}{4}$, $y = x^2 -$ maxsus yechim. **2.** $y = cx - c^3$, $27y^2 = 4x^3 -$ maxsus yechim.

3. $y = \dfrac{x^2}{c} - x + c$, $y = x$ va $y = -3x -$ maxsus yechimlar ($x \neq 0$ da).

4. $x = cy^2 - 2c^2$, $y^4 - 8x = 0 -$ maxsus yechim.

5. $x = ce^y + c^2$, $4x + e^{2y} = 0 -$ maxsus yechim.

6. $\left\{ x = c\dfrac{(2+\ln p)\ln p}{2}, y = cp\ln p \right\}$, maxsus yechim yo‘q.

7. $x = 2c - \ln(c - y)$, $y = \dfrac{x}{2} - \dfrac{1+\ln 2}{2} -$ maxsus yechim.

8. $y = c\,\mathrm{tg}x - c^2$, $y = \dfrac{1}{4}\mathrm{tg}^2 x -$ maxsus yechim.

9. $y = cx + \dfrac{c}{\sqrt{1+c^2}}$, $x^{2/3} + y^{2/3} = 1 -$ maxsus yechim.

10. $27(y-c)^2 = 4(x-c)^3$, $y = x - 1 -$ maxsus yechim.

11. $y = 2c^3 x + c^2$, $y = \pm\dfrac{\sqrt{3}}{9}\cdot\dfrac{1}{x}$ $-$maxsus yechimlar ($z = y^2$ almashtirish bajaring).

12. $y = 2c^2(x+c)^2$, $y = 0$ va $y = x^4/8$ $-$ maxsus yechimlar.

8. Aralash tenglamalar

1. $(1-4ce^x)y = ce^x(x-1)$. **2.** $y = x/(c-2x)$. **3.** $y^2 + x^2 - 4\ln x = c$. **4.** $x^4y^2 - x^6 = c$. **5.** $x + (1+x^2)\ln y = c$ **(**$u = \ln y$**).** **6.** $y = 2/(cx^3 + 3x)$. **7.** $(2y+x)^2 = c\left(4 - 3cx^2\right)$.

8. $(y^2-1)x = yc$. **9.** $x^2 - y^2 - cy^3 = 0,\ y = 0$. **10.** $4y\ln x + y^4 = c$ $(x > 0)$. **11.** Cheksiz ko‘p yechim $y = x^3 + cx^2$. **12.** $e^{4x/3}\left(8y^{2/3} - 4x + 3\right) = c$.

13. $y^2 - y + x^2 = cx$ **(** $\mu = 1/x^2$ **).** **14.** $\exp(2x^2/3)\left(2y^{2/3} - 1\right) = c$. **15.** $y^2(\ln y + \ln x) = c$

16. $2y^3 + 3xy + c = 0$. **17.** $\sin y = (x-1)e^x c$. **18.** $2y^2 = x + 2x^3 c$.

19. $4x\cos y + y^4 = c$. **20.** $4y^{-2} - 1 - 2x^2 - 4e^{2x^2}c = 0$, $y = 0$. **21.** $y = 0$ va $y = x + 1$ yechimlardan boshqa yechimlar parametrik ko‘rinishda quyidagicha beriladi:

$x=\dfrac{c}{(1-p)^2}-1,\ y=\dfrac{cp^2}{(1-p)^2}$. **22.** $x^2/y^2+2\ln|y|=c$. **23.** $xe^{\cos y}-\int\limits_0^y e^{\cos t}\cos t\,dt=c$.

24. $y=\dfrac{1}{2}\operatorname{arctg}\dfrac{2c_1(e^x-1)}{1+c_1^2-2c_1^2e^x+c_1^2e^{2x}}$, $y=\dfrac{1}{2}\operatorname{arctg}\dfrac{c_2-1-2c_2e^x+c_2e^{2x}}{c_2+1-2c_2e^x+c_2e^{2x}}$.

25. $y=x-2-\sqrt{2(x-2)^2c^2+1}\Big/c$. **26.** $\sqrt{y-2x-1}-2=c\exp\left(\dfrac{x-2\sqrt{y-2x-1}}{4}\right)$.

27. $(2\sin y-x)\sqrt{\sin y+x}=c$. **28.** $y+1=c\exp\left(-\dfrac{2}{\sqrt{3}}\operatorname{arctg}\dfrac{x+2y+2}{\sqrt{3}x}\right)$.

29. $(x-4y+3)^3=c(x-y+2)^3(3y-1)$, $x+2y\neq-1$.

30. $y=4/\left(3e^{x^2}\sqrt{\pi}\operatorname{erf}(x)-6x+ce^{x^2}\right)$, $\operatorname{erf}(x)=\dfrac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}ds$.

31. $x^2y^2+2y+2x^2=c$. **32.** $y=(x\cos x-\sin x+c)/x^2$.

33. Rikkati tenglamasi, $y=\pm1/x-$ xususiy yechim.

34. $\mu=1/(xy)$. **35.** Rikkati tenglamasi, $y=x-$ xususiy yechim.

36. $y=c_1\left(x-\sqrt{x^2-1}\right)\exp\left(x^2+\sqrt{x^2-1}\right), y=c_2\left(x+\sqrt{x^2-1}\right)\exp\left(x^2-\sqrt{x^2-1}\right)$.

37. $x=4p+\arcsin p,\ y=2p^2+\arcsin p+c$.

38. $y=c_1\left(x^2-x\sqrt{x^2-2}-1\right)\exp\left(x^2+x\sqrt{x^2-2}\right)$,

$y=c_2\left(x^2+x\sqrt{x^2-2}-1\right)\exp\left(x^2-x\sqrt{x^2-2}\right)$.

39. $y=\pm x$, $y=\dfrac{1}{2c}\left(c^2x^2+1\right)$. **40.** $y=\exp\left(\dfrac{c}{x}-\dfrac{1}{4}x^3\right)$. **41.** $y=0$, $y=1/(x+c)^2$.

42. Cheksiz koʻp yechim $y=x^2+cx$. **43.** $y=-x^4/16, y=cx^2+4c^2$.

44. $y=x-\exp\left((xc)^{-1}\right)$. **45.** $4y^2\cos^2x+4x\cos^2x-\sin(2x)-2x=c$.

46. $y=-\ln(x+cx^2)$. **47.** $4y^2-4x^3\ln y-3x^3=c$. **48.** $x+\ln y=c\ln^2y$.

49. $cxe^{2y}-xe^y=1$. **50.** $y=9\Big/\sqrt[3]{e^{9x^2}\sqrt{\pi}\operatorname{erf}(3x)-36x^3-6x+ce^{9x^2}}$.

51. $y=-\ln(cx-x\ln x)$. **52.** $y=\exp\left(\dfrac{x^2+c}{2x}\right)$. **53.** $1/x=2-y^2+c\exp(-y^2/2)$.

54. $2x+\ln y = cy$. **55.** Bitta yechim: $y = e^{x}\int\limits_{x}^{+\infty} e^{-(1+t^2)}dt$.

56. $1\pm\sqrt{1+y^2}+y\ln\left(\sqrt{1+y^2}\pm y\right)=(x+c)y$.

57. $x^2 = y-\dfrac{1}{2}+ce^{-2y}$. **58.** $x^2y^2-2x^3y-x^4=c$.

59. $y+\operatorname{arctg}\dfrac{y}{x}=c$. **60.** $y+x=\operatorname{tg}(y+c)$. **61.** $4xy^3+x^2=c$. **62.** $y=\ln x+c/x$.

63. $y=c/\cos x-1/2$. **64.** $y^2-e^{x^2}/(2x+c)=0$. **65.** $y=2(\sin x-1)+ce^{-\sin x}$.

66. $y(x)=1,\ \sqrt{1-x^2}-2\sqrt{1-y}=c,\ -1<x<1,\ y<1$. **67.** $y^2(x+y)=c$.

68. $\ln y^2+x/y+x^2=c$. **69.** $\dfrac{e^{2y}}{2x^2}-\int\limits^{y}\dfrac{e^{2s}\ln s}{s}ds=c$. **70.** $y=\dfrac{\cos x}{2\sin x+c}$.

71. $y=0;\ x=\dfrac{\ln p+c}{4\sqrt{p}},\ y=\sqrt{p}\left(1-\dfrac{\ln p+c}{4}\right)$. **72.** $y+x^2+2=ce^{y}$.

73. $y=\dfrac{1}{x}\left(\operatorname{tg}\left(\frac{1}{2}\ln x-c\right)-1\right)$. **74.** $\mu=1/x^2$. **75.** Agar $e^{-x}=t$ desak, $t=t(y)$ ga nisbatan chiziqli tenglama. **76.** $\operatorname{arctg}(8x+2y)=2(y-c)$. **77.** Rikkati tenglamasi, $y=kx+b-$ xususiy yechim. **78.** $xy(c-\ln^2 y)=2$. **79.** $ce^{2y}=2y-2x^2-1$ ($x\to t=x^2$). **80.** $y=(x^2+c)\ln x$. **81.** $x+4\sin y+4=ce^{\sin y}$. **82.** $\left(4y+ce^{2x}\right)^2=4x^2$.

83. $x=(\ln t+c)/t^2,\ y=(2\ln t+2c+1)/t\quad (t\in\mathbb{R})$. **84.** $xy-\ln xy^3=c$.

85. $(y-x+1)^5(10y-5x+2)=c$. **86.** $x^2=(x^2-y)\ln(xc)$. **87.** $y=x^2\left(\operatorname{tg}(\ln(xc))+1\right)$. **88.** $\ln\dfrac{y+x}{x+3}=1+\dfrac{c}{y+x}$. **89.** $x=-y^2\ln(cx)$. **90.** $x^2\left(\sqrt{x^2y^4+1}-x^2y\right)=c$.

91. $y=0,\ x=cy^3+y^2$. **92.** $x^2=ce^{2y}+2y$. **93.** $y=\dfrac{2}{x}+\dfrac{4}{cx^5-x}$.

94. $\cos y=(x^2-1)\ln\left((x^2-1)c\right)$. **95** $y^2=c(x+1)^2-2(x+1)$. **96.** $y=\dfrac{1}{x}+\dfrac{1}{cx^{2/3}+x}$.

97. $2x\sqrt{y}=x\arcsin x+\sqrt{1-x^2}+c$. **98.** $y=x+2+\dfrac{4}{ce^{4x}-1},\ y=x+2$.

99. $(x-c)^2+y^2=c, x=y^2-1/4-\text{maxsus yechim}$.

100. $y=e^{x+c}-\frac{1}{2}c$, $y=\frac{1}{2}\left(x+1+\ln 2\right)-$maxsus yechim.

101. $\ln(ye^{-x})-ye^{-x}+x=c$. **102.** $2\exp\left(-\frac{1}{2}y^2\right)-x\int\limits_1^{+\infty}\exp\left(-\frac{s}{2}y^2\right)s^{-1}ds=2xc$.

103. $y^2=2x^2-2x+1+e^{-2x}c$. **104.** $y=0,\ y^{5/3}=(x+y^2)c$. **105.** $y=\mathrm{th}\frac{cx-1}{3x}$.

106. $y^5+cy+x=0$. **107.** $x^2\cos y-y=c$. **108.** $cy^2=xy-2$. **109.** $(2xy-1)y^2=cx$.

110. $\sin y=\ln((x-3)c)$. **111.** $(x-1)^2e^{2x}y=c+(2x-3)e^{2x}$.

112. $(x+1)^2y=x\left(x^2+2x\ln(cx)-1\right)$. **113.** $y^2=ce^{2x}-4\sin x-2\cos x$.

114. $y^5=(5xy+1)c$. **115.** $\sqrt{y+x}+\sqrt{x}=c$, $y=-x$.

116. $4x^2\sqrt{y}=2x\,\mathrm{arctg}x-\ln(x^2+1)+c$. **117.** $x=2p-e^p$, $y=p^2-(p-1)e^p+c$.

118. $y=0$; $\mathrm{arcsh}\left(\frac{x}{y}\right)=x+c$. **119.** $y=-x+\mathrm{tg}(x+c)$. **120.** $y=x+1+\frac{2ce^{2x}}{1-ce^{2x}}$.

121. $y=x+2\mathrm{arctg}\frac{c-x-2}{c-x}$. **122.** $y=(x+1)^2c-x-1$.

123. $y=\ln\left(2x+e^{2c}/2\right)-c$, $y=\ln\left(2e^{2c}x+1/2\right)-c$, $y=\ln(4x)/2$ - maxs. yech.

124. $y=-\frac{1}{e^x}+\frac{3c}{e^x(c-2e^{3x})}$. **125.** $y=\sin x+\frac{c}{1-cx}$. **126.** $xe^y+(x^2-1)y=c$. **127.** $y^3+x^3\ln y+x=c$. **128.** $x^2-2x^3y+2xe^y+y^2=c$. **129.** $x\sin xy=c$.

130. $2y+x^4+3=e^yc$. **131.** $y=-1+x(y+x^2)c$. **132.** $y\sin x-x^2=c$.

133. $(x^2-y^2-xy^3)c=xy$. **134.** $\left(c+\int\limits_0^x e^{-t^2}dt\right)\cos y=e^{-x^2}$. **135.** $xy^2+2e^x-3e^y=c$. **136.** $6x+2y^3+1=\mathrm{tg}(6x+c)$. **137.** $xy+\sin x+e^y=c$. **138.** $(4x+3+2\ln y=cy^2$.

139. $x^2y^2+y^2+x^3-x=c$. **140.** $\left(x(x^2+y^2)+\sin x\right)c=x$.

141. $\mathrm{arctg}x+2\mathrm{arctg}y+x+y=c$ ($\mu=(1+x^2)^{-1}(1+y^2)^{-1}$).

142. $\arcsin x+\arcsin y+\sqrt{1-x^2}\sqrt{1-y^2}=c$. **143.** $x^2+y^2+\sqrt{xy}=c$.

144. $4e^{x/y}+\ln y=c$. **145.** $y^3=\left(3ye^{-x}+1\right)^2c$. **146.** $y=c_1e^x$, $y=c_2e^{2x}$.

147. $\sin y+1/\cos x=c$. **148.** $e^{2y}+e^{2x}=c$. **149.** $y^3=(3x+c)x^3$.

150. $y(\ln y+2\ln x+3)=c$. **151.** $2(y-2)\sqrt{y+1}-3x^2=c$. **152.** $x=y+c\sqrt{1+y^2}$.

153. $x=(y-2)e^y+e^{y/2}c$. **154.** $y=\exp\left(x^2+\dfrac{c}{x^2}\right)$. **155.** $y=x,\ 2(x+c)(x-y)^2=3$. **156.** $y=\ln\left((xe^x-e^x+c)/x^2\right)$.

9. Yuqori tartibli differensial tenglamalar. Ularning tartibini pasaytirish va yechish

5. $y=c_1x(\ln x-1)+c_2x+c_3+\dfrac{x^2}{2}$. **6.** $y=c_1\sin x+c_2x+c_3$.

7. $c_1^2(c_2+e^{-x})^2=4(1-c_1e^{-x}y^2)$. **8.** $\left(\displaystyle\int\frac{dy}{\sqrt{c_1-\sin^4 y}}\right)^2=(x+c_2)^2$. **9.** $c_1y^2+c_2y+c_3=x$, $y=c$. **10.** $y=c_1\exp(x^4+c_2x)$. **11.** $y=-\dfrac{2c_1^2}{\mathrm{ch}^2(c_1x+c_2)}$.

12. $y=c_1\exp(c_2x^2+c_3x-\sin x)$. **13.** $y=c_2(x^2+c_1^2)\exp\left(-\dfrac{2}{c_1}\mathrm{arctg}\dfrac{x}{c_1}\right)$.

14. $y=c_1\exp\left(c_2\mathrm{arctg}\dfrac{c_2\ln x}{2}\right)$. **15.** $y=\dfrac{1}{x}\left(3+c_1\mathrm{tg}\left(\dfrac{c_1(c_2-\ln x)}{2}\right)\right)$.

16. $y=2c_1\mathrm{th}\left(c_1(1-c_2x^2)/x^2\right)$. **17.** $y=x\ln\left(c_1-\dfrac{c_2}{x}\right)$. **18.** $y=x\left(2c_1\mathrm{arcth}\dfrac{x}{c_1}+c_2\right)$, $y=0$.

19. $y=x\left(c_1+\arcsin\dfrac{c_2}{x}\right)$. **20.** $y=e^{x^3/3}\displaystyle\int(2x+c_1)e^{-x^3/3}dx+c_2e^{x^3/3}$.

21. $\displaystyle\int\frac{dy}{c_1+y^3}=\frac{x^2+c_2}{6}$; $\mu=\dfrac{1}{x^2}$. **22.** $y=c_2e^{c_1x}+\dfrac{1}{c_1}$; $\mu=\dfrac{1}{y^2}$.

23. $y=\dfrac{c_2}{c_1}e^{c_1x}+\dfrac{c_1\cos x-\sin x}{1+c_1^2}$, $y=\dfrac{1}{c_2-\cos x}$, $y=0$; $\mu=\dfrac{1}{y^2}$.

24. $y-2\ln y-x=0,\ y\in(0,2)$. **25.** $y=4x^2/(x^2+1)$. **26.** $y=1/(x+1)$.

27. $\displaystyle\int_0^y\frac{\exp(\cos 2t)}{\cos t}dt+e\cdot x=0$.

10. O‘zgaruvchan koeffitsientli chiziqli differensial tenglamalar

1. Ha. **2.** Yo‘q. **3.** Ha. **4.** Yo‘q. **5.** Yo‘q. **6.** Ha. **7.** Yo‘q. **8.** a) Ha. b)Yo‘q.

9. $y''-\frac{2x}{x^2-1}y'+\frac{x}{x^2-1}y=0$; $x\in(-\infty;-1)$, $x\in(-1;1)$, $x\in(1;+\infty)$.

10. $y''-\frac{2x+1}{2x(x+1)}y'-\frac{1}{4x(x+1)}y=0$; $x\in(0;+\infty)$

11. $y''-\frac{4(x+1)}{2x+1}y'+\frac{4}{2x+1}y=0$; $x>-1/2$, $x<-1/2$.

12. $y''-3\text{ctg}\,x\cdot y'+(3\text{ctg}^2x+1)y=0$; $\sin x\neq 0$.

13. $y''-\frac{1}{x(\ln x-1)}y'+\frac{1}{x^2(\ln x-1)}y=0$; $x>0$.

14. $2x^3y'''-x^2y''+2xy'-2y=0$; $x>0$.

15. $x^3(3-2\ln x)y'''+2x^2y''-4xy'+4y=0$; $x>0$.

16. $(2x^3-9x^2+6x)y'''-(6x^3-21x^2+6)y''+$

$$+(4x^3-42x+18)y'-(12x^2-36x+12)y=0.$$

11. *n* - tartibli chiziqli o‘zgarmas koeffitsientli differensial tenglamalar

1. $y=c_1e^x+c_2e^{2x}$. **2.** $y=c_1e^{-x}+c_2e^{2x}$. **3.** $y=c_1e^x\cos x+c_2e^{2x}\sin x$.

4. $y=c_1e^{2x}\cos 3x+c_2e^{2x}\sin 3x$. **5.** $y=c_1e^{-x}+c_2xe^{-x}$. **6.** $y=c_1e^{-x}+c_2e^x+c_3e^{2x}$.

7. $y=c_1e^{3x}+c_2e^x\cos x+c_3e^x\sin x$. **8.** $y=c_1e^{3x}+c_2e^{2x}+c_3xe^{2x}$.

9. $y=c_1\cos 2x+c_2\sin 2x+c_3x\cos 2x+c_4x\sin 2x$.

10. $y=c_1e^{-x}+c_2xe^{-x}+c_3e^{2x}\cos 3x+c_4e^{2x}\sin 3x$.

11. $y=c_1e^x+c_2e^{3x}+x^2e^x/2$. **12.** $y=y=c_1e^{-2x}+c_2e^{-3x}+x(x^2-3x+9)e^{-2x}$.

13. $y=c_1e^{2x}+c_2xe^{2x}+2x^3+6x^2+9x+6+x^3e^{2x}$.

14. $y=c_1e^{-3x}+c_2e^{2x}-\frac{\cos x+7\sin x}{10}+xe^{2x}-\frac{1}{5}e^{2x}$.

15. $y=c_1+c_2x+c_3e^{-x}+\cos x-\sin x$.

16. $y=c_1\cos x+c_2\sin x+c_3e^x-(x+2)(\cos x+1)-(x-1)\sin x$

17. $y=c_1e^{-x}+c_2e^x+c_3xe^x+e^{-x}\sin x$.

18. $y=c_1e^{-x}+c_2e^{-x}\cos x+c_3e^{-x}\sin x+2x-4-(1+x)e^{-x}\cos x+(2-x)e^{-x}\sin x$.

19. $y=y=c_1+c_2x+c_3\cos x+c_4\sin x-3\cos x-x\sin x$.

20. $y=c_1e^{-x}+c_2xe^{-x}+c_3e^{x}+c_4xe^{x}+x^2+x+5$.

21. $y=2-\cos x-x\cos x+2\sin x-x\sin x$.

22. $y=2x+3-3e^{x}-xe^{x}+e^{2x}$. **23.** $y=2x^2+5x+5-4e^{-x}$.

24. $y=3e^{-x/2}\cos\left(\dfrac{\sqrt{3}}{2}x\right)+\sqrt{3}e^{-x/2}\sin\left(\dfrac{\sqrt{3}}{2}x\right)+3xe^{x}-2e^{x}$.

12. Chegaraviy masalalar

1. $y=\dfrac{2}{e^2-1}e^{-x}-\dfrac{e^2+1}{e^2-1}e^{x}+xe^{x}$. **2.** $y=\dfrac{4}{e^2+1}e^{2x}-x^2-2x+\dfrac{e^2-3}{e^2+1}$.

3. $y=\cos x+x\sin x-\pi(\cos x+\sin x)/2$. **4.** $\pi\cos x-\dfrac{\pi}{2}\sin x+x$.

5. $y=2x\ln x-x-\dfrac{e(e-1)}{x}$. **6.** $G(x,\xi)=-\dfrac{1}{\cos 1}\begin{cases}\sin x\cos(1-\xi), & 0\le x\le\xi,\\ \cos(1-x)\sin\xi, & \xi\le x\le 1.\end{cases}$

7. $G(x,\xi)=\begin{cases}1-e^{x}, & 0\le x\le\xi,\\ e^{x}(e^{-\xi}-1), & \xi\le x\le 1.\end{cases}$ **8.** $G(x,\xi)=\dfrac{1}{W(\xi)}\begin{cases}y_1(x)y_2(\xi), & 0\le x\le\xi,\\ y_2(x)y_1(\xi), & \xi\le x\le 1,\end{cases}$

$$W(\xi)=2(\cos^2\xi\cdot\cos 1+\sin^2\xi\cdot\sin 1),\ y_1(x)=\cos x-\sin x,$$
$$y_2(x)=(\cos 1-\sin 1)\cos x+(\cos 1+\sin 1)\sin x.$$

9. $G(x,\xi)=\dfrac{1}{\ln 2}\begin{cases}\ln x\cdot(\ln\xi-\ln 2), & 0\le x\le\xi,\\ (\ln x-\ln 2)\ln\xi, & \xi\le x\le 1.\end{cases}$ **10.** $G(x,\xi)=\begin{cases}-\mathrm{arctg}x, & 0\le x\le\xi,\\ -\mathrm{arctg}\xi, & \xi\le x\le 1.\end{cases}$

13. Differensial tenglamalarning normal sistemasi

1. $x=c_1+2c_2e^{-3t}$, $y=2c_1+c_2e^{-3t}$. **2.** $x=(c_1+c_2t)e^{-3t}$, $y=(2c_1-c_2+2c_2t)e^{-3t}$.

3. $x=c_1e^{t}+c_2e^{-t}-1+\dfrac{1}{2}\sin t$, $y^2-c_1e^{t}+c_2e^{-t}-t-\dfrac{1}{2}\cos t=0$.

4. $x=c_1+c_2e^{-t}+\dfrac{c_2^2}{4c_1}e^{-2t}+t$, $y=\dfrac{4c_1}{c_2^2e^{-2t}+2c_1c_2e^{-t}}$.

5. $x=c_1+c_2e^{t}\cos t+c_3e^{t}\sin t$, $y=c_1+c_2e^{t}\sin t-c_3e^{t}\cos t$,

$z=-c_1+c_2e^{t}\sin t-c_3e^{t}\cos t$.

6. $x=c_2t+c_3$, $y=-c_2t+c_2-c_3$, $z=c_1e^{t}-c_2$. **10.** $2xy-t^2=c$. **11.** $x-y+z=c$.

12. $y=c_1z$, $2(x^3-y^3)+3x^2y^2-6x=c_2$. **13.** $x=c_1(c_2-2c_1t)^{-1}, y=(c_2-2c_1t)^{-1}$.

14. $x=\dfrac{c_1-1}{(c_1-1)c_2e^{(c_1-1)t}+1}, y=c_1-\dfrac{c_1-1}{(c_1-1)c_2e^{(c_1-1)t}+1}$.

15. $x=\sqrt{c_1e^t+c_2e^{-t}-1},\ y=\sqrt{c_1e^t-c_2e^{-t}-1}$.

16. $x=c_1e^t-c_2te^t+2c_3^2e^{2t},\ y=c_2e^t-c_3^2e^{2t},\ z=c_3e^t$.

17. $x=\dfrac{c_1}{(c_2-t)(-c_1\ln|c_2-t|+1)},\ y=\dfrac{1}{c_2-t},\ z=\dfrac{c_3}{c_2-t}$.

18. $x=\dfrac{c_2(c_1c_3-c_2-c_1c_2t)}{(c_3-c_2t)(c_1c_3-2c_2-c_1c_2t)},\ y=\dfrac{1}{c_3-c_2t},\ \ z=\dfrac{c_2^2}{(c_3-c_2t)(c_1c_3-2c_2-c_1c_2t)}$.

14. Normal ko'rinishdagi chiziqli differensial tenglamalar sistemasi

1. Har qanday oraliqda chiziqli erkli. **2.** Har qanday oraliqda chiziqli bo'g'liq.

3. Har qanday oraliqda chiziqli erkli. **4.** Har qanday oraliqda chiziqli erkli.

5. Har qanday oraliqda chiziqli bo'g'liq.

6. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1/t & 0 \\ 0 & 1/t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. **7.** $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1/(t-1) & e^{-t}/(1-t) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

8. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \dfrac{1}{\cos t - t\sin t}\begin{pmatrix} -2\sin t & \cos t + t\sin t \\ \cos t & -t\cos t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

9. $\begin{cases} \dfrac{dx_1}{dt} = \dfrac{1}{t(t-1)}x_2 + \dfrac{1}{t^2(1-t)}x_3, \\ \dfrac{dx_2}{dt} = \dfrac{1}{2(1-t^{1/2})}x_1 + \dfrac{4t^{3/2}-3t-2t^{1/2}+1}{2t(1-t)(1-t^{1/2})}x_2 + \dfrac{1}{t(1-t)}x_3, \\ \dfrac{dx_3}{dt} = \dfrac{t}{2(1-t^{1/2})}x_1 + \dfrac{1}{2(t^{1/2}-1)}x_2 + \dfrac{2}{t}x_3. \end{cases}$

10. $x=c_1e^{2t}-c_2e^{-t}-t,\ y=3c_2e^{-t}+t-1$. **11.** $x=c_1(e^{2t}-te^{2t})+c_2e^{2t}-2e^t-t-1$,

$y=-c_1te^{2t}+c_2e^{2t}-e^t+t$. Mos bir jinsli sistemani yo'qotish usuli yordamida yeching.

12. $x=(c_1+c_2)\cos t+(c_2-c_1)\sin t+\big(t+\ln|\cos t|\big)\cos t+\big(t-\ln|\cos t|\big)\sin t$,

$y=c_1\cos t+c_2\sin t+t\sin t+\ln|\cos t|\cdot\cos t$. **13.** $x=c_1t+c_2+t^2(2\ln t-3)$,

$y=-c_1t+c_1-c_2+t^2(3-2\ln t)+4t(\ln t-1)$.

14. $x=t^2(c_1t+c_2+3\ln|t|+3)+1,\ y=t(3c_1t+2c_2+6\ln|t|+8)$.

15. $x = c_1 t + c_2 + 3/t$, $x = 2c_1 t + c_2 + 2/t$. Mos bir jinsli sistemada $t = e^{\tau}$ $(t = -e^{\tau})$ almashtirish bajaring.

16. $x = c_1 e^t + 5c_2 e^{3t} - c_3 e^{-t} - 8e^{2t}$, $y = 2c_2 e^{3t} - 2c_3 e^{-t} - e^{2t}$, $z = 2c_2 e^{3t} + 2c_3 e^{-t} - 2e^{2t}$.

15. Chiziqli o‘zgarmas koeffitsientli normal differensial tenglamalar sistemasi

1. $a)$. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{2t} & te^{2t} \\ e^{2t} & (1+t)e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, e^{At} = \begin{pmatrix} e^{2t}(1-t) & te^{2t} \\ -te^{2t} & e^{2t}(1+t) \end{pmatrix}$.

$b)$. $\begin{pmatrix} x \\ y \end{pmatrix} = e^{2t} \begin{pmatrix} \sin 3t & \cos 3t \\ -\cos 3t & \sin 3t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, $e^{At} = e^{2t} \begin{pmatrix} \cos 3t & -\sin 3t \\ \sin 3t & \cos 3t \end{pmatrix}$.

2. $a)$. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & t & 1 \\ 1 & 3t & 3 \\ -1 & -1-2t & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, e^{At} = \begin{pmatrix} 1+t & -t & -t \\ 3t & 1-3t & -3t \\ -2t & 2t & 1+2t \end{pmatrix}$.

$b)$. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^t \begin{pmatrix} 1 & \sin t & \cos t \\ 0 & -\cos t & \sin t \\ 1 & \cos t & -\sin t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, $e^{At} = e^t \begin{pmatrix} \cos t & 1-\cos t-\sin t & 1-\cos t \\ \sin t & \cos t-\sin t & -\sin t \\ -\sin t & 1-\cos t+\sin t & 1+\sin t \end{pmatrix}$.

$c)$. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{2t} \begin{pmatrix} 1 & e^t & e^{2t} \\ -3 & e^t & -e^{2t} \\ 1 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$,

$$e^{At} = \frac{e^{2t}}{2} \begin{pmatrix} 1+2e^t-e^{2t} & -1+e^{2t} & -2-2e^t+4e^{2t} \\ -3+2e^t-e^{2t} & 3-e^{2t} & 6-2e^t-4e^{2t} \\ 1-e^{2t} & -1+e^{2t} & -2+4e^{2t} \end{pmatrix}.$$

3. $\chi(\lambda) \overset{def}{=} \det(A-\lambda E) = \lambda^2 - 4\lambda + 4$, $e^{At} = e^{2t} \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}$.

4. $\chi(\lambda) = \lambda^2 - 4\lambda + 5$, $e^{At} = \begin{pmatrix} 2e^t - e^{4t} & e^t - e^{4t} \\ -2e^t + 2e^{4t} & -e^t + 2e^{4t} \end{pmatrix}$.

5. $\chi(\lambda) = \lambda^2 + 1$, $e^{At} = \begin{pmatrix} \cos t + \sin t & 2\sin t \\ -\sin t & \cos t - \sin t \end{pmatrix}$.

6. $\chi(\lambda)=-\lambda^3-2\lambda^2+\lambda+2$, $e^{At}=\begin{pmatrix} 2e^t-2e^{-t}+e^{-2t} & e^{-t}-e^{-2t} & -2e^t+3e^{-t}-e^{-2t} \\ e^t-e^{-2t} & e^{-2t} & -e^t+e^{-2t} \\ e^t-2e^{-t}+e^{-2t} & e^{-t}-e^{-2t} & -e^t+3e^{-t}-e^{-2t} \end{pmatrix}$.

7. $\chi(\lambda)=-\lambda^3+4\lambda^2-5\lambda+2$,

$$e^{At}=\begin{pmatrix} -2e^{2t}+3e^t+3te^t & -3te^t & -6e^{2t}+6e^t+6te^t \\ te^t & -(t-1)e^t & 2te^t \\ e^{2t}-e^t-te^t & te^t & 3e^{2t}-2e^t-2te^t \end{pmatrix}.$$

8. $\chi(\lambda)=-\lambda^3+4\lambda^2-6\lambda+4$,

$$e^{At}=e^t\begin{pmatrix} 2e^t-\cos t-2\sin t & \cos t+\sin t-e^t & \sin t \\ 2e^t-2\cos t-4\sin t & 2\cos t+2\sin t-e^t & 2\sin t \\ 2e^t-2\cos t+\sin t & \cos t-\sin t-e^t & \cos t \end{pmatrix}.$$

9. $J=\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$, $S=\frac{1}{4}\begin{pmatrix} 1 & 3 \\ -3 & 3 \end{pmatrix}$,

$$e^{tA}=Se^{tJ}S^{-1}=\frac{1}{4}\begin{pmatrix} 3e^{2t}+e^{-2t} & e^{2t}-e^{-2t} \\ 3e^{2t}-3e^{-2t} & e^{2t}+3e^{-2t} \end{pmatrix}.$$

10. $J=\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, $S=\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$, $e^{tA}=e^{2t}\begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix}$.

11. $J=\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$, $S=\frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ i & -i \end{pmatrix}$, $e^{tA}=\begin{pmatrix} \cos t+\sin t & -2\sin t \\ \sin t & \cos t-\sin t \end{pmatrix}$.

12. $J=\begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $S=\begin{pmatrix} 1 & -2 & 2 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{pmatrix}$, 6- masalaga qarang.

13. $J=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $S=\begin{pmatrix} -2 & 3 & 3 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}$, 7- masalaga qarang.

14. $J=\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1-i & 0 \\ 0 & 0 & 1+i \end{pmatrix}$, $S=\frac{1}{5}\begin{pmatrix} -3 & 4+2i & 4-2i \\ 0 & 5i & -5i \\ 6 & -3+i & -3-i \end{pmatrix}$,

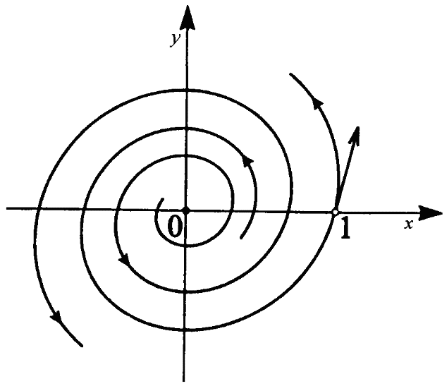
$$e^{At}=\frac{1}{5}\begin{pmatrix} -3e^{-t}+8e^{t}\cos t+4e^{t}\sin t & 2e^{-t}-2e^{t}\cos t-6e^{t}\sin t & -4e^{-t}+4e^{t}\cos t+2e^{t}\sin t \\ 2e^{t}\sin t & e^{t}\cos t-e^{t}\sin t & e^{t}\sin t \\ 6e^{-t}-6e^{t}\cos t+2e^{t}\sin t & -4e^{-t}+4e^{t}\cos t+2e^{t}\sin t & 8e^{-t}-3e^{t}\cos t+e^{t}\sin t \end{pmatrix}$$

15. $J=\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $S=\begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, $e^{At}=\frac{1}{3}\begin{pmatrix} e^{5t}+2e^{-t} & e^{5t}-e^{-t} & e^{5t}-e^{-t} \\ e^{5t}-e^{-t} & e^{5t}+2e^{-t} & e^{5t}-e^{-t} \\ e^{5t}-e^{-t} & e^{5t}-e^{-t} & e^{5t}+2e^{-t} \end{pmatrix}$.

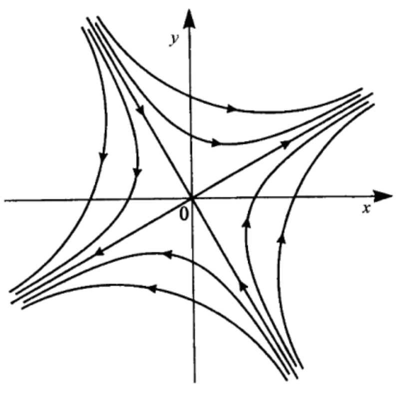
16. $\begin{cases} x=3t+2 \\ y=t+1 \end{cases}$. **17.** $\begin{cases} x=e^{2t}(t^3-3t^2) \\ y=e^{2t}t^3 \end{cases}$. **18.** $\begin{cases} x=\frac{9}{5}\cos t-\frac{12}{5}\sin t \\ y=\frac{4}{5}\cos t+\frac{3}{5}\sin t \end{cases}$.

16. Tekislikda avtonom sistemalar

1. Noturg‘un fokus. $\lambda_{1,2}=\left(3\pm i\sqrt{15}\right)/2$, $\mathrm{Re}\,\lambda_{1,2}=3/2>0$.



2. Egar. $\lambda_1=-3$, $\boldsymbol{h}_1=\begin{pmatrix} 1 \\ -2 \end{pmatrix}$; $\lambda_2=2$, $\boldsymbol{h}_2=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Traektoroyalar uchun $\boldsymbol{h}_1$ va $\boldsymbol{h}_2$ vektorlar aniqlovchi to‘g‘ri chiziqlar asimptotalardir.



3. Markaz. $\lambda_1=i$, $\lambda_2=-i$. $\mathrm{Re}\,\lambda_{1,2}=0$.

4. Turg‘un tugun. $\lambda_1 = -1$, $\boldsymbol{h}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$; $\lambda_2 = -2$, $\boldsymbol{h}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Traektoroyalar $\boldsymbol{h}_1$ vektorga urinadi.

5. Turg‘un fokus. $\lambda_{1,2} = \left(-3 \pm i\sqrt{7}\right)/2$, $\operatorname{Re}\lambda_{1,2} = -3/2 < 0$.

6. Noturg‘un tugun. $\lambda_{1,2} = 2 \pm \sqrt{2} > 0$.

7. Egar. $\lambda_{1,2} = \pm\sqrt{5}$.

8. Dikritik tugun (noturg‘un). $\lambda_{1,2} = 2 > 0$, $\boldsymbol{h}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\boldsymbol{h}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

9. Aynigan tugun (turg‘un). $\lambda_{1,2} = -2$, $\boldsymbol{h}_1 = \boldsymbol{h}_2 = \boldsymbol{h} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

10. Egar. $\lambda_{1,2} = \left(5 \pm \sqrt{33}\right)/2$.

11. $(1;1) -$ egar, $\lambda_{1,2} = \pm 1$, $\boldsymbol{h}_{1,2} = \begin{pmatrix} \mp 1 \\ 1 \end{pmatrix}$.

$(2;2) -$ noturg‘un (dikritik) tugun, $\lambda_{1,2} = 1$, $\boldsymbol{h}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\boldsymbol{h}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

12. $(1;1) -$ egar, $\lambda_1 = -2, \lambda_2 = 3$, $\boldsymbol{h}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\boldsymbol{h}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$(1;-1) -$ turg‘un fokus, $\lambda_{1,2} = \left(-3 \pm i\sqrt{15}\right)/2$, .

13. $(-7;-2) -$ turg‘un tugun, $\lambda_1 = -3, \lambda_2 = -2$, $\boldsymbol{h}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\boldsymbol{h}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

$(5;1) -$ noturg‘un fokus, $\lambda_{1,2} = \left(1 \pm i\sqrt{11}\right)/2$, .

14. $(1;0) -$ turg‘un fokus, $\lambda_{1,2} = -1 \pm i$.

$(-1;-2) -$ egar, $\lambda_{1,2} = -1 \pm \sqrt{3}$, $\boldsymbol{h}_{1,2} = \begin{pmatrix} \pm\sqrt{3} \\ 3 \end{pmatrix}$.

15. $(1;0) -$ noturg‘un fokus, $\lambda_{1,2} = \left(1 \pm i\sqrt{3}\right)/2$.

16. $(0;0) -$ egar, $\lambda_{1,2} = -1 \pm \sqrt{2}$, $\boldsymbol{h}_{1,2} = \begin{pmatrix} \pm\sqrt{2} \\ 1 \end{pmatrix}$.

$(-1/3;-2/9) -$ turg‘un tugun, $\lambda_{1,2} = \dfrac{-4 \pm \sqrt{7}}{3}$, $\boldsymbol{h}_{1,2} = \begin{pmatrix} 1 \pm \sqrt{7} \\ 1 \end{pmatrix}$.

17. $(0;0)$ – egar, $\lambda_{1,2}=\pm\sqrt{7}$, $\boldsymbol{h}_{1,2}=\begin{pmatrix}1\pm\sqrt{7}\\ 2\end{pmatrix}$.

$(-7/4;-7/16)$ – turg‘un fokus, $\lambda_{1,2}=\dfrac{-7\pm i3\sqrt{7}}{4}$.

18. $(3;-1)$ – egar, $\lambda_1=1,\lambda_2=-2$, $\boldsymbol{h}_1=\begin{pmatrix}1\\ 0\end{pmatrix},\boldsymbol{h}_2=\begin{pmatrix}1\\ 3\end{pmatrix}$.

$(-3;-7)$ – turg‘un fokus, $\lambda_{1,2}=\dfrac{-1\pm i\sqrt{7}}{2}$.

19. $(-2;1)$ – noturg‘un fokus, $\lambda_{1,2}=5\pm i\sqrt{23}$.

$(4;-2)$ – turg‘un tugun, $\lambda_1=-8,\lambda_2=-12$, $\boldsymbol{h}_1=\begin{pmatrix}-1\\ 1\end{pmatrix},\boldsymbol{h}_2=\begin{pmatrix}-1\\ 2\end{pmatrix}$

20. $(0;-1)$ – noturg‘un fokus, $\lambda_{1,2}=\left(1\pm i\sqrt{95}\right)/6$.

17. Differensial tenglamalar yechimlarining turg‘unligi

1. Asimptotik turg‘un. **2.** Noturg‘un. **3.** Asimptotik turg‘un.

4. Noturg‘un. **5.** Noturg‘un. **6.** Noturg‘un. **7.** Turg‘un, asimptotik turg‘un emas.

8. Asimptotik turg‘un. **9.** Turg‘un. **10.** Noturg‘un. **11.** Asimptotik turg‘un.

12. Noturg‘un. **13.** Asimptotik turg‘un. **14.** Noturg‘un.

15. $(a;-a;\sqrt{a-1})$ – noturg‘un; $(a;-a;-\sqrt{a-1})$ – asimptotik turg‘un ($a\approx 1{,}981$).

16. $(0;0;1)$ – noturg‘un; $\left(\frac{1}{4};\frac{1}{4};\frac{1}{2}\right)$ – noturg‘un.

17. $(0;0;1)$ – noturg‘un; $\left(-\frac{5}{16};-\frac{5}{16};\frac{1}{4}\right)$ – noturg‘un; $\left(\frac{5}{4};-\frac{5}{4};\frac{3}{2}\right)$ – noturg‘un;

18. Aniq musbat

$$v(x_1,x_2,x_3)=\left(x_1-\frac{1}{2}x_2-\frac{1}{2}x_3\right)^2+\frac{3}{4}x_2^2+\frac{3}{4}x_3^2-\frac{1}{2}x_2x_3=$$

$$=\left(x_1-\frac{1}{2}x_2-\frac{1}{2}x_3\right)^2+\frac{3}{4}\left(x_2-\frac{1}{3}x_3\right)^2+\frac{2}{3}x_3^2.$$

19. Aniq musbat. $v(x,y)=x^2-4x^3+y^4=x^2(1-4x)+y^4\geq\frac{1}{2}x^2+y^4$, agar $|x|\leq\frac{1}{8}$ bo‘lsa. **20.** Asimptotik turg‘un $v=x^2+y^2$. **21.** Asimptotik turg‘un $v=x^2+y^4$.

22. Noturg‘un $v=x^2+y^4$, $\dot{v}>0$. **23.** Noturg‘un $v=y^2-x^2$, $|y|>|x|$ da $v>0$ hamda $\dot{v}>0$ (Chetayev teor.). **24.** Turg‘un $v=x^2+y^2$. **25.** Turg‘un $v=x^4+2y^2$.

26. Noturg‘un $v=x^2+y^2$, $\dot{v}>0$.

18. Differensial tenglamalar yechimlarini qatorlar yordamida qurish

1. $$y=c_1\left(1-\frac{1}{2}x^2+\ldots+(-1)^n\frac{5\cdot 9\cdot\ldots\cdot(4n-3)}{2\cdot 3\cdot\ldots\cdot(2n)}x^{2n}+\ldots\right)+$$
$$+c_2\left(x-\frac{1}{2}x^3+\ldots+(-1)^n\frac{3\cdot 7\cdot\ldots\cdot(4n-1)}{2\cdot 3\cdot\ldots\cdot(2n+1)}x^{2n+1}+\ldots\right),\quad R=+\infty.$$

2. $y=c_1y_1(x)+c_2y_2(x)$, bunda
$$y_1(x)=1-\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{24}x^4-\frac{7}{120}x^5-\frac{1}{120}x^6+\ldots,$$
$$y_2(x)=x-\frac{1}{2}x^2+\frac{1}{12}x^4-\frac{1}{120}x^5-\frac{1}{120}x^6+\ldots;\ R=+\infty.$$

3. $y=x-\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{12}x^4+\frac{1}{60}x^5+\ldots,\ R=1.$

4. $y=1+x-\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{5}{32}x^4-\ldots,\ R=1.$

5. $$y=c_0+c_1x+\frac{c_0}{12}x^4+\sum_{n=5}^{\infty}a_nx^n,\ a_n=\frac{1}{n^2-n}a_{n-4}\ (n\geq 5),$$
$$a_0=c_0,\ a_1=c_1,\ a_2=0,\ a_3=0,\ a_4=\frac{c_0}{12};\ R=+\infty.$$

6. $y=1-x-\frac{1}{3}x^3-\frac{1}{30}x^5-\frac{1}{45}x^6-\ldots,\quad R\geq R_0=\frac{\sqrt{2}\left(\sqrt{3}-\sqrt{2}\right)}{16}\approx 0{,}028.$

$|x|<r,\ |y-1|<r,\ |y'+1|<r\ (r>0)$ bo‘lganda

$$|f(x,y,y')|=\left|-y'^2+y^2+x^2\right|\leq 2(1+r)^2+r^2=3r^2+4r+2=M.$$

$\min\left\{r,\frac{r}{4M}\right\}=\frac{r}{4(3r^2+4r+2)}$ funksiyaning maksimumi $R_0=\frac{\sqrt{2}\left(\sqrt{3}-\sqrt{2}\right)}{16}$.

7. $y = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{3}x^4 + \frac{29}{120}x^5 + \ldots$, $R \geq R_0 \approx 0{,}183$.

8. $y_1 = 1 - \frac{2}{3}x + \frac{1}{5}x^2 - \ldots$, $y_2 = x^{-1/2}\left(1 - \frac{3}{2}x + \frac{5}{8}x^2 - \ldots\right)$.

9. $y_1 = x\left(-\frac{1}{15}x^3 + \frac{1}{180}x^6 + \ldots\right)$, $y_2 = x^{-1}\left(1 + \frac{1}{3}x^3 - \frac{1}{36}x^6 + \ldots\right)$.

10. $y_1 = x^2(1 - 4x + 4x^2 - \ldots)$, $y_2 = y_1(x)\ln x + x^2\left(8x - 12x^2 + \frac{176}{27}x^3 + \ldots\right)$.

19. Kichik parametr metodi

1. $x = \dfrac{1}{(2t+1)^{1/2}} + \dfrac{(2t+1)^{5/2}(5t-1) + 36}{35(2t+1)^{3/2}}\mu + O(\mu^2)$.

2. $x = 1 + x - e^{x-1} + \left(-x^2 - 2x - 2 + (5 - x^2)e^{x-1} + e^{2(x-1)}\right)\mu + O(\mu^2)$.

3. $x = \arcsin\dfrac{\sqrt{2}e^t}{2} - \dfrac{(2 - e^{2t})e^t}{3}\mu + O(\mu^2)$.

4. $$x = a\cos t + b\sin t + \left(-\frac{4a^2 + 11b^2}{30} + \frac{2a^2 + 8b^2}{15}\cos t - \right.$$
$$\left. - \frac{4ab}{15}\sin t - \frac{b^2}{6}\cos 2t + \frac{ab}{6}\sin 2t + \frac{b^2 - a^2}{120}\cos 4t - \frac{ab}{60}\sin 4t\right)\mu + O(\mu^2).$$

5. $x = 1 + e^t + \left(\frac{1}{2}e^{2t} + 2e^t t - 3e^t - t + \frac{5}{2}\right)\mu + O(\mu^2)$.

6. $x = 2e^{2t} - 2e^{-t} + \left(\frac{1}{5}e^{4t} + 3e^{2t} - 4e^t - \frac{6}{5}e^{-t} + 2e^{-2t}\right)\mu + O(\mu^2)$,

$y = 2e^{2t} + 2e^{-t} + \left(\frac{3}{10}e^{4t} + \frac{3}{2}e^{2t} + \frac{6}{5}e^{-t} - 3e^{-2t}\right)\mu + O(\mu^2)$.

7. $\dfrac{t^3 - t}{2}$. **8.** $\dfrac{(t+e)e^t}{1 + (e-1)e^t}$. **9.** e^{t-1}. **10.** $2e^{2t} - e^t - t - 1$.

11. $\frac{1}{6}\left(e^{2t} - 4e^{-t} - 3e^{-2t}\right)$. **12.** $t\int_t^1 \frac{e}{s}\exp\left(-\frac{1}{s^2}\right)ds$.

20. Birinchi tartibli xususiy hosilali differensial tenglamalar

1. $\Phi\left(\frac{x+y}{u}\exp\left(\frac{1}{u}\right), \frac{x-y}{u}\exp\left(-\frac{1}{u}\right)\right) = 0$. **2.** $u = (x-y)\varphi\left(\frac{x-y}{x-z}, \frac{x-z}{y-z}\right)$.

3. $\Phi\left(y(x-z), y(x-u), y\exp\left(\frac{1}{u^2}\right)\right) = 0$. **4.** $\sqrt{u^2 + 1} - u = \frac{1}{x}\varphi(x^4 + y^2)$.

5. $u=\dfrac{x+y}{1+(x+y)\varphi\left(\dfrac{x^2-y^2}{x}\right)}$. **6.** $\Phi(\frac{u}{x}, x^3u-2y^2)=0$.

7. $u=\Phi(x^2+y^2)$; $u=x^2+y^2-1$. **8.** $u=x-y+\Phi\left(\frac{y}{x}\right)$; $u=x-y+2\dfrac{y^2}{x^2}-\dfrac{y}{x}$.

9. $u=\Phi\left(\dfrac{y}{x^2-y^2}\right)\cdot\exp\left(\dfrac{x}{x^2-y^2}\right)$; $u=\dfrac{x^2-y^2}{3y}\exp\left(\dfrac{x-2y}{x^2-y^2}\right)$.

10. $u=\Phi\left(-\dfrac{y^2}{4}+z, \dfrac{3x-2y^2+3z}{3\sqrt{y}}\right)$; $u=\dfrac{7}{6}-\dfrac{y^2}{2}+2z+\dfrac{(3x-y^2+3z)\sqrt{y}}{3y}$.

11. $u=\ln x+\Phi\left(\dfrac{z}{x}, xz^2+3z+y\right)$; $u=\ln x+\dfrac{yz}{x}+z^3+\dfrac{3z^2}{x}-\dfrac{3z}{x}-1$.

12. $u=\Phi(z^2+2\sin y, xy+\sin y)$; $u=2xy-z^2$.

13. $u=\Phi(e^z-y, xy+z)$; $u=xy-y+z+e^z$.

14. $u=\Phi(z^2-\ln x, xy-z)$; $u=2\sqrt{z^2-\ln x}+xy-z$.

15. $\ln u=xy+\varphi\left(\dfrac{y}{x}\right)$; $u=\dfrac{y^2}{x^2}\exp\left(xy-\dfrac{y}{x}\right)$. **16.** $u=\dfrac{1}{\varphi(xy)-x-y}$; $u=\dfrac{1}{1-x-y+2xy}$. **17.**

$\Phi((x-y)u, (4y-u)u)=0$; $u=\dfrac{4y}{1+3(x-y)^2}$.

18. $\Phi(x^2-u^2, yu)=0$; $u=\sqrt{\dfrac{1+x^2}{1+y^2}}$.

19. $\Phi\left(x-y^2, y+\dfrac{1}{u}\right)=0$; $u=\dfrac{1}{\varphi(x-y^2)-y}, \varphi(0)=0$ (yechim cheksiz ko‘p). **20.**

$\Phi\left(x-y^2, y+\dfrac{1}{u}\right)=0$; yechim mavjud emas.

ILOVALAR

Differensiallash (hosila hisoblash) qoidalari

Bir o'zgaruvchining funksiyasi

0. Differensial: $df(x) = f'(x)dx$

1. O'zgarmas: $\frac{d}{dx}c = (c)' = 0$

2. O'zgarmas ko'paytuvchi: $\left(cf(x)\right)' = cf'(x)$

3. Yig'indi: $\left(f(x) + g(x)\right)' = f'(x) + g'(x)$.

2-3. Hosilaning chiziqliligi: $\left(c_1 f(x) + c_2 g(x)\right)' = c_1 f'(x) + c_2 g'(x)$

4. Ko'paytma: $\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$

5. Nisbat: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

6. Murakkab funksiya: $\left(f(g(x))\right)' = f'(g(x))g'(x)$

Elementar funksiyalar

7. Darajali: $\left(x^\mu\right)' = \mu x^{\mu-1}$.

8. Ko'rsatkichli: $\left(a^x\right)' = a^x \ln a$, $\left(e^x\right)' = e^x$.

9. Logarifmik: $\left(\log_a |x|\right)' = \frac{1}{x \ln a}$, $\left(\ln|x|\right)' = \frac{1}{x}$.

Trigonometrik:

10. $\left(\sin x\right)' = \cos x$. **11.** $\left(\cos x\right)' = -\sin x$.

12. $\left(\mathrm{tg} x\right)' = \frac{1}{\cos^2 x}$. **13.** $\left(\mathrm{ctg} x\right)' = -\frac{1}{\sin^2 x}$.

Teskari triginometrik:

14. $\left(\arcsin x\right)' = \frac{1}{\sqrt{1-x^2}}$. **15.** $\left(\arccos x\right)' = -\frac{1}{\sqrt{1-x^2}}$.

16. $\left(\mathrm{arctg}\, x\right)' = \frac{1}{1+x^2}$. **17.** $\left(\mathrm{arcctg}\, x\right)' = -\frac{1}{1+x^2}$.

Hiperbolik

$$\left(\operatorname{sh} x=\frac{e^{x}-e^{-x}}{2}\ (=\sinh x),\ \ \operatorname{ch} x=\frac{e^{x}+e^{-x}}{2}\ (=\cosh x),\right.$$

$$\left.\operatorname{th} x=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\ (=\operatorname{tgh} x=\tanh x),\ \operatorname{cth} x=\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}\ (=\operatorname{ctanh} x)\right):$$

18. $(\operatorname{sh} x)'=\operatorname{ch} x$. **19.** $(\operatorname{ch} x)'=\operatorname{sh} x$. **20.** $(\operatorname{th} x)'=\dfrac{1}{\operatorname{ch}^2 x}$. **21.** $(\operatorname{cth} x)'=-\dfrac{1}{\operatorname{sh}^2 x}$.

Teskari hiperbolik

$$\left(\operatorname{Arsh} x=\ln\left(x+\sqrt{x^2+1}\right),\ \operatorname{Arch} x=\ln\left(x+\sqrt{x^2-1}\right),\right.$$

$$\left.\operatorname{Arth} x=\frac{1}{2}\ln\frac{1+x}{1-x}\ ,\ \operatorname{Arcth} x=\frac{1}{2}\ln\frac{x+1}{x-1}\right):$$

22. $(\operatorname{Arsh} x)'=\dfrac{1}{\sqrt{x^2+1}}$. **23.** $(\operatorname{Arch} x)'=\dfrac{1}{\sqrt{x^2-1}}$.

24. $(\operatorname{Arth} x)'=\dfrac{1}{1-x^2}$. **25.** $(\operatorname{Arcth} x)'=\dfrac{1}{1-x^2}$.

Ko'p o'zgaruvchining funksiyalari

$u=u(x_1,x_2,\ldots,x_n),\ v=v(x_1,x_2,\ldots,x_n)$ funksiyalar uchun

26. $du=\dfrac{\partial u}{\partial x_1}dx_1+\dfrac{\partial u}{\partial x_2}dx_2+\ldots+\dfrac{\partial u}{\partial x_n}dx_n=u'_{x_1}dx_1+u'_{x_2}dx_2+\ldots+u'_{x_n}dx_n$.

27. Yig'indi: $d(u+v)=du+dv$.

28. O'zgarmas ko'paytuvchi: $d(c\cdot u)=c\cdot du$.

29. Ko'paytma: $d(uv)=vdu+udv$.

30. Nisbat: $d\left(\dfrac{u}{v}\right)=\dfrac{vdu-udv}{v^2}$.

31. Murakkab funksiya:

$$u=u(x_1(t_1,t_2,\ldots t_m),x_2(t_1,t_2,\ldots t_m),\ldots,x_n(t_1,t_2,\ldots t_m))$$

funksiyalar uchun

$$\frac{\partial u}{\partial t_j}=\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_j}+\frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_j}+\ldots+\frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_j}\ \ (j=\overline{1,n}).$$

Integrallar jadvali

1. $\int u^{\mu}du=\frac{u^{\mu+1}}{\mu+1},\mu\neq-1.$ **2.** $\int \frac{1}{u}du=\ln|u|.$

3. $\int e^{u}du=e^{u}.$ **4.** $\int a^{u}du=\frac{1}{\ln a}a^{u}.$

5. $\int \sin u\,du=-\cos u.$ **6.** $\int \cos u\,du=\sin u.$

7. $\int \frac{1}{\cos^2 u}du=\mathrm{tg}u.$ **8.** $\int \frac{1}{\sin^2 u}du=-\mathrm{ctg}u.$

9. $\int \mathrm{sh}u\,du=\mathrm{ch}u.$ **10.** $\int \mathrm{ch}u\,du=\mathrm{sh}u.$

11. $\int \frac{1}{\mathrm{ch}^2 u}du=\mathrm{th}u.$ **12.** $\int \frac{1}{\mathrm{sh}^2 u}du=-\mathrm{cth}u.$

13. $\int \frac{1}{\sqrt{a^2-u^2}}du=\arcsin\frac{u}{a}.$ **14.** $\int \frac{1}{\sqrt{u^2-a^2}}du=\ln\left|u+\sqrt{u^2-a^2}\right|.$

15. $\int \frac{1}{\sqrt{a^2+u^2}}du=\ln\left|u+\sqrt{a^2+u^2}\right|.$ **16.** $\int \frac{1}{a^2+u^2}du=\frac{1}{a}\mathrm{arctg}\frac{u}{a}.$

17. $\int \frac{1}{a^2-u^2}du=\frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right|.$

Integrallash qoidalari

1. O‘zgarmas ko‘paytuvchi: $\int cf(x)dx=c\int f(x)dx.$

2. Yig‘indi: $\int\left(f(x)+g(x)\right)dx=\int f(x)dx+\int g(x)dx.$

3. Bo‘laklab integrallash: $\int udv=uv-\int vdu.$

4. O‘zgaruvchini almashtirish:

$$\int f(x)dx=\int f(\varphi(t))\varphi'(t)dt,\ x=\varphi(t)\ .$$

Ba’zi funksiyalarni integrallash

Ratsional funksiyalarni integrallash

Ratsional funksiya ko‘phadlar nisbatidan iborat:

$R(x)=\dfrac{P_m(x)}{Q_n(x)}$, $P_m(x)$, $Q_n(x)-$ko‘phadlar, $\deg P_m(x)=m$, $\deg Q_n(x)=n$. $\int R(x)dx$ integralni hisoblash uchun dastlab $R(x)=\dfrac{P_m(x)}{Q_n(x)}$ ratsional funksiyaning butun qismini ajratamiz:

$$\frac{P_m(x)}{Q_n(x)}=S(x)+\frac{P_k(x)}{Q_n(x)},\ S(x)-\text{ko‘phad},\ k<n.$$

$$\left(\int\frac{P_m(x)}{Q_n(x)}dx=\int S(x)dx+\int\frac{P_k(x)}{Q_n(x)}dx\right)$$

Bu yerda, agar $m<n$ bo‘lsa, $S(x)=0$ va $k=m$ bo‘ladi.

Endi bu yerdagi to‘g‘ri ratsional funksiya $\dfrac{P_k(x)}{Q_n(x)}$ ni integrallash uchun uni eng oddiy ratsional kasrlar (eng oddiy kasrlar) yig‘indisi ko‘inishida ifodalash kerak. Buning uchun $Q_n(x)$ ko‘phadni keltirilmas haqiqiy ko‘paytuvchilarga ajratish lozim:

$$Q_n(x)=a_n(x-x_1)^{s_1}(x-x_2)^{s_2}\ldots(x^2+p_1x+q_1)^{k_1}(x^2+p_2x+q_2)^{k_2}\ldots,$$
$$s_1+s_2+\ldots+2k_1+2k_2+\ldots=n,\ p_1^2-4q_1<0,\ p_2^2-4q_2<0,\ldots$$

($x=x_j$ sonlar $Q_n(x)$ ko‘phadning s_j karrali ildizlari ($j=1,2,...$); $x^2+p_lx+q_l$ ko‘phadlar haqiqiy ildizga ega emas ($l=1,2,...$)).

Bundan foydalanib, ushbu

$$\begin{aligned}\frac{P_k(x)}{Q_n(x)}=&\frac{A_1}{x-x_1}+\frac{A_2}{(x-x_1)^2}+\ldots+\frac{A_{s_1}}{(x-x_1)^{s_1}}+\\&+\frac{B_1}{x-x_2}+\frac{B_2}{(x-x_2)^2}+\ldots+\frac{B_{s_2}}{(x-x_2)^{s_2}}+\\&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+\\&+\frac{M_1x+N_1}{x^2+p_1x+q_1}+\frac{M_2x+N_2}{(x^2+p_1x+q_1)^2}+\ldots+\frac{M_{l_1}x+N_{l_1}}{(x^2+p_1x+q_1)^{l_1}}+\\&+\frac{U_1x+F_1}{x^2+p_2x+q_2}+\frac{U_2x+F_2}{(x^2+p_2x+q_2)^2}+\ldots+\frac{U_{l_2}x+F_{l_2}}{(x^2+p_2x+q_2)^{l_2}}+\\&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

yoyilmani topish kerak (u yoki bu usul yordamida). Bu yerdagi eng oddiy kasrlar odatdagicha integrallanadi.

$$\int \frac{A_1}{x-x_1}dx = A_1 \ln|x-x_1| \ ,\int \frac{A}{(x-a)^s}dx = -\frac{A}{(s-1)(x-a)^{s-1}}, \quad s = 2,3,4\ldots$$

$$\int \frac{M_1x+N_1}{x^2+p_1x+q_1}dx = \int \frac{M_1(x+p_1/2)+N_1-p_1/2}{(x+p_1/2)^2+(4q_1-p_1^2)/4}dx =$$

$$= \int \frac{M_1(x+p_1/2)}{(x+p_1/2)^2+(4q_1-p_1^2)/4}dx + \int \frac{N_1-p_1/2}{(x+p_1/2)^2+(4q_1-p_1^2)/4}dx =$$

$$= \frac{M_1}{2}\ln(x^2+p_1x+q_1)+(N_1-p_1/2)\int \frac{1}{u^2+a^2}du$$

$$(u = x+p_1/2, a = \sqrt{4q_1-p_1^2}/2).$$

Ushbu $\int \frac{M_1x+N_1}{(x^2+p_1x+q_1)^l}dx$, $l \geq 2$, integralni hisoblash yuqoridagi shakl almashtirishlardan so‘ng $I_l = \int \frac{1}{(u^2+a^2)^l}du$ integralni hisoblashga keltiriladi. Oxirgi integral esa bo‘laklab integrallash yordamida hosil qilinuvchi ushbu

$$I_l = \frac{u}{2a^2(l-1)(u^2+a^2)^{l-1}} + \frac{2l-3}{2a^2(l-1)}I_{l-1}$$

rekurrent formulaga ko‘ra topiladi. Ratsional funksiyaning integrali elementar funksiyadan iborat bo‘ladi.

Trigonometrik funksiyalar qatnashgan ifodalarni integrallash. $\int R(\sin x, \cos x)dx$ ($R(u,v)-$ ratsional funksiya, ya’ni u,v o‘zgaruvchilarning ko‘phadlari nisbati) ko‘rinishdagi integralni hisoblash $\mathrm{tg}\frac{x}{2} = t$ almashtirish yordamida ratsional funksiyani integrallashga keltiriladi.

Ba’zi hollarda soddaroq almashtirishlardan foydalanish mumkin.

I. Agar $R(-\sin x, \cos x) = -R(\sin x, \cos x)$ bo‘lsa, $\cos x = t$ almashtirishdan foydalanish maqsadga muvofiq.

II. Agar $R(\sin x, -\cos x) = R(\sin x, \cos x)$ bo‘lsa, $\sin x = t$ deyish kerak.

III. Agar $R(-\sin x, -\cos x) = R(\sin x, \cos x)$ bo‘lsa, $\mathrm{tg}x = t$ almashtirishni ishlatish kerak.

Radikal (ildiz) qatnashgan ifodalarni integrallash

I. $\int x^m(a+bx^n)^p dx$, $m,n,p-$ratsional sonlar, $a,b-$noldan farqli o'zgarmaslar, quyidagi uch holda elementar funksiyadan iborat bo'ladi:

1) $p-$butun son; bu holda m va n kasrlarning umumiy maxrajini N deb, $z=x^N$ almashtirish bajarish kerak.

2) $\frac{m+1}{n}-$butun son; N bilan p kasrning maxrajini belgilab, $a+bx^n=z^N$ deymiz.

3) $\frac{m+1}{n}+p-$ butun son; N bilan p kasrning maxrajini belgilaymiz va, $ax^{-n}+b=z^N$ almashtirish bajaramiz.

II. $\int R\left(x,\sqrt[m]{\frac{\alpha x+\beta}{\gamma x+\delta}}\right)dx$ ($R(u,v)-$ ratsional funksiya, $m-$natural son, $\alpha,\beta,\gamma,\delta-$ o'zgarmas sonlar) ko'rinishdagi integralni hisoblash uchun

$$\sqrt[m]{\frac{\alpha x+\beta}{\gamma x+\delta}}=t, \text{ ya'ni } \frac{\alpha x+\beta}{\gamma x+\delta}=t^m$$

deb, t o'zgaruvchiga o'tish kerak. Bunda t ning ratsional funksiyasini integrallashga kelamiz.

III. $\int R\left(x,\sqrt{ax^2+bx+c}\right)dx$ integralni hisoblashda quyidagi Eyler almashtirishlaridan foydalanish mumkin:

1) $a>0$ holda $\sqrt{ax^2+bx+c}=\pm x\sqrt{a}+t$;

2) $b^2-4ac\geq 0$ bo'lganda $\pm(x-x_1)t$, bunda x_1 bilan ax^2+bx+c ning ildizi belgilangan;

3) $c>0$ da $\sqrt{ax^2+bx+c}=\pm\sqrt{c}+xt$.

Elementar funksiyalarning umumiy ta'rifi va xossalari

Ma'lumki, asosiy elementar funksiyalar deb quyidagi funksiyalarga aytilgan:

– o'zgarmas funksiya, $f(x)=c$, $c=const$;

– darajali funksiya, $f(x)=x^\mu$, $\mu=const$;

– ko'rsatkichli funksiya, $f(x)=a^x$ $(a=const>0, a\neq 1)$;

– logarifmik funksiya, $f(x)=\log_a x$ $(a=const>0, a\neq 1)$;

– trigonometrik funksiyalar, $f(x)=\sin x$, $f(x)=\cos x$, $f(x)=\mathrm{tg}\, x$, $f(x)=\mathrm{ctg} x$;

– teskari trigonometrik funksiyalar, $f(x)=\arcsin x$, $f(x)=\arccos x$, $f(x)=\mathrm{arctg}\,x$, $f(x)=\mathrm{arcctg}\,x$.

Asosiy elementar funksiyalar hamda ular orqali qo‘shish, ayirish, ko‘paytirish, bo‘lish va kompozitsiya olish (murakkab funksiya tuzish) amallarini chekli marta ishlatib hosil qilingan funksiyalar elementar funksiyalar deyiladi.

Masalan: $f(x)=x$ va $g(x)=c$, ($c-$o‘zgarmas son) asosiy elementar funksiyalardan chekli sondagi ko‘paytirish va qo‘shish amallari yordamida $P(x)=a_n x_{10}^n+...+a_1x+a_0$ elementar funksiyani (ko‘phadni, yoki polinomni) tuzish mumkin. Ko‘phadlar nisbati kasr-ratsional funksiya deyiladi. U ham elementar funksiyadir. $\sin\left(e^{\arccos(2x+1)}\right)$ funksiya ham elementar. Matematik analiz kursida quyidagi teorema isbotlanadi.

Teorema. Har qanday elementar funksiya o‘z aniqlanish to‘plamida uzluksizdir.

Elementar bo‘lmagan funksiyalar ham mavjud. Masalan. x ning butun qismi $f(x)=[x]-$elementar bo‘lmagan funksiya, chunki u $\forall x\in\mathbb{R}$ da aniqlangan, lekin $m\in\mathbb{Z}$ butun nuqtalarda 1-tur uzilishga ega. Analiz kursida ushbu

$$f(x)=\int_0^x e^{-t^2}dt,\ f(x)=\int_0^x \frac{\sin t}{t}dt,\ f(x)=\int_1^x \frac{t}{\ln t}dt,\ f(x)=\int_1^x \frac{e^t}{t}dt$$

funksiyalarning ham elementar emasligi e’tirof etilgan.

Uzluksiz funksiyaar yig‘indisi, ayirmasi, ko‘paytmasi, bo‘linmasi (maxraj nolga aylanmaganda) va kompozitsiyasi (mavjud bo‘lganda) uzluksiz funksiyadir. Bunda funksiyalar bir yoki bir necha argumentlarga bog‘liq bo‘lishi mumkin. Masalan, $f(x,y)=xy$, $g(x)=\sqrt{x^2+1}$, $h(x)=\mathrm{arctg}\,x$ funksiyalar uzluksiz bo‘lgani uchun ularning kompozitsiyasi bo‘lmish $u(x,y)=f(g(x),h(y))=\sqrt{x^2+1}\,\mathrm{arctg}\,y$ funksiya ham uzluksizdir.

Oshkormas funksiya va $C^k(U,V)$ sinflar to‘g‘risida ma’lumotlar

Ma’lumki, X to‘plamda aniqlangan $f:X\to\mathbb{R}$ (haqiqiy) funksiya barcha $x\in X$ larga bittadan $y=f(x)\in\mathbb{R}$ sonni mos qo‘yadi. Funksiyani turli usullar bilan berish mumkin, masalan, analitik usulda, ya’ni funksiya qiymatini hisoblash formulasi yordamida:

$f(x)=x^2-x+1,\ f(x)=\dfrac{2\sin x}{x^4+1}$, ….(bu funksiyalar uchun $x\in X=\mathbb{R}$). Ba’zan funksiyalarni oshkormas ko‘rinishda, ya’ni biror tenglamaning yechimi sifatida berishga to‘g‘ri keladi. Aytaylik, $y+e^x y-\sqrt{x}+2=0$ tenglama berilgan bo‘lsin. Tushunarliki, har bir $x\ge 0$ son uchun bu tenglama bir dona $y=\dfrac{\sqrt{x}-2}{e^x+1}$ yechimga ega, ya’ni berilgan tenglama $y=f(x)$, $f(x)=\dfrac{\sqrt{x}-2}{e^x+1}$ $(x\ge 0)$, funksiyani bir qiymatli aniqlaydi. Bu holda $y+e^x y-\sqrt{x}+2=0$ tenglama $y=\dfrac{\sqrt{x}-2}{e^x+1}$ funksiyani oshkormas ko‘rinishda (aniqlagan) bergan deyiladi.

Endi umumiyroq $F(x,y)=0$ tenglamani qaraylik ($F(x,y)-$ikki haqiqiy argunentning haqiqiy funksiyasi). Agar biror $X\subset\mathbb{R}$ to‘plamdagi har bir $x\in X$ uchun bu tenglama yagona $y=f(x)$ yechimga ega bo‘lsa ($F(x,f(x))=0,\ x\in X$), u holda X da aniqlangan $y=f(x)$ funksiya $F(x,y)=0$ tenglama bilan oshkormas ko‘rinishda berilgan deb yuritiladi. Agar $F(x,y)=0$ tenglama berilgan x larda $y$ga nisbatan kamida ikkita yechimga ega bo‘lsa, bu tenglamani qanoatlantiradigan $y=f(x)$ funksiyalar ko‘p bo‘ladi va ularning bittasini ajratish uchun yechimdan u yoki bu shartlarni talab qilish kerak bo‘ladi.

Masalan, $y^2-x^2=0$ tenglamani qaraylik. Bu tenglama har bir $x\ne 0$ uchun ikkita $y=\pm x$ yechimga ega. Ixtiyoriy $E\subset\mathbb{R}$ to‘plamga ko‘ra ushbu

$$f_E(x)=\begin{cases} x, \text{ agar } x\in E \text{ bo'lsa,}\\ -x, \text{ agar } x\notin E \text{ bo'lsa}\end{cases}$$

funksiyani tuzaylik. Ravshanki, har qanday $E\subset\mathbb{R}$ uchun $y=f_E(x)$, $x\in\mathbb{R}$, funksiya qaralayotgan tenglamaning yechimi, ya’ni $y^2-x^2=0$ tenglama cheksiz ko‘p $y=f_E(x)$ yechimlarga ega. Bu tenglamaning uzluksiz yechimlari to‘rtta: $y=x$, $y=-x$, $y=|x|$ va $y=-|x|$; C^1 sinfga tegishli yechimlari esa ikkita: $y=x$ va $y=-x$. Agar tenglamaning $x=1$ ga yetarlicha yaqin x lar uchun $y=1$ ga yaqin y yechimlarini izlasak, u holda, ravshanki, bir dona $y=x$ yechimni topamiz.

Oshkormas funksiya haqidagi teorema $F(x,y)=0$ tenglamaning biror $y=f(x)$ funksiyani oshkormas ko‘rinishda aniqlashi uchun yetarli shartlarni beradi.

***Teorema* (*oshkormas funksiya haqidagi*).** Aytaylik,

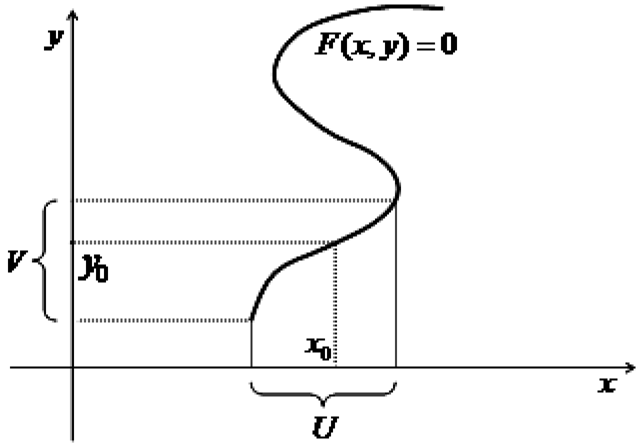
1^0. $F(x_0, y_0)=0$,

2^0. F funksiya $(x_0, y_0)\in\mathbb{R}^2$ nuqtaning biror atrofida C^1 sinfga tegishli va

3^0. $\left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0)} \neq 0$

bo‘lsin.U holda $x_0\in\mathbb{R}$ nuqtaning shunday $U\subset\mathbb{R}$ va $y_0\in\mathbb{R}$ nuqtaning shunday $V\subset\mathbb{R}$ atroflari mavjudki, har qanday $x\in U$ uchun $F(x,y)=0$ tenglama yagona $y=f(x)\in V$ yechimga ega va $f(x)\in C^1(U,V)$ bo‘ladi.

Geometrik nuqtai nazardan bu teorema quyidagini anglatadi. Tekislikda $F(x,y)=0$ tenglamani qanoatlantiruvchi nuqtalar to‘plami (x_0, y_0) nuqtasining biror $U\times V$ ko‘rinishdagi atrofida $y=f(x)$ oshkor ko‘rinishda ifodalanadi (** rasm).



** rasm.

Endi oshkormas funksiya haqidagi teoremaning umumlashgan ko‘rinishini keltiramiz. Bu teoremada ushbu

$$\begin{cases} F_1(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n)=0, \\ F_2(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n)=0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ F_n(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n)=0 \end{cases} \tag{1}$$

sistemadan $y_1, y_2, \dots, y_n$ o‘zgaruvchilarni $x_1, x_2, \dots, x_m$ o‘zgaruvchilarning silliq funksiyalari sifatida topib olish uchun yetarli shartlar keltiriladi.

***Teorema* (*oshkormas funksiyalar haqidagi*).** Faraz qilaylik, quyidagi uch shart bajarilgan bo‘lsin:

1^0. $\begin{cases} F_1(x_1^0, x_2^0, \ldots, x_m^0, y_1^0, y_2^0, \ldots, y_n^0) = 0 \\ F_2(x_1^0, x_2^0, \ldots, x_m^0, y_1^0, y_2^0, \ldots, y_n^0) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F_n(x_1^0, x_2^0, \ldots, x_m^0, y_1^0, y_2^0, \ldots, y_n^0) = 0 \end{cases}$,

2^0. $F_j \ (j = 1, 2, \ldots, n)$ funksiyalar $(x_1^0, x_2^0, \ldots, x_m^0, y_1^0, y_2^0, \ldots, y_n^0) \in \mathbb{R}^{m+n}$ nuqtaning biror atrofida C^1 sinfga tegishli,

3^0. $(x_1^0, x_2^0, \ldots, x_m^0, y_1^0, y_2^0, \ldots, y_n^0) \in \mathbb{R}^{m+n}$ nuqtada

$$\begin{vmatrix} \dfrac{\partial F_1}{\partial y_1} & \dfrac{\partial F_1}{\partial y_2} & \cdots & \dfrac{\partial F_1}{\partial y_n} \\ \dfrac{\partial F_2}{\partial y_1} & \dfrac{\partial F_2}{\partial y_2} & \cdots & \dfrac{\partial F_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial F_n}{\partial y_1} & \dfrac{\partial F_n}{\partial y_2} & \cdots & \dfrac{\partial F_n}{\partial y_n} \end{vmatrix} \neq 0.$$

U holda $(x_1^0, x_2^0, \ldots, x_m^0) \in \mathbb{R}^m$ nuqtaning shunday $U \subset \mathbb{R}^m$ va $(y_1^0, y_2^0, \ldots, y_n^0) \in \mathbb{R}^n$ nuqtaning shunday $V \subset \mathbb{R}^n$ atroflari mavjudki, har qanday $(x_1, x_2, \ldots, x_m) \in U$ uchun (1) tenglamalar sistemasi yagona

$$\begin{cases} y_1 = f_1(x_1, x_2, \ldots, x_m), \\ y_2 = f_2(x_1, x_2, \ldots, x_m), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_n = f_n(x_1, x_2, \ldots, x_m), \end{cases}, \ (y_1, y_2, \ldots, y_n) \in V,$$

yechimga ega va bu yerdagi $f_1, f_2, \ldots, f_n$ funksiyalar $C^1(U, V)$ sinfga tegishli bo‘ladi.

Oshkormas funksiya haqidagi teoremaning xususiy holi bo‘lgan teskari funksiya haqidagi teoremani ham keltiraylik.

Qulaylik uchun yozuvda vektorlardan foydalanamiz.

$\boldsymbol{f}$ vektor-funksiyaning koordinata funksiyalarini $(f_1, f_2, \ldots, f_n)$ bilan belgilaylik, ya’ni $\boldsymbol{f} = (f_1, f_2, \ldots, f_n)$ bo‘lsin.

Teorema (teskari funksiya haqidagi). Aytaylik, $G-$ $\mathbb{R}^n$ fazodagi ochiq to‘plam, $\boldsymbol{f}: G \to \mathbb{R}^n$ uzluksiz differensiallanuvchi vektor-funksiya va $\det \boldsymbol{f}'(\boldsymbol{a}) \neq 0,\ \boldsymbol{a} \in G$, bo‘lsin. U holda $\boldsymbol{a}$ nuqtani o‘z ichiga olgan shunday $U \subset G$ ochiq to‘plam va $\boldsymbol{f}(\boldsymbol{a})$ nuqtani o‘z ichiga olgan shunday V ochiq to‘plamlar mavjudki, $\boldsymbol{f}: U \to V$ funksiya $\boldsymbol{f}^{-1}: V \to U$ uzluksiz differensiallanuvchi teskari funksiyaga ega va $\forall\, \boldsymbol{y} \in V$ uchun

$$(\boldsymbol{f}^{-1})'(\boldsymbol{y}) = [\boldsymbol{f}'(\boldsymbol{f}^{-1}(\boldsymbol{y}))]^{-1}.$$

Chekli orttirmalar haqidagi quyidagi Lagranj teoremasi ham ko‘p qo‘llaniladi.

Teorema. (Lagranj formulasi). $G \subset \mathbb{R}^n$ ochiq to‘plam va $f: G \to \mathbb{R}$ differensiallanuvchi haqiqiy funksiya berilgan bo‘lsin. Agar $x \in G$ va $x + h \in G$ nuqtalarni tutashtiruvchi $\{x + \tau h \mid 0 \le \tau \le 1\}$ kesma G da joylashsa, u holda shunday $\theta \in (0;1)$ son mavjudki, uning uchun $f(x+h) - f(x) = f'(x + \theta h) \cdot h$ ya’ni

$$f(x^1 + h^1, ..., x^n + h^n) - f(x^1, ..., x^n) = \sum_{i=1}^{n} D_i f(x + \theta h) \cdot h^i \qquad (2)$$

formula o‘rinli bo‘ladi. (2) formula **Lagranj formulasi** deb ataladi.

Teorema. (Ko‘p o‘zgaruvchili funksiyaning differensiallanuvchi bo‘lishi uchun yetarli shart). $G \subset \mathbb{R}^n$ ochiq to‘plam, $a \in G$ va $f: G \to \mathbb{R}$ funksiya a nuqtaning biror atrofida $D_1 f, ..., D_n f$ xususiy hosilalarga ega bo‘lsin. Agar bu xususiy hosilalar a nuqtada uzluksiz bo‘lsa, f funksiya shu a nuqtada differensiallanuvchi bo‘ladi.

Natija 1. Agar $f: G \to \mathbb{R}$ funksiyaning $D_1 f, ..., D_n f$ xususiy hosilalari $G \subset \mathbb{R}^n$ ochiq to‘plamda uzluksiz bo‘lsa, u holda f vektor–funksiya G da differensiallanuvchidir.

Natija 2. (Vektor–funksiyaning differensiallanuvchi bo‘lishi uchun yetarli shart). $G \subset \mathbb{R}^n$ ochiq to‘plam, $a \in G$ va $f: G \to \mathbb{R}^m$ funksiyaning koordinata funksiyalari nuqtaning biror atrofida $D_j f^i$, $i = \overline{1,m}$, $j = \overline{1,n}$, xususiy hosilalarga ega bo‘lsin. Agar bu xususiy hosilalar a nuqtada uzluksiz bo‘lsa, u holda f funksiya shu nuqtada differensiallanuvchi bo‘ladi.

Natija 3. Agar $f: G \to \mathbb{R}^m$, $f = (f^1, ..., f^m)^t$ funksiya uchun barcha $D_j f^i$, $i = \overline{1,m}$, $j = \overline{1,n}$, xususiy hosilalar $G \subset \mathbb{R}^n$ ochiq to‘plamda uzluksiz bo‘lsa, u holda f funksiya G da differensiallanuvchidir.

Agar $f:G\to\mathbb{R}^m$ funksiya uchun barcha $D_j f^i,\ i=\overline{1,m},\ j=\overline{1,n},$ xususiy hosilalar $G\subset\mathbb{R}^n$ ochiq to‘plamda uzluksiz bo‘lsa, u holda bu funksiya G da **uzluksiz differensiallanuvchi** deyiladi. G da uzluksiz differensiallanuvchi bo‘lgan barcha $f:G\to\mathbb{R}^m$ vektor-funksiyalar to‘plamini (sinfini) $C^1(G;\mathbb{R}^m)$ bilan belgilaymiz. Qisqalik uchun $C^1(G;\mathbb{R})=C^1(G)$ deymiz.

Matritsa va determinantlarga oid tayanch ma’lumotlar

$A=[a_{ij}]_{n\times m}$ matritsa quyidagi sonlar jadvalini anglatadi:

$$A=[a_{ij}]=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} \ (n\times m \text{ o‘lchamli matritsa})$$

Bu yerdagi a_{ij} sonlar A matritsaning elementlari deb ataladi.

$A=[a_{ij}]_{n\times m}$ va $B=[b_{ij}]_{m\times k}$ matritsalar quyidagicha ko‘paytiriladi:

$$AB=[d_{ij}]_{m\times k},\ d_{ij}=\sum_{q=1}^{m} a_{iq}b_{qj};$$

d_{ij} ni hosil qilish uchun A matritsaning i- satri B matritsaning j- ustuniga skalyar ko‘paytirilgan.

Bir xil o‘lchamli matritsalarni qo‘shish mumkin. Bunda matritsalarning mos elementlari qo‘shiladi. Matritsa songa (o‘ngdan yoki chapdan) ko‘paytirilganda, uning barcha elementlari shu songa ko‘paytiriladi.

Matritsalarni ko‘paytirish amali quyidagi xossalarga ega:

$$(AB)C=A(BC)$$
$$A(B+C)=AB+AC$$
$$(B+C)A=BA+CA$$

Bu yerda matrits*a*larning o‘lchamlari mos amallarni bajarish uchun mos keladi deb faraz qilinadi.

Ushbu

$$A=[a_{ij}]_{n\times m}=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}$$ matritsaning transpozitsiyasi deb

$$A^T=[a_{ji}]_{m\times n}=\begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}$$

matritsaga aytiladi. Quyidagilar soddagina tekshiriladi:

$$\left(A^T\right)^T=A\,,\left(A+B\right)^T=A^T+B^T,\,\left(\lambda A\right)^T=\lambda A^T,\,\left(AB\right)^T=A^T B^T$$

Satrlar soni ustunlar soniga teng bo‘lgan matritsa kvadrat matritsa deyiladi. Kvadrat matritsa uchun determinant tushunchasi aniqlangan. $A=[a_{ij}]_{n\times n}$ kvadrat matritsaning determinanti det*A* bilan belgilanadi:

$$\det A=\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

va u quyidagi xossalarga ega:

1. Agar matritsada nollardan iborat satr (ustun) mavjud bo‘lsa, uning determinanti nolga teng.

2. Matritsaning ixtiyoriy ikki satrining (ustunining) o‘rinlari almashtirilsa, determinantning ishorasi almashadi.

3. Matritsaning biror satriga (ustuniga) boshqa satrlarning (ustunlarning) chiziqli kombinatsiyasi qo‘shilsa , uning determinanti o‘zgarmaydi.

4. Matritsaning biror satri (ustuni) biror songa ko‘paytirilsa, determinantning qiymati ham shu songa ko‘payadi.

5. Agar matritsaning biror satri (ustuni) ikkita satrning yig‘indisi sifatida ifodalangan bo‘lsa , u holda bu matritsaning determinanti o‘sha satrni qo‘shiluvchi satrlar bilan almashtirishdan hosil bo‘lgan ikkita matritsaning determinantlari yig‘indisiga teng.

6. Transpozitsiyalashda determinantning qiymati o‘zgarmaydi: $\det A=\det A^T$.

7. $\det A=0$ bo‘lishi uchun A matritsaning satrlari (ustunlari) chiziqli bog‘langan bo‘lishi yetarli va zarurdir.

8. $A=[a_{ij}]$ matritsaning i- satri va j- ustunini yo‘qotishdan (chizib tashlashdan) hosil bo‘lgan M_{ij} **matritsa minor** deb ataladi. a_{ij} elementning **algebraik to‘ldiruvchisi** deb

$$A_{ij}=(-1)^{i+j}\det M_{ij}$$

songa aytiladi. Determinantni i-satr bo‘yicha yoyish:

$$\det A=\sum_{k=1}^{n}a_{ik}A_{ik},\quad i=\overline{1,n}.$$

Determinantni j- ustun bo‘yicha yoyish:

$$\det A=\sum_{k=1}^{n}a_{kj}A_{kj},\quad j=\overline{1,n}.$$

«Begona» algebraik to‘ldiruvchilarga ko‘paytirish:

$$\sum_{k=1}^{n}a_{ik}A_{jk}=\sum_{k=1}^{n}a_{kj}A_{ki}=0\ ,\ i\neq j\ .$$

Keltirilgan formulalar Laplas formulalari deb ataladi.

9. A va B $n\times n$-kvadrat matritsalar uchun

$$\det\left(AB\right)=\det A\cdot\det B$$

tenglik o‘rinlidir.

Ushbu

$$E=\begin{pmatrix}1 & 0 & 0 & \dots & 0\\ 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 1\end{pmatrix}$$

$n\times n$ kvadrat matritsa **birlik matritsa** deyiladi.

A kvadrat matritsaning teskarisi deb $AX=XA=E$ tenglamalarni qanoatlantiruvchi X matritsaga aytiladi. A ning teskarisi A^{-1} bilan belgilanadi. Agar teskari matritsa mavjud bo‘lsa, u yagonadir .

10. A^{-1} ning mavjud bo‘lishi uchun $\det A\neq 0$ bo‘lishi yetarli va zarurdir.

11. Teskari matritsa quyidagi xossalarga ega:

$$\left(A^{-1}\right)^{-1}=A;\ (AB)^{-1}=B^{-1}A^{-1};\ \det A^{-1}=(\det A)^{-1}.$$

12. $A=[a_{ij}]$ matritsaning teskarisi $A^{-1}=[\alpha_{ij}]$ quyidagi formulaga ko‘ra hisoblanishi mumkin:

$$\alpha_{ij}=\frac{1}{\det A}A_{ji},$$

bu yerda A_{ji} son A dagi a_{ji} elementning algebraik to‘ldiruvchisi.

13. $A-$ $n\times n$ matritsa va $\mathbf{b}-$ vektor berilgan, $\mathbf{x}=(x^1, x^2,..., x^n)^T$ vektor esa noma’lum bo‘lsin. Agar $\det A\neq 0$ bo‘lsa , u holda $A\mathbf{x}=\mathbf{b}$ chiziqli tenglamalar sistemasi yagona $\mathbf{x}=A^{-1}\mathbf{b}$ yechimga ega.

14. Agar chiziqli tenglamalar sistemasining o‘ng tomoni nollardan iborat bo‘lgan ustundan iborat bo‘lsa , u holda ushbu

$$A\mathbf{x}=0,\ 0=(0,0,...,0)^T$$

bir jinsli sistema $\det A=0$ holdagina notrivial, ya’ni $\mathbf{x}\neq 0$ yechimga ega bo‘ladi.

15. *Matritsaning Jordan (kanonik) ko‘rinishi*.

Bizga kompleks sonlardan tuzilgan ushbu

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\in \mathbb{M}_{n\times n}(\mathbb{C})$$

kvadrat matritsa berilgan bo‘lsin. Agar λ son uchun

$$A\mathbf{v}=\lambda\mathbf{v} \tag{1}$$

shartni qanoatlantiruvchi nol-vektordan farqli $\mathbf{v}\in\mathbb{C}^n$ mavjud bo‘lsa, u holda bu yerdagi λ soni A matritsaning xos soni (xarakteristik soni, xos qiymati), $\mathbf{v}$ vektor esa shu λ ga mos kelgan xos vektor (xarakteristik vektor) deyiladi. A matritsa $\mathbf{v}$ xos vektorni λ xos songa ko‘paytiradi.

(1) tenglamani

$$(A-\lambda E)\mathbf{v}=0 \tag{2}$$

ko‘rinishda yozaylik ($E\in M_{n\times n}(\mathbb{C})-$birlik matritsa). Bu tenglama $\mathbf{v}$ xos vektorning komponentalariga nisbatan chiziqli bir jinsli tenglamalar sistemasini ifodalaydi. U notrivial ($\mathbf{v}\neq 0$) yechimga ega bo‘lishi uchun

$$\det(A-\lambda E)=0 \tag{3}$$

shartning bajarilishi yetarli va zarurdir. $P_A(\lambda)\equiv\chi_A(\lambda)=\det(A-\lambda E)$ ko‘phad A matritsaning xarakteristik ko‘phadi, (3) tenglama esa A matritsaning xarakteristik tenglamasi deyiladi. Bu tenglama λ ga nisbatan $n-$darajali algebraik tenglamadir. Demak, oliy algebraning asosiy teoremasiga ko‘ra $\lambda_1,\lambda_2,\ldots,\lambda_m$ turli xos sonlar mavjud bo‘lib, ularning umumiy soni $k_1,k_2,\ldots,k_m$ karralilikIari bilan birgalikda n ta bo‘ladi, ya’ni

$$P_A(\lambda)=(-1)^n(\lambda-\lambda_1)^{k_1}(\lambda-\lambda_2)^{k_2}\cdots(\lambda-\lambda_m)^{k_m},\ \ k_1+k_2+\ldots+k_m=n, k_j\ge 1.$$

E birlik matritsa bir dona $\lambda=1$ (n karrali) xos songa ega va har qanday noldan farqli vektor uning xos vektor bo‘ladi.

Ushbu

$$J_{\lambda,n}=\begin{pmatrix}\lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda\end{pmatrix}\in \mathbb{M}_{n\times n}(\mathbb{C})$$

Jordan katagi (yozilmagan elementlar nolga teng) deb ataluvchi matritsa ham bir dona $\lambda=1$ (n karrali) xos songa ega, lekin uning bir dona xos vektori (skalyar ko‘paytuvchi aniqligida) bor xolos.

Agar A haqiqiy matritsa $\lambda=\mu+i\nu$ $\left(\{\mu,\nu\}\subset\mathbb{R}\right)$ kompleks xos son va unga mos $\mathbf{v}=\mathbf{u}+i\mathbf{w}$ kompleks xos vektorga ega bo‘lsa, u $\bar{\lambda}=\mu-i\nu$ qo‘shma kompleks xos son va unga mos $\bar{\mathbf{v}}=\mathbf{u}-i\mathbf{w}$ qo‘shma kompleks xos vektorga ham ega bo‘ladi.

Agar $\lambda_1,\lambda_2,\ldots,\lambda_s-$ turli xos sonlar bo‘lsa, u holda ularga mos $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_s$ xos vektorlar chiziqli erkli bo‘ladi. Agar $A\in M_{n\times n}(\mathbb{C})$ kompleks matritsaning n ta turli kompleks xos sonlari mavjud bo‘lsa, u holda ularga mos xos vektorlar $\mathbb{C}^n$ chiziqli fazoning bazisini tashkil etadi.

Agar $A\in M_{n\times n}(\mathbb{R})$ haqiqiy matritsa n ta turli haqiqiy $\lambda_1,\lambda_2,\ldots,\lambda_n$ xos sonlarga ega bo‘lsa, u holda ularga mos $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n$ haqiqiy xos vektorlar $\mathbb{R}^n$ haqiqiy chiziqli fazoning bazisini tashkil etadi.

Agar biror $\Lambda=\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$ diagonal va biror S teskarilanuvchi matritsalar uchun

$$S^{-1}AS=\Lambda \tag{4}$$

bo‘lsa, u holda A matritsa diagonallashtiriluvchi, S esa diagonallashtiruvchi matritsa deb yuritiladi. Diagonallashtirilish sharti (4) ni

$$AS = S\Lambda \tag{5}$$

ko‘rinishda yozish mumkin. Bu tenglikdan Λ matritsa diagonalida A matritsaning xos sonlari joylashganligini, S matritsaning ustunlari esa mos xos vektorlardan iborat ekanligini ko‘ramiz:

$$S = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n], \; A\mathbf{v}_k = \lambda_k \mathbf{v}_k, k = 1, \ldots, n.$$

S matritsa teskarilanuvchi bo‘lgani uchun bu $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ xos vektorlar chiziqli erkli bo‘lishi kerak. Shunday qilib, A diagonallashtiriluvchi matritsa bo‘lishi uchun uning n ta chiziqli erkli xos vektorlarga ega bo‘lishi yetarli va zarurdir. Ma’lumki, har qanday simmetrik haqiqiy matritsaning barcha xos sonlari haqiqiy va u diagonallashtiriluvchi bo‘ladi.

Har qanday matritsa ham diagonallashtiriluvchi bo‘lavermaydi. Lekin ixtiyoriy matritsani Jordan matritsasi ko‘rinishiga keltirish mumkin.

k uzunlikli Jordan zanjiri deb ushbu

$$\begin{aligned} &A\mathbf{x}_1 = \lambda\mathbf{x}_1, \\ &A\mathbf{x}_2 = \lambda\mathbf{x}_2 + \mathbf{x}_1, \\ &A\mathbf{x}_3 = \lambda\mathbf{x}_3 + \mathbf{x}_2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &A\mathbf{x}_k = \lambda\mathbf{x}_k + \mathbf{x}_{k-1}. \end{aligned} \tag{6}$$

shartlarni qanoatlantiruvchi noldan farqli $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ vektorlarga aytiladi (skalyar ko‘paytuvchi aniqligida). Tushunarliki, $\mathbf{x}_1$ vektor λ xos songa mos kelgan xos vektordir. Qolgan $\mathbf{x}_2, \ldots, \mathbf{x}_k$ vektorlar (shu xos songa mos kelgan) umumlashgan xos vektorlar deb yuritiladi. Agar A matritsa k_j karrali $\lambda = \lambda_j$ xos songa ega bo‘lsa, u holda shu xos songa mos kelgan va uzunliklarining yig‘indisi k_j ga teng bo‘lgan Jordan zanjirlari mavjud bo‘ladi; bu zanjirlar soni $\lambda = \lambda_j$ xos songa mos kelgan chiziqli erkli vektorlar soni, ya’ni $\dim\{\mathbf{x} \mid (A - \lambda_j E)\mathbf{x} = 0\}$ ga teng. Barcha xos sonlarga mos kelgan Jordan zanjirlarini topib va ularni birlashtirib, Jordan bazisi tuziladi. Jordan bazisi $k_1 + k_2 + \ldots + k_m = n$ dona chiziqli erkli vektorlarni tashkil etadi. Jordan bazisi vektorlarining komponentalarini S matritsaning ustunlarida joylashtiramiz. U holda ushbu

$$J = S^{-1}AS \tag{7}$$

matritsa A matritsaning Jordan (kanonik) ko‘rinishini beradi. Bu J Jordan matritsasi

$$J = \mathrm{diag}(J_{\lambda_1,n_1},\ldots,J_{\lambda_2,n_2},\ldots,J_{\lambda_m,n_m}) = \begin{pmatrix} J_{\lambda_1,n_1} & & & & \\ & \ddots & & & \\ & & J_{\lambda_2,n_2} & & \\ & & & \ddots & \\ & & & & J_{\lambda_m,n_m} \end{pmatrix}$$

ko‘rinishga ega; diagonalda Jordan kataklari joylashgan, $n_1 + n_2 + \ldots + n_m = n,$ turli kataklar diagonallarida bir xil yoki turli xos sonlar turadi, diagonallarida bir xil λ_j son joylashgan kataklar soni ushbu $A\mathbf{x} = \lambda_j \mathbf{x}$ sistemaning chiziqli erkli yechimlari soniga teng, bo‘sh o‘rinlarda nollar yozilgan deb hisoblanadi.

(7) formuladan va determinantlar xossalaridan foydalanib quyidagilarni yozamiz:

$$P_J(\lambda) = \det(J - \lambda E) = \det(SAS^{-1} - S\lambda ES^{-1}) = \det(S(A - \lambda E)S^{-1}) =$$
$$= \det S \cdot \det(A - \lambda E) \cdot \det S^{-1} = \det(A - \lambda E) \cdot \det S^{-1} \cdot \det S = \det(A - \lambda E) = P_A(\lambda).$$

Demak, $J = SAS^{-1}$ va A matritsalarning xarakteristik tenglamalari bir xil. Shuning uchun bu matritsalarning xos sonlari (karralilikIari bilan birgalikda) ham bir xil.

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