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FIZIKADA KOMPLEKS O'ZGARUVCHILI FUNKSIYALAR

O‘QUV QO‘LLANMA

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Ushbu o‘quv qo‘llanma 5110200 – Fizika va astronomiya o‘qitish metodikasi, 5140200 – Fizika, 5140400 – Astronomiya bakalavr ta’lim yo‘nalishlarining o‘quv rejasidagi matematika va tabiiy-ilmiy fanlar blokiga tegishli fanlarning o‘quv dasturilari talablari asosida tayyorlangan bo‘lib, unda amaliy mashg‘ulotlarini o‘z ichiga olgan ma’lumotlar berilgan.

O‘quv rejalardagi matematika va tabiiy-ilmiy fanlar blokiga tegishli fanlarning xususiyatlaridan kelib chiqib, qo‘llanmada “Oliy matematika” kursining kompleks o’zgaruvchili funksiyalar bo’limiga oid asosiy tushuncha va tasdiqlar yoritilgan bo‘lib, ularning fizik mazmuni va tatbiqlari ko‘rsatilgan. Mavzular bo‘yicha talabalar mustaqil ishi uchun topshiriqlar va ularni yechilish usullari hamda talabalar uchun foydali bir qator tavsiyalar berilgan.

Qo‘llanma oliy ta’lim muassalarining fizika, astronomiya hamda texnika yo‘nalishlari talabalari, tayanch doktorantlari va ilmiy izlanuvchilar uchun mo’ljallangan.

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O‘zbekiston Respublikasi Oliy va o‘rta maxsus ta’lim vazirligi Chirchiq davlat pedagogika instituti kengashining 2019 yil 9 dekabrdagi 5-sonli qaroriga asosan 5110200 – Fizika va astronomiya o‘qitish metodikasi, 5140200 – Fizika, 5140400 – Astronomiya ta‘lim yo‘nalishlari bo‘yicha tahsil olayotgan talabalar uchun o‘quv qo‘llanma sifatida nashr qilishga tavsiya etilgan.

O‘quv qo‘llanma O‘zbekiston Respublikasi Oliy va o‘rta maxsus ta’lim vazirligining 2020-yil 06-oktyabrdagi 522-sonli buyrug’iga asosan nashr etishga ruxsan berilgan (Ro’yxatga olish raqami 522-013)

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SO‘Z BOSHI

Mamlakatimizda kadrlar tayyorlashning sifat darajasini oshirish, xalqaro standartlar asosida oliy malakali mutaxassislar tayyorlash, o'quv jarayoniga xalqaro ta'lim standartlariga asoslangan ilg'or pedagogik texnologiyalarni joriy etish borasida olib borilayotgan keng islohotlar natijasida oliy ta'lim muassasalarida o'qitish sifat-samaradorligi ortib, pedagog va talabalarda kreativ, yangicha matematik tafakkur rivojlanishiga sabab bo'lmoqda. Shu bilan bir qatorda talabalar mustaqil ijodiy o'quv, biluv faoliyatini faollashtirish zaruriyati ham yuzaga kelmoqda.

Yosh mutaxasislarni har taraflama rivojlangan komil inson qilib tarbiyalashda matematika alohida o’ringa ega. Matematikani o'rganish jarayonida inson faoliyatining barcha sohasi uchun zarur qobilyatlar: ijodiy fikrlash, mantiqiy mushohada, fazoviy tasavvur, tahliliy mulohaza, abstrakt tafakkur shakllanib boradi.

Ushbu o’quv qo’llanma matematika va fizikaning muhim qismlaridan biri
bo'lgan kompleks argumentli funksiyalar tushunchasini o‘rganishga, tahlil qilishga bag’ishlangan. Bu o’quv qo’llanmadan fizika, matematika va texnika oliy ta’lim muassalari talabalari va tayanch doktorantlari foydalanishi mumkin. Undagi kompleks sonlar, kompleks funksiyaning hosilasi, integralli, analitik funksiyalar, yassi vektor maydonlar tushunchasi, ulardagi muhim matematik amallar sodda tilda bayon etilishiga harakat qilingan.

O’quv qo’llanma sakkiz bobdan tashkil topgan.

Birinchi bobda kompleks sonlarga oid tushunchalar, ular ustida amallarga tegishli ma’lumotlar, kompleks sonlarning tregonametrik, ko’rsatkichli va logarifmik shakillari keltirilgan.

Ikkinch bobda kompleks argumentli elementar funksiyalar, aniqlanish sohasi, egri chiziqlarni kompleks shakldagi tenglamalari, ayrim trigonometrik funksiyalarning kompleks shakldagi ko’rinishlari, kompleks sohadagi giperbolik funksiyalar va teskari trigonometrik funksiyalarga tegishli amallar bayon qilingan.

Uchunchi bobda kompleks argumentli funksiyaning hosilasi, analitik funfsiyalar va yassi vektor maydonlarni hisoblashda ularni qo’llanilishi haqida so’z yuritilgan.

To'rtinch bobda kompleks argumentli funksiyalardan tekislikdagi biror chiziq buylab olingan egri chiziqli integrallar misollar yordamida tushintirilgan.

Beshinch bobda ba`zi kompleks argumentli funksiyalardan olingan egri chiziqli integrallarni Koshi formulasidan foydalanib hisoblash, **Oltinchi bobda** hadlari kompleks sonlardan yoki kompleks argumentli funsiyalardan iborat bo‘lgan qatorlar jumladan, praktikada ko‘p ishlatiladigan darajali qatorlar, Loran qatorlariga doir misollar bilan shug‘ullanganmiz.

Ettinchi bobda maxsus nuqtalar va ularning klassifikasiyasi **Sakkizinchi bobda** qoldiqlar nazariyasi, integrallarni chegirmalar nazariyasi yordamida hisoblash keltirilgan. Har bir mavzular misollar bilan bayon qilingan va mustaqil ishlash uchun mashqlar berilgan.

Har bir bob tegishli paragraflarga bo'lingan bo'lib, har bir paragraf mavzuga taalluqli asosiy ta'riflar, tasdiqlar, teoremalarni o‘z ichiga oladi, shuningdek, ularning har biri an'anaviv misollarni batafsil tahlil yordamida yechish orqali namoyish qilingan.

O'ylaymizki, o'quv qullanma o‘z o'quvchilarini topadi va boshqa mavjud o‘quv adabiyotlari qatorida kompleks o'zgaruvchili funksiyalar kursi bo‘yicha ularga bilimlarini oshirishga ko‘mak beradi.

Mualliflar.

I BOB

KOMPLEKS SONLAR USTIDA AMALLAR
1.1-§ KOMPLEKS SON MODULI VA ARGUMENTINING GEOMETRIK MA'NOSI

Kompleks son deb $z = x + iy$ ko'rinishdagi songa aytiladi. Bunda x va y ixtiyoriy haqiqiy sonlardir. ***Mavhum birlik*** deb nomlanadigan kattalik i esa, $i^2 = -1$ tenglikni qanoatlantiradi.

Haqiqiy x va y sonlarni z komleks sonning mos ravishda ***haqiqiy*** va ***mavhum*** qismining koeffisiyenti deyiladi va $x = \mathrm{Re}\, z$, $y = \mathrm{Im}\, z$ ko'rinishda yoziladi (lotincha reabis – haqiqiy , imaginarius – mavhum demakdir).

$x + iy$ son kompleks sonning algebraik formasi (shakli) deyiladi.

Komleks son $\bar{z} = x - iy$ esa, $z = x + iy$ ga nisbatan ***qo'shma komleks son*** deyiladi va kompleks son belgisi ustiga chiziqcha bilan yoziladi.

To'g'ri burchakli Dekart koordinatalar sistemasi XOY ni tanlab, uning absissalar o'qiga $z = x + iy$ ning haqiqiy qismi X ni, ordinatalar o'qiga esa mavhum qismining koeffisienti Y ni joylashtirsak, tekislikda (x, y) nuqtaga ega bo'lamiz.

Ana shu nuqta $z = x + iy$ kompleks sonning geometrik tasviri deb qabul qilingan.

Shunday qilib, har bir kompleks songa tekislikda birgina nuqta va aksincha, tekislikdagi har bir nuqta uchun bitta kompleks son mos keladi.

OX o'q – haqiqiy o'q, OY – mavhum o'q, XOY tekislik esa **kompleks tekislik** deyiladi.

Komleks sonning uch xil ko'rinishi mavjud:

$z = x + iy$ – ***algebraik*** korinish:

$z = r(\cos\varphi + i\sin\varphi)$ – ***trigonometrik*** ko'rinish.

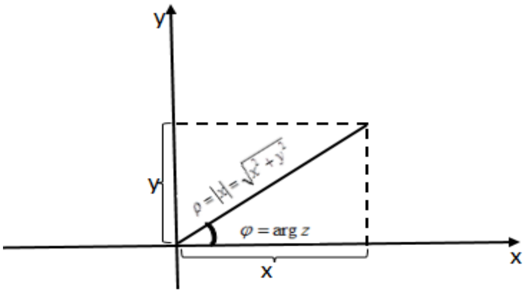
$z = re^{i\varphi}$ – esa ***ko'rsatkichli*** ko'rinishidir.

Bu yerda $r = |z|$ va $\varphi = \arg z$ kattaliklar mos ravishda komleks sonning ***moduli*** va ***argumenti*** deyiladi.

Ta'rif bo'yicha, z komleks sonning moduli quydagi ifodaga tengdir:

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2} \qquad (1.1)$$

Oxirgi ifodadan kompleks sonning modulini geometrik ma’nosi kelib chiqadi: Kompleks tekislikda kordinata boshidan z nuqtagacha bo’lgan masofaga tengdir (1.1-rasmga qarang).



1.1-rasm

Kompleks sonning argumenti esa, haqiqiy o’q yo’nalishi bilan z nuqtaga mos radius-vektor orasidagi burchakdir. Birgina z kompleks songa cheksiz ko’p burchaklar mos keladi, shuning uchun argumentning aniqlilik chegarasi $2\pi k$ ga teng:

$$Argz = \arg z + 2\pi k .$$

$\arg z -$ ***argumentning bosh qiymati*** deyiladi:

$$\arg z = \varphi = arctg\frac{y}{x}, \quad -\pi < \arg z \le \pi$$

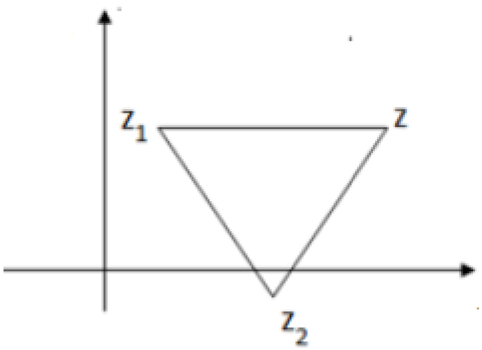
1-misol. Quydagi munosabatlarning geometrik o’rnini aniqlang

A. $|z - z_1| = |z - z_2|$

B. $|z - 2| + |z + 2| = 5$

Yechilishi: Komleks son modulining geometrik ma’nosidan foydalanib,

a) tenglikning chap qismi z nuqtadan qo’zg’almas z_1 nuqtagacha bo’lgan, o’ng qismi esa, z nuqtadan qo’zg’almas z_2 nuqtagacha bo’lgan masofaga tengligini topamiz. (1.2-rasm)



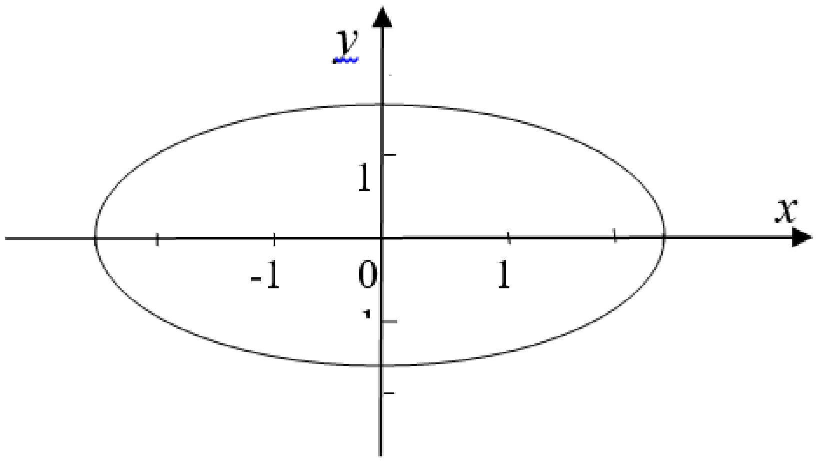
1.2-rasm

Bu masofalarning bir –biriga tengligi esa, z nuqtalar z_1 va z_2 nuqtalardan bir xil uzoqlikda joylashganligini ko'rsatadi. Hamma z nuqtalarning geometrik o'rni esa, z_1 va z_2 nuqtalarni tutashtiruvchi kesmaning o'rtasidan o'tkazilgan perpendikulyarda yotadi.

в) Bu munosabatning hadlariga nazar tashlaylik:

$|z-2|-z$ nuqtadan kordinatalari $x=2,\ \ y=0$ nuqtagacha bo'lgan masofa,

$|z+2|-z$ nuqtadan kordinatalari $x=-2,\ \ y=0$ nuqtagacha bo'lgan masofa.



1.3-rasm

Bu geometrik munosabatlar shunday z nuqtalarning o'rnini ifodalaydiki, bunda fiksirlangan $z_1=2$ va $z_2=-2$ nuqtalardan z nuqtagacha bo'lgan masofalar yig'indisi doimiy va 5 ga tengdir. Bunday nuqtaning geometrik o'rni fokuslari $z_1=2$ va $z_2=-2$ nuqtalarda joylashgan ellipsdan iborat bo'ladi (1.3-rasm). Ellipsning xossalarini eslatib o'tamiz: ellipsning ixtiyoriy nuqtasidan ellips fokuslarigacha bo'lgan masofa doimiydir.

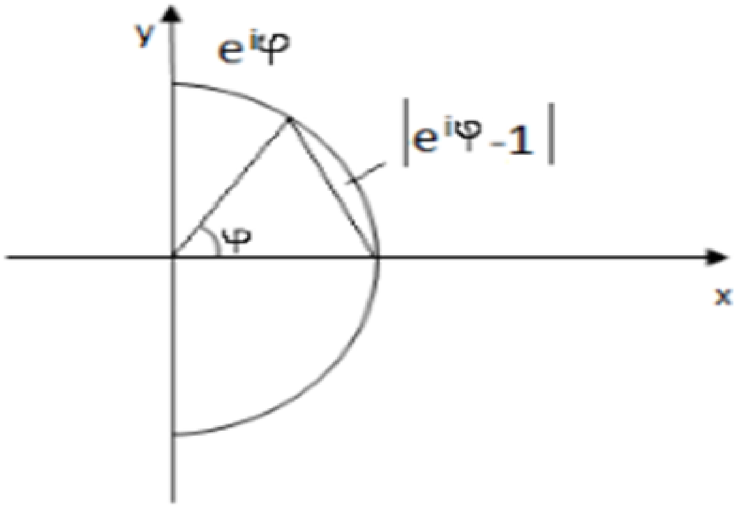
2-misol. Geometrik mulohazalardan kelib chiqib, quydagi tengsizlikni isbotlang: $\left|\frac{z}{|z|}-1\right| \le |\arg z|$

Yechilishi: Tengsizlikni kompleks sonning ko'rsatgichli ko'rinishidan $z=|z|\,e^{i\varphi}$ foydalanib quydagi ko'rinishda yozib olamiz:

$$\left|e^{i\varphi}-1\right| \le |\arg z|\,.$$

$\left|e^{i\varphi}-1\right| \le |\arg z|$ funksiyani kompleks tekislikda chizamiz (1.4-rasm).

Bu masofa $z=e^{i\varphi}$ nuqta bilan (bu nuqta radiusi birga teng aylanada yotadi) $z=1$ nuqtani birlashtiruvchi vatar bo'ladi. Ma'lumki, vatarning uzunligi shu vatarga to'g'ri kelgan yoyning uzunligidan doim kichikdir.



1.4-rasm.

Oxirgi ifodada birlik radiusli aylana bo'lganligidan yoyning uzunligi $\left|e^{i\varphi}-1\right| \approx \varphi$ ga teng. Shuni isbot qilish kerak edi.

Mashqlar

Quyidagi kompleks sonlarning haqiqiy va mavhum qismlari, argumentning bosh qiymatini hamda modulini toping.

1.	$z=(2+3i)(3-i)$	4.	$z=\frac{1}{1+4i}+\frac{1}{4-i}$
2.	$z=(1-i)^3-(1+i)^3$	5.	$z=\left(\frac{1}{i}-\frac{1}{3i}\right)^2$
3.	$z=(1-2i)(1+4i)$	6.	$z=\frac{(1+i)(3+i)}{3-i}-\frac{(1-i)(3-i)}{3+i}$

Quydagi munosabatlarni qanoatlantiruvchi kompleks sonlarning geometrik o'rni ko'rsatilsin.

7.	$\left\|z-i\right\|<4$	12.	$\left\|z+4-5i\right\|=3$
8.	$\left\|z+4i\right\|\geq 3$	13.	$\operatorname{Re} z^2=a$
9.	$\operatorname{Re} z<-5$	14.	$\operatorname{Im} z^2=a$
10.	$\operatorname{Im} z\geq 3$	15.	$\alpha<\arg z<\beta$
11.	$\operatorname{Re}\bar{z}+\operatorname{Im} z<2$	16.	$\left\|z-4\right\|+\left\|z+2\right\|=10$

1.2-§ KOMPLEKS SONLAR USTIDA AMALLAR

Agar berilgan

$$z_1=x_1+iy_1 \quad \text{va} \quad z_2=x_2+iy_2$$

kompleks sonlarda

$$x_1=x_2 \quad \text{va} \quad y_1=y_2$$

bo'lsa, bu kompleks sonlar teng deyiladi.

Har qanday ikkita kompleks sonni quyidagicha qo'shish va ayirish qabul qilingan:

$$\alpha \pm \beta = (a+ib) \pm (c+id) = (a \pm c) + i(b \pm d) \qquad (1.2)$$

Masalan:

$$(3-4i)+(-8+2i)=(-8+3)+i(-4+2)=-5-2i;$$

$$\left(1+i\sqrt{3}\right)-\left(6-\frac{1}{2}i\right)=(1-6)+i\left(\sqrt{3}+\frac{1}{2}\right)=-5+i\left(\sqrt{3}+\frac{1}{2}\right).$$

Har qanday ikkita kompleks sonni ko'paytirish ko'phadlarni ko'paytirish qoidasiga binoan bajariladi, ya'ni

$$\alpha \cdot \beta = (a+ib)(c+id) = ac + iad + ibc + i^2 bd$$

Ma'lumki, $i^2=-1$, shuning uchun

$$\alpha\beta = (a+ib)(c+id) = (ac-bd) + i(ad+bc) \qquad (1.3$$

Quyidagi:

$$\alpha = a+ib \text{ va } \bar{\alpha} = a-ib$$

kompleks sonlar o'zaro qo'shma sonlar deyiladi. Qo'shma sonlarning yig'indisi ham haqiqiy son bo'ladi :

$$\alpha + \bar{\alpha} = 2a \text{ va } \alpha\bar{\alpha} = a^2 + b^2 \qquad (1.4)$$

Masalan :

$$(-5+2i)+(-5-2i)=-10$$

$$(-5+2i)(-5-2i)=25+4=21$$

Ikkita $\alpha = a+ib$ va $\beta = c+id$ kompleks sonning birini ikkinchisiga bo'lish uchun kasrning surat va maxrajini $\bar{\beta} = c-id$ qo'shma kompleks songa ko'paytiramiz. U holda

$$\frac{\alpha}{\beta} = \frac{a+ib}{c+id} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2} \qquad (1.5)$$

Masalan :

$$\frac{1+i}{1-i} = \frac{(1+i)^2}{(1+i)(1-i)} = \frac{1+2i+i^2}{1-i^2} = \frac{2i}{2} = i;$$

$$\frac{1}{i} = \frac{(-i)}{i(-i)} = \frac{-i}{1} = -i.$$

Endi to'rt amal yordamida yechiladigan misollarning ayrim namunalari bilan tanishamiz .

1-misol. i ning turli darajalarini hisoblang:

Yechilishi:

$$i=\sqrt{-1}, \quad i^2=-1; \; i^3 = i \cdot i^2 = -i; \; i^4 = \left(i^2\right)^2 = (-1)^2 = 1; \; i^5 = i \cdot i^4 = i;$$

$$i^6 = \left(i^3\right)^2 = (-i)^2 = i^2 = -1$$

va hokazo. Shunday qilib, i ni ixtiyoriy musbat butun darajaga ko'tarsak quyidagi to'rtta sondan biri kelib chiqadi:

$$i, -1, \quad -i, 1.$$

Umumiy holda buni quyidagicha yozish mumkin :

$$i^{4k} = 1,\; i^{4k+1} = i\;;\; i^{4k+2} = -1,\; i^{4k+3} = -i \qquad (1.6)$$

bu yerda $k = 0,1,2...$.

2-misol. Ushbu tenglamani yeching :

$$(1+2i)x + (3-5i)y = 1-3i\,.$$

Yechilishi: Dastlab qavslarni ochib tenglamaning chap tomonida haqiqiy va mavhum qismlarni ajratamiz: $(x+3y)+i(2x-5y) = 1-3i$.

Ikkita kompleks sonning tengligi ta'rifiga asoslanib, bularning haqiqiy qismlarini o'zaro hamda mavhum qismlarini o'zaro hamda mavhum qismlarini o'zaro tenglashtirib , ushbu tenglamalar sistemasini hosil qilamiz :

$$\begin{cases} x+3y = 1 \\ 2x-5y = -3 \end{cases}$$

Bu sistemani yechib, $x = -\dfrac{4}{11},\quad y = \dfrac{5}{11}$ larni topamiz.

3-misol. Ushbu tenglamani yeching: $z^2 + 1 = -2\bar{z}.$

Yechilishi: Kompleks sonni $x + i\,y$ ko'rinishida yozib olamiz .

$$1 + (x + i\,y)^2 = -2(x - i\,y)$$

Qavslarni ochib chiqsak:

$$1 + x^2 + i\,2\,x\,y - y^2 = -2x + i\,2\,y$$

Endi haqiqiy va mavhum qismlarini o'zaro tenglashtirib, quydagi tenglamalar sistemasiga ega bo'lamiz.

$$\begin{cases} x^2 - y^2 = -2x - 1 \\ 2\,xy = 2y \end{cases} \Rightarrow \begin{cases} x^2 - y^2 = -2x - 1 \\ x = 1 \end{cases} \Rightarrow \begin{cases} y = \pm 2 \\ x = 1 \end{cases}.$$

4-misol. Ushbu $\dfrac{\alpha}{\beta} = \dfrac{1+itg\alpha}{1-itg\alpha}$ kasrni soddalashtiring.

Yechilishi:

$$\frac{\alpha}{\beta} = \frac{1+itg\alpha}{1-itg\alpha} = \frac{1+2itg\alpha - tg^2\alpha}{1+tg^2\alpha} = \frac{1-tg^2\alpha}{1+tg^2\alpha} + i\frac{2tg\alpha}{1+tg^2\alpha} = \cos 2\alpha + i\sin 2\alpha$$

5-misol. $\dfrac{\alpha}{\beta} = \dfrac{(1+i)^n}{(1-i)^{n-2}}$ kompleks sonni hisoblang (n – natural son).

Yechilishi:

$$\frac{\alpha}{\beta}=\frac{(1+i)^n}{(1-i)^{n-2}}=\frac{(1+i)^n(1-i)^2}{(1-i)^n}=\left(\frac{1+i}{1-i}\right)^n(1-i)^2 .$$

Bundan $\frac{1+i}{1-i}=\frac{(1+i)^2}{1+1}=i,\quad (1-i)^2=1-2i-1=-2i$ bo'lganligi sababli

$$\frac{\alpha}{\beta}=i^n(-2i)=-2i^{n+1}.$$

6-misol. Quyidagi tenglamalar sistemasini yeching:

$$\begin{cases}(3-i)x+(4+2i)y=2+6i\\(4+2i)x-(2+3i)y=5+4i.\end{cases}$$

Yechilishi: Sistema yechimni determinantlar yordamida topish osonroq. Ma'lumki,

$$x=\frac{D_1}{D};\quad y=\frac{D_2}{D},$$

$$D=\begin{vmatrix}3-i & 4+2i\\4+2i & -(2+3i)\end{vmatrix},\quad D_1=\begin{vmatrix}2+6i & 4+2i\\5+4i & -(2+3i)\end{vmatrix},\quad D_2=\begin{vmatrix}3-i & 2+6i\\4+2i & 5+4i\end{vmatrix}.$$

Hisoblab, quyidagi yechimni hosil qilamiz $x=1+i,\quad y=i$. Mana shu misoldan ko'rinadiki, chiziqli tenglamalar sistemasining koeffitsientlari kompleks bo'lsa, uning yechimi ham, umuman aytganda, kompleks sonlar bo'lishi mumkin ekan.

Mashqlar

Misollarni yeching:

17. a) $\frac{a+ib}{a-ib}$; b) $(1+2i)^5$

18. $\frac{(1+2i)^2-(1-i)^3}{(3+2i)^2-(2+i)^2}$ 19. $\frac{(1+i)^9}{(1-i)^7}$

20. $(1+2i)^6$ 21. $\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2$

22, $\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3$ 23. $\frac{(-2+2i)^5}{(-1+i)^3}+2i-5$

24. $\frac{(1+i)^n}{2(1-i)^{n-3}}\quad (n-butun\ son)$

25. Ushbu $(b-8i)x+(7+3i)y=2-i$ tenglamadan x va y ning haqiqiy qiymatlarini toping.

26. x va y ning $(7+2i)x-(5-4i)y=-1-i$ tenglamani qanoatlantiruvchi qiymatlarini toping.

27. Ushbu chiziqli tenglamalar sestemasini yeching:

$$\left.\begin{array}{l}(2+i)x-(3+i)y=i\\(3+i)x+(2-i)y=i\end{array}\right\}$$

28. Ushbu kvadrat tenglamani yeching:

$$x^2-(4+3i)x+1+5i=0$$

29. Ushbu bikvadrat tenglamani yeching:

$$x^4-30x^2+289=0$$

30. Tenglamani yeching:

a) $(4+2i)x+5(1-\frac{3}{5}i)y=13+i$

b) $\frac{1}{z-i}+\frac{2+i}{1+i}=\sqrt{2}$

1.3-§. KOMPLEKS SONNING TRIGONOMETRIK SHAKLI

$\alpha=a+ib$ kompleks sonning ikki xil geometrik ma'nosi bor.

a) u tengsizlikda (a,b) nuqtani tasvirlaydi:

b) u (0,0) nuqta bilan (a,b) nuqtani tutashtiruvchi vektorni tasvirlaydi (1.5-rasm).

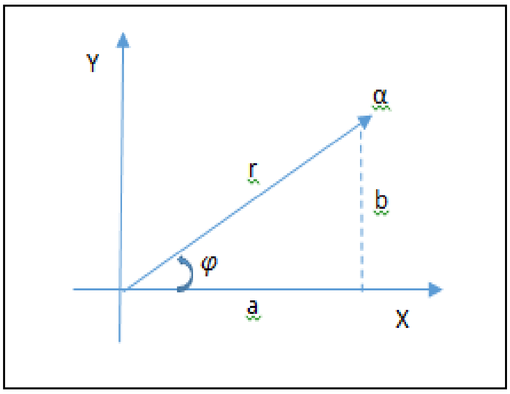
Ma'limki tekislikda har bir kompleks songa bitta nuqta va aksincha, har bir nuqtaga bitta kompleks son mos keladi. 1.5-rasmdan quyidagilarni ko'rish mumkin:

$$a=r\cos\varphi,\quad b=r\sin\varphi,\quad r=\sqrt{a^2+b^2} \qquad (1.7)$$

$$tg\varphi=\frac{b}{a},\quad \varphi=arctg\frac{b}{a}$$

Bu yerda ishtirok etayotgan r son $\alpha=a+ib$ ning moduli φ esa uning argumenti (yoki fazasi) deyiladi, quyidagicha yoziladi:

$$r=|\alpha|=\sqrt{a^2+b^2} \text{ va } \varphi=\arg\alpha \qquad (1.8)$$



1.5-rasm.

Shakldan ko'rinadiki, $0\le r<+\infty$ va $0\le\varphi<2\pi$; ba'zan $-\pi<\varphi<\pi$ chegaralar ham ishlatilib, ikkala chegara ham bir maqsadga olib keladi. (1.7) ga muvoffiq α kompleks son ushbu

$$\alpha = a + ib = r\left(\cos\varphi + i\sin\varphi\right) \qquad (1.9)$$

ko’rinishga ega bo’lib, u kompleks sonning trigonometrik shakl deyiladi. Kompleks sonlar ustida amallar bajarishda asosan (1.9) forma ishlatiladi. Matematik analiz kursidagi Eylerning

$$e^{ix} = \cos x + i\sin x \qquad (1.10)$$

Formulasidan foydalanib, (1.9) ni

$$\alpha = re^{i\varphi} \qquad (1.11)$$

ko’rinishida yozish mumkin. (1.11) Kompleks sonning *ko’rsatkichli shakl* deyiladi. Bu formulada

$$e = \lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m \qquad (1.12)$$

bo’lib, u irratsional sondir

$2 < e < 3$ yoki $e = 2{,}718281828459045...$

Ta’rif. α kompleks soni deb ma’lum bir tartibda berilgan a va b haqiqiy sonlar juftiga aytiladi:

$$\alpha = (a, b).$$

Shunday qilib, bitta kompleks sonni quydagicha

$$\alpha = a + ib, \quad \alpha = \mathrm{r}\left(\cos\varphi + i\sin\varphi\right), \quad \alpha = re^{i\varphi} \qquad (1.13)$$

uch xil shaklda yozish mumkin bo’lib so’ngi ikkita ifoda ko’pincha qo’l keladi. Har qanday kompleks sonni bir shakldan ikkinchi shaklga o’tkazish mumkin. Buning uchun (1.7) munosabatlardan foydalanishga to’g’ri keladi. Ko’pincha kompleks sonni trigonometrik shaklga keltirish talab qilinadi, bunday vaqtlarda 1.5- rasmga murojaat qilinib, kompleks songa tegishli vektorning qaysi chorakda (kvadratda) yozishiga e’tibor qilish tavsiya etiladi.

Izoh. φ burchakka uning davrini ham qo’shib, quyidagicha yoziladi:

$$Arg\alpha = arg\alpha + 2k\pi = \varphi + 2k\pi, \quad k = 0, \pm 1, \pm 2, ... \qquad (1.14).$$

1-misol. Ushbu 1+*i* sonni trigonometrik shaklga keltiring.

Yechilishi: a=1 va b=1 bo’lgani sabab $r = |1+i| - \sqrt{1^2 + 1^2} = \sqrt{2},$ $tg\varphi = \frac{a}{b} = 1$ bo’lib, $1+i$ ga tegishli vektor tekislikning birinchi choragida yotgani uchun $\varphi = \frac{\pi}{4}$ bo’ladi. Demak, $1 + i = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$. Buning ko’rsatkichli formasi esa, (18) ga muvofiq $1 + i = \sqrt{2}e^{\frac{\pi}{4}t}$ ko’rinishga ega bo’ladi.

2-misol. $-1+i$ sonni trigonometrik shaklga keltiring.

Yechilishi: . $a=-1,\ b=1,\ r=|-1+i|=\sqrt{1^2+1^2}=\sqrt{2}\quad tg\varphi=\frac{b}{a}=-1$

$\varphi=arctg(-1)$. Berilgan songa tegishli vektor ikkinchi chorakda yotgani sababli $\varphi=\frac{3\pi}{4}$ bo'ladi. Demak,

$$-1+i=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=\sqrt{2}e^{i\frac{3\pi}{4}}\ .$$

3-misol. $-1-i$ sonni trigonometrik shaklga keltiring.

Yechilishi:

$$a=-1\ ,\ b=-1\ ,\ r=\sqrt{2};\quad tg\varphi=\frac{b}{a}=1.$$

Lekin mos vektor uchunchi chorakda yotgani uchun $\varphi=\frac{5\pi}{4}$ bo'ladi. Agar istasak buning o'rniga manfiy $\left(-\frac{3\pi}{4}\right)$ burchakni olishimiz ham mumkin. Ikkalasi ham bir xil natija beradi, chunki

$$\cos\left(-\frac{3\pi}{4}\right)=\cos\left(2\pi-\frac{3\pi}{4}\right)=\cos\left(\frac{5\pi}{4}\right),$$

$$\sin\left(-\frac{3\pi}{4}\right)=\sin\left(2\pi-\frac{3\pi}{4}\right)=\sin\left(\frac{5\pi}{4}\right).$$

Demak.

$$-1-i=\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)=\sqrt{2}e^{i\frac{5\pi}{4}}$$

4-misol. $1-i$ sonni trigonometrik shaklga keltirirng.

Yechilishi: $a=1,\quad b=-1,\qquad r=\sqrt{2};\ tg\varphi=\frac{-1}{1}=-1$, mos vektori to'rtinchi chorakka joylashganligi uchun $\varphi=\frac{7\pi}{4}$ bo'ladi. Istasak buning o'rniga manfiy $\left(-\frac{\pi}{4}\right)$ burchakni olishimiz ham mumkin. Demak,

$$1-i=\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)=\sqrt{2}e^{i\frac{7\pi}{4}}\ .$$

5-misol. (-1) sonni trigonometrik shaklga keltiring.

Yechilishi: $a=-1,\quad b=0\ ,\ r=|-1|=1;\ tg\varphi=\frac{0}{-1}=0\ ,$

Mos vektor OX haqiqiy o'qning manfiy tomoniga joylashgan, shu sababli $\varphi=\pi$ bo'ladi. Demak,

$$-1=\cos\pi+i\sin\pi=e^{i\pi}$$

6-misol. +1 sonni trigonometrik shaklga keltiring.

Yechilishi: Bu songa mos vektor haqiqiy o'qning musbat tomonida joylashganligi sababli $\varphi=0,\quad r=1$ demak $1=\cos 0+i\sin 0=e^0$.

7-misol. $1+i\sqrt{3}$ sonni trigonometrik shaklga keltiring.

Yechilishi: $a=1,\quad b=\sqrt{3}\quad r=\sqrt{1+(\sqrt{3})^2}=2$; $tg\varphi=\frac{b}{a}=\sqrt{3}$

Mos vektor birinchi chorakda bo'lgani uchun $\varphi=\frac{\pi}{3}$. Demak,

$$1+i\sqrt{3}=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=2e^{i\frac{\pi}{3}}$$

8-misol. $-1+i\sqrt{3}$ sonni trigonometrik shaklga keltiring.

Yechilishi: $a=-1,\quad b=\sqrt{3},\quad r=2;\ tg\varphi=-\frac{\sqrt{3}}{1}=-\sqrt{3}$

vektor ikkinchi chorakka joylashganligi sababli $\varphi=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$.

Demak, $-1-i\sqrt{3}=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)=2e^{i\frac{2\pi}{3}}$

9-misol. $-1-i\sqrt{3}$ sonni trigonometrik ko'rinishga keltiring.

Yechilishi: $a=-1$, $b=-\sqrt{3}$, $r=2$; $tg\varphi=\sqrt{3}$

Mos vektor uchunchi chorakka joylashganligi uchun

$$\varphi=\pi+\frac{\pi}{3}=\frac{4\pi}{3}\quad \text{yoki}\quad -\pi+\frac{\pi}{3}=-\frac{2\pi}{3}$$

bo'ladi. Demak, $-1-i\sqrt{3}=2\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)=2e^{i\frac{4\pi}{3}}$

10-misol. $\frac{-1+i\sqrt{3}}{2}$ sonni trigonometrik ko'rinishda ifodalang.

Yechilishi: $a=-\frac{1}{2},\ b=\frac{\sqrt{3}}{2}$, $r=1$; $tg\varphi=\frac{b}{a}=-\sqrt{3}$

Mos vektor ikkinchi chorakda, shu sababli $\varphi=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$

Demak, $-\frac{1}{2}+i\frac{\sqrt{3}}{2}=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}=e^{i\frac{2\pi}{3}}$

11-misol. $-2-5i$ sonni tregonometrik shaklga keltiring.

Yechilishi: $a=-2$, $b=-5$, $r=\sqrt{29}$; $tg\varphi=\frac{-5}{-2}=2.5$

$$-2-5i=\sqrt{29}\left[\cos\left(arctg\frac{5}{2}\right)+i\sin\left(arctg\frac{5}{2}\right)\right].$$

12-misol. $1+\cos\alpha+i\sin\alpha$ sonni trigonometrik shaklga keltiring

Yechilishi: $a=1+\cos\alpha$, $b=\sin\alpha$,

$$r=\sqrt{(1+\cos\alpha)^2+\sin^2\alpha}=\sqrt{2(1+\cos\alpha)}=2\sqrt{\frac{1+\cos\alpha}{2}}=2\cos\frac{\alpha}{2};$$

$$tg\varphi=\frac{\sin\alpha}{1+\cos\alpha}\Rightarrow\varphi=arctg\left(\frac{\sin\alpha}{1+\cos\alpha}\right)=arctg\left(\frac{2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}{2\cos^2\left(\frac{\alpha}{2}\right)}\right)=arctg\left(tg\left(\frac{\alpha}{2}\right)\right)=\frac{\alpha}{2}$$

Demak, $1+\cos\alpha+i\sin\alpha=2\cos\left(\frac{\alpha}{2}\right)\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)$

Mashqlar

Quydagi kompleks sonlarni trigonometrik shaklga keltiring: 31. i; 32. $-i$; 33. -2; 34. $\frac{1}{2}+\frac{\sqrt{3}}{2}i$; 35. $\frac{1}{2}-i\frac{\sqrt{3}}{2}$; 36. $-\cos\varphi-i\sin\varphi$; 37. $1-\cos\alpha$; 38. $-\sin\alpha+i(1+\cos\alpha)$.

1.4-§. KOMPLEKS SONLARNI KO'PAYTIRISH VA DARAJAGA KO'TARISH

Trigonometrik shaklda berilgan kompleks sonlani o'zaro ko'paytirish va darajaga ko'tarish quydagicha bajariladi:

$$\alpha\cdot\beta=r(\cos\varphi_1+i\sin\varphi_1)\rho(\cos\varphi_2+i\sin\varphi_2)=r\rho\big(\cos(\varphi_1+\varphi_2)+i\sin(\varphi_1+\varphi_2)\big) \qquad (1.15)$$

ya'ni ikki kompleks sonni ko'paytmasi yana kompleks son bo'lib, uning moduli ko'paytuvchining modullarining ko'paytmasiga, argumenti ko'paytuvchilar argumentlarining yig'indisiga tengdir.

Agar $\alpha=\beta$ bo'lsa

$$\alpha^2=\left[r(\cos\varphi+i\sin\varphi)\right]=r^2(\cos2\varphi+i\sin2\varphi)$$

bo'ladi. Kompliks sonning n-darajasi esa quyidagicha ifodalanadi:

$$\alpha^n=\left[r(\cos\varphi+i\sin\varphi)\right]^n=r^n(\cos n\varphi+i\sin n\varphi) \qquad (1.16).$$

Xususan α ning moduli $r=1$ bo'lsa, (1.16) dan:

$$(\cos\varphi+i\sin\varphi)^n=\cos n\varphi+i\sin n\varphi. \qquad (1.17)$$

Bu tenglik Muavr formulasi deb ataladi. Buning chap tomonidagi qavslarni Nyuton binomi bo'yicha ochib, so'ngra tenglikning ikki tomonidagi haqiqiy qisimlarni o'zaro hamda mavhum qismlarni o'zaro tenglashtirish natijasida trigonometryaga doir turli formulalarni keltirib chiqarish mumkin. Misol uchun (1.17) da n =2 bo'lsin, u holda

$$\cos^2\varphi+2i\sin\varphi\cos\varphi-\sin^2\varphi=\cos2\varphi+i\sin2\varphi$$

bu erdan

$$\cos2\varphi=\cos^2\varphi-\sin^2\varphi \text{ va } \sin2\varphi=2\sin\varphi\cos\varphi \text{ va h.k.}$$

Agar algebraik shaklda berilgan kompleks sonni darajaga ko'tarish talab qilinsa, dastlab uni trigonometrik shaklga keltirib olish maqsadga muvofiqdir.

1-misol. $(1+i)^{25}$ ni hisoblang.

Yechilishi: dastlab $1+i$ ni tregonometrik shaklga keltirib olamiz:

$$(1+i)^{25}=\left|\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right|^{25}=$$

$$=\left(\sqrt{2}\right)^{25}\left(\cos\frac{25\pi}{4}+i\sin\frac{25\pi}{4}\right)=\left(\sqrt{2}\right)^{25}\left|\cos\left(6\pi+\frac{\pi}{4}\right)+i\sin\left(6\pi+\frac{\pi}{4}\right)\right|=$$

$$=\left(\sqrt{2}\right)^{25}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\left(\sqrt{2}\right)^{25}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)=2^{12}(1+i)\ .$$

Demak,

$$(1+i)^{25}=2^{12}(1+i)$$

2-misol. $\left(\frac{1+i\sqrt{3}}{1-i}\right)^{20}$ ni hisoblang.

Yechilishi: Dastlab kasrning surat va mahrajida turgan kompleks sonlarni trigonometrik shakilga keltirib olamiz.

$$1+i\sqrt{3}=2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})\ ;$$

$$1-i=\sqrt{2}\left[\cos\left(-\frac{\pi}{4}+i\sin\left(-\frac{\pi}{4}\right)\right)\right]=\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)\ ,$$

Bulardan

$$\frac{1+i\sqrt{3}}{1-i}=\frac{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}{\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)}=\sqrt{2}\,\frac{\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}{\cos^2\frac{\pi}{4}+\sin^2\frac{\pi}{4}}.$$

Bu ifodaning suratidagi ko'paytma quyidagicha hisoblanadi:

$$\alpha\cdot\beta=\rho\left(\cos\gamma+i\sin\gamma\right)\cdot r\left(\cos\theta+i\sin\theta\right)=\rho\cdot r\quad\left[\cos(\theta+\gamma)+i\sin(\theta+\gamma)\right]$$

Natijada

$$\frac{1+i\sqrt{3}}{1-i}=\sqrt{2}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)$$

kelib chiqadi. Endi buning ikki tomonini Muavr formulasiga asosan 20-darajaga ko'tarib, ixchamlashtirilsa quyidagi natija hosil bo'ladi:

$$2^9\left(1-i\sqrt{3}\right)$$

Agar $x+iy$ kompleks sonning n-darajali ildizlarni topish lozim bo'lsa, dastlab uni trigonometrik shaklga keltirib olish zarur.

3-misol. Quydagi tenglikning to'g'riligini isbot qiling:

$$\left(\sqrt{3}-i\right)^n=2^n\left(\cos\frac{n\pi}{6}-i\sin\frac{n\pi}{6}\right),\ (n\text{- butun son})$$

Yechilishi: Dastlab asosdagi kompleks sonni trigonometrik shaklga keltirib olamiz:

$$a=\sqrt{3}\ ,b=-1\ ,r=\sqrt{a^2+b^2}=2\ ;\ tg\varphi=\frac{b}{a}=-\frac{1}{\sqrt{3}}\ .$$

Kompleks songa tegishli vektor to'rtinchi chorakka joylashganligi uchun $\varphi=-\frac{\pi}{6}$ (yoki $\varphi=2\pi-\frac{\pi}{6}$ deb olish ham mumkin). U holda:

$$\left(\sqrt{3}-i\right)^n=2^n\left[\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right]^n=2^n\left[\cos\left(-\frac{n\pi}{6}\right)+i\sin\left(-\frac{n\pi}{6}\right)\right]=$$
$$=2^n\left(\cos\frac{n\pi}{6}-i\sin\frac{n\pi}{6}\right).$$

4-misol. Quyidagi $\omega_1=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$ va $\omega_2=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$.
Kompleks sonlar berilgan bo'lib, ularning yig'indisi $\omega_1^n+\omega_2^n$ (*n*-butun son) ni hisoblash talab qilinadi.

Yechilishi: Dastlab kompleks sonlarni trigonometrik shaklga yozib olamiz:

$$\omega_1=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\quad \text{va}\quad \omega_2=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\ .$$

Endi Muavr formulasiga muvofiq ularni darajaga ko'tarib, so'ngra qo'shaylik, u holda

$$\omega_1^n+\omega_2^n=\left(\cos\frac{2n\pi}{3}+i\sin\frac{2n\pi}{3}\right)+\left(\cos\frac{4n\pi}{3}+i\sin\frac{4n\pi}{3}\right)=$$
$$=\left(\cos\frac{2n\pi}{3}+\cos\frac{4n\pi}{3}\right)+i\left(\sin\frac{2n\pi}{3}+\sin\frac{4n\pi}{3}\right).$$

Ma'lumki,

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\quad \text{va}\quad \sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\ .$$

Bularga asosan quydagi natija hosil bo'ladi. $\omega_1^n+\omega_1^n=2\cos\frac{n\pi}{3}$. Chunki,

$\cos(n\pi)=(-1)^n$, $\sin n\pi=0$, $n=0,\pm1,\pm2...$.

5-misol. $\alpha=\dfrac{\left(1-i\sqrt{3}\right)(\cos\varphi+i\sin\varphi)}{2(1-i)(\cos\varphi-i\sin\varphi)}$ ni hisoblang.

Yechilishi: Dastlabki kasrning bir qismini soddalashtirib olaylik, ya'ni

$$\frac{\cos\varphi+i\sin\varphi}{\cos\varphi-i\sin\varphi}=\frac{(\cos\varphi+i\sin\varphi)^2}{\cos^2\varphi+\sin^2\varphi}=\cos2\varphi+i\sin2\varphi$$

So'ngra, $1-i\sqrt{3}=2\left[\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right]$, $1-i=\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)$.

U holda $\dfrac{1-i\sqrt{3}}{1-i}=\sqrt{2}\left[\cos\left(\frac{\pi}{4}-\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{4}-\frac{\pi}{3}\right)\right]=\sqrt{2}\left[\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)\right]$.

Yuqoridagilarni inobatga olamiz va (1.17)-ifodadan foydalanamiz

$$\alpha=\frac{\sqrt{2}}{2}\left[\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)\right](\cos2\varphi+i\sin2\varphi)=\frac{\sqrt{2}}{2}\left[\cos\left(2\varphi-\frac{\pi}{12}\right)+i\sin\left(2\varphi-\frac{\pi}{12}\right)\right].$$

6-misol. Hisoblang: $\omega=(1+\cos\alpha+i\sin\alpha)^n$.

Yechilishi: Dastlab qavslar ichidagi kompleks sonni terigonametrik shaklga keltirib olamiz (bu esa bizga 3-§ dagi 12-misoldan ma'lum):

$$1+\cos\alpha+i\sin\alpha=2\cos\frac{\alpha}{2}(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}).$$

Endi (1.17) formulaga muvofiq ushbu natijaga erishamiz:

$$\omega=2^n\cos^n\frac{\alpha}{2}(\cos\frac{\pi\alpha}{2}+i\sin\frac{\pi\alpha}{2}).$$

8-misol: $\omega=\left(\frac{1+itg\alpha}{1-itg\alpha}\right)^n$ ni hisoblang

Yechilishi: Kasrning surat va maxrajini $\cos\alpha$ ko'paytirilsa

$$\omega=\left(\frac{\cos\alpha+i\sin\alpha}{\cos\alpha-i\sin\alpha}\right)^n=\frac{\cos n\alpha+i\sin n\alpha}{\cos n\alpha-i\sin n\alpha}$$

hosil bo'ladi uni $\cos n\alpha$ ga qisqartirsak

$$\omega=\frac{1+itgn\alpha}{1-itgn\alpha}$$

natijaga erishamiz. Bu misolni yechishda biz quydagi munosabatdan foydalandik:

$$(\cos\alpha-i\sin\alpha)^n=(\cos(-\alpha)+i\sin(-\alpha))^n=\cos(-n\alpha)+i\sin(-n\alpha)=\cos(n\alpha)-i\sin(n\alpha)\ .$$

9-misol. Agar $x=\cos\theta+i\sin\theta$ bo'lsa ,quydagicha tenglikni isbot qiling.

$$x^m+\frac{1}{x^m}=2\cos m\theta$$

Yechilishi: x ni ko'rinishida izlab ko'raylik. Bundan

$$\frac{1}{x}=\frac{1}{\cos\theta+i\sin\theta}=\frac{\cos\theta-i\sin\theta}{\cos^2\theta+\sin^2\theta}=\cos(-\theta)+i\sin(-\theta)$$

$$x^m=(\cos\theta+i\sin\theta)^m=\cos m\theta+i\sin m\theta\ ,$$

$$\frac{1}{x^m}=\cos m\theta-i\sin m\theta\ .$$

Natijada $x^m+\frac{1}{x^m}=2\cos m\theta$. Tenglik isbot qilindi.

10-misol. $\sin^3 x$ funksiyani $\cos kx$, $\sin kx$ lar orqali ifoda qiling.

Yechilishi: Quydagicha ish ko'ramiz:

$$\alpha=\cos x+i\sin x \text{ va } \alpha^{-1}=\frac{1}{\alpha}=\cos x-i\sin x,$$

$\alpha^m=\cos mx+i\sin mx$ va $\alpha^{-m}=\cos mx-\sin mx$, bulardan esa

$$\alpha-\alpha^{-1}=2i\sin x \text{ va } \alpha+\alpha^{-1}=2\cos x,$$

$$\alpha^m-\alpha^{-m}=2i\sin mx \text{ va } \alpha^m+\alpha^{-m}=2\cos mx.$$

Shularga asosan

$$(a+b)^n = a^n + C_n^1 a^{n-1} b + C_n^2 a^{n-2} b^2 + \ldots + C_n^k a^{n-k} b^k + \ldots + C_n^{n-1} a b^{n-1} + b^n$$

$$\sin^3 x = \left(\frac{\alpha - \alpha^{-1}}{2i}\right)^3 = \frac{1}{8i^3}\left(\alpha^3 - 3\alpha^2\alpha^{-1} + 3\alpha\alpha^{-2} - \alpha^{-3}\right) = -\frac{1}{8i}\left(\alpha^3 - \alpha^{-3} - 3\left(\alpha - \alpha^{-1}\right)\right) =$$

$$-\frac{1}{8i}\left(2i\sin 3x - 3\cdot 2i\sin x\right) = \frac{1}{4}\left(3\sin x - \sin 3x\right)$$

Demak, $\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$.

Mashqlar

Quyidagi ifodalarni hisoblang:

39. $\frac{\left(1 - i\sqrt{3}\right)^6}{\left(1 + i\sqrt{3}\right)^4} + (1+i)(3-i)$;

40. $\frac{(1 + ictg\varphi)^5}{(1 - ictg\varphi)^5}$;

41. $\frac{\left(1 + i\sqrt{3}\right)(\cos\theta + i\sin\theta)}{(1+i)(\cos\theta - i\sin\theta)}$;

42. $\frac{\left(1 + i\sqrt{3}\right)^{15}}{\left(1 + i\sqrt{3}\right)^{10}}$;

43. $(1+i)\left(1 + i\sqrt{3}\right)(\cos\varphi + i\sin\varphi)$;

44. Quyidagi tenglikning to'g'riligini isbot qiling:

$(1+i)^n = 2^{\frac{n}{2}}\left(\cos\left(\frac{\pi n}{2}\right) + i\sin\left(\frac{\pi n}{2}\right)\right)$.

1.5-§. KOMPLEKS SONDAN ILDIZ CHIQARISH

Agar $x+iy$ kompleks sonning n-darajali ildizlarini topish lozim bo'lsa, dastlab uni trigonometrik shaklga keltirib olish zarur.

Kompleks sondan n-darajali ildiz chiqarish formulasi quyidagidan iborat.

$$w_k = \sqrt[n]{z} = \sqrt[n]{r(\cos\gamma + i\sin\gamma)} = \sqrt[n]{r}\left(\cos\frac{\gamma + 2k\pi}{n} + i\sin\frac{\gamma + 2k\pi}{n}\right), \qquad (1.18)$$

k=0,1,2...,n-1

Ma'lumki, $\cos\frac{\gamma + 2k\pi}{n} = \cos\frac{\gamma + 2(k+n)\pi}{n}$,

Shuning uchun $k = n,\ n+1, \ldots$ deb faraz qilinsa, oldingi ildiz kelib chiqaveradi. Shunday qilib $w_0, w_1, w_2, \ldots, w_{n-1}$ ildizlar hosil bo'ladi, ya'ni $x+iy$ sonning n-darajali ildizlari faqat n ta bo'ladi

1-misol. $W=\sqrt[3]{-8}$ ni hisoblang.

Yechilishi: Kompleks son $|-8|$ ni trigonometrik ko'rinishga keltiraylik.

$$-8=8(\cos\pi+i\sin\pi),$$

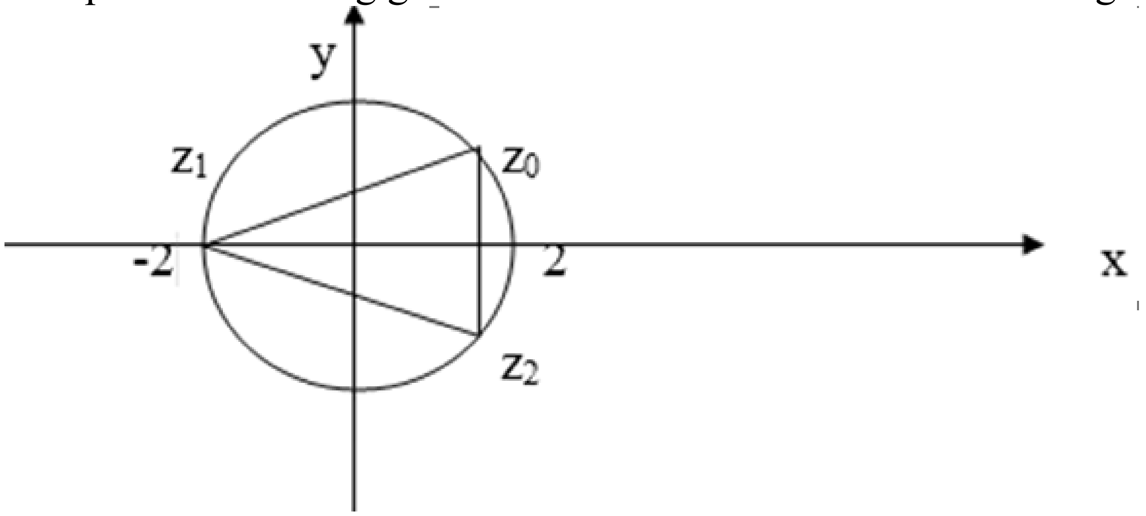
u holda $\sqrt[3]{-8}=\sqrt[3]{8}\left(\cos\frac{\pi+2k\pi}{3}+i\sin\frac{\pi+2k\pi}{3}\right),\quad k=0\ ,\ 1,\ 2,\ \ldots$ deb,

$$W_0=2\left(\cos\pi/3+i\sin\pi/3\right)=1+i\sqrt{3},$$

$$W_1=2\left(\cos\pi+i\sin\pi\right)=-2,$$

$$W_2=1-i\sqrt{3}$$

Bu kompleks sonlarning geometrik o'rni 1.6- rasimda ko'rsatilgan.



1.6-rasm

5-misol. $W_K=\sqrt{\frac{\sqrt{3}-i}{-1+i}}$ ni hisoblang.

Yechilishi: Avval ildiz ostidagi kasrni trigonometrik ko'rinishga keltirib olamiz:

$$\sqrt{3}-i=2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right),$$

$$-1+i=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right),$$

$$\frac{\sqrt{3}-i}{-1+i}=\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right),$$

Demak, bulardan

$$W_k=\sqrt[u]{2}\left(\cos\frac{(1+24k)\pi}{48}+i\sin\frac{(1+24k)\pi}{48}\right),\quad k=0.1.2.3..$$

6-misol. Quyidagi yig'ndi hisoblansin. $\sin\theta+\sin2\theta+......+\sin n\theta.$

Yechlishi. Dastlab, quyidagi yig'indini ko'raylik.

$$S_N=(\cos\theta+i\sin\theta)+(\cos2\theta+i\sin2\theta)+.......+(\cos n\theta+i\sin n\theta).$$

Muavr formulasiga ko'ra

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

quyidagilarni topamiz

$$S_n = (\cos\theta + i\sin\theta) + (\cos\theta + i\sin\theta)^2 + ... + (\cos\theta + i\sin\theta)^n$$

Bu ifoda esa, asosi $q = \cos\theta + i\sin\theta$ va birinchi hadi $a_1 = \cos\theta + i\sin\theta$ dan iborat bo'lgan kamayib boruvchi geometrik progressiya n ta hadining yig'indisidir. Ma'lumki, uning hadlari yig'indisi ushbuga teng:

$$q + q^2 + ... + q^n = \frac{a_1 - a_1 q^n}{1-q} = \frac{a_1(1-q^n)}{1-q}$$

Demak

$$S_n = \frac{(\cos\theta + i\sin\theta) - (\cos n\theta + i\sin n\theta)(\cos\theta + i\sin\theta)}{1-(\cos\theta + i\sin\theta)}$$

Haqiqiy va mavhum qismlarini ajratsak

$$S_n = \frac{\cos\theta - \cos n\theta\cos\theta + \sin n\theta\sin\theta + i(\sin\theta - \sin n\theta\cos\theta - \cos n\theta\sin\theta)}{1-\cos\theta - i\sin\theta}$$

$$S_n = \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\cos\left(\frac{n+1}{2}\right)\theta + i\frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\sin\left(\frac{n+1}{2}\right)\theta$$

Bu ifodadan

$$\sin\theta + \sin 2\theta + ... + \sin n\theta = \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\sin\left(\frac{n+1}{2}\right)\theta$$

ekanligini topamiz.

Mashqlar

Ildizning barcha qiymatlarini toping;

45. $\sqrt{-1+i\sqrt{3}}$
46. $\sqrt[3]{-1-i}$
47. $\sqrt[6]{1+i\sqrt{3}}$
48. $\sqrt[5]{(2-2i)^4}$
49. $\sqrt[3]{-1+i}$
50. $\sqrt{2-2\sqrt{3}i}$
51. $\sqrt[5]{\sqrt{2}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})}$
52. $\sqrt{-2+2i\sqrt{3}}$

Javoblar

1. $\operatorname{Re} z = 9$, $\operatorname{Im} z = 7$, $\arg z = arctg\frac{7}{9}$, $|z| = \sqrt{130}$; 2. $\operatorname{Re} z = 0$, $\operatorname{Im} z = 16$, $\arg z = 90^\circ$ $|z| = 16$; 3. $\operatorname{Re} z = 9$, $\operatorname{Im} z = 2$, $\arg z = arctg\frac{2}{9}$, $|z| = \sqrt{85}$; 4. $\operatorname{Re} z = \frac{5}{17}$, $\operatorname{Im} z = \frac{3}{17}$, $\arg z = arctg\frac{3}{5}$, $|z| = \frac{\sqrt{34}}{17}$; 5. $\operatorname{Re} z = \frac{4}{9}$, $\operatorname{Im} z = 0$, $\arg z = 0^\circ$, $|z| = \frac{4}{9}$; 6. $\operatorname{Re} z = 0$, $\operatorname{Im} z = 2.8$, $\arg z = 90^\circ$, $|z| = 2.8$; 7.Markazi i nuqtada va $R = 4$ bo'lgan aylananing ichi, ya'ni doira; 8. Markazi $(-3i)$ nuqtada va radiusi 3 bo'lgan doiraning tashqarisi (aylana ham shu sohaga kiradi); 9. $x = -5$ to'g'ri chiziqning chap tamoni; 10. $y = 3$ to'g'ri chiziq va uning yuqori qismi; 11. $x + y = 2$ to'g'ri chiziq bilan chegaralangan va koordinatalar boshini o'z ichiga olgan tekislik; 12.Radiusio 3 ga va markazi $-4+5i$ nuqtada bo'lgan aylana; 13. $x^2 - y^2 = a$ giperbolalar oilasi; 14. $xy = \frac{a}{2}$ giperbolalar oilasi; 15. Haqiqiy o'qning musbat tomoni bilan α va β burchak tashkil qiluvchi to'g'ri chiziqlar orasidagi burchakning ichi, burchakning uchi koordinata boshiga joylashgan; 16. Markazi $z_0 = 1$ nuqtada katta va kichik yarim o'qlari mos ravishda 5 va 4 ga teng bo'lgan mavhum o'q bo'yicha siqilgan ellipsl; 17. $\frac{a^2 - b^2}{a^2 + b^2} + i\frac{2ab}{a^2 + b^2}$; 18. $\frac{44 - 5i}{318}$; 19. 2; 20. $117 + 44i$; 21. $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$ 22. 1; 23. $-5 - 62i$; 24. $(-1+i)\cdot i^n$ 25. $x = \frac{13}{71}, y = \frac{11}{71}$; 26. $x = -\frac{9}{38}, y = -\frac{5}{38}$; 27. $x = \frac{6+13i}{41}, y = -\frac{6+13i}{41}$; 28. $x_1 = 3+2i, x_2 = 1+i$; 29. $\pm 4 \pm i$; 30. $x = 2, y = 1$; 31. $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$; 32. $\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$; 33. $2(\cos\pi + i\sin\pi)$; 34. $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$; 35. $\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$; 36. $\cos(\pi+\varphi) + i\sin(\pi+\varphi)$ 37. $2\sin\frac{\alpha}{2}\left(\cos\frac{\pi-\alpha}{2} + i\sin\frac{\pi-\alpha}{2}\right)$; 38. $2\cos\frac{\alpha}{2}\left(\cos\frac{\pi+\alpha}{2} + i\sin\frac{\pi+\alpha}{2}\right), 0 < \alpha < \pi$; 39. $2 + 2i(1+\sqrt{3})$; 40. $\cos(\pi - 10\varphi) + i\sin(\pi - 10\varphi)$; 41. $\sqrt{2}\left[\cos\left(26 + \frac{\pi}{12}\right) + i\sin\left(26 + \frac{\pi}{12}\right)\right]$; 42. $2^{10}i$; 43. $2\sqrt{2}\left[\cos\left(\varphi + \frac{7\pi}{12}\right) + i\sin\left(\varphi + \frac{7\pi}{12}\right)\right]$; 44. $\frac{1}{2}\left(2^{n-1} - 2^{\frac{n}{2}}\sin\frac{\pi n}{4}\right)$; 45. $\pm\frac{\sqrt{2}}{2}(1 + i\sqrt{3})$; 46. $\sqrt[10]{2}\left(\cos\left(\frac{\pi}{4} + \frac{2\pi k}{5}\right) + i\sin\left(\frac{\pi}{4} + \frac{2\pi k}{5}\right)\right), k = 0,1,2$;

47. $\sqrt[6]{2}\left(\cos\left(\frac{\pi}{18} + \frac{\pi k}{3}\right) + i\sin\left(\frac{\pi}{18} + \frac{\pi k}{3}\right)\right), k = \overline{0.5}$;

48. $2\sqrt[5]{2}\left(\cos\left(\frac{(2k-1)\pi}{5} + i\sin\frac{(2k-1)\pi}{5}\right)\right), k = \overline{0.4}$;

49. $\frac{\sqrt[3]{4}}{2}(1+i), \sqrt[6]{2}\left(-\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right), \sqrt[6]{12}\left(\sin\frac{\pi}{12}-i\cos\frac{\pi}{12}\right)$; 50. $\pm(\sqrt{3}-i)$ 51. $\sqrt[10]{2}\left(\cos\frac{(4k+1)\pi}{10}+i\sin\frac{(4k+1)\pi}{10}\right), k=\overline{0.4}$; 52. $\pm\left(1+i\sqrt{3}\right)$.

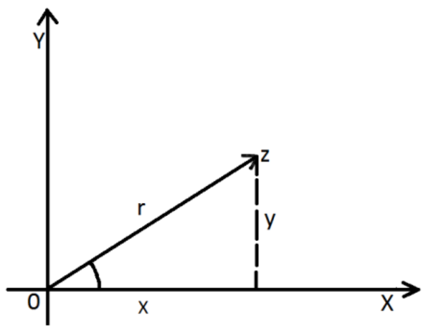
II BOB

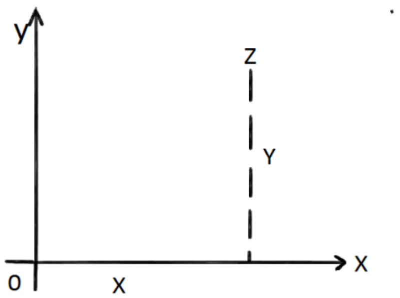
KOMPLEKS ARGUMENTLI ELEMENTAR FUNKSIYALAR

Bu bobda kompleks argumentli elementar funksiyalarga tegishli misollar yechish bilan shug'ullanamiz. Bunga tegishli nazariy ma'lumotlarni shu sohaga doir kitoblardan o'qish tavsiya etiladi, chunki bizning vazifamiz o'sha nazariyani mustahkamlashdan iborat.

2.1-§ FUNKSIYANING TA'RIFI

Komleks $z = x + iy$ sonning geometrik tasviri tekislikdagi (x, y) nuqtadan iborat ekanligi ma'lum (2.1-rasm).

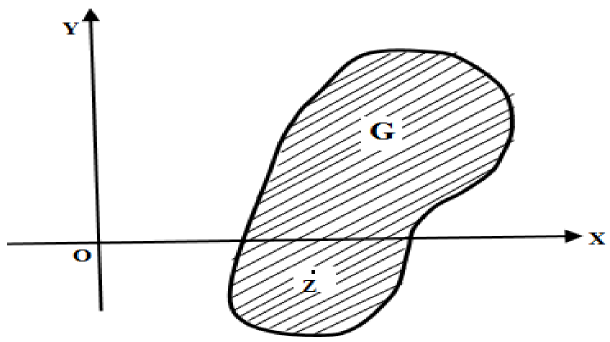




2.1-rasm

Har bir songa tekislikda bitta nuqta va aksincha, tekislikdagi har bir nuqtaga bitta kompleks son mos kelishini ilgari ham aytib o'tgan edik. Barcha kompleks sonlar to'plami bilan tekislikning barcha nuqtalarini to'plami orasida o'zoro bir qiymatlik moslik o'rnatilgan. Bu tekislik kompleks tekislik yoki z tekislik deb ataladi.

Kompleks tekislikning birorta E to'plami berilgan bo'lsin. Buning geometrik ma'nosi z tekislikdagi E nuqtalar to'plamidan iborat. Agar E to'plami ochiq va bog'lamli bo'lsa bu to'plam soha deb ataladi .Sohani G, B, D harflarining biri orqali belgilaymiz (2.2-rasm).



2.2-rasm

Ta'rif. Agar E to'plamdan olingan har bir $z = x + iy$ kompleks songa biror qonun bo'yicha aniq bir $\omega = u + iv$ kompleks son mos kelsa u holda E to'plamda funksiya berilgan deyiladi va bu quydagicha yoziladi: $\omega = f(z)$ (2.1). Bu tarifdan ko'rinadiki ω funksiyani (2.1) ko'rinishdan tashqari yana quydagicha yozish mumkin ekan:

$$\omega = u(x, y) + iv(x, y) \qquad (2.2)$$

Odatda ω funksiyaning bir ko'rinishidan ikkinchi ko'rinishiga o'tish mumkin bo'lib, uning uchun doimo $z = x + iy$ esda saqlash tavsiya etiladi. Ma'lumki, $z = x + iy$ erkli o'zgaruvchi, ya'ni argument, ω esa erksiz o'zgaruvchi bo'lib, funksiya deyiladi, funksiyaning aniqlanish sohasi deb Z tekislikdagi shunday nuqtalar to'plami E ga aytiladiki, bu to'plamdan olingan har qanday z_0 nuqtaga chekli $f(z_0)$ kompleks son mos keladi. (2.2) dagi $u(x, y)$ berilgan $f(z)$ funksiyaning haqiqiy qismi, $v(x.y)$ esa mavhum qismi deyiladi.

Agar (2.1) yoki (2.2) funksiya berilgan bo'lsa uning E to'plamdan olingan har qanday nuqtaga mos keluvchi qiymatini topish mumkin. Buning uchun $z = x + iy$ ning qiymatini (2.1) yoki (2.2) qo'yib, ω topiladi.

1-misol. $\omega = z^2$ funksiyaning aniqlanish sohasini toping.

Yechilishi: Bu misoldan ko'rinadiki, $z = x + iy$ ga ixtiyoriy qiymat berish mumkin, chunki har qanday sonning kvadrati mavjud. Quydagicha belgilaymiz:

$$f(z) = z^2$$

U holda misol uchun

$$f(i+1) = (i+1)^2 = i^2 + 2i + 1 = 2i,\ \ i^2 = -1, i = \sqrt{-1};\ f(-i) = (-i)^2 = i^2 = -1;$$

$$f\left(\frac{1}{1-i}\right) = \left(\frac{1}{1-i}\right)^2 = \frac{1}{1-2i+i^2} = -\frac{1}{2i} = \frac{i}{2}$$

Shunday qilib, $\omega = z^2$ funksiyaning aniqlanish sohasi tekislikning har qanday chegaralangan qismidan iborat ekan.

2-misol. $f(z) = \frac{1}{z}$ funksiya berilgan bo'lsa, tekislikning noldan farqli har qanday qismi bu funksiyaning aniqlanish sohasidir. *Masalan:*

$$f\left(\frac{1}{i}\right) = \frac{1}{\frac{1}{i}} = i, \quad f(-1+i) = \frac{1}{-1+i} = \frac{-1-i}{2} = -\frac{i+1}{2};$$

$$f(3-5i) = \frac{1}{3-5i} = \frac{3+5i}{34} = \frac{3}{34} + i\frac{5}{34}.$$

3-misol. $f(z) = \frac{z-i}{z+i}$ funksiya uchun $z = -i$ nuqta uning aniqlanish sohasiga kirmaydi. Boshqa nuqtalar sohaga kiradi. *Masalan:*

$$f(i) = \frac{i-i}{i+i} = \frac{0}{2i} = 0;\ f(-1) = \frac{-1-i}{-1+i} = \frac{1+i}{1-i} = i.$$

4-misol. $f(z)=z^2+z$ funksiya berilgan. Bu funksiyaning ba'zi xususiy qiymatlarini topaylik:

$$f(1+i)=(1+i)^2+1+i=1+3i;$$

$$f(2-i)=(2-i)^2+2-i=5(1-i);$$

$$f(-1)=(-1)^2-1=0;\ f(i)=i^2+i=-1+i.$$

5-misol. $f(z)=x^2-iy^2$ berilgan. Uning ba'zi xususiy qiymatlarini topaylik:

$$f(1+i)=1-i(1)^2=1-i;\quad f(-i)=i(-i)^2=-i;\quad f(5-3i)=5^2-i(-3)^2=25-9i,$$

chunki bunda $x=5$ va $y=-3$.

6-misol. $\omega=z^2$ funksiyaning haqiqiy va mavhum qismlarini toping.

Yechilishi: $\omega=f(x+iy)=(x+iy)^2=x^2+2ixy-y^2=x^2-y^2+2ixy$ demak, funksiyaning haqiqiy va mavhum qismlari:

$$u=x^2+y^2,\quad v=2xy.$$

7-misol. $\omega=\frac{1}{z},(z\neq 0)$ funksiyaning haqiqiy va mavhum qismlarini toping.

Yechilishi. z o'rniga $z=x+iy$ni qo'yib, maxrajni komplekislikdan ozod qilamiz:

$$\omega=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}.$$

Demak,

$$u=\frac{x}{x^2+y^2},\quad v=-\frac{y}{x^2+y^2}$$

8-misol. $\omega=\frac{z+i}{z-i}$ $(z\neq i)$ ning haqiqiy va mavhum qismlarini toping.

Yechilishi:

$$\omega=\frac{(x+iy)+i}{(x+iy)-i}=\frac{x+i(y+1)}{x+i(y-1)}=\frac{\left[(x+i(y+1))(x-i(y-1))\right]}{x^2+(y-1)^2}=\frac{x^2+y^2-1}{x^2+(y-1)^2}+\frac{x(y+1)-x(y-1)}{x^2+(y-1)^2}$$

,

shunday qilib,

$$u=\frac{x^2+y^2-1}{x^2+(y-1)^2}\ \text{ va }\ v=\frac{2x}{x^2+(y-1)^2}$$

9-misol. $\omega=\bar{z}-iz^2$ funksiya berilgan. u va v larni toping.

Yechilishi: $z=x+iy$ va $\bar{z}=x-iy$ lar o'zaro qo'shma sonlardir.

$$\omega=(x-iy)-i(x+iy)^2=x-iy-i\left(x^2+2ixy-y^2\right)=x+2xy-i\left(x^2-y^2+y\right)$$

Demak, $u=x+2xy$, $v=-\left(x^2-y^2+y\right)$.

Mashqlar.

Quyidagi misollarni yuqoridagi usul yordamida yeching.

1. $f(z)=z^2$ funksiya berilgan. Ushbu xususiy qiymatlarni toping:

a) $f(2+3i)$; b) $f(-1-i)$; c) $f(a+ib)$.

2. $f(z)=\frac{z}{\overline{z}}$ funksiya berilgan. Quyidagilarni toping:

a) $f(1+i)$; b) $f(-i)$; c) $f(-2+4i)$.

3. $f(z)=(z+i)^2$ funksiya berilgan. Quyidagilarni toping:

a) $f(i)$; b) $f(-i)$; c) $f(7+6i)$.

4. $\omega=z^2+i$ funksiyaning haqiqiy va mavhum qismlarini ajrating.
5. $\omega=iz^2-\overline{z}$ funksiyaning haqiqiy va mavhum qismlarini ajrating.
6. $\omega=\frac{1+i\overline{z}}{1+iz}$ funksiyaning haqiqiy va mavhum qismlarini ajrating.
7. $\omega=\frac{z}{\overline{z}}$ funksiyaning haqiqiy va mavhum qismlarini ajrating.

2.2-§. FUNKSIYANING ANIQLANISH SOHASI

Berilgan har qanday

$$\omega=u+iv=f(z)$$

funksiyaning aniqlanish sohasi G ga teng. Ma'lumki, agar chegara ham sohaga tegishli bo'lsa, u yopiq soha deyilib u $\overline{G}$ ko'rinishida yoziladi. Masala yechishda soha berilgan bo'ladi, yoki funksiyaning o'ziga qarab sohani aniqlash talab qilinadi.

Misollar

1-misol. Ushbu $|z-\alpha|<R$ (2.3) tengsizlik qanday sohani bildirishini aniqlang, bu yerda $z=x+iy$ va $\alpha=a+ib$

Yechilishi: Ma'lumki

$$z-\alpha=(x+iy)-(a+ib)=(x-a)+i(y-b),$$

$$|z-\alpha|=|(x-a)+i(y-b)|=\sqrt{(x-a)^2+(y-b)^2}<R$$

yoki

$$(x-a)^2+(y-b)^2<R^2$$

Bu markazi (a,b) nuqtada joylashgan R radiusli aylananing ichki nuqtalari to'plamidan, ya'ni doiradan iborat (2.5-rasm). Agar $\alpha=0,\quad R=1$ bo'lsa, unga birlik doira deyiladi va (2.3) dan:

$$|z|<1 \tag{2.4}$$

2-misol. Ushbu

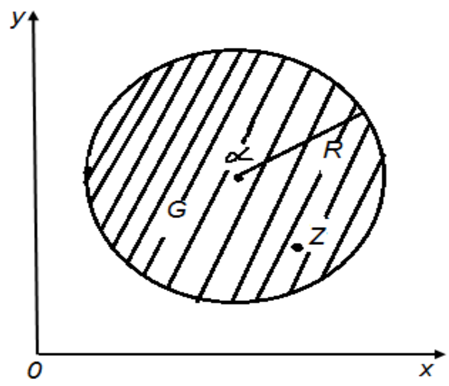
$$\operatorname{Re} z \ge 1 \qquad (2.5)$$

tengsizlikni qanoatlantiruvchi nuqtalar qanday to'plamni aniqlaydi?

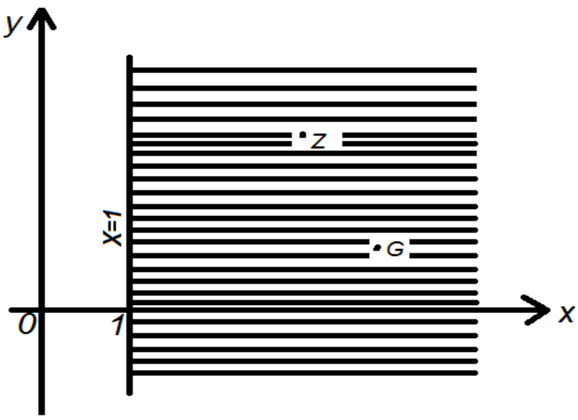
Yechilish: Ma'lumki,

$$z = x + iy, \quad \operatorname{Re} z = \operatorname{Re}(x + iy) = x \ge 1 .$$

Bu esa $x = 1$ to'g'ri chiziq va uning o'ng tomonida yotuvchi nuqtalar to'plamidir (2.6-rasm).



2.5-rasm



2.6-rasm

3-misol. Ushbu

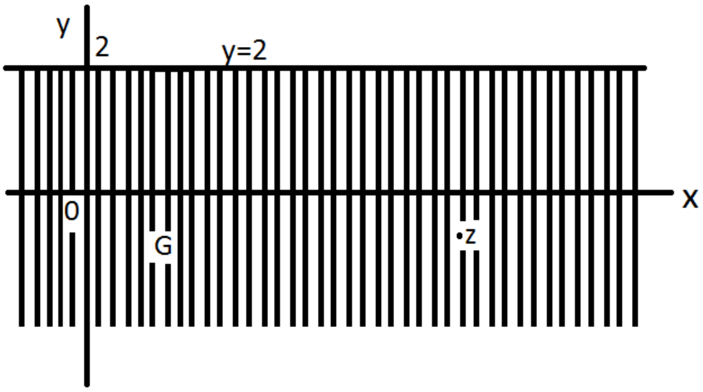
$$Im z < 2 \qquad (2.6)$$

tengsizlikni qanoatlantiruvchi nuqtalar to'plami qanday sohani aniqlaydi?

Yechilishi: Ma'lumki,

$$\operatorname{Im} z = \operatorname{Im}(x + iy) = y < 2 .$$

Bu esa $y = 2$ to'g'ri chiziqning ostida yotuvchi nuqtalar sohasidan iborat (2.7-rasm).

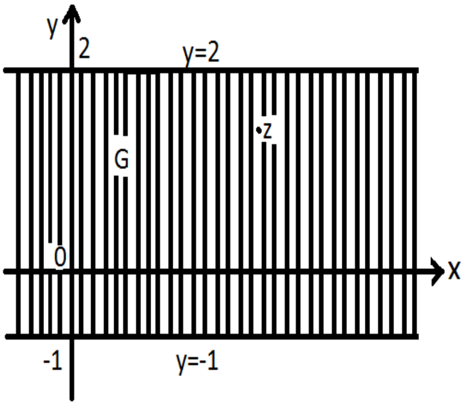


2.7-rasm

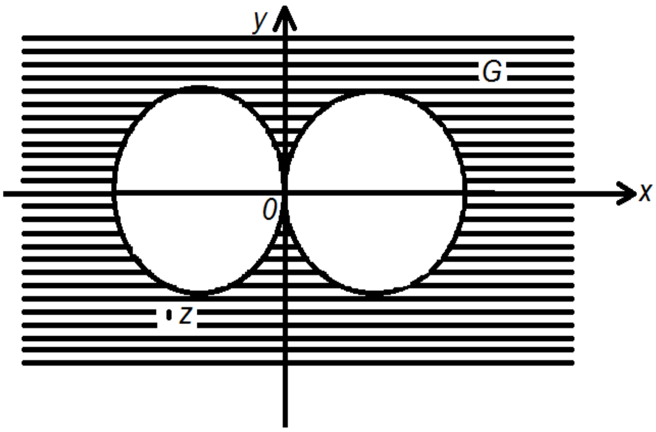
4-misol. Ushbu $-1 < \operatorname{Im} z < 2$ (2.7) tengsizlikni qanoatlantiruvchi nuqtalar to'plami qanday sohani tashkil etadi?

Yechilishi: 2-misolga ko'ra

$$-1 < y < 2 .$$



2.8-rasm



2.9-rasm

Bu esa $y=-1$ va $y=2$ to’g’ri chiziqlar orasidagi nuqtalar to’plamidir. (2.8-rasm).

5-misol. Ushbu

$$|z^2-1| \geq a^2, \quad a>0 \tag{2.8}$$

tengsizlikni qanoatlantiruvchi nuqtalar to’plamini aniqlang.

Yechilishi:

$$z^2-1=(x+iy)^2-1=(x^2-y^2-1)+2ixy\,;$$

$$|z^2-1|=\sqrt{(x^2-y^2-1)^2+4x^2y^2}=\sqrt{|(x^2-y^2)-1|^2+4x^2y^2} \geq a^2$$

yoki bu yerdan

$$(x^2+y^2)^2-2(x^2-y^2) \geq a^4-1,$$

bu esa Bernulli limpiskatasi va uning tashqarisidan iborat (2.9-rasm).

6-misol. Ushbu

$$4 \leq |z-1|+|z+1| \leq 8$$

tengsizliklarni qanoatlantiruvchi nuqtalar to’plamini aniqlang.

Yechilishi:

$$|z-1|=|(x-1)+iy|=\sqrt{(x-1)^2+y^2}\,,$$

$$|z+1|=|(x+1)+iy|=\sqrt{(x+1)^2+y^2}\,.$$

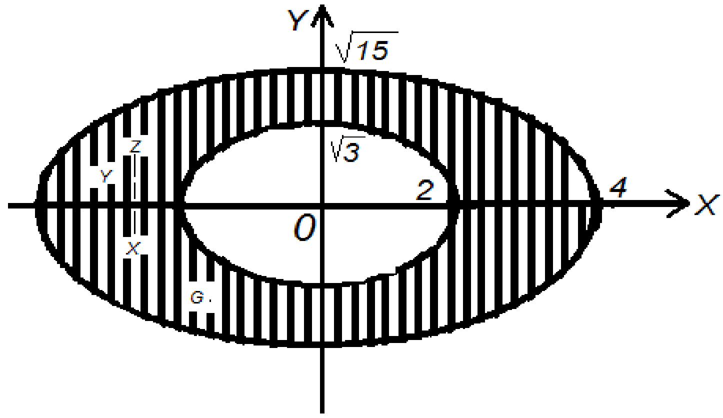
U holda

$$4 \leq \sqrt{(x-1)^2+y^2}+\sqrt{(x+1)^2+y^2} \leq 8$$

Radikallar yig’ndisini bir marta 4 ga ikkinchi marta 8 ga tenglab olib, so’ngra irratsionallikdan ozod qilsak, quyidagi ellipslarni hosil qilamiz:

$$\frac{x^2}{4}+\frac{y^2}{3}=1 \quad \text{va} \quad \frac{x^2}{16}+\frac{y^2}{15}=1.$$

Demak, izlanayotgan G to'plam bu ellipslar orasidagi halqadan iborat (2.10-rasm).



2.10-rasm

7-misol. Ushbu

$$\frac{1}{4} < \operatorname{Re}\left(\frac{1}{z}\right) + \operatorname{Im}\left(\frac{1}{z}\right) < \frac{1}{2} \qquad (2.9)$$

Tengsizlikni qanoatlantiruvchi nuqtalar to'plamini aniqlang.

Yechilishi:

$$\frac{1}{z} = \frac{1}{x-iy} = \frac{x+iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}$$

Bu yerdan

$$\operatorname{Re}\left(\frac{1}{z}\right) = \frac{x}{x^2+y^2} \quad \text{va} \quad \operatorname{Im}\left(\frac{1}{z}\right) = \frac{y}{x^2+y^2}.$$

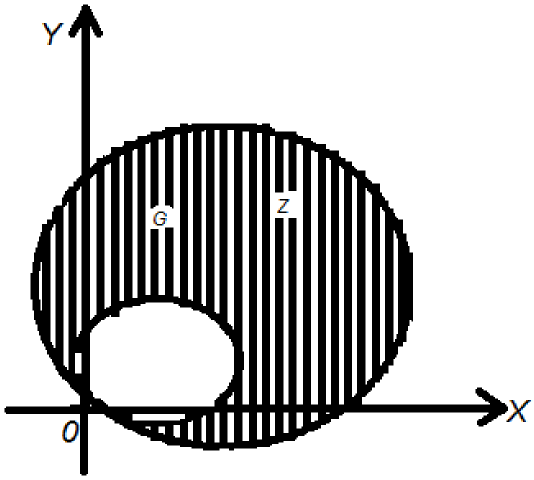
Ikkalasining yig'indisini navbat bilan $\frac{1}{4}$ ga, so'ngra $\frac{1}{2}$ ga tenglashtirib, ya'ni

$$\frac{x+y}{x^2+y^2} = \frac{1}{4} \quad \text{va} \quad \frac{x+y}{x^2+y^2} = \frac{1}{2}$$

deb olinsa, quyidagi aylanalar kelib chiqadi:

$$(x-1)^2 + (y-1)^2 = 2 \quad \text{va} \quad (x-2)^2 + (y-2)^2 = 8.$$

Demak, izlanayotgan G to'plam bu aylanalar orasidagi halqadan iborat ekan (2.11-rasm), bu yerda $r = \sqrt{2}$ va $R = \sqrt{8} = 2\sqrt{2} = 2r$.



2.11-rasm

Mashqlar

Quyidagi tengsizliklarni qanoatlantiruvchi nuqtalar to'plamini aniqlang.

8. $\operatorname{Re} z \le -1$ tengsizlik qanday sohani bildiradi?

9. $\operatorname{Im} > 1$

10. $-1 \le \operatorname{Re} z \le 1$

11. $1 \le \operatorname{Im} z \le 2$

12. $1 \le |z + 2 + i| \le 2$

13. $|z-1| < |z-i|$

14. $|z-a| < |1-az|$ a -haqiqiy son bo'lib, $|a| \ne 1.$ $|a| \ne 1$

15. $\operatorname{Im}\left(\frac{1}{z}\right) < -\frac{1}{2}$

16. $\left|\frac{z-3}{z-2}\right| \ge 1$

2.3-§. BA'ZI EGRI CHIZIQLAR

Tekislikdagi egri chiziq turli tenglamalar orqali ifoda qilinishi mumkin. Birgina egri chiziqning o'zini dekart koordinatalari sistemasida, qutb koordinatalar sistemasida, parametrik shaklda yoki vektor orqali berish mumkin. Ana shu egri chiziqni kompleks shakldagi tenglama orqali ham ifoda qilish mumkin. Misol uchun, agar tekislikdagi egri chiziqning ushbu

$$x = x(t), \quad y = y(t), \quad (t_0 \le t \le T) \tag{2.10}$$

parametrik tenglamalari berilgan bo'lsa, uni

$$z = x + iy = x(t) + iy(t) = z(t),$$

ya'ni

$$z = z(t), \quad (t_0 \le t \le T) \qquad (2.11)$$

Agar tekislikdagi chiziqning dekart koordinatalari sistemasidagi

$$y = f(x), \quad (a \le x \le b) \qquad (2.12)$$

tenglamasi berilgan bo'lsa,

$$z = x + iy \quad \text{va} \quad \overline{z} = x - iy$$

lardan foydalanib

$$x = \frac{z + \overline{z}}{2} \quad \text{va} \quad y = \frac{z - \overline{z}}{2i}$$

qiymatlarni (2.12) ga eltib qo'yiladi:

$$\frac{z - \overline{z}}{2i} = f\left(\frac{z + \overline{z}}{2}\right) \qquad (2.13)$$

1-misol. Ellipsning Dekart koordinatalar sistemasidagi tenglamasi berilgan:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Bu tenglamani kompleks ko'rinishga keltiring.

Yechilishi: x, y larning yuqoridagi z va $\overline{z}$ lar orqali ifodasi qo'yilsa, ellipsning kompleks shakldagi tenglamasi

$$\frac{(z + \overline{z})^2}{4a^2} - \frac{(z - \overline{z})^2}{4b^2} = 1 \qquad (2.14)$$

ko'rinishga ega bo'ladi.

Agar o'sha ellips ushbu

$$x = a\cos t, \quad y = b\sin t, \quad (0 \le t \le 2\pi)$$

parametrik tenglamalar orqali berilgan bo'lsa, uning kompleks shakldagi tenglamasi:

$$z = a\,\mathrm{cost} - \mathrm{ibsint}, \quad (0 \le t \le 2\pi) \qquad (2.15)$$

ko'rinishni oladi. Xususiy holda, agar $b = a = r$ bo'lsa, ellipsdan aylana hosil bo'lib, uning tenglamasi Eyler formulasiga muvofiq ixcham

$$z = r(\cos t + i\sin t) = re^{it}, \quad (0 \le t \le 2\pi) \qquad (2.16)$$

ko'rinishga keladi. Bu aylananing birinch xil tenglamasini (14) ga asosan quyidagicha yozish mumkin:

$$(z + \overline{z})^2 - (z - \overline{z})^2 = (2r)^2 \qquad .17)$$

bu yerda $2r -$ aylana diametri.

2-misol. Ushbu

$$\mathrm{Re}\, z^2 = a^2 \qquad (2.18)$$

tenglamani qanoatlantiruvchi nuqtalar qanday to’plamni aniqlaydi?

Yechilishi:

$$z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy,$$

$$\operatorname{Re} z^2 = x^2 - y^2 = a^2.$$

bu teng yoqli giperbolaning tenglamasidir.

3-misol. Ushbu

$$|z-i| = |z+2| \qquad 2.19)$$

tenglama qanday chiziqni ifodalaydi?

Yechilishi: Ma’lumki,

$$|z-i| = |x+i(y-1)| = \sqrt{x^2+(y-1)^2},$$

$$|z+2| = |x+2+iy| = \sqrt{(x+2)^2+y^2}.$$

Bunga asosan $\sqrt{x^2+(y-1)^2} = \sqrt{(x+2)^2+y^2}$,

Bu tenglikni ikkala tomonini kvadratga ko’tarib ixchamlashtirilsa, ushbu

$$4x+2y+3=0$$

to’g’ri chiziq tenglamasi kelib chiqadi. Bu chiziq $z_0=-2$ va $z_1=-i$ nuqtalarni tutashtiruvchi kesmaning o’rtasidan o’tadigan perpindikulyar ekanligini ko’rsatish mumkin.

4-misol. Ushbu

$$\arg(z-i) = \frac{\pi}{4} \qquad (2.20)$$

tenglama qanday chiziqni ifodalaydi?

Yechilishi: Ma’lumki,

$$z-i = x+i(y-1),$$

$$tg\varphi = \frac{y-1}{x}$$

Berilgan misolda $\varphi = \frac{\pi}{4}$.

Demak,

$$\frac{y-1}{x} = tg\frac{\pi}{4} = 1 \quad \text{yoki} \quad y-x-1=0$$

Bu esa $z_0 = i$ nuqtadan o’tib, Ox o’qning musbat tomoni bilan $\frac{\pi}{4}$ burchak hosil qiladigan to’g’ri chiziqdir.

5-misol. Ushbu

$$2z\bar{z} + (2+i)z + (2-i)\bar{z} = 2 \qquad (2.21)$$

tenglama qanday chiziqni ifodalaydi?

Yechilishi: Ma’lumki,

$$2z\overline{z}=2(x+iy)(x-iy)=2(x^2+y^2),$$

$$(2+i)z+(2-i)\overline{z}=2(z+\overline{z})+i(z-\overline{z})=4x-2y.$$

Shunga asosan berilgan tenglamaning chap tomoni quyidagi

$$2(x^2+y^2)+4x-2y=2$$

yoki

$$(x+1)^2+\left(y-\frac{1}{2}\right)^2=\frac{9}{4}$$

aylana tenglamasini ifodalaydi.

6-misol. Ushbu

$$|z|-3\operatorname{Im}z=6 \qquad (2.22)$$

Tenglama qanday chiziqni ifodalaydi.

Yechilishi: Ma'lumki,

$$|z|=|x+iy|=\sqrt{x^2+y^2}, \quad \operatorname{Im}z=\operatorname{Im}(x+iy)=y$$

Shu sababli

$$\sqrt{x^2+y^2}=3(2+y),$$

buni irratsionallikdan ozod qilib, ixchamlashtirgandan so'ng giperbola tenglamasi

$$\frac{\left(y+\frac{9}{4}\right)^2}{\left(\frac{3}{4}\right)^2}-\frac{x^2}{\left(\frac{3\sqrt{2}}{2}\right)^2}=1$$

kelib chiqadi. Demak, (2.22) giperbolani aniqlar ekan.

7-misol. To'g'ri chiziqning ushbu

$$Ax+By+C=0$$

tenglamasini kompleks shaklga keltiring.

Yechilishi: Ma'lumki,

$$x+iy \text{ va } \overline{z}=x-iy.$$

$$x=\frac{z+\overline{z}}{2} \text{ va } y=\frac{z-\overline{z}}{2i}$$

Bularni berilgan tenglamaga qo'yib chiqamiz, u holda

$$A\cdot\frac{z+\overline{z}}{2}+B\cdot\frac{z-\overline{z}}{2i}+C=0$$

yoki

$$(A+iB)\overline{z}+(A-iB)z+2C=0$$

$A+iB=a$ *va* $A-iB=\overline{a}$ deb belgilab, ixchamlashtirgandan so'ng:

$$a\overline{z}+\overline{a}z+2C=0$$

8-misol. Aylananing $x^2+y^2+2x-2y=0$ tenglamasini kompleks shaklga keltiring.

Yechilishi: Ma'lumki,

$$z\overline{z}+(1+i)z+(1-i)\overline{z}=0$$
$$x^2+y^2-(x+iy)(x-iy)=z\overline{z}$$
$$2x=z+\overline{z}\ ,\quad 2y=\frac{z-\overline{z}}{i}=i(\overline{z}-z)$$

Demak,

$$z\overline{z}+(z+\overline{z})z+(z-\overline{z})i=0 \quad \text{yoki} \quad z\overline{z}+(1+i)z+(1-i)\overline{z}=0$$

Mashqlar

Quyidagi tenglamalarning har biri qanday chiziqni ifodalashini ayting.

17. $\mathrm{Re}\left(\frac{1}{z}\right)=1$;
18. $\mathrm{Im}\left(\overline{z^2-\overline{z}}\right)=2-\mathrm{Im}\,z$;
19. $z^2-\overline{z^2}=1$;
20. $|z-i|-|z+2|=2$;
21. $3|z|-\mathrm{Re}\,z=12$;
22. $\mathrm{Re}\left(z^2-\overline{z}\right)=0$;
23. $\mathrm{Re}(1+z)=|z|$;
24. $z=t+\frac{i}{t}$;
25. $z=t^2+\frac{i}{t^2}$;

2.4-§. TRIGONOMETRIK FUNKSIYALAR

Bu paragrafda ayrim trigonometrik funksiyalarning kompleks shakldagi ko'rinishlari bilan tanishamiz. Dastlab uchta funksiyani ta'rif orqali kiritib, boshqa trigonometrik funksiyalarni bulardan hosil qilamiz.

e^z, $\cos z$ va $\sin z$ deb mos ravishda quyidagi darajali qatorlarni qabul qilamiz;

$$e^z=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+...+\frac{z^n}{n!}+... \tag{2.23}$$

$$\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+...+(-1)^n\frac{z^{2n}}{(2n)!}+... \tag{2.24}$$

$$\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+...+(-1)^n\frac{z^{2n+1}}{(2n+1)!}+... \tag{2.25}$$

Bularda $z = x + iy$ - kompleks o'zgaruvchi va $e = \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t$ va $e = 2,718281828459045$ $z = x + iy$ ixtiyoriy son bo'lgani uchun (2.23) ning ikki tomonida z o'rniga iz qo'yib chiqamiz, u holda

$$e^{iz} = 1 + (iz) + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \ldots + \frac{(iz)^n}{n!}.$$

Ma'lumki, $i = \sqrt{-1}$ $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, shuning uchun (2.24) va (2.25) ga asosan Eylerning mashhur formulasi

$$e^{iz} = \cos z + i\sin z \qquad (2.26)$$

kelib chiqadi. (2.26)

$$e^{-iz} = \cos(-z) + \sin(-z) \;,$$

ammo (2.24) va (2.25) muvofiq

$$\cos(-z) = \cos z \quad \text{va} \quad \sin(-z) = -\sin z$$

bo'lgani uchun

$$e^{-iz} = \cos z - i\sin z \qquad (2.27)$$

Endi (2.26) va (2.27) larni o'zaro qo'shib, so'ngra o'zaro ayirsak, ushbu formulalarni hosil qilamiz:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{va} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad (2.28)$$

Bulardan esa

$$tgz = \frac{\sin z}{\cos z} = \frac{1}{i}\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$$

va

$$ctgz = \frac{\cos z}{\sin z} = i\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \qquad (2.29)$$

formulalar hosil qilinadi. (2.28) va (2.29) lar kompleks sohadagi trigonometrik funksiyalar deyiladi. Elementar matematikada

$$|\sin x| \le 1 \quad \text{va} \quad |\cos x| \le 1$$

edi, lekin kompleks sohada bu shart bajarilmasligi ham mumkin, masalan, (2.28)ga muvofiq,

$$\cos i = \frac{e^{-1} + e}{2} \approx \frac{3,086}{2} \approx 1,54 > 1$$

$$\sin i = \frac{e - e^{-1}}{2i} \approx 1,17520i\,, \quad |\sin i| \approx 1,1752 > 1.$$

Elementar matematikada trigonometriyaga doir barcha formulalar bu yerda ham o'z kuchida qoladi. Ularning to'g'riligini (2.28) va (2.29)larga asoslanib isbot qilish mumkin.

e^z ko'rsatkichli funksiya ko'p ishlatiladi. $z = x + iy$ bo'lgani uchun (2.26) ga muvofiq bu funksiyani ushbu

$$\omega = e^x(\cos y + i\sin y) \qquad (2.30)$$

ko'rinishga keltirish mumkin. ω ning moduli va argumenti quyidagilardan iborat:

$$|\omega| = |e^z| = e^e, \quad \arg\omega = \arg e^z = y, \qquad (2.31)$$

chunki,

$$|\cos y + i\sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1 \qquad e^2 > 0$$

Eyler formulasidan foydalanib ko'rsatkichli funksiyaning argumentning mavhum qismiga nisbatan davriy ekanligini ko'rsatish mumkin, ya'ni

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z(\cos 2\pi + i\sin 2\pi) = e^z$$

Chunki,

$$\cos 2\pi = 0 \quad \text{va} \quad \sin 2\pi = 0$$

Demak,

$$e^{z+2\pi i} = e^z = \omega \qquad (2.32)$$

bo'lib, uning o'zgaruvchining mavhum qismiga nisbatan davri $2\pi i$ ga teng ekan. Umuman, o'zgaruvchiga nisbatan davriy funksiyalar ikki davrli yoki elliptik funksiyalar deb ataladi. Bunday funksiyalar to'g'risidagi ma'lumotni $[2]$ dan va u yerda ko'rsatilgan adabiyotlardan olishingiz mumkin.

1-misol. $e^{-\frac{\pi}{2}i}$ ni hisoblang.

Yechilishi: (2.26) formulaga muvofiq

$$e^{-\frac{\pi}{2}i} = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = 0 - i\cdot 1 = -i$$

demak,

$$e^{-\frac{\pi}{2}i} = -i$$

2-misol. e^{2+i} ni hisoblang.

Yechilishi: (2.30) formulaga muvofiq

$$e^{2+i} = e^2\cdot e^i = e^2(\cos 1 + i\sin 1)$$

demak,

$$|e^{2+i}| = e^2, \quad \arg e^{2+i} = 1$$

3-misol. e^{-3-4i} ni hisoblang

Yechilishi:

$$e^{-3-4i} = e^{-3}\cdot e^{-4i} = e^{-3}\left[\cos(-4) + i\sin(-4)\right] = e^{-3}\left[\cos(2\pi-4) + i\sin(2\pi-4)\right],$$

demak,

$$|e^{-3-4i}| = e^{-3} = \frac{1}{e^3} \quad \text{va} \quad \arg e^{-3-4i} = 2\pi - 4$$

4-misol. $\cos(2+i)$ ning haqiqiy va mavhum qismlarini toping.

Yechilishi:

(2.28) ga muvofiq $\cos(2+i)=\frac{1}{2}\left[e^{i(2+i)}+e^{-i(2+i)}\right]=\frac{1}{2}(e^{-1}\cdot e^{2i}+e\cdot e^{-2i})$. Eyler formulasiga kura

$$e^{2i}=cos2+i\sin 2 \quad \text{va} \quad e^{-2i}=\cos 2-i\sin 2$$

Bularga asosan, ixchamlashdan so'ng

$$\cos(2+i)=\frac{1}{2}\left[\left(e+e^{-1}\right)\cos 2-i\left(e-e^{-1}\right)\sin 2\right]$$

natijaga kelamiz.

5-misol. Ushbu tenglikning to'g'riligini isbot qiling: $\cos\left(\frac{\pi}{2}-z\right)=\sin z$

Yechilishi:

Tenglamaning chap tomonini (2.28) ga asosan ko'rsatkichli funksiya orqali ifodalaymiz.

$$\cos\left(\frac{\pi}{2}-z\right)=\frac{1}{2}\left[e^{i\left(\frac{\pi}{2}-z\right)}+e^{-i\left(\frac{\pi}{2}-z\right)}\right]=\frac{1}{2}\left(e^{\frac{\pi}{2}i}\cdot e^{-iz}+e^{-\frac{\pi}{2}i}\cdot e^{iz}\right),$$

ma'lumki,

$$e^{\frac{\pi}{2}i}=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=i \quad \text{va} \quad e^{-\frac{\pi}{2}i}=-i;$$

shunga ko'ra

$$\cos\left(\frac{\pi}{2}-i\right)=\frac{1}{2}\left(ie^{-iz}-ie^{iz}\right)=\frac{1}{2i}\left(e^{iz}-e^{-iz}\right)=\sin z .$$

6-misol. Quyidagi ayniyatni isbotlang:

$$\sin 2z=2\sin z\cos z$$

Yechilishi: (2.28) formulaga murojaat etamiz:

$$\sin 2z=\frac{1}{2i}(e^{2iz}-e^{-2iz})=\frac{1}{2i}\left[\left(e^{iz}\right)^2-\left(e^{-iz}\right)^2\right]=\frac{1}{2i}\left(e^{iz}+e^{-iz}\right)\left(e^{iz}-e^{-iz}\right)=$$

$$=2\cdot\frac{e^{iz}-e^{-iz}}{2i}\cdot\frac{e^{iz}+e^{-iz}}{2}=2\sin z\cos z$$

7-misol. Ushbu formulaning to'g'riligini isbot qiling.

$$\cos\left(z_1+z_2\right)=\cos z_1\cos z_2-\sin z_1\sin z_2$$

Yechilishi: (2.28) formulaga ko'ra

$$\cos\left(z_1+z_2\right)=\frac{1}{2}\left[e^{i\left(z_1+z_2\right)}+e^{-i\left(z_1+z_2\right)}\right]=\frac{1}{2}\left(e^{iz_1}\cdot e^{iz_2}+e^{-iz_1}\cdot e^{-iz_2}\right)$$

qavs ichidagi ifodaning qiymatlarini qo'yib ixchamlashtirilsa, masala hal bo'ladi.

Agar $z_2=z_1=z$ bo'lsa o'sha berilgan tenglikning xususiy holiga ega bo'lamiz, ya'ni

$$\cos 2z=\cos^2 z-\sin^z z, \quad z=x+iy$$

8-misol. Ushbu formulaning to'g'riligini isbot qiling.

$$tg2z = \frac{2tgz}{1 - tg^2 z}$$

Yechilishi: Tenglikni isbot qilish uchun so'nggi ikkita formuladan foydalanamiz:

$$tg2z = \frac{\sin 2z}{\cos 2z} = \frac{2\sin z \cos z}{\cos^2 z - \sin^2 z} = \frac{2\frac{\sin z}{\cos z}}{1 - \frac{\sin^2 z}{\cos^2 z}} = \frac{2tgz}{1 - tg^2 z}$$

9-misol. Ushbu tenglamani yeching.

$$\cos z = 2$$

Yechilish: (2.28) formuladan foydalanamiz:

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = 2 \quad \text{yoki} \quad e^{iz} + e^{-iz} = 4 \Rightarrow e^{2iz} - 4e^{iz} + 1 = 0$$

$e^{iz} = \omega$ deylik, u holda kvadrat tenglama hosil bo'lib,

$$\omega^2 - 4\omega + 1 = 0 \quad \text{dan} \quad \omega = 2 \pm \sqrt{3},$$

demak,

$$\omega = e^{iz} = 2 \pm \sqrt{3}, \quad iz = Ln\left(2 \pm \sqrt{3}\right) \quad z = \frac{1}{i} Ln\left(2 \pm \sqrt{3}\right)$$

10-misol. $\omega = \sin z$ ning haqiqiy va mavhum qismlarini toping.

Yechilishi: Oldingi misollardan ma'lumki,

$$\omega = u + iv = \sin z = \sin(x + iy) = \sin x \cos(iy) + \cos x \sin(iy)$$

(2.28) formulaga muvofiq

$$\cos(iy) = \frac{e^{-y} + e^{y}}{2} \quad \text{va} \quad \sin(iy) = \frac{e^{-y} - e^{y}}{2i}$$

bularga asosan

$$u = \frac{1}{2}\left(e^{y} + e^{-y}\right)\sin x \quad \text{va} \quad v = \frac{1}{2}\left(e^{y} - e^{-y}\right)\cos x$$

11-misol. $\omega = e^{\overline{z^2}}$ ning haqiqiy va mavhum qismlarini toping.

Yechilishi:

$\bar{z} = x - iy$, $\overline{z^2} = (x - iy)^2 = x^2 - y^2 - 2ixy$, demak,

$$\omega = e^{\overline{z^2}} = e^{x^2 - y^2} \cdot e^{-2ixy} = e^{x^2 - y^2}\left[\cos(-2xy) + i\sin(-2xy)\right] = e^{x^2 - y^2}\left(\cos(2xy) - i\sin(2xy)\right).$$

Shunday qilib,

$$u = e^{x^2 - y^2}\cos(2xy) \quad \text{va} \quad v = -e^{x^2 - y^2}\sin(2xy)$$

Mashqlar

26. Hisoblang. a) e^{2-2i}; b) e^{-3+i}; c) e^{-2-5i}; d) e^{-1+iz}.

27. $\sin(3 - i)$ ning qiymatini toping.

28. Ushbu tenglikning to'g'riligini isbot qiling:

$$\sin\left(\frac{\pi}{2} - z\right) = \cos z$$

29. Ushbu formulaning to'g'riligini isbot qiling
$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \sin z_2 \cos z_1$

30. Ushbu formulaning to'g'riligini isbot qiling: $ctg2z = \dfrac{ctg^2 z - z}{2ctgz}$ $\quad z = x + iy$

31. Ushbu tenglamani yeching: $\sin z = 3$

32. $\omega = u + iv = \cos z$ ning haqiqiy va mavhum qismlarini toping:

33. $\omega = e^{z^{z^2}}$ ning haqiqiy va mavhum qismlarini toping.

2.5-§. GIPERBOLIK FUNKSIYALAR

Endi kompleks sohadagi giperbolik funksiyalar bilan tanishamiz. Ular quyidagilardan iborat:

$$shz = \frac{e^z - e^{-z}}{2}; \qquad chz = \frac{e^z + e^{-z}}{2};$$

$$thz = \frac{shz}{chz} = \frac{e^z - e^{-z}}{e^z + e^{-z}}; \qquad cthz = \frac{chz}{shz} = \frac{e^z + e^{-z}}{e^z - e^{-z}}. \qquad (2.33)$$

bu yerda $z = x + iy$ - ixtiyoriy kompleks son agar $y = 0$ bo'sa, $z = x$ bo'lib, (2.33) dan bizga haqiqiy analizdan ma'lum bo'lgan giperbolik funksiyalar kelib chiqadi. (2.33) ning ba'zi xususiy qiymatlarini hisoblaymiz:

$$sh0 = 0, \quad ch0 = 1, \quad th0 = 0, \quad cth0 = \infty.$$

$$sh1 = \frac{e - e^{-1}}{2}, \quad ch1 = \frac{e + e^{-1}}{2}, \quad th1 = \frac{e - e^{-1}}{e + e^{-1}}, \quad cth1 = \frac{e + e^{-1}}{e - e^{-1}},$$

bu yerda

$$e = 2,71828, \qquad e^{-1} = \frac{1}{e} = 0,3679$$

Oldingi paragrafda tekshirilgan kompleks sohadagi trigonometrik funksiyalar bilan (2.33) giperbolik funksiyalar orasida munosabat o'rnatish mumkin. Masalan, (2.28)da z o'rniga iz qo'yib chiqsak, quyidagi munosabatlar hosil bo'ladi:

$$\cos iz = \frac{1}{2}\left(e^{i^2 z} + e^{-i^2 z}\right) = \frac{1}{2}(e^{-z} + e^z) = chz;$$

$$tgiz = \frac{\sin iz}{\cos iz} = \frac{ishz}{chz} = ithz; \quad ctgiz = \frac{\cos iz}{\sin iz} = \frac{1}{i}ctgz;$$

demak,

$$shz = \frac{1}{i}\sin iz; \quad chz = \cos iz; \quad thz = \frac{1}{i}tgiz; \quad cthz = ictgiz. \qquad (2.34)$$

Shunday qilib, (2.34) - trigonometrik funksiyalardan giperbolik funksiyalarga o'tish formulalaridir. Aksincha, (2.33) da z o'rniga iz

qo'yib chiqsak, kompleks sohadagi giperbolik funksiyalardan shu sohadagi trigonometrik funksiyalarga o'tish formulalarini hosil

$$shiz=\frac{e^{iz}-e^{-iz}}{2}=i\frac{e^{iz}-e^{-iz}}{2i}=i\sin z;\quad chiz=\frac{e^{iz}+e^{-iz}}{2}=\cos iz;$$

$$thiz=\frac{shiz}{chiz}=\frac{i\sin z}{\cos z}=itgz;\quad cthiz=\frac{chiz}{shiz}=\frac{\cos z}{i\sin z}=\frac{1}{i}ctgz;$$

demak,

$$\sin z=\frac{1}{i}shiz;\quad \cos z=chiz;\quad tgz=\frac{1}{i}thiz;\quad ctgz=icthiz \qquad (2.35)$$

Masalan,

$$\sin i=\frac{1}{i}sh(-1);\quad \cos i=ch(-1);\quad tgi=\frac{1}{i}th(-1);\quad ctgi=icth(-1).$$

1-misol. $sh(-2+i)$ ning qiymatini hisoblang.

Yechilishi: (2.33) ga muvofiq:

$$sh(-2+i)=\frac{1}{2}\left[e^{-2+i}-e^{-(-2+i)}\right]=\frac{1}{2}\left(e^{-2}\cdot e^{i}-e^{2}\cdot e^{-i}\right),\quad e^{i}=\cos 1+i\sin 1 \quad \text{va} \quad e^{-i}=\cos 1-i\sin i,$$

bularga asosan:

$$sh(-2+i)=i\sin 1\cdot\frac{e^{2}+e^{-2}}{2}-\cos 1\cdot\frac{e^{2}-e^{-2}}{2}=i\sin 1ch2-\cos 1sh2.$$

2-misol. chi ning qiymatini hisoblang.

Yechilishi: $z=i$ desak, (2.33) dan:

$$chi=\frac{e^{i}+e^{-i}}{2}=\frac{1}{2}\left[(\cos 1+i\sin 1)+(\cos 1-i\sin 1)\right]=\cos 1$$

3-misol. $\cos(1+i)$ ning qiymatini hisoblang.

Yechilishi: (2.35) ga muvofiq

$$\cos(1+i)=chi(1+i)=ch(-1+i)=\frac{1}{2}\left[e^{-1+i}+e^{-(-1+i)}\right]=\frac{1}{2}\left[e^{-1}(\cos 1+i\sin 1)+e(\cos 1-i\sin 1)\right]=$$

$$=\cos 1\cdot\frac{e+e^{-1}}{2}-i\sin 1\cdot\frac{e-e^{-1}}{2}$$

bu yerda (2.33) ga ko'ra $\cos(1+i)=\cos 1\cdot ch1-i\sin 1\cdot sh1$.

4-misol. $\sin(1+i)$ ning qiymatini hisoblang.

Yechilishi:

$$\sin(1+i)=\frac{1}{2i}\left[e^{i(1+i)}-e^{-i(i+1)}\right]=\frac{1}{2i}\left(e^{-1}\cdot e^{i}-e\cdot e^{-i}\right)=\frac{1}{2i}\left[e^{-1}(\cos 1+i\sin 1)-e(\cos 1-i\sin 1)\right]=$$

$$=\frac{1}{2i}\left[-\cos 1\left(e-e^{-1}\right)+i\sin 1\left(e+e^{-1}\right)\right]=\sin 1\cdot\frac{e+e^{-1}}{2}+i\cos 1\cdot\frac{e-e^{-1}}{2}$$

Demak, $\sin(1+i)=\sin 1\cdot ch1+ish1\cdot\cos 1$.

5-misol. $tg(1+i)$ ning qiymatini hisoblang.

Yechilishi: Yuqoridagi misoldan foydalanamiz.

$$tg(1+i)=\frac{\sin(1+i)}{\cos(1+i)}=\frac{\sin 1\cdot ch1+ish1\cdot\cos 1}{\cos 1\cdot ch1-i\sin 1\cdot sh1},$$

maxrajni komplekslikdan ozod qilib, so'ngra ixchamlashtirilsa, ushbu natija kelib chiqadi:

$$tg(1+i)=\frac{\sin 1\cdot\cos 1+ishi\cdot chi}{\cos^2 1+sh^2 1}$$

6-misol. Ushbu tenglikning to'g'riligini isbotlang. $ch^2z-sh^2z=1$, $z=x+iy$

Yechilishi: (2.33) formulalardan foydalanamiz:

$$ch^2z-sh^2z=\left(\frac{e^z+e^{-z}}{2}\right)^2-\left(\frac{e^z-e^{-z}}{2}\right)=\frac{1}{4}\left[\left(e^{2z}+2+e^{-2z}\right)-\left(e^{2z}+2+e^{2z}\right)\right]=\frac{1}{4}\cdot 4=1$$

7-misol. $\sin(2-i)$ ning qiymatini toping.

Yechilishi: (2.28) dan foydalanamiz:

$$\sin(2-i)=\frac{1}{2i}\left[e^{i(2-i)}-e^{-i(2-i)}\right]=\frac{1}{2i}\left(ee^{2i}-e^{-1}\cdot e^{-2i}\right)=\frac{1}{2i}\left[e(\cos 2+i\sin 2)-e^{-1}(\cos 2-i\sin 2)\right]=$$

$$=\frac{1}{2i}\left[\left(e-e^{-1}\right)\cos 2+i\left(e+e^{-1}\right)\sin 2\right]=\sin 2\cdot\frac{e+e^{-1}}{2}-i\cos 2\cdot\frac{e-e^{-1}}{2}$$

(2.33) ni nazarga olsak, ushbu natija hosil bo'ladi:

$$\sin(2-i)=\sin 2\cdot ch1-i\cos 2\cdot sh1$$

8-misol. $\cos(2-i)$ ning qiymatini hisoblang.

Yechilishi: Bu yerda ham (2.28) dan foydalanamiz:

$$\cos(2-i)=\frac{1}{2}\left[e^{i(2-i)}+e^{-i(2-i)}\right]=\frac{1}{2}\left(ee^{2i}+e^{-1}e^{-2i}\right)=\frac{1}{2}\left[\cos 2\left(e+e^{-1}\right)+i\sin 2\cdot sh1\right]$$

(2.33) ni e'tiborga olib, ushbu natijani hosil qilamiz:

$$\cos(2-i)=\cos 2\cdot ch1+i\sin 2\cdot sh1.$$

9-misol. $tg(2-i)$ ning qiymatini aniqlang.

Yechilishi: Yuqoridagi misollardan foydalanamiz:

$$tg(2-i)=\frac{\sin(2-i)}{\cos(2-i)}=\frac{\sin 2\cdot ch1-i\cos 2\cdot sh1}{\cos 2\cdot ch1+i\sin 2\cdot sh1},$$

maxrajni komplekslikdan ozod qilamiz, u holda

$$tg(2-i)=\frac{\left(\sin 2\cdot\cos 2-ish1sh^2 1\right)}{\cos^2 2\cdot ch^2 1+\sin^2 2\cdot sh^2 1},$$

tenglikni ixchamlashtirish maqsadida tubandagi ishlarni bajaramiz:

$$\cos^2 2ch^2 1+\sin^2 2sh^2 1=\cos^2 2ch^2 1+\left(1-\cos^2 2\right)sh^2 1=\cos^2 2\left(ch^2 1-sh^2 1\right)+sh^2 1=\cos^2 2+sh^2 1$$

chunki $ch^2z-sh^2z=1$, shuningdek,

$$sh1ch1=\frac{\left(e-e^{-1}\right)}{2}\cdot\frac{\left(e+e^{-1}\right)}{2}=\frac{1}{2}\frac{\left(e^2-e^{-2}\right)}{2}=\frac{1}{2}\sin 2;$$

$$\sin 2\cos 2=\frac{1}{2}\cdot 2\sin 2\cos 2=\frac{1}{2}\sin 4,$$

bularga asosan, oxirida:

$$tg(2-i)=\frac{(\sin 4 - ish2)}{2(\cos 2 + sh^2 1)}$$

10-misol. $sh\left(\ln 3+\frac{\pi}{4}i\right)$ ning qiymatini hisoblang .

Yechilishi: (2.33) ga muvofiq . $shz=\frac{1}{2}\left(e^z-e^{-z}\right)$,

$$sh\left(\ln 3+\frac{\pi}{4}i\right)=\frac{1}{2}\left(e^{\ln 3+\frac{\pi}{4}i}-e^{-\ln 3-\frac{\pi}{4}i}\right)=\frac{1}{2}\left(e^{\ln 3}\cdot e^{\frac{\pi}{4}i}-e^{-\ln 3}\cdot e^{-\frac{\pi}{4}i}\right);$$

Ma'lumki,

$$e^{\ln 3}=3;\quad e^{-\ln 3}=e^{\ln 3^{-1}}=3^{-1}=\frac{1}{3};\quad e^{\pm\frac{\pi}{4}i}=\cos\frac{\pi}{4}\pm i\sin\frac{\pi}{4};\quad \cos\frac{\pi}{4}=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}.$$

Demak ,

$$sh\left(\ln 3+\frac{\pi}{4}i\right)=\frac{1}{2}\left[\left(3-\frac{1}{3}\right)\cos\frac{\pi}{4}+i\left(3+\frac{1}{3}\right)\sin\frac{\pi}{4}\right]=\frac{\sqrt{2}}{6}(5i+4).$$

11-misol. $ch(\ln 3+\frac{\pi}{4}i)$ ning qiymatini hisoblang.

Yechilishi:

$$ch\left(\ln 3+\frac{\pi}{4}i\right)=\frac{1}{2}\left[\left(3+\frac{1}{3}\right)\cos\frac{\pi}{4}+i\left(3-\frac{1}{3}\right)\sin\frac{\pi}{4}\right]=\frac{\sqrt{2}}{6}(5+4i).$$

12-misol. $th\left(\ln 3+\frac{\pi}{4}i\right)$ ning qiymatini toping.

Yechilishi: Oldingi ikki misoldan foydalanamiz:

$$th\left(\ln 3+\frac{\pi}{4}i\right)=\frac{sh\left(\ln 3+\frac{\pi}{4}i\right)}{ch\left(\ln 3+\frac{\pi}{4}i\right)}=\frac{4+5i}{5+4i};$$

mahrajni komplekslikdan ozod qilib topamiz :

$$th\left(\ln 3+\frac{\pi}{4}i\right)=\frac{40}{41}+i\frac{9}{41}$$

13-misol. Quyidagi tenglikning to'g'riligini isbot qiling: $sh2z=2shzchz$

Yechilishi:

$$sh2z=\frac{1}{2}\left(e^{2z}-e^{-2z}\right)=\frac{1}{2}\left[\left(e^z\right)^2-\left(e^{-z}\right)^2\right]=\frac{1}{2}\left(e^z+e^{-z}\right)\left(e^z-e^{-z}\right)=2\cdot\frac{e^z-e^{-z}}{2}\cdot\frac{e^z+e^{-z}}{2}=$$

$$=2shzchz$$

14-misol. Quyidagi tenglikning to'g'riligini isbot qiling : $ch2z=sh^2z+ch^2z$

Yechilishi. $ch2z=\frac{e^{2z}+e^{-2z}}{2}=\frac{2e^{2z}+2e^{-2z}}{4}=\frac{2e^{2z}+2e^{-2z}+2-2}{4}=$

$$=\frac{\left(e^{2z}+2+e^{-2z}\right)+\left(e^{2z}-2+e^{-2z}\right)}{4}=\left(\frac{e^{z}+e^{-z}}{2}\right)^2+\left(\frac{e^{z}-e^{-z}}{2}\right)^2=ch^2z+sh^2z$$

15-misol. Quyidagi tenglikning to'g'riligini isbot qiling:

$$th2z=\frac{2thz}{1+th^2z}$$

Yechilishi: Kasrning surat va maxrajini bo'lamiz:

$$th2z=\frac{sh2z}{ch2z}=\frac{2shzchz}{ch^2z+sh^2z}=\frac{2thz}{1+th^2z};$$

16-misol. Ushbu tenglamani yeching: $chz=i$

Yechilishi:

$$chz=\frac{e^{z}+e^{-z}}{2}=i,\quad e^{z}+e^{-z}=2i,\quad e^{2z}-2ie^{z}+1=0;\qquad w=e^{z}\Rightarrow \mathrm{w}^2-2\mathrm{iw}+1=0;$$

$$w=i+\sqrt{i^2-1}=i+i\sqrt{2},\qquad e^{z}=w=\left(1\pm\sqrt{2}\right);\quad z=\ln\left(1+\sqrt{2}\right)+\frac{\pi}{2}i+2\pi ki,\qquad k=0,\pm1,\pm2,...$$

Demak, tenglamaning ildizlari : $z_k=\ln\left(1+\sqrt{2}\right)+\left(2k+\frac{1}{2}\right)\pi i$

Mashqlar

Ifodalarning qiymatini hisoblang.

34. $\cos(2+i)$

35. $sh\frac{\pi i}{2}$

36. $cth(2+i)$

37. $ctg\left(\frac{\pi}{4}-i\ln 2\right)$

38. $sh(-2+i)$

39. $\sin(x+iy)$

40. $\cos(x+iy)$

Quyidagi tenglikning to'g'riligini isbot qiling:

41. $\sin i\cdot ch1=\cos i\cdot sh1$

Tenglamalarni yeching.

42. $e^{z}+i=0$

43. $shiz=-i$

2.6-§. TESKARI TRIGONOMETRIK FUNKSIYALAR

Bu paragrafda kompleks sohadagi teskari trigonometrik funksiyalar bilan tanishamiz. Ma'lumki,

$$z=\sin w,\quad z=\cos w,\quad z=tgw,\quad z=ctgw \qquad (2.35)$$

funksiyalar trigonometrik funksiyalar deyilib, ularda ishtirok etayotgan argument (erkli o'zgaruvchi) ni funksiya deb qarasak, (2.35) mos ravishda esa uning funksiyasi (erksiz o'zgaruvchi) dir. Aksincha, agar

$$w = Arc\sin z, \quad w = Arc\cos z, \quad w = Arctgz, \quad w = Arcctgz, \tag{2.36}$$

kabi yozilib bular, teskari trigonometrik funksiyalar deyiladi. Agar z son berilgan bo'lsa, (2.36) dagi teskari funksiyaning qiymatlarini topishga imkon beradigan formulalarni keltirib chiqarish mumkin . Shu formulalarni topaylik:

a). (2.35) ga asosan

$$z = \sin w = \frac{1}{2i}\left(e^{iw} - e^{-iw}\right), \quad e^{iw} - e^{-iw} = 2iz;$$

Buning ikki tomonini e^{iw} ga ko'paytirib, e^{iw} ga nisbatan kvadrat tenglama hosil qilamiz, uni yechsak,

$$\left(e^{iw}\right)^2 - 2ize^{iw} - 1 = 0, \quad e^{iw} = iz \pm \sqrt{(iz)^2 + 1},$$

Tenglikning ikki tomonini logarfmlaymiz .

$$iw = Ln\left(iz \pm \sqrt{1 - z^2}\right),$$

bu yerda w o'rniga uning (2.36) dagi ifodasini keltirib qo'yamiz.

$$Arc\sin z = \frac{1}{i}Ln\left(iz \pm \sqrt{1 - z^2}\right) \tag{2.37}$$

ildiz belgisi oldida $\pm$ borligini esda saqlaymiz:

b) Yana (2.35) muvofiq, $z = \cos w = \frac{1}{2}\left(e^{iw} + e^{-iw}\right)$,

$$e^{iw} + e^{-iw} = 2z; \quad e^{2iw} - 2ze^{iw} + 1 = 0,$$

Bu kvadrat tenglamani yechsak:

$$iw = Ln\left(z \pm \sqrt{z^2 - 1}\right) \quad e^{iw} = z \pm \sqrt{z^2 - 1},$$

(2.36) ga asosan : $Arc\cos z = \frac{1}{i}Ln\left(z \pm \sqrt{z^2 - 1}\right);$ (2.38)

d) (2.35) va (2.36) ga asosan ;

$$z = tgw = \frac{1}{i}\frac{e^{iw} - e^{iw}}{e^{iw} + e^{-iw}}, \quad e^{iw} - e^{-iw} = iz\left(e^{iw} + e^{-iw}\right);$$

bo'lib, o'xshash hadlarni guruhlasak,

$$(1 - iz)e^{iw} = (1 + iz)e^{-iw}, \quad e^{2iw} = \frac{1 + iz}{1 - iz};$$

bu tenglikning ikki tomonidan logarifm olib, (2.36)dan qiymati qo'yilsa,

$$Arcctgz = \frac{1}{2i}Ln\frac{1 + iz}{1 - iz} \tag{2.39}$$

formula kelib chiqadi.

g) Yuqoridagi usul bo'yicha

$$Arcctgz = \frac{1}{2i} Ln \frac{z+i}{z-i} \qquad (2.40)$$

formulani ham topish mumkin. Shunday qilib, to'rtala teskari trigonometrik funksiya ham logarifm orqali ifoda qilinar ekan. Sonning logarifmini hisoblash usuli oldingi bobda ma'lum.

1-misol. $Arc\sin i$ ning barcha qiymatlarini hisoblang.

Yechilishi: (2.37) formulaga $z = i$ ni qo'yamiz:

$$Arc\sin i = \frac{1}{i} Ln\left(-1 \pm \sqrt{2}\right),$$

ushbu $t_2 = -1 - \sqrt{2}$, $t_1 = -1 + \sqrt{2}$ belgilashlarni kiritamiz. U holda

$$Lnt_1 = ln\left(-1+\sqrt{2}\right) = Ln\left|-1+\sqrt{2}\right| + i\varphi_1 + 2k\pi i = ln\left(-1+\sqrt{2}\right) + 2k\pi i;,$$

$$Lnt_2 = Ln\left(-1-\sqrt{2}\right) = ln\left|-1-\sqrt{2}\right| + i\varphi_2 + 2\pi ki = ln\left(-1-\sqrt{2}\right) + \pi i + 2k\pi i;$$

chunki t ga tegishli vektor Ox o'qning musbat tomonida joylashganligi uchun $\varphi_1 = 0$ t esa chap tomonida joylashganligi uchun $\varphi_2 = \pi$ bo'ladi. Shunday qilib

$$Arc\sin i = 2k\pi - i\ln\left(\sqrt{2}-1\right),$$

$$Arc\sin i = \left(2k+1\right)\pi - i\ln\left(\sqrt{2}+1\right), \quad k = 0, \pm 1, \pm 2... ,$$

chunki $\frac{1}{i} = -i$ $k = 0$ bo'lganda $Arc\sin i$ ning bosh qiymati $\arcsin i$ hosil bo'ladi.

2-misol. $Arc\cos(-i)$ ning barcha qiymatlarini hisoblang.

Yechilishi: (2.38) formulaga $z = -i$ ni qo'yib chiqamiz

$$Arc\cos\left(-i\right) = \frac{1}{i} Ln\left(-i \pm \sqrt{2}\right) = \frac{1}{i} Ln\left(-1+\sqrt{2}\right) i .$$

Ko'rinib turibdiki,

$$t_1 = -1 + \sqrt{2} > 0, \quad \varphi_1 = \frac{\pi}{2} + 2k\pi,$$

$$t_2 = \left(-1\right) - \sqrt{2} < 0, \quad \varphi_2 = -\frac{\pi}{2} + 2k\pi, \quad k = 0, \pm 1, \pm 2, ...,$$

shu sababli

$$Ln\left(it_1\right) = Ln\left(\sqrt{2}-1\right)i = \ln\left(\sqrt{2}-1\right) + \frac{\pi}{2}i + 2k\pi i,$$

$$Ln\left(it_2\right) = Ln\left(-\sqrt{2}-1\right)i = \ln\left(\sqrt{2}+1\right) - \frac{\pi}{2}i + 2k\pi i$$

Demak, izlanayotgan qiymatlar ikkita bo'lib, ular quyidagilardan iborat ekan:

$$\left(2k+\frac{1}{2}\right)\pi - i\ln\left(\sqrt{2}-1\right); \quad \left(2k-\frac{1}{2}\right)\pi - i\ln\left(\sqrt{2}+1\right); \quad k = 0, \pm 1, \pm 2, ... ;$$

$k=0$ desak, $Arc\cos(-i)$ ning bosh qiymati, ya'ni $\arccos(-i)$ kelib chiqadi.

3-misol. $Arctg5i$ ning barcha qiymatlarini toping.

Yechilishi: (2.39) formulaga $z=5i$ ni qo'yib chiqamiz, u holda

$$Arctg5i=\frac{1}{2i}Ln\frac{1+i(5i)}{1-i(5i)}=\frac{1}{2i}Ln\frac{-4}{6}=\frac{1}{2i}Ln\frac{-2}{3};$$

ma'lumki, $\left|-\frac{2}{3}\right|=\frac{2}{3}$; $\arg\left(-\frac{2}{3}\right)=\pi$;

Demak , $Ln\left(-\frac{2}{3}\right)=\ln\frac{2}{3}+\pi i+2k\pi i$;

shu sababli

$$Arctg5i=\left(k+\frac{1}{2}\right)\pi+i\ln\frac{3}{2};\quad k=0,\pm1,\pm2,...\ ;$$

$k=0$ da $arctg5i$ hosil bo'ladi.

4-misol. $Arcctg(-2i)$ ning barcha qiymatlarini toping.

Yechilishi: (2.40) formulaga $z=-2i$ ni qo'yib chiqamiz, u holda $Arcctg(-2i)=\frac{1}{2i}Ln\frac{-2i+1}{-2i-1}=\frac{1}{2i}Ln\frac{1}{3}$; bo'lib,

$$\left|-\frac{1}{3}\right|=\frac{1}{3},\quad \arg\left(\frac{1}{3}\right)=0\quad Ln\frac{1}{3}=\ln\frac{1}{3}+2k\pi i;$$

demak,

$$Arcctg(-2i)=k\pi+\frac{1}{2}\ln3\quad k=0,\pm1,\pm2,...\ ;$$

5-misol. $Arcctg(1+2i)$ ning barcha qiymatlarini toping.

Yechilishi: (2.39)formulaga $[z=1+2i]$ ni qo'yib chiqamiz,

Ya'ni $Arcctg(1+2i)=\frac{1}{2i}Ln\frac{1+i(1+2i)}{1-i(1+2i)}=\frac{1}{2i}\ln\frac{i-1}{3-i}$,

Kasrning maxrajini komplekslikdan ozod qilib, uning moduli va argumentini topaylik:

$$\frac{i-1}{3-i}=-\frac{2}{5}+\frac{1}{5}i;\quad \left|-\frac{2}{5}+\frac{1}{5}\right|=\sqrt{\frac{4}{25}+\frac{1}{25}}=\frac{1}{\sqrt5};$$

$$\arg\left(-\frac{2}{5}+\frac{1}{5}\right)i=\varphi;\ tg\varphi=\frac{1}{5}:\left(-\frac{2}{5}\right)=-\frac{1}{2};$$

$$\varphi=arctg\left(-\frac{1}{2}\right)=-arctg\frac{1}{2}=-arcctg2;\quad Ln\left(-\frac{2}{5}+\frac{1}{5}\right)=\ln\frac{1}{\sqrt5}-iarcctg2+2k\pi i;$$

U holda, $Arctg(1+2i)=k\pi-\frac{1}{2}arctg2+\frac{1}{4}\ln5$; bo'ladi.

6-misol. $Arc\sin\left(\sqrt2-1\right)$; ning barcha qiymatlarini hisoblang.

Yechilishi: (2.37) ga $\sqrt2-1$ ni qo'yib chiqamiz, ya'ni

$$w = Arc\sin\left(\sqrt{2}-1\right) = \frac{1}{i}Ln\left(1+i\sqrt{2}\pm\sqrt{1-\left(\sqrt{2}-1\right)^2}\right)$$

buni soddalashtirsak, $w=\frac{1}{i}Ln\left(1+i\sqrt{2}\pm\sqrt[4]{8}\cdot\sqrt{i}\right)$ hosil bo'lib,

$$\varepsilon_m = \sqrt{i} = \sqrt{\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}} = \cos\frac{\frac{\pi}{2}+2m\pi}{2}+i\sin\frac{\frac{\pi}{2}+2m\pi}{2}, \quad m=0,1.$$

$$\varepsilon_0 = \cos\frac{\pi}{4}+i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}(1+i) \quad \varepsilon_1 = \cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}(1+i) \;;$$

Bu yerda, yozuvni qisqartirish maqsadida, ε_0 ildiz bilan chegaralanamiz. U holda ba'zi soddalashtirishlardan so'ng

$$\omega = \frac{1}{i}\left[\ln\left(1+\sqrt[4]{2}\right)+\ln\sqrt{1+\sqrt{2}}+iarctg\sqrt[4]{2}+2k\pi i\right],$$

Yoki bundan

$$\omega = 2k\pi + arcctg\sqrt[4]{2} - i\ln\left(1+\sqrt[4]{2}\right) - \frac{1}{2}i\ln\left(1+\sqrt{2}\right), \quad k=0,\pm1,\pm2,\ldots,$$

kelib chiqadi.

Mashqlar

Quyidagi teskari trigonometrik funkasiyalarning barcha qiymatlarini hisoblang.

44 $Arc\sin\frac{1}{2}$;

45. $Arc\cos\frac{1}{2}$;

46. $Arc\cos 2$;

47. $Arc\sin 2$;

48. $Arctg 2i$;

49. $Arctg\frac{i}{3}$;

50. $Arc\sin\frac{\pi i}{3}$

2.7-§. TESKARI GIPERBOLIK FUNKSIYALAR

Giperbolik funksiyalar oldingi paragraflardan bizga ma'lum bo'lib, ular quyidagilardan iborat:

$$z = sh\omega = \frac{e^{\omega}-e^{-\omega}}{2}, \; z = ch\omega = \frac{e^{\omega}+e^{-\omega}}{2}, \; z = th\omega = \frac{e^{\omega}-e^{-\omega}}{e^{\omega}+e^{-\omega}}, \; z = cth\omega = \frac{e^{\omega}+e^{-\omega}}{e^{\omega}+e^{-\omega}} \quad (2.41)$$

bu yerda $\omega-$ argument, ya'ni erkli o'zgaruvchi, z esa uning funksiyasi, ya'ni erksiz o'zgaruvchi deb faraz qilingan. Teskarisini faraz qilsak, ya'ni $\omega = u + iv$ ni funksiya, $z = x + iy$ ni argument deb olsak, (2.41) dan quyidagilar kelib chiqadi:

$$\omega = Arshz, \quad \omega = Archz, \quad \omega = Arthz, \quad \omega = Arcthz \tag{2.42}$$

$z = x + iy$ bo'lganda (2.42) dagi funksiyalarni logarifmik funksiya orqali ifodalash mumkin.

a) $\omega = Arshz$ ga doir formulani keltirib chiqarish uchun (2.41) ga murojaat qilamiz:

$$\frac{e^{\omega} - e^{-\omega}}{2} = z, \quad e^{\omega} - e^{-\omega} - 2z = 0$$

Bu esa e^{ω} ga nisbatan kvadrat tenglama bo'lib, uning yechimi

$$e^{\omega} = z \pm \sqrt{z^2 + 1}$$

dan iborat. Endi buning ikki tomonidan logarifm olib, so'ngra ω o'rniga (2.42) dagi qiymat qo'yilsa,

$$Arshz = Ln\left(z + \sqrt{z^2 + 1}\right) \tag{2.43}$$

formula hosil bo'ladi. $\omega = Archz$ ga doir formula ham shu usulda topiladi, ya'ni

$$\frac{e^{\omega} + e^{-\omega}}{2} = z, \quad e^{\omega} + e^{-\omega} - 2z = 0, \quad e^{2\omega} - 2ze^{\omega} + 1 = 0$$

$$e^{\omega} = z \pm \sqrt{z^2 - 1}; \quad \omega = Ln\left(z \pm \sqrt{z^2 - 1}\right),$$

demak,

$$Archz = Ln\left(z \pm \sqrt{z^2 - 1}\right) \tag{2.44}$$

b) $\omega = Arcthz$ ga doir formula ham ana shu usulda keltirib chiqariladi, ya'ni

$$\frac{e^{\omega} - e^{-\omega}}{e^{\omega} + e^{-\omega}} = z, \quad e^{\omega} - e^{-\omega} = z\left(e^{\omega} + e^{-\omega}\right), \quad e^{2\omega} - 1 = z\left(e^{2\omega} + 1\right),$$

$$e^{2\omega} = \frac{1+z}{1-z}, \quad 2\omega = Ln\frac{1+z}{1-z},$$

demak,

$$Arthz = \frac{1}{2} Ln\frac{1+z}{1-z} \tag{2.45}$$

g) $\omega = Arcthz$ ga doir formula ham ana shu usulda keltirib chiqariladi, ya'ni

$$\frac{e^{\omega} + e^{-\omega}}{e^{\omega} - e^{-\omega}} = z, \quad e^{\omega} + e^{-\omega} = z\left(e^{\omega} - e^{-\omega}\right), \quad e^{2\omega} + 1 = z\left(e^{2\omega} - 1\right),$$

$$e^{2\omega} = \frac{z+1}{z-1}, \quad 2\omega = Ln\frac{z+1}{z-1},$$

demak,

$$Arcthz = \frac{1}{2} Ln\frac{z+1}{z-1} \tag{2.46}$$

1-misol. $Arshi$ ning barcha qiymatlarini hisoblang.

Yechilishi: (2.43) ga ko'ra

$$Arshi = Ln\left(i+\sqrt{i^2+1}\right) = Lni,\quad i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2},\ \varphi=\frac{\pi}{2},\ r=|i|=1,\ \ln 1=0\,.$$

Logarifmning formulasiga muvofiq,

$$Arsh(i) = Lni = \left(2k+\frac{1}{2}\right)\pi i,\quad k=0,\pm1,\pm2,....$$

2-misol. $Arch(-i)$ ning barcha qiymatlarini hisoblang.

Yechilishi: (2.44) ga ko'ra

$$Arch(-i) = Ln\left(-i+\sqrt{(-i)^2+1}\right) = Ln(-i);\ -i=\cos\left(-\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right),\ r=|-i|=1$$

Shularga asosan

$$Arch(-i) = Ln(-i) = \left(2k-\frac{1}{2}\right)\pi i\,.$$

3-misol. $Arth(-i)$ ning barcha qiymatlarini hisoblang.

Yechilishi: (2.45) formulaga ko'ra

$$Arth(1+i) = \frac{1}{2}Ln\frac{1+(1+i)}{1-(1+i)} = \frac{1}{2}Ln(-1+2i);\ |-1+2i|=\sqrt{5},\ \ tg\varphi\frac{2}{-1}=-2,\ \ \varphi=\pi-arctg2;$$

$$Arth(1+i) = \frac{1}{2}\left[\ln\sqrt{5}+i(\pi-arcrtg2)+2k\pi i\right] = \frac{\ln 5}{4}+i(2k\pi+\pi-arctg2),\ k=0,\pm1,\pm2,...\ .$$

4-misol. $Arcth(-1+i)$ ning barcha qiymatlarini hisoblang.

Yechilishi: (2.46) formulaga murojaat qilamiz:

$$Arcth(-1+i) = \frac{1}{2}Ln\frac{(-1+i)+1}{(-1+i)-1} = \frac{1}{2}Ln\left(\frac{1}{5}-\frac{2}{5}i\right);\left|\frac{1}{5}-\frac{2}{5}i\right| = \frac{1}{\sqrt{5}},\quad tg=-2,\quad \varphi=-arctg2$$

$$Arcth(-1+i) = \frac{1}{2}\left(\ln\frac{1}{\sqrt{5}}-iarctg2+2k\pi i\right) = i\left(k\pi-\frac{1}{2}arctg2\right)-\frac{\ln 5}{4},\quad k=0,\pm1,\pm2,...$$

Mashqlar

Quyidagi teskari giperbolik funksiyaning barcha qiymatlarini hisoblab toping.

51. $Arch(-1)$; 52. $Arth(i)$; 53. $Arch(2i)$ 54. $Arth(1-i)$

Javoblar

1.a) $-5+2i$; b) $2i$; c) a^2-b^2+2iab; 2. a) i; b) -1; c) $-\frac{3+4i}{5}$; 3. a) -4; b) 0; c) $98i$; 4. $u=x^2-y^2$, $v=1+2xy$; 5. $u=-x-2xy$, $v=x^2-y^2+y$; 6. $u=\frac{(1+x)(1+y)+xy}{(1+x)^2+y^2}$, $v=\frac{x(1+x)-y(1+y)}{(1+x)^2+y^2}$, 7. $u=\frac{x^2-y^2}{x^2+y^2}$, $v=\frac{2xy}{x^2+y^2}$; 8. $x=-1$ to'g'ri chiziqning chap tomoni; 9. $y=1$ to'g'ri chiziqning yuqorisi; 10. $x=-1,\ y=1$ to'g'ri chiziqning orqasi; 11. $y=1,\ y=2$ to'g'ri chiziqning orqasi; 12. Markazi $(-2-i)$ nuqtada joylashgan $R_1=1$ va $R_2=2$ radiusli aylanalar orasi; 13. $y=x$ to'g'ri chiziqning osti; 14**.** Birlik doira; 15. $x^2+(y-1)^2=1$ aylananing ichi; 16. $x=\frac{5}{2}$ to'g'ri chiziqning chap tomoni; 17. Aylana $\left(x-\frac{1}{2}\right)^2+y^2=\frac{1}{4}$; 18. Giperbola: $xy=-1$; 19. Giperbola: $x^2-y^2=\frac{1}{2}$; 20.Oy o'qdagi -1 dan $-\infty$ gacha olingan nur; 21. Ellips: $\frac{\left(x-\frac{3}{2}\right)^2}{\left(\frac{3}{2}\right)^2}+\frac{y^2}{\left(3\sqrt{2}\right)^2}=1$; 22. Giperbola $\left(x-\frac{1}{2}\right)^2-y^2=\frac{1}{4}$; 23. Parabola: $y^2=2x+1$; 24. Giperbola: $y=\frac{1}{x}$; 25. Giperbolaning birinchi choragidagi tarmog'i: $y=\frac{1}{x}(x>0)$; 26. a) $e^2(\cos 2-i\sin 2)$; b) $e^{-3}(\cos 1+i\sin 1)$; c) $e^{-2}\left[\cos(2\pi-5)+i\sin(2\pi-5)\right]$; d) $-\frac{1}{e}$; 27. $\frac{e+e^{-1}}{2}\sin 3+i\frac{e-e^{-1}}{2}\cos 3$; 31. $\frac{1}{i}Ln\left(3+2\sqrt{2}\right)i$; 32. $\omega=\frac{1}{2}\left(e^y+e^{-y}\right)\cos x-\frac{1}{2}i\left(e^y-e^{-y}\right)\sin x$; 33. $\omega=e^{x^2-y^2}(\cos 2xy)$; 34. $\cos 2ch1-i\sin 2sh1$; 35.t; 36. $\frac{sh8-i\sin 4}{2\left(ch^2 4-\cos^2 2\right)}$; 37. $\frac{8+15i}{17}$; 38. $\left(2k-\frac{1}{2}\right)\pi i,\ k=0,\pm1,\pm2,...$; 39. $\left(k-\frac{\pi}{2}\right)\pi,\ k=0,\pm1,\pm2,...$; 40. $i\sin 1ch2-\cos 1sh2$; 41. $\sin xchy+i\cos xshy$; 42. $\cos xchy-i\sin xshy$; 44. $\frac{\pi}{6}+2k\pi$; 45. $2k\pi\pm\frac{\pi}{3}$; 46. $2k\pi\pm i\ln\left(2+\sqrt{3}\right)$; 47. $\left(2k+\frac{1}{2}\right)\pi-\ln\left(2\pm\sqrt{3}\right)$; 48. $\left(k+\frac{1}{2}\right)\pi i\frac{\ln 3}{2}$; 49. $k\pi+\frac{i}{2}\ln 2$; 50. $2k\pi-i\ln\left(-\frac{\pi}{3}+\sqrt{1+\frac{\pi^2}{9}}\right)$; 51. $(2k+1)\pi i$; 52. $\left(k\pi+\frac{\pi}{4}\right)i$; 53. $\ln\left(\sqrt{5}\pm2\right)+\left(2k\pm\frac{1}{2}\right)\pi i$; 54. $\frac{1}{4}\ln 5+\left[\frac{1}{2}arctg2+\left(k+\frac{1}{2}\right)\pi\right]i.$

III BOB

KOMPLEKS ARGUMENTLI FUNKSIYALARNING HOSILASI

Bu bobda biz kompleks argumentli funksiyaning hosilasiga doir mashqlar yechish bilan shug'ullanamiz. Hosila argumentining va modulining geometrik ma'nolari haqida so'z yuritmamiz, ularni kelgusi boblardan biriga qoldiramiz. Bu temaga tegishli nazariy masalalrni [1] dan chuqurroq o'rganishni o'quvchilarga tavsiya etamiz.

3.1-§. HOSILANING TA'RIFI

Biror G sohada aniqlangan, bir qiymatli va uzluksiz funklsiya berilgan bo'lsin:

$$\omega = f(z) \text{ yoki } \omega = u(x,y) + iv(x,y) \tag{3.1}$$

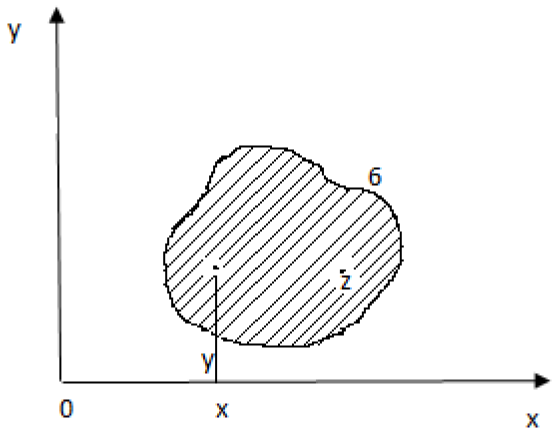
Bu yerda $z = x + iy$ - kompleks argument bo'lib, G sohaga tegishlidir (3.1-rasm). G soha ichida ixtiyoriy bir o'zgarmas $z_0 = x_0 + iy_0$ nuqta olaylik. Ma'lumki,

$$\Delta z = z - z_0, \quad \Delta\omega = f(z) - f(z_0) = f(z_0 + \Delta z) - f(z) \tag{3.2}$$

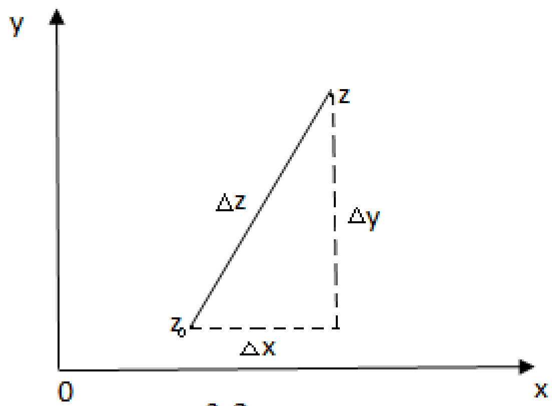
orttirmalar deyiladi (3.2-rasm). Ularni quyidagicha yozish ham mumkin:

$$\Delta z = \Delta x + i\Delta y, \quad \Delta\omega = \Delta u + i\Delta v, \quad \Delta x = x - x_0, \quad \Delta = y - y_0 \tag{3.3}$$

$$\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0), \quad \Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0) \tag{3.4}$$



3.1-rasm



3.2-rasm

Ma'lumki, *z* nuqta G sohaga tegishli bo'lib, tayinlangan z_0 nuqtaga ixtiyoriy ravishda yaqinlashishi mumkin yoki $\Delta z = z - z_0$ nolga ixtiyoriy ravishda intilishi mumkin.

Ta'rif Agar Δz har qanday yo'l bilan nolga intilganda ham $\frac{\Delta w}{\Delta z}$ nisbat aniq birgina chekli limitga intilsa, o'sha limit $f(z)$ funksiyaning z_0 nuqtadagi hosilasi deyilib, bu quydagicha yoziladi:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}.$$

Bu ta'rif matematik analiz kursidagi hosila ta'rifiga o'xshash bo'lgani sababli undagi asosiy formulalar bu joyda ham o'z kuchini saqlaydi (3.1-jadval).

3.1-jadval.

1. $c' = \frac{dc}{dz} = 0$, bu yerda $C-$ixtiyoriy o'zgarmas kompleks son.	7. $(e^z)' = e^z$, 8. $(e^{mz})' = me^{mz}$.
2. $z' = \frac{dz}{dz} = 1$	9. $(\ln z)' = \frac{1}{z}, \quad z \neq 0$,
3. $(z^n)' = nz^{n-1}$	10. $(\sin z)' = \cos z$,
4. $\left(\frac{1}{z}\right)' = -\frac{1}{z'}, \ z \neq 0$	11. $(\cos z)' = -\sin z$,
5. $(z^{-n})' = -\frac{n}{z^{n+1}}, \quad z \neq 0$	12. $(tgz)' = \frac{1}{\cos^2 z} = 1 + tg^2 z$
6. $(\sqrt{z})' = \frac{1}{2\sqrt{z}}$	13. $(ctgz)' = -\frac{1}{\sin^2 z} = -(1 + ctg^2 z)$.

Agar funksiya $w = u + iv$ formada berilgan bo'lsa, undan hosila olish uchun quydagi to'rtta formulaning biridan foydalanishga to'g'ri keladi:

$$w' = f'(z) = \frac{du}{dx} + i\frac{dv}{dx} = \frac{dv}{dy} - i\frac{du}{dy} = \frac{du}{dx} - i\frac{du}{dy} = \frac{dv}{dy} + i\frac{dv}{dx}. \qquad (3.5)$$

Berilgan (1) funksiyaning biror $z \in G$ nuqtada hosilaga ega bo'lishi uchun quydagi

$$\frac{du}{dx} = \frac{dv}{dy}, \qquad \frac{du}{dy} = -\frac{dv}{dx} \qquad (3.6)$$

Shartlarning bajarilishi zarur va yetarli bo'lib, ular Koshi-Riman (ba'zan Eyler-Dalamber) shartlari deyiladi.

Kompleks argumentli funksiyalardan xususiy $\frac{dw}{d\overline{z}}$, $\frac{dw}{dz}$ hosilalar w ning haqiqiy va mavhum qismlarining xususiy hosilalari orqali ifodalanadi:

$$\frac{dw}{dz}=\frac{1}{2}\left(\frac{dw}{dx}-i\frac{dw}{dy}\right)=\frac{1}{2}\left(\frac{du}{dx}+\frac{dv}{dy}\right)+\frac{1}{2}\left(\frac{dv}{dx}+\frac{du}{dy}\right),$$

$$\frac{dw}{d\overline{z}}=\frac{1}{2}\left(\frac{dw}{dx}+i\frac{dw}{dy}\right)=\frac{1}{2}\left(\frac{du}{dx}-\frac{dv}{dy}\right)+\frac{1}{2}\left(\frac{dv}{dx}+\frac{du}{dy}\right),$$

(3.6) Koshi-Riman sharti esa $\frac{dw}{d\overline{z}}=0$ ga ekvivalentdir. Agar $w=0$ bo'lsa, ya'ni

$$u_x=v_y,\ u_y=\pm v_x \text{ bo'lsa,}$$

$$\frac{dw}{dz}=w'=u_x-iu_y=v_y+iv_x=u_x+iv_x=v_y-iu_y.$$

1-misol. $w=z^2$ funksiyaning hosilasini toping.

Yechilishi: Hosilalar jadvalidan foydalanamiz , u holda

$$w'=(z^2)'=2z$$

Buning to'g'riligini yuqoridagi formulalarning (3.6) biri orqali tekshirib ko'raylik.

$$w=z^2=(x+iy)^2=x^2-y^2+2ixy,\ u=x^2-y^2$$

$$v=2xy,\ \frac{\partial u}{\partial x}=2x,\ \frac{\partial u}{\partial y}=-2y,\ \frac{\partial v}{\partial x}=2y,\ \frac{\partial v}{\partial y}=2x$$

bulardan esa,

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=2x,\ \frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}=-2y$$

ya'ni, analitiklik shartlari bajariladi: Demak, hosila mavjud ekan va o'sha hosilani quyidagicha topamiz.

$$w'=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=2x+2iy=2(x+iy)=2z$$

ya'ni, $(z^2)'=2z$

2-misol. $w=3x^2-5y^2+i2xy$ funksiyaning hosilasini toping.

Yechilishi: Dastlab (3.6) shartlarni tekshirib ko'raylik:

$$u=3x^2-5y^2,\quad v=2xy;\quad \frac{du}{dx}=6x$$

$$\frac{du}{dy}=-10y,\quad \frac{dv}{dx}=2y,$$

$$\frac{dv}{dy}=2x;\quad \frac{du}{dx}=\frac{dv}{dy}\ \text{dan}\ 6x=2x,\ 4x=0,\quad x=0;$$

$$\frac{du}{dy}=-\frac{dv}{dx}\ \text{dan}\quad -10y=-2y,\ 8y=0,\quad y=0,$$

Demak, funksiya bittagina (0,0), ya'ni $z=x_0+iy_0=0$ nuqtagina hosilaga ega ekan. Shu hosilani topamiz:

$$w'=f'(0)=\left(\frac{du}{dx}+i\frac{dv}{dx}\right)_{z-0}=(6x+2iy)_{z=0}=0.$$

3-misol. $w=5xy-6x+9y+i(x^2-y^2)$ funksiyaning hosilasini toping.

Yechilishi: Yuqoridagi usul bo'yicha ish ko'ramiz, ya'ni

$$u=5xy-6x+9y,\quad v=x^2-y^2;$$

$$\frac{du}{dx}=5y-6,\quad \frac{du}{dy}=5x+9,$$

$$\frac{dv}{dx}=2x,\ \frac{dv}{dy}=-2y;\ \frac{du}{dx}=\frac{dv}{dy.}\ \text{dan}\ 5y-6=-2y;\quad y=\frac{6}{7};$$

$$\frac{du}{dy}=-\frac{dv}{dx}\ \text{dan}\quad 5x+9=-2x,\quad x=-\frac{9}{7}.$$

Demak, $z_0=-\frac{9}{7}+i\frac{6}{7}$ nuqtadagina hosila mavjud bo'lib, u quydagidan iborat:

$$f'\left(-\frac{9}{7}+i\frac{6}{7}\right)=\left(\frac{du}{dx}+i\frac{dv}{dx}\right)_{z_0}=\left(5y-6+2ix\right)_{z_0}=5\cdot\frac{6}{7}+i2\left(-\frac{9}{7}\right)=\frac{30-18i}{7}$$

$$==5\cdot\frac{6}{7}-6+i2\left(-\frac{9}{7}\right)=-\frac{12}{7}=i\frac{18}{7}$$

4-misol. $w=x-iy$ funksiyaning hosilasini toping.

Yechilishi: Bu funksiya $z=x+iy$ ga qo'shmadir. Misolda berilishiga ko'ra: $u=x,\ \ v=-y$. Topamiz:

$$\frac{du}{dx}=1,\quad \frac{du}{dy}=0,\quad \frac{dv}{dx}=0,\quad \frac{dv}{dy}=-1,$$

Bunda ushbu $\frac{du}{dx}=\frac{dv}{dy}$ Shart buziladi chunki $1=-1$ bo'lib qoladi. Demak, $w=x-iy=\bar{z}$ funksiya hech qanday nuqtada hosilaga ega emas.

Lekin funksiya uzluksiz. Odatda funksiyaning uzluksizligi quydagidan ma’lum bo’ladi:

$$\lim_{\Delta z \to 0} \Delta w = 0.$$

Bizning misolda esa,

$$\Delta z = \Delta x + i\Delta y. \quad \Delta w = \Delta x - i\Delta y, \quad \lim_{\Delta z \to 0}(\Delta x - i\Delta y) = 0.$$

Shunday qilib, funksiya uzluksiz bo’lsa ham hosilaga ega bo’lmay qolishi mumkin, lekin aksincha, funksiya nuqtada hosilaga ega bo’lsa, u shu nuqtada albatta uzluksiz bo’ladi. Aytib o’tish kerakki, $w = \overline{z}$ funksiyadan $w_{\overline{z}}$ hosila mavjud bo’lib, $w_z = 0$ tenglik esa qoshma Koshi-Riman sistemasini qanoatlantiradi:

$$\frac{du}{dx} = -\frac{dv}{dy}, \qquad \frac{du}{dy} = \frac{dv}{dx}.$$

5-misol. $w = e^z$ funksiyaning hosilasi funksiyaning o’ziga tengligini tekshirib ko’ring.

Yechilishi: Buning uchun quydagicha ish ko’ramiz:

$$r = x + iy, \quad w = e^z \cdot e^{iy} = e^x(\cos y + i\sin y),$$

$$u = e^x \cos y, \qquad v = e^x \sin y,$$

$$\frac{du}{dx} = (e^x \cos y)'_x = e^x \cos y, \quad \frac{du}{dy} = (e^x \cos y)'_y = -e^x \sin y,$$

$$\frac{dv}{dx} = (e^x \sin y)'_x = e^x \sin y, \quad \frac{dv}{dy} = (e^x \sin y)'_y = e^x \cos y,$$

$$\frac{du}{dx} = \frac{dv}{dy} = e^x \cos y, \qquad \frac{du}{dy} = -\frac{dv}{dx} = -e^x \sin y,$$

Demak, Koshi-Riman shartlari bajarilganligi uchun e^z barcha nuqtalarda hosilaga ega bo’lib, quydagicha topiladi:

$$(e^z)' = \frac{du}{dx} + i\frac{dv}{dx} = e^x \cos y + ie^x \sin y = e^x(\cos y + i\sin y) = e^x \cdot e^{iy} = e^{x+iy} = e^z.$$

Shunday qilib, $(e^z)' = e^z$.

6-misol. $w = e^{iz}$ funksiyaning hosilasini toping.

Yechilishi: Oldingi misolga muvofiq:

$$w' = (e^{iz})' = e^{iz}(iz)' = ie^{iz}; \qquad (e^{iz})' = ie^{iz}.$$

7-misol. $w' = e^{-iz}$ funksiyaning hosilasini toping.

Yechilishi: $w' = (e^{-iz})' = e^{iz}(-iz)' = -ie^{-iz}$.

3.2-§. HOSILALARNI HISOBLASHGA DOIR MASHQLAR

Hosilaga doir asosiy formulalar 3.1-jadvalda berilgan ulardan ba’zilarini keltirib chiqarishni misol tariqasida ko’rib chiqaylik.

1-misol. $w=\sin z$ funksiyaning hosilasini toping.

Yechilishi. Eyler formulasidan foydalanib topamiz:

$$w=\sin z=\frac{e^{iz}-e^{-iz}}{2i},\quad w'=(\sin z)=\frac{1}{2i}(e^{iz}-e^{-iz})'=\frac{1}{2i}\left[(e^{iz})'-(e^{-iz})'\right]=$$

$$=\frac{1}{2i}(ie^{iz}+ie^{-iz})=\frac{e^{iz}+e^{-iz}}{2}=\cos z,$$

demak,

$$(\sin z)'=\cos z.$$

2-misol. $w=\cos z$ funksiya hosilasini toping.

Yechilishi:

$$w=\cos z=\frac{1}{2}(e^{iz}+e^{-iz}),\quad w'=\frac{1}{2}(e^{iz}+e^{-iz})'=\frac{1}{2}(ie^{iz}-ie^{-iz})=$$

$$=\frac{1}{2}(e^{iz}-e^{-iz})=-\frac{e^{iz}-e^{-iz}}{2i}=-\sin z.$$

demak,

$$(\cos z)'=-\sin z.$$

3-misol. $w=tgz$ funksiyaning hosilasini toping.

Yechilishi:

$$w=tg\,z=\frac{\sin z}{\cos z},\quad w'=(tgz)'=\left(\frac{\sin z}{\cos z}\right)'=\frac{\cos^2 z+\sin^2 z}{\cos^2 z}=\frac{1}{\cos^2 z}=1+tg^2 z.$$

Demak,

$$(tg\,z)'=\frac{1}{\cos^2 z}=1+tg^2 z.$$

4-misol. $w=ctg\,z$ funksiyaning hosilasini toping.

Yechilishi:

$$w=ctgz=\frac{\cos z}{\sin z};\quad w'=(ctgz)'=\frac{-\sin^2 z-\cos^2 z}{\sin^2 z}=-\frac{1}{\sin^2 z}=-(1+ctg^2 z)$$

Demak,

$$(ctgz)'=-\frac{1}{\sin^2 z}=-(1+ctg^2 z).$$

5-misol. $w = shz$ (giperbolik kosinus)ning hosilasini toping.

Yechilishi: Ta’rifga ko’ra

$$w = shz = \frac{e^z - e^{-z}}{2}; \quad w' = (shz)' = \frac{1}{2}(e^z - e^{-z})' = \frac{1}{2}(e^z + e^{-z}) = chz.$$

Demak, $(shz)' = chz.$
(3.7)

6-misol. $w = chz$ (giperbolik kosinus)ning hosilasini toping.

Yechilishi:

$$w = chz = \frac{e^z + e^{-z}}{2}; \quad w' = (chz)' = \frac{1}{2}(e^z + e^{-z})' = \frac{1}{2}(e^z - e^{-z}) = shz.$$

Demak,

$$(chz)' = shz \tag{3.8}$$

7-misol. $w = th\,z$ (giperbolik tangens)ning hosilasini toping.

Yechilishi: Kasrning hosilasi formulasiga muvofiq,

$$w' = (th\,z)' = \left(\frac{shz}{chz}\right)' = \frac{(shz)'chz - (chz)'shz}{ch^2 z} = \frac{ch^2 z - sh^2 z}{ch^2 z} = \frac{1}{ch^2 z}.$$

Demak,

$$(th\,z)' = \frac{1}{ch^2 z}. \tag{3.9}$$

8-misol. $w = cth\,z$ (giperbolik kotangens)ning hosilasini toping.

Yechilishi: Oldingi misolga o’xshash topamiz:

$$w' = (cth\,z)' = \left(\frac{chz}{shz}\right)' = \frac{(chz)'shz - (shz)'chz}{sh^2 z} = \frac{sh^2 z - ch^2 z}{sh^2 z} = \frac{-1}{sh^2 z}.$$

Demak,

$$(cth\,z)' = -\frac{1}{sh^2 z} \tag{3.10}$$

3.3-§. ANALITIK FUNKSIYALAR

Ta`rif. Agar $w = f(z)$ funksiya E sohaning z_0 nuqtasida va uning atrofida ham differensiallanuvchi bo‘lsa, u shu nuqtada analitik deyiladi.

Ta`rif. Agar $w = f(z)$ funksiya E sohaning z_0 nuqtasida hosilaga ega bo‘lib, uning atrofida hosilaga ega bo‘lmasa, u holda funksiya z nuqtada monogen deyiladi.

Demak, funksiya biror nuqtada monogen bo‘lishidan, uning shu nuqtada analitik bo‘lishi kelib chiqmaydi.

Ta`rif. Agar $f(z)$ funksiya E sohaning barcha nuqtalarida hosilaga ega bo‘lsa, u funksiya Eda analitik deyiladi.

Ta`rif. $f(z)$ funksiya analitik bo‘lgan nuqtalar uning to‘g‘ri nuqtasi, analitik bo‘lmagan nuqtalar esa maxsus nuqtalar deyiladi.

1-misol. $w=\frac{1}{z}=f(z)$ funksiyaning ba’zi nuqtalardagi hosilasini toping.

Yechilishi: $f(z)$ ning hosilasini va bu hosilaning $\pm i, 1+i, -1+3i$ nuqtalardagi qiymatlarini topaylik:

$$f'(z)=\left(\frac{1}{z}\right)=-\frac{1}{z^2};\quad f'(i)=-\frac{1}{i^3}=1;\quad f'(-i)=1;$$

$$f'(1+i)=-\frac{1}{(1+i)^2};\quad f'(-1+3i)=-\frac{1}{(-1+3i)^2}\ldots$$

O’ng tomonidagi kasrlarni qanday hisoblash o’quvchiga ma’lum. Bu misoldan ko’rinadiki, hosiladagi z nol qiymatni qabul qilolmaydi. $\frac{1}{z}$ ning nol nuqtada hosilasi yo’q; shu sababli $z=0$ nuqta funksiyaning maxsus nuqtasidir. Demak, berilgan funksiya $z=0$ dan boshqa barcha nuqtalarda analitik funksiya ekan.

2-misol. $f(z)=z\overline{z}$ funksiyaning hosilasini toping.

Yechilishi: $f(z)=z\overline{z}=(x+iy)(x-iy)=x^2+y^2,$

$$u=x^2+y^2,\ v=0,\ \frac{du}{dx}=2x,\ \frac{du}{dy}=2y,\ \frac{dv}{dx}=0,\ \frac{dv}{dy}=0.$$

Koshi-Riman shartlariga muvofiq

$$2x=0,\quad x=0,\quad 2y=0,\quad y=0,\quad z=0.$$

$$f'(0)=\left(\frac{du}{dx}+i\frac{dv}{dx}\right)_{z=0}=(2x+i\cdot 0)_{x=0}=0.$$

Demak, berilgan funksiya faqat birgina $z=0$ nuqtada hosilaga ega bo’lib, boshqa nuqtalarda hosilaga ega emas, demak, analitik ham emas.

3-misol. $f(z)=z^3$ ning analitikligini tekshiring.

Yechilishi:

$$f'(z)=(z^3)'=3z^2;\quad f'\left(\frac{1}{1}\right)=3\left(\frac{1}{1}\right)^2=-3;\quad f'(2-3i)=3(2-3i)^2,$$

Demak, har qanday chegaralangan G sohada bu funksiya analitik ekan.

4-misol. $f(z)=\dfrac{1}{z+1}$ funksiyaning analitikligini tekshiring.

Yechilishi:

$$f'(z)=\left(\frac{1}{z+i}\right)'=-\frac{1}{(z+i)^2},\qquad f'(i)=-\frac{1}{(1+i)^2}=\frac{1}{4};...$$

Demak, $z=-i$ dan boshqa barcha nuqtalarda funksiya analitik bo'lib, $-i$ maxsus nuqtadir.

5-misol. $f(z)=e^z$ funksiyaning analitikligini tekshiring.

Yechilishi: Bu yerda z ga har qanday qiymat berish mumkin bo'lganligi sababli funksiya tekislikning barcha chekli nuqtalarida analitikdir.

6-misol. $f(z)=zRez$ funksiyaning analitikligini tekshiring.

Yechilishi: $f(z)=\overline{z}Rez=(x-iy)x=x^2-ixy;$

$$u=x^2,\quad v=-xy;\quad \frac{du}{dx}=2x,\qquad \frac{du}{dy}=0,$$

$$\frac{dv}{dx}=-y,\qquad \frac{dv}{dy}=-x;$$

$$\frac{du}{dx}=\frac{dv}{dy}\ \text{dan}\ \ 2x=-x,\quad 3x=0;\qquad x=0,$$

$$\frac{du}{dy}=-\frac{dv}{dx}\ \text{dan}\ \ 0=y,\quad y=0;\quad z_0=x_0+iy_0=0,$$

Demak, funksiya $z=0$ nuqtada manogen bo'lib, boshqa nuqtalarda uning hosilasi yo'q. $z=0$ nuqtadagi hosilasi:

$$f'(0)=\left(\frac{du}{dx}+i\frac{dv}{dx}\right)_{z=0}=(2x-iy)_{z=0}=0,$$

7-misol. $f(z)=|z|Re\overline{z}$ funksiyani tekshiring.

Yechilishi: $f(z)=|z|Re\overline{z}=\sqrt{x^2+y^2}\cdot x;$

$$u=x\sqrt{x^2+y^2},\quad v=0,\qquad \frac{dv}{dx}=\frac{dv}{dy}=0,$$

$$\frac{du}{dx}=\sqrt{x^2+y^2}+\frac{x^2}{\sqrt{x^2+y^2}}=\frac{dv}{dy}=0;$$

Bundan, $x=y=0$ kelib chiqadi. Demak, birgina $z=0$ nuqtada hosila mavjud.

8-misol. $f(z)=\sin 3z-i$ funksiyani tekshiring.

Yechilishi:

$$f'(z)=(\sin 3z-i)'=(\sin 3\text{z})'=3\cos 3\text{z}=\frac{3}{2}(e^{3iz}+e^{-3iz}),$$

Bu yerda z ga har qanday qiymat berish mumkin, demak, berilgan funksiya har qanday chegaralangan sohada analitikdir.

9-misol. Agar funksiya $w=u+iv$ ko'rinishida berilgan bo'lsa, hosilaga ega bo'lishi uchun yuqoridagi (3.6) Koshi-Riman shartlari ya'ni

$$\frac{du}{dx}=\frac{dv}{dy}, \quad \frac{du}{dy}=-\frac{dv}{dx}$$

bajarilishi zarur va yetarli edi. Bu shartlar Dekart koordinatalar sistemasida berilgan. Bu shartlarning qutb koordinatalar sistemasida qanday ifodalanishini tekshiraylik. Ma'lumki, (3.6) da $u=u(x,y),\ v=v(x,y)$ edi. Endi $x=r\cos\varphi,\ \text{y}=\text{r}\sin\varphi,\ \text{r}=\sqrt{x^2+y^2},\ \ \varphi=arctg\frac{x}{y}$ lardan foydalanib, $u=u(r\cos\varphi, r\sin\varphi),\ \ v=v(r\cos\varphi, r\sin\varphi)$ kabi yozish mumkin. Shularga asosan

$$\frac{\delta u}{\delta x}=\frac{\delta u}{\delta r}\cdot\frac{\delta r}{\delta x}+\frac{\delta u}{\delta\varphi}\cdot\frac{\delta\varphi}{\delta x}, \quad \frac{\delta u}{\delta y}=\frac{\delta u}{\delta r}\cdot\frac{\delta r}{\delta y}+\frac{\delta u}{\delta\varphi}\cdot\frac{\delta\varphi}{\delta y},$$

$$\frac{\delta v}{\delta x}=\frac{\delta v}{\delta r}\cdot\frac{\delta r}{\delta x}+\frac{\delta v}{\delta\varphi}\cdot\frac{\delta\varphi}{\delta x}, \quad \frac{\delta v}{\delta y}=\frac{\delta v}{\delta r}\cdot\frac{\delta r}{\delta y}+\frac{\delta v}{\delta\varphi}\cdot\frac{\delta\varphi}{\delta y};$$

bularning birinchisini hisoblab chiqaylik:

$$\frac{\delta r}{\delta x}=(\sqrt{x^2+y^2})'_x=\frac{x}{\sqrt{x^2+y^2}}=\frac{r\cos\varphi}{r}=\cos\varphi,$$

$$\frac{\delta\varphi}{\delta x}=\left(arctg\,\frac{y}{x}\right)'_x=\frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}}=-\frac{y}{x^2+y^2}=-\frac{r\sin\varphi}{r^2}=-\frac{\sin\varphi}{r},$$

u holda

$$\frac{\delta u}{\delta x}=\cos\varphi\frac{\delta u}{\delta r}-\frac{1}{r}\sin\varphi\frac{\delta u}{\delta\varphi};$$

shunga o'xshash

$$\frac{\delta r}{\delta y}=\left(\sqrt{x^2+y^2}\right)'_y=\frac{y}{\sqrt{x^2+y^2}}=\sin\varphi,$$

$$\frac{\delta\varphi}{\delta y}=\left(arctg\,\frac{y}{x}\right)'_y=\frac{x}{x^2+y^2}=\frac{\cos\varphi}{r};$$

u holda

$$\frac{\delta u}{\delta y}=\sin\varphi\frac{\delta u}{\delta r}+\frac{1}{r}\cos\varphi\frac{\delta u}{\delta\varphi};$$

$$\frac{\delta v}{\delta x}=\cos\varphi\frac{\delta v}{\delta r}-\frac{1}{r}\sin\varphi\cdot\frac{\delta v}{\delta\varphi};$$

$$\frac{\delta v}{\delta y}=\sin\frac{\delta v}{\delta r}+\frac{1}{r}\cos\varphi\frac{\delta v}{\delta\varphi}.$$

Topilganlarni Koshi-Riman tenglamalariga qo'yib chiqamiz, u holda

$$\left.\begin{array}{l}\cos\varphi\frac{\delta u}{\delta r}-\frac{1}{r}\sin\varphi\frac{\delta u}{\delta\varphi}=\sin\varphi\frac{\delta v}{\delta r}+\frac{1}{r}\cos\varphi\frac{\delta v}{\delta\varphi},\\ \sin\varphi\frac{\delta u}{\delta r}+\frac{1}{r}\cos\varphi\frac{\delta u}{\delta\varphi}=-\cos\varphi\frac{\delta v}{\delta r}+\frac{1}{r}\sin\frac{\delta v}{\delta\varphi}\end{array}\right\}$$

Ikki noma'lumli ikkita chiziqli tenglamalar sistemasiga ega bo'lamiz. Birinchi tenglamaning ikkala tomonini $\cos\varphi$ ga, ikkinchisini esa $\sin\varphi$ ga ko'paytirib bir-biriga qo'shamiz. U holda $\left(\cos^2\varphi+\sin^2\varphi\right)\frac{\delta u}{\delta r}=\frac{1}{r}\left(\cos^2\varphi+\sin^2\varphi\right)\frac{\delta v}{\delta\varphi}$, ya'ni $\frac{\delta u}{\delta r}=\frac{1}{r}\frac{\delta v}{\delta\varphi}$;

Endi birinchi tenglamani $\sin\varphi$ ga, ikkinchisini $\cos\varphi$ ga ko'paytirib bir-biridan ayiramiz, natijada

$$\frac{1}{r}\frac{\delta u}{\delta\varphi}=-\frac{\delta v}{\delta r}$$

hosil bo'ladi.

Shunday qilib, qutb koordinatalari istemasida Koshi-Riman tenglamasi quydagicha ifodalanar ekan.

$$\frac{\delta u}{\delta r}=\frac{1}{r}\frac{\delta v}{\delta\varphi},\quad \frac{\delta v}{\delta r}=\frac{1}{r}\frac{\delta u}{\delta\varphi}, \tag{3.11}$$

bu yerda $u=u(r,\varphi)$ va $v=v(r,\varphi)$.

Mashqlar

Quyudagi kompleks argumentli funksiyalarning hosilalarini toping.

1. $\sin(2e^z)$; 2. e^{ch_z}; 3. ze^{-z}; 4. $\frac{e^z}{z}$; 5. $\frac{z\cos z}{1+z^2}$; 6. $\frac{e^z+1}{e^z-1}$; 7. $\frac{1}{tgz+ctgz}$; 8. $\left(e^z-e^{-z}\right)^{-2}$.

Quyudagi funksiyalar hosilaga ega yoki ega emasligini tekshiring.

a) ω=e^{z^2}; b) ω=$\bar{z}$ Jmz; v) ω=$|z|$Jmz; g) ω=$|z|\bar{z}$.

Quyida berilgan funksiyalarning analitik yoki analitik emasligini tekshiring.

9. a) $w=z^2\cdot\bar{z}$ b) $w=z^3+2z$

10. a) $w=z\cdot e^z$ b) $w=z^2+5z$

11. a) $w=|z|\operatorname{Re}\bar{z}$ b) $w=z^2+3\bar{z}^2$

12 a) $w = z \cdot |\bar{z}|$ b) $w = z^2 + 5z$

13. a) $w = \bar{z}\,\mathrm{Re}\,z$ b) $w = 2z^3 - 3z$

14 a) $w = \bar{z}\,\mathrm{Im}\,z$ b) $w = \bar{z}^2 - 2z$

15. a) $w = |z| - \mathrm{Im}\,z$ b) $w = z - 3z^2$

16. a) $w = \bar{z} - 2z$ b) $w = 3\bar{z}^2 - 2z$

17. Hosilalarni toping:

$a)\, ze^{-z}, \quad b)\, \frac{z\cos z}{1+z^2}, \quad c)\, \frac{1}{tgz + ctgz}$

18. Berilgan haqiqiy va mavhum qisimlarning z_0 nuqtadagi qiymatiga binoan funksiyani o'zini toping:

$$a)\; u(z,y) = x^2 - y^2 2x, \quad f(1) = 2i - 1.$$

$$b)\; v(x,y) = 2(ch\sin y - xy),\; f(0) = 0.$$

19. Qanday a,b larda funksiya analitik bo'ladi?

$f(z) = \cos x(chy + ashy) + i\sin x(chy + bshy).$

3.4-§. GARMONIK FUNKSIYALAR HAQIDAGI TUSHUNCHALAR

Kompleks argumentli funksiyaning ikki xil shakldа yozilishi ma'lum:

$$\omega = f(z) \;\text{ va }\; \omega = u(x,y) + iv(x,y) \tag{3.12}$$

bu yerda

$$u(x,y) = \mathrm{Re}\, f(z) \;\text{ va }\; v(x,y) = \mathrm{Im}\, f(z) \tag{3.13}$$

Ma'lumki, ushbu

$$\Delta\omega = \frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2} = 0 \tag{3.14}$$

Laplas tenglamasini qanoatlantiradigan har qanday ikki argumentli $\omega = \omega(x,y)$ funksiya garmonik funksiya deyiladi. Garchi $u(x,y)$ va $v(x,y)$ funksiyalar garmonik bo'lsa-da, (3.12) funksiya analitik bo'lmay qolishi mumkin. Agar funksiya analitik bo'lsa, u holda (3.14) ni qanoatlantiruvchi $u(x,y)$ va $v(x,y)$ lar o'zaro qo'shma garmonik funksiyalar deyiladi. Uning uchun Koshi-Riman shartlari bajarilishi kerak. Agar noma'lum analitik funksiyaning haqiqiy yoki mavhum qismi ma'lum bo'lsa, u holda analitik funksiyaning o'zini topish mumkin. Shulardan biz ikkita usulni ko'rsatib o'tish bilan chegaralanamiz.

Birinchi usul. Yuqorida, hosila ta'rifidan, kompleks argumentli funksiyalardan olinadigan xosilalarning barcha xossalarini saqlashini uqtirib o'tgan edik. Shuning uchun ham haqiqiy o'zgaruvchi z ga nisbatan olingan hosilalar jadvali sifatida analitik funksiyalar uchun qo'llash mumkin. Shunday ekan, boshlang'ich funksiya tushunchasi va analitik funksiyadan olingan aniqmas integral tushunchasini ham kiritish mumkin, ya'ni agar $F'(z)=f(z)$ bo'lsa, $F(z)$ analitik funksiya $f(z)$ analitik funksiya uchun boshlang'ich funksiya deyiladi va $F(z)=\int f(z)dz+C$ (C-kompleks o'zgarmas son) kabi yoziladi. Bu to'g'rida batafsil [1] dan ma'lumot olish mumkin. Ko'rgan edikki, agar $z=u(x,y)+iv(x,y)$ golomorf bo'lsa, $\omega_{\bar z}=0$ va

$$\omega'_z=f'(z)=u_x-iu_y=v_y+iv_x=u_x+iv_x=v_y-iu_y,$$

unda

$$f(z)=\int_{z_0}^{z}(u_x-iu_y)dz=\int_{z_0}^{z}(v_x+iv_x)dz$$

Shunday qilib, berilgan $u(x,y)$ yoki $v(x,y)$ ga asosan $f(z)$ funktsiyaning o`zini topish mumkin.

1-misol. Analitik funktsiyaning haqiqiy qismi

$$u=\frac{x}{x^2+y^2} \quad \text{va} \quad f(\pi)=\frac{1}{\pi}$$

berilgan bo'lib funktsiyani o'zini topish talab qilinadi. π nuqtadagi qiymati berilgan bo'lib, o'zini topish talab qilinadi.

Yechilishi: $u_x=\dfrac{y^2-x^2}{\left(x^2+y^2\right)^2};\qquad u_y=-\dfrac{2yx}{\left(x^2+y^2\right)^2}.$

$$f'(z)=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}+i\frac{2xy}{\left(x^2+y^2\right)^2}=\frac{y^2+2ixy-x^2}{\left(x^2+y^2\right)^2}=\frac{(y+ix)^2}{\left(x^2+y^2\right)^2},\ y+ix=i(x-iy)=i\bar z$$

va $x^2+y^2=z\bar z$ ekanligadan foydalansak

$$f'(z)=\frac{-\bar z^2}{z^2\bar z^2}=-\frac{1}{z^2},$$

U holda

$$f(z)=-\int\frac{1}{z^2}dz+c=\frac{1}{z}+c,\quad f(\pi)=\frac{1}{\pi}+c=\frac{1}{\pi}$$

Ya'ni, $c=0$. Demak, $f(z)=\dfrac{1}{z}$ ekan.

2-misol.Analitik funksiyaning mavhum qismi

$$v(x,y)=\operatorname{arctg}\frac{y}{x}\quad (x>0),\quad f(1)=0$$

Berilgan funksiyaning o'zini toping.

Yechilishi:

$$\frac{\partial v}{\partial x}=-\frac{y}{x^2+y^2}, \quad \frac{\partial v}{\partial y}=\frac{x}{x^2+y^2},$$

$$f'(z)=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}=\frac{x-iy}{x^2+y^2}=\frac{\bar{z}}{z\bar{z}}=\frac{1}{z},$$

$$f(z)=\int\frac{1}{z}dz+C=lnz+C.$$

$f(1)=C=0$ dan $f(z)=lnz$

3-misol. Analitik funktsiyaning haqiqiy qismi berilgan:

$$u(x,y)=2sinx\ chy-x, \quad f(0)=0.$$

Funktsiyaning o'zini toping.

Yechilishi:

$$\frac{\partial u}{\partial x}=2\cos xchy-1, \quad \frac{\partial u}{\partial y}=2\sin x\,shy$$

$$f'(z)=2\cos xchy-1-2i\sin xshy=2(\cos xchy-i\sin xshy)-1.$$

Ammo (2.33) formulalarga asosan:

$$\cos iz=\frac{e^{-z}+e^{z}}{2}=chz,$$

$$\sin iz=\frac{e^{-z}-e^{z}}{2i}=-ishz.$$

Shuning uchun

$$\frac{\partial u}{\partial x}=2\cos x\cos iy-1,$$

$$\frac{\partial u}{\partial y}=i2\sin x\sin iy.$$

U holda

$$f'(z)=2(\cos x\cos iy-i\sin x\sin iy)-1=2\cos(x+iy)-1=2\cos z-1$$

$$f(\mathrm{z})=\int(2\cos z-1)dz+C=2\sin z-z+C$$

$$f(0)=C=0.$$

$$f(\mathrm{z})=2\,\mathrm{sinz}-\mathrm{z}.$$

4-misol. Analitik funksiya $f(z)$ ning mavhum qismi berilgan:

$$v(x,y)=-2\sin 2x\cdot\mathrm{sh}\,2y+y, \quad f(0)=0$$

Funksianing o'zini toping .

Yechilishi: $\frac{\partial v}{\partial x}=-4\cos 2x\,\mathrm{sh}\,2y, \quad \frac{\partial v}{\partial y}=-4\sin 2x\,ch2y+1.$

$$f'(z)=-4\sin 2xch2y+1-i4\cos 2xsh2y=-4(\sin 2x\cos i2y+\cos 2x\cdot\sin i2y)+1=$$
$$-4\left[\sin 2(x+iy)\right]+1=-4\sin 2z+1.$$

$$f(z)=-4\int\sin 2z\cdot dz+\int dz+c=\frac{4\cos 2z}{2}+z+c.$$

$$f(0)=2+c=2, \quad c=0.$$

Demak $f(z) = 2\cos 2z + z.$

5-misol. Funktsiyaning haqiqiy qismi berilgan: $u = x^2 + 2x - y^2$. Funktsiyaning garmonikligini tekshiring.

Yechilishi: Undan ikki marta xususiy hosila olib, (3.14) Laplas tenglamasiga qo'yib chiqamiz:

$$\frac{\partial u}{\partial x} = 2x + 2,\ \frac{\partial^2 u}{\partial x^2} = 2;\ \frac{\partial u}{\partial y} = -2y,\ \frac{\partial^2 u}{\partial y^2} = -2;$$

Shunga asosan

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + (-2) = 0$$

Demak, berilgan funktsiya garmonik ekan.

6-misol. Funktsiyani garmonikligini tekshiring:

Yechilishi:

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2},\ \frac{\partial^2 u}{\partial x^2} = \frac{2\left(y^2 - x^2\right)}{\left(x^2 + y^2\right)^2};$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2},\ \frac{\partial^2 u}{\partial y^2} = \frac{2\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2};$$

Shularga asosan:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Funksiya garmonik ekan.

7-misol. Funktsiyani garmonikligini tekshiring: $u = 2e^x \cos y$.

Yechilishi:

$$\frac{\partial u}{\partial x} = 2e^x \cos y,\quad \frac{\partial^2 u}{\partial x^2} = 2e^x \cos y;$$

$$\frac{\partial u}{\partial y} = -2e^x \sin y,\quad \frac{\partial^2 u}{\partial y^2} = -2e^x \cos y;$$

bularga asosan

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Demak, berilgan funktsiya garmonik ekan.

Quyidagi juft u(x, y) va *v*(x, y) garmonik funktsiyalarning uzaro qo'shma bo'lish yoki bo'lmasligini tekshiring.

8-misol. $u = 3\left(x^2 - y^2\right),\ v = 3x^2 y - y^3.$

Yechilishi: Bularning garmonik ekanligiga shubha bulmaganligi uchun Koshi-Riman shartlari bajarilishini tekshirish kifoya:

$$\frac{\partial u}{\partial x} = 6x,\ \frac{\partial u}{\partial y} = -6y,\ \frac{\partial v}{\partial x} = 6xy,\ \frac{\partial v}{\partial y} = 3x^2 - 3y^2,$$

bundan ko'rinadiki, shartlar bajarilmaydi. Demak, ular qo'shma emas.

9-misol. $u = e^x \cos y + 1,\ v = 1 + e^x \sin y$.

Yechilishi: Bu yerda ham o'sha shartlarni tekshirib ko'ramiz:

$$\frac{\partial u}{\partial x} = e^x \cos y,\ \frac{\partial u}{\partial y} = -e^x \sin y;\ \frac{\partial v}{\partial x} = e^x \sin y;\ \frac{\partial v}{\partial y} = e^x \cos y,$$

E'tibor qilsak, Koshi-Riman shartlarining bajarilishini ko'ramiz. Demak, berilgan funktsiyalar qo'shma garmonik ekan.

10-misol. Ushbu ko'rinishga ega bo'lgan garmonik funktsiya mavjudmi:

$$u = \varphi\left(x^2 + y^2\right)?$$

Yechilishi: Funktsiyadan ikkinchi tartibli xususiy hosilalar olib ularni (3.14) Laplas tenglamasiga qo'yib chiqamiz, u holda

$$x^2 + y^2 = t$$

$$u = \varphi(t);\quad \frac{\partial u}{\partial x} = \varphi'(t)\frac{\partial t}{\partial x} = \varphi'(t)\left(x^2 + y^2\right)'_x = 2x\varphi'(t);\quad \frac{\partial^2 u}{\partial x^2} = 2\varphi'(t) + 4x^2\varphi''(t);$$

$$\frac{\partial u}{\partial y} = 2y\varphi'(t),\quad \frac{\partial^2 u}{\partial y^2} = 2\varphi'(t) + 4y^2\varphi''(t);$$

bularni (3.14) ga qo'yib soddalashtiramiz, natijada

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\varphi'(t) + 4\left(x^2 + y^2\right)\varphi''(t) = 0;\quad t\varphi''(t) + \varphi'(t) = 0;$$

quyidagicha belgilash kiritamiz:

$$\varphi'(t) = \psi(t)$$

Natijada

$$\frac{\psi'(t)}{\psi(t)} = -\frac{1}{t}\ \text{ yoki }\ \frac{d\psi}{\psi} = -\frac{dt}{t};$$

$$ln\psi(t) = -\ln t + lnC_1;\ \psi(t) = \frac{C_1}{t};\ \ \varphi'(t) = \frac{C_1}{t};\ \ \varphi(t) = c_1 \ln t + c_2$$

bu yerda $t = \varphi\left(x^2 + y^2\right)$ bo'lgani sababli tekshirilayotgan garmonik funksiya ko'rinishga ega bo'lar ekan. C_1 va C_2 lar ixtiyoriy o'zgarmas haqiqiy sonlar bo'lgani uchun shu ko'rinishga ega bo'lgan garmonik funktsiyalar cheksiz ko'p ekan.

Ikkinchi usul. Noma'lum analitik funksiyaning haqiqiy yoki mavhum qismi berilgan bo'lsa, uning o'zini ikkinchi usul bilan topish ham mumkin. Uning uchun Koshi-Riman (Eyler—Dalamber) shartlaridan foydalanishga to'g'ri keladi. Buni misollar orqali ko'rsatamiz.

11-misol. Funksiyaning haqiqiy qismi va qo'shimcha shart berilgan:

$$u(x, y) = 2e^x \cos y,\quad f(0) = 2$$

$\omega = f(z)$ analitik funktsiyani ikkinchi usul bilan toping.

Yechilishi: Koshi—Riman

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},\quad \text{(A)}$$

shartlaridan foydalanib, *u* ma'lum bo'lgani sababli v ni topish mumkin :

$$\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}=\left(2e^x\cos y\right)_x^{'}=2e^x\text{cosy},$$

bundan *u* buyicha integral olsak

$$v=2e^x\int\cos y dy+\varphi(x)=2e^x\sin y+\varphi(x).$$

Tenglikning ikki tomonidan *x* bo'yicha xususiy hosila olib, (A) shartlarning ikkinchisidan foydalanamiz, u holda

$$\frac{\partial v}{\partial x}=2e^x\sin y+\varphi^{'}(x)=-\frac{\partial u}{\partial y}=-\left(2e^x\cos y\right)_y=2e^x\sin y,$$

mos hadlar o'zaro yo'qotilib,

$$\varphi^{'}(x)=0,\quad \varphi(x)=C$$

kelib chiqadi. Demak, $v=2e^x\sin y+C.$ Bularga asosan

$$w=f(z)=u+iv=2e^x\cos y+i2e^x\sin y+Ci=2e^x(\cos y+i\sin y)+iC=2e^x*e^{iy}+iC=$$
$$=2e^{x+iy}+iC=2e^z+iC.$$

Endi boshlangich shartdan foydalanib C ni topamiz, ya'ni $f(0)=2=2e^0+iC,$ bundan $C=0$. Shunday qilib, $w=f(z)=2e^z$

12-misol. Mavhum qismi $v=ln\left(x^2+y^2\right)+x-2y$ bo'lgan analitik funksiyani ikkinchi usul bilan toping.

Yechilishi: Oldingi misolga o'xshash

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=\frac{2y}{x^2+y^2}-2,\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=-\left[\frac{2x}{x^2+y^2}+1\right]=\frac{-2x}{x^2+y^2}-1;$$

bularning birinchisidan *x* bo'yicha integral olib *u(x, y)* ni topaylik:

$$u(x,y)=2y\int\frac{dx}{x^2+y^2}-2x+\varphi(y)=2arctg\frac{x}{y}-2x+\varphi(y),$$

buning ikki tomonidan u buyicha hosila olib, yuqoridagi qiymatiga tenglashtiramiz, u holda

$$\frac{\partial u}{\partial y}=2\frac{\left(\frac{x}{y}\right)^{'}}{1+\left(\frac{x}{y}\right)^2}+\varphi^{'}(y)=2\frac{-\frac{x}{y^2}}{1+\frac{x^2}{y^2}}+\varphi^{'}(y)=\frac{-2x}{x^2+y^2}+\varphi^{'}(y)=\frac{-2x}{x^2+y^2}-1$$

ikki tomondagi mos hadlar o'zaro yuqoladi va $\varphi^{'}(y)=-1,$ bundan $\varphi(y)=-y+C,$

hosil buladi. Bularga asosan $u=2arctg\frac{x}{y}-2x-y+C.$

Demak,

$$w = f(z) = u + iv = 2arctg\frac{x}{y} - 2x - y + C + i\left[in\left(x^2 + y^2\right) + x - 2y\right].$$

Buni ixchamlashtirish maqsadida quyidagi formuladan foydalanamiz:

$$Arctgz = -\frac{i}{2}Ln\frac{1+iz}{1-iz}, \quad z = x + iy.$$

U holda $2arctg\frac{x}{y} = -iLn\frac{1+i\frac{x}{y}}{1-i\frac{x}{y}} = -iLn\frac{y+ix}{y-ix} = iLn\frac{y-ix}{y+ix} = iLn\frac{-(x+iy)}{x-iy} = iLn\frac{-z}{\bar{z}};$

shunga muvofiq

$$2arctg\frac{x}{y} + iln\left(x^2 + y^2\right) = i\left[Ln\left(-\frac{z}{\bar{z}}\right) + Lnz\bar{z}\right] = iLn\frac{-z \cdot z\bar{z}}{\bar{z}} = iL\text{n}\left(-z^2\right) =$$

$$= iLn(iz)^2 = 2iLniz = 2i\left[Lnz + Lni\right] = 2iLnz + 2iLni,$$

so’ngi hadni C ichiga kiritib yuboramiz, natijada

$$f(z) = 2iLnz - (2-i)z + C$$

13-misol. Funksiyaning haqiqiy qismi berilgan:

$$u = x^2 - y^2 + xy$$

Analitik funksiyani toping.

Yechilishi: Bunday masalalarni yechishda Koshi-Riman shartining istalgan formulasidan foydalanish mumkin:

$$\frac{dv}{dy} = \frac{du}{dx} = 2x + y; \quad \frac{dv}{dx} = -\frac{du}{dy} = -(-2y + x) = 2y - x;$$

bularning birinchisidan y bo'yicha integral olamiz:

$$v = \int (2x + y)dy + \varphi(x) = 2xy + \frac{y^2}{2} + \varphi(x);$$

endi tenglikdan x bo’yich hosila olib, yuqoridagi ikkinchi tenglikdan foydalansak:

$$\frac{dv}{dx} = 2y + \varphi'(x) = 2y - x \Rightarrow \varphi'(x) = -x \Rightarrow \varphi(x) = -\frac{x^2}{2} + C$$

Demak,

$$v = 2xy + \frac{y^2}{2} - \frac{x^2}{2} + C; \quad f(z) = u + iv = \left(x^2 - y^2 + xy\right) + i\left(2xy + \frac{y^2 - x^2}{2} + C\right) =$$

$$= \frac{1}{2}\left[(2-i)x^2 - (2-i)y^2 + 2i(2-i)xy\right] + iC = \frac{1}{2}(2-i)z^2 + iC.$$

14-misol. Berilgan $f(z)$ funksiyaning analitik bo’lishi uchun a, b, c constantalar qanday bo’lishi kerak?

$$f(z) = x + ay + i(bx + cy) = u + iv.$$

Yechilishi: (3.6) Koshi-Riman shartidan foydalanamiz.

$$\frac{\partial u}{\partial x}=1,\ \frac{\partial v}{\partial y}=c,\ \frac{\partial u}{\partial y}=a,\ \frac{\partial v}{\partial x}=b,$$

Ya'ni $c=1,\quad a=-b$ bo'lishi kerak.

Demak, koeffisentlarning shunday qiymatlaridagina funksiya analitik bo'ladi .

$$f(z)=x+ay+i(-ax+y)=z-iaz=z(1-ia).$$

15-misol. Qutb kordinatalar sestemasida funksiyaning haqiqiy qismi berilgan:

$$u=r\varphi\cos\varphi+r\ln r\sin\varphi.$$

Analitik funksiyani toping.

Yechilishi: Buning uchun Koshi-Rimanning qutb kordinatalari orqali ifoda qilingan ushbu shartlaridan foydalanamiz:

$$\frac{du}{dr}=\frac{1}{r}\frac{dv}{d\varphi},\quad \frac{dv}{dr}=-\frac{1}{r}\frac{du}{d\varphi}.$$

Berilganiga ko'ra:

$$\frac{dv}{dr}=-\frac{1}{r}\frac{du}{d\varphi}=-\frac{1}{r}(r\cos\varphi-\varphi r\sin\varphi+r\ln r\cos\varphi)=\varphi\sin\varphi-(1+\ln r)\cos\varphi;$$

$$\frac{dv}{d\varphi}=r\frac{du}{dr}=r(\varphi\cos\varphi+\ln r\sin\varphi+\sin\varphi)=r\left[\varphi\cos\varphi+(1+\ln r)\sin\varphi\right];$$

bularning birinchisidan r bo'yicha integral olamiz:

$$v=r\varphi\sin\varphi-\cos\varphi\int(1+\ln r)\,dr+F(\varphi)=r\varphi\sin\varphi-r\ln r\cos\varphi+F(\varphi).$$

Tenglikning ikki tomonidan φ bo'yicha hosila olsak:

$$\frac{dv}{d\varphi}=r\sin\varphi+r\varphi\cos\varphi+r\ln r\sin\varphi+F'(\varphi)=r\left[\varphi\cos\varphi+(r+\ln r)\sin\varphi\right],$$

Bundan

$$F'(\varphi)=0 \text{ , yoki } F(\varphi)=C.$$

Demak,

$$v=r\varphi\sin\varphi-r\ln r\cos\varphi+C.$$

$$\omega=f(z)=u+iv=(r\varphi\cos\varphi+r\ln r\sin\varphi)+i(r\varphi\sin\varphi-r\ln r\cos\varphi+C)=$$

$$=r\varphi(\cos\varphi+i\sin\varphi)-ir\ln r(\cos\varphi+i\sin\varphi)+iC=r(\varphi-i\ln r)e^{i\varphi}+iC,$$

chunki Eyler formulasiga muvofiq:

$$\cos\varphi+i\sin\varphi=e^{i\varphi}$$

16-misol. Agar u va v funksiyalar qo'shma garmonik bo'lsa quyidagilarning ham garmonik bo'lishini isbot qiling:

$$U=au-bv,\quad V=bu+av$$

bu yerda $a,\ b$ lar o'zgarmas sonlar.

Yechilishi: Berilgan u,v funksiyalar garmonik bo'lganligi sababli ular Laplas tenglamasini qanoatlantiradi:

$$\Delta u=\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=0,\quad \Delta v=\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}=0.$$

Bularga asosan

$$\Delta U=\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}=\left(a\frac{d^2u}{dx^2}-b\frac{d^2v}{dx^2}\right)+\left(a\frac{d^2u}{dy^2}-b\frac{d^2v}{dy^2}\right)=a\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}\right)-b\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}\right)=$$

$$=a\cdot 0-b\cdot 0=0$$

Huddi shuningdek

$$\Delta V=\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}=\left(b\frac{d^2u}{dx^2}+a\frac{d^2v}{dx^2}\right)+\left(b\frac{d^2u}{dy^2}+a\frac{d^2v}{dy^2}\right)=b\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}\right)+a\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}\right)=$$

$$=b\cdot 0+a\cdot 0=0$$

Demak, u, v lar Laplas tenglamasini qanoatlantirganligi uchun ular ham garmonik funksiyalar ekan.

17-misol. Agar u va v funksiyalar o'zaro qo'shma garmonik bo'lsa, quyidagilar ham garmonik bo'lishini isbot qiling:

$$U=e^u\cos v,\quad V=e^u\sin v$$

Yechilishi: Buning uchun bular Laplas tenglamasini qanoatlantirishi lozim. Birinchi funksiyani garmonik ekanligini isbotlash bilan chegaralanamiz; ikkinchisi ham xuddi shunday isbot qilinadi. U dan ikki marta x bo'yicha, so'ngra ikki marta y bo'yicha hosila olib, Laplas teglamasiga qo'yamiz, u holda

$$\frac{dU}{dx}=\left(e^u\cos v\right)'_x=\left(e^u\right)'_x\cos v+e^u\left(\cos v\right)'_x=e^u\cos v\frac{du}{dx}-e^u\sin v\frac{dv}{dx}=e^u\left(\frac{du}{dx}\cos v-\frac{dv}{dx}sinx\right);$$

bundan yana bir marta x buyicha hosila olib ixchamlashtirilsa:

$$\frac{d^2U}{dx^2}=e^u\left[\left(\left(\frac{du}{dx}\right)^2-\left(\frac{dv}{dx}\right)^2\right)\cos v-2\frac{du}{dx}\frac{dv}{dx}\sin v+\frac{d^2u}{dx^2}\cos v-\frac{d^2v}{dx^2}\sin v\right]$$

kelib chiqadi. Huddi mana shu usulda y bo'yicha ikkinchi tartibli xususiy hosila olinsa x o'rnida y paydo bo'ladi, yani

$$\frac{d^2U}{dy^2}=e^u\left[\left(\left(\frac{du}{dy}\right)^2-\left(\frac{dv}{dy}\right)^2\right)\cos v-2\frac{du}{dy}\frac{dv}{dy}\sin v+\frac{d^2u}{dy^2}\cos v-\frac{d^2v}{dy^2}\sin v\right]$$

Bularni o'zaro qo'shib nolga tengligini ko'rsatamiz

$$\Delta U=\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}-e^u\left[\left(\frac{du}{dy}\right)^2-\left(\frac{dv}{dx}\right)^2\right]\cos v+e^u\left[\left(\frac{du}{dy}\right)^2-\left(\frac{dv}{dy}\right)^2\right]\sin v+$$

$$+e^u\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}\right)\cos v-e^u\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}\right)\sin v=0$$

chunki so'ngi ikki yig'indining nolga tengligi u, v larning garmonikligidan ma'lum, oldingi ikkita ayirmaning nolga tengligini quyidagicha isbot qilamiz:

$$\left(\frac{du}{dx}\right)^2-\left(\frac{dv}{dx}\right)^2=0 \text{ bundan } \frac{du}{dx}=\frac{dv}{dx},\quad \frac{d^2u}{dx^2}=\frac{d^2v}{dx^2}$$

$$\left(\frac{du}{dy}\right)^2-\left(\frac{dv}{dy}\right)^2=0 \text{ bundan } \frac{du}{dy}=\frac{dv}{dy},\quad \frac{d^2u}{dy^2}=\frac{d^2v}{dy^2}$$

so'ngi ikki tenglikni o'zaro qo'shsak, yig'indi Laplas tenglamasiga asosan nolga teng bo'ladi. Demak, berilgan U funksiya garmonik ekan.

Mashqlar

Quyidagi funksiyalarning garmonik bo'lishi yoki bo'lmasligini Laplas tenglamasi orqali tekshiring, agar garmonik bo'lsa ularni toping. 20. $u=\varphi(x)$; 21. $u=\varphi(ax+by)$, a, b-o'zgarmas sonlar; 22. $u=\varphi(xy)$; 23. $u=\varphi\left(\frac{x}{y}\right)$; 24. $u=\varphi\left(x^2+y^2\right)$; 25. $u=\varphi\left(\frac{x^2+y^2}{x}\right)$; 26. $u=\varphi\left(x+\sqrt{x^2+y^2}\right)$; 27. $u=\varphi\left(x^2+y\right)$; 28. $u=\varphi\left(x^2-y^2\right)$.

Haqiqiy yoki mavhum qismi berilgan analitik funksiya topilsin: 29. $u=x^2-y^2+2y,\ f(i)=2i-1$; 30. $v=2(chx\sin y-xy),\ f(0)=0$; 31. $v=-2\sin 2xsh2y+y,\ f(0)=2$; 32. $v=2\cos xchy-x^2+y^2,\ f(0)=2$; 33. $u=x^2-y^2+5x+y-\frac{y}{x^2+y^2}$; 34. $v=3+x^2-y^2-\frac{y}{2\left(x^2+y^2\right)}$; 35. $u=e^1\left(x\cos y-y\sin y\right)+2\sin xshy+x^3-3xy^2+y$;

Agar u va v lar o'zaro qo'shma garmo'nik funksiyalar bo'lsa, quydagilarning ham garmonik bo'lishini isbot qiling:

a) $U=e^{a^2-v^2}\cos 2uv$ va $V=e^{u^2v^2}\sin 2uv$;

b) $U=e^{uv}\cos\frac{u^2-v^2}{2}$ va $V=e^{uv}\sin\frac{v^2-u^2}{2}$.

3.5-§. YASSI VEKTOR MAYDONLARNI HISOBLASHDA ANALITIK FUNKSIYALARNING QO`LLANILISHI

Fizika va texnikaning turli masalalarida, u yoki bu fizik xususiyatning vektor maydonini hisoblash imkoniyati juda muhimdir. Masalan, elektrostatikada *E* elektr maydon kuchlanganligining vektor maydoni, gidrodinamikada - suyuqlik zarralari A tezligining maydoni va boshqalar haqida gapirishimiz mumkin. Analitik funktsiyalar yordamida bunday maydonlarni tekis, potentsial va statsionar bo'lganda o'rganish nisbatan osonlashadi. Agar (x, y, z) fazoning har bir nuqtasida A vektori *Oxy* tekisligiga parallel va uchinchi koordinata z ga bog'liq bo'lmasa, A maydoni yassi deb ataladi. Agar A vektor vaqtga bog`liq bo`lmay, faqat nuqtaning koordinatalariga bog`liq bo`lsa, A maydon statsionar (yoki barqaror) deb nomlanadi. Nihoyat, A = A (x, y) statsionar yassi maydon potentsialga ega deb ataladi, agar A vektorning gradienti bo'lgan $\varphi = \varphi(x, y)$ funktsiya mavjud bo'lsa, ya'ni

$$A = grad\varphi \tag{3.15}$$

E elektrostatik maydon potensialga ega bo`lib, elektrostatikada (3.15) o`rniga

$$E = -grad\varphi \tag{3.16}$$

ko`rinishida yozish qabul qilingan. Agar suyuqlik ideal, siqilmaydigan va uning harakati uyurmasiz bo`lsa, harakatdagi suyuqlik zarralari tezligining maydoni potensialga ega bo`ladi.

Agar mana shu shartlarning hammasi bajarilsa, u holda E yoki A maydonlarni z kompleks sonlar tekisligida z kompleks o`zgaruvchining funksiyasi sifatida qarash mumkin. Masalan, tezlik vektori o`rniga

$$A = A(z) = A_x(x, y) + iA_y(x, y) = |A|e^{i\alpha} \tag{3.17}$$

kompleks tezlik haqida gapirish mumkin. Bunda $A_x(x, y)$ va $A_y(x, y)$ haqiqiy funksiyalar mos ravishda *Ox* va *Oy* koordinata o`qlaridagi A tezlikning proyeksiyalarini ifodalaydi; $|A|$ kattalik – A vektorning moduli, $\alpha$ – *Ox* o`qi bilan A vektor orasidagi burchak. Xuddi shunday E kuchlanganlik vektori o`rniga

$$E = E(z) = E_x(x, y) + iE_y(x, y) = |E|e^{i\beta} \tag{3.18}$$

kompleks o`zgaruvchili kuchlanganlik sifatida qarash mumkin.

Gidrodinamikada quyidagi isbotlangan: Agar ideal va siqilmaydigan suyuqlik uyurmasiz harakatda bo`lsa, u holda $\varphi = \varphi(x, y)$ tezlik potensiali x va y erkli o`zgaruvchilarning garmonik funksiyasi hisoblanadi:

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0. \tag{3.19}$$

Xuddi shunday $f = f(x, y)$ elektrostatik potensial uchun elektr zaryadidan erkli bo`lgan sohada quyidagi tenglik o`rinli

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \tag{3.20}$$

Ma’lumki, $\varphi(x, y)$ garmonik funksiyani bilgan holda (o`zgarmas qo`shiluvchi aniqlikda) hech bo`lmaganda lokal ma’noda $\psi(x, y)$ qo`shma garmonik funksiyani qurish mumkin. Gidrodinamikada $\varphi(x, y)$ funksiya tok funksiyasi deb ataladi va uning fizik ma'nosi shundaki, $\psi(x_2, y_2) - \psi(x_1, y_1)$ farq silindrsimon sirt orqali 1m balandlikdan 1 sekundda oqayotgan suyuqlik hajmiga teng, ya'ni $\psi(x_2, y_2) - \psi(x_1, y_1)$ farq oqim yoki boshi (x_1, y_1) nuqtada va oxiri (x_2, y_2) nuqtada bo`lgan yoy orqali suyuqlik sarfi hisoblanadi. Bu ikki funksiyani birlashtirsak,

$$\varphi(x, y) + \mathrm{i}\psi(x, y) = \Phi(\mathrm{z}), \qquad (3.21)$$

suyuqlik oqimining kompleks potensiali deb nomlangan analitik funksiyani hosil qilamiz.

Kompleks potensial va kompleks tezlik orasidagi bog`lanishni ko`rib chiqamiz. (3.15) va (3.17) formulalardan

$$A = A(z) = A_x(x, y) + iA_y(x, y) = \frac{\partial \varphi}{\partial x} + i\frac{\partial \varphi}{\partial y}$$

ega bo`lamiz. Koshi-Riman shartiga ko`ra $\frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}$ va

$$A(z) = \frac{\partial \varphi}{\partial y} - i\frac{\partial \psi}{\partial x} = \overline{\Phi^{'}(z)} \qquad (3.22)$$

kelib chiqadi.

Bu yerda yassi gidrodinamikaning asosiy formulalaridan biri olingan: kompleks tezlik tezliklarning kompleks potensialidan olindan kompleks qo`shma hosila kattaligi hisoblanadi. Tezliklar potensialini bilgan holda (3.22) formuladan kompleks tezlikni oson topish mumkin.

Xuddi shunday, elektrostatik maydon bo`lgan hol uchun $f(x, y)$ garmonik funksiya bo`yicha $g(\mathrm{x}, \mathrm{y})$ qo`shma garmonik funkisiyani qurish mumkin, uning fizik ma'nosi shundan iboratki, $g(\mathrm{x}_2, \mathrm{y}_2) - \mathrm{g}(\mathrm{x}_1, y_1)$ farq boshi $(x_1, y_2)$ nuqtada va oxiri $(x_2, y_2)$ nuqtada bo`lgan yoyni kesuvchi elektrostatik maydonning kuch chiziqlari sonini beradi. Bu ikki funksiyani birlashtirib,

$$\mathrm{f}(x, y) + \mathrm{ig}(x, y) = F(\mathrm{z}), \qquad (3.23)$$

elektrostatik maydonning kompleks potensialini topamiz. $F(\mathrm{z})$ kompleks potensial bilan kompleks kuchlanganlik orasidagi bog`lanish

$$F(\mathrm{z}) = -\overline{F^{'}(z)}, \qquad (3.24)$$

formula orqali ifodalanadi, bu formula (3.22) ga o`xshash.

Demak, ma'lum shartlar bilan yassi vektor maydonning kompleks potensiali z kompleks o`zgaruvchining analitik funksiyasi hisoblanadi. Bu kompleks potensialni bilgan holda vektor maydonga ta'luqli barcha kattaliklarni aniqlash imkonini beradi. Biroq shuni ta'kidlash lozimki, har qanday berilgan analitik funksiya biror fizik real maydonning kompleks potensiali ekanligini anglatmaydi. Bu yerda cheklanishlar mavjud bo`lib, buni esa bu kitobda biz qarab chiqmaymiz.

1-misol. Juda kichik diametrli cheksiz uzun togri chiziqli otkazgich Oxy tekislikni koordinatalar boshida togri burchak ostida kesib o’tsin va q kulon zaryadni bir metrga kochirsin, bunda bu otkazgich 1 m radiusli yerga tutashtirilgan silindr bilan o'ralgan bo'lib, silindr bilan otkazgich orasidagi bo'shliq dielektrik sindruvchanligi ε bolgan dielektrik bilan to`ldirilgan. Elektrostatikadan ma’lumki, elektrostatik maydondan hosil qilingan potensial $f(x,y)=-\frac{q}{2\pi\varepsilon}\ln\frac{1}{\sqrt{x^2+y^2}}$ ko`rinishga ega bo`ladi. $f(x,y)=-\frac{q}{2\pi\varepsilon}\ln\frac{1}{r}$ uchun qo`shma garmonik funksiya $-\frac{q}{2\pi\varepsilon}=\mathrm{Arg}z+C$ bo`ladi, bundan $C=0$ ekanligini hisobga olib, berilgan silindr kondensatorning potensiali

$$F(z)=\frac{q}{2\pi\varepsilon}Log\frac{1}{z} \tag{3.25}$$

ekanligini topamiz.

Bu misolda (3.25) kompleks potensial z kompleks o`zgaruvchining ko`pma’noli analitik funksiyasi hisoblanadi. $0<|z|<1.$ ikki bog`lamli sohada (halqada) qo`shma garmonik funksiyani izlaganimizni anglatadi.

Ko`rilgan juda sodda misolda kompleks potensial va analitik funksiyalarning qo`llanilishi hech qanday afzallikni bermaydi. Biroq (3.25) formuladan komform akslantirish yordamida turli tipdagi kondensatorlar haqida elektostatikaning yetarlicha murakkab bo`lgan masalalarining qator yechimlarini olish mumkin.

Faraz qilaylik, $f(u,v)$ funksiya $w=u+iv.$ kompleks sonlar tekisligining biror G sohasida berilgan bo`lsin va $w=w(z)$ funksiya $z=x+iy$ tekislikning biror D sohasini G sohaga komform akslantirishini anglatsin, hamda bunda f(u,v) funksiya $\varphi(x,y)=f(u(x,y),v(x,y)).$ funksiyaga o`tsin. Hisoblashlar shuni ko`rsatadiki, Bu funksiyalarning Laplas operatori

$$\Delta f(u,v)=|\omega'(z)|^2\Delta\varphi(x,y) \tag{3.26}$$

tenglik orqali bog`langan. Bundan agar $f(u,v)$ - garmonik funksiya $(\Delta f(u,v)=0)$ bo`lsa, u holda hosil qilingan $\varphi(x,y)$ garmonik bo`ladi, ya’ni $\Delta f=0$ Laplas tenglamasi komform almashtirishlarga nisbatan invariant bo`ladi. G sohada zaryadlardan holi bo`lgan elektrostatik maydon D sohadagi xuddi shunday maydonga o`tishi kelib chiqadi. Shuni eslatib o`tamizki, komform almashtirishlarga nisbatan invariantlilik faqat Laplas tenglamasi uchun o`rinlidir. Masalan, $\Delta f+$

$cf = 0$ eng sodda Gelmgols tenglamasi (bunda $c \neq 0$ - o`zgarmas koeffitsiyent) o`zgaruvchi koeffitsiyentli

$$\Delta\varphi + \frac{c}{\left|w'(z)\right|^2}\varphi = 0.$$

tenglamaga o`tadi.

2-misol. Oxy teksligining koordinatalar boshidan farqli birlik aylananing ichki ixtiyoriy $z_0 = x_0 \ (0 < x_0 < 1),$ nuqtasida to`g`ri burchak ostida kesib o`tuvchi silindr bilan o`ralgan o`tkazgich bo`lsin. Bunday kondensatorning kompleks potensialini toping.

Laplas tenglamasining invariantliligidan foydalanamiz. $w = e^{i\alpha}\dfrac{z - x_0}{1 - x_0 z}$ -x_0 nuqtani koordinatalar boshiga o`tkazuvchi birlik aylananing komform akslantirishi bo`lsin. U holda $\alpha = Arg\, w'(x_0) = 0,$ deb olib,

$$F(z) = \frac{q}{2\pi\varepsilon} Log \frac{1 - x_0 z}{z - x_0}.$$

qidirilgan yechimni hosil qilamiz.

Endi biroz boshqacha yondoshamiz.

3-misol. Yerdan h balandlikda bo`lgan zaryadlangan o`tkazgich maydonining kompleks potensialini toping (3.3-rasm).

Tasvirlar metodidan foydalanamiz. Agar yer sirtida, ya’ni Ox o`qida nolga aylanuvchi, $(0, \mathrm{h})$ nuqtadan o`tuvchi o`tkazgich maydoning potensialini qursak, masala yechilgan hisoblanadi. Bunga esa $(0, -\mathrm{h})$ simmetrik nuqtada –q zaryad tasvirini qo`yish bilan erishish mumkin. Zaryadlarning yig`indi potensiali

$$\varphi F(\mathrm{z}) = \frac{q}{2\pi\varepsilon} Log \frac{1}{z - ih} - \frac{q}{2\pi\varepsilon} Log \frac{1}{z + ih} = \frac{q}{2\pi\varepsilon} Log \frac{z + ih}{z - ih}. \qquad (3.27)$$

ko`rinishga ega bo`ladi. Uning haqiqiy qismi

$$\frac{q}{2\pi\varepsilon} \ln\left|\frac{z + ih}{z - ih}\right|$$

Ox o`qida nolga teng bo`ladi. (3.27) formula masala yechimini beradi.

4-misol. Ikkita o'zaro parallel va U_0 potensiallar farqiga ega bolgan o'tkazuvchan silindrlar z=x+iy tekislikni togri burchak ostida kesib otsin. Ular orasida hosil bo`lgan maydonning kompleks potensialini toping.

Biz (3.27) formulani qo`llashimiz mumkin. O`tkazgichlar sirti, yani 3.4-rasmdagi r radiusli aylanalar (3.27) maydon uchun ekvipotensial hisoblanmaydi. Zaryadlarni $(0, \mathrm{h})$ va $(0, -\mathrm{h})$ nuqtalarga emas, r radiusli doiralarga nisbatan simmetrik bo`lgan $(0, \eta)$ va $(0, -\eta)$

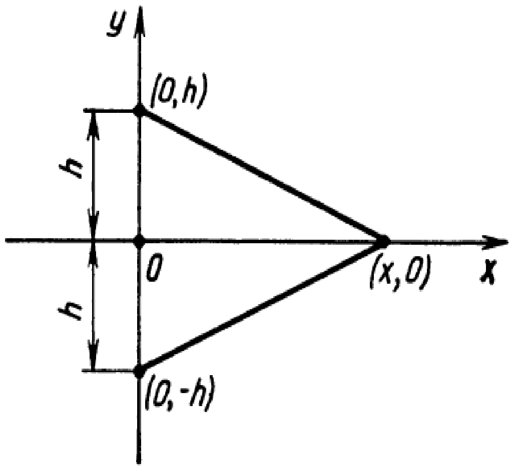
nuqtalarga qo`yamiz. η sonini simmetriya shartidan topish mumkin, ya’ni

$$(h+\eta)(h-\eta)=r^2,$$

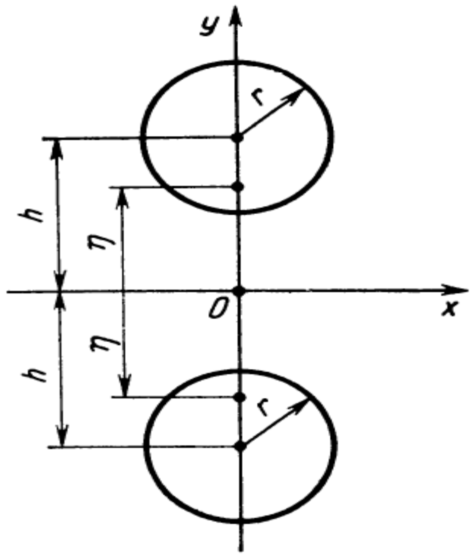
bundan $\eta=\sqrt{h^2-r^2}$, va qidirilayotgan potensial

$$F(z)=k\,Log\frac{z+i\sqrt{h^2-r^2}}{z-i\sqrt{h^2-r^2}}, \qquad (3.28)$$

ko`rinishga ega bo`ladi, bunda k – hozircha noma’lum haqiqiy o`zgarmas.



3.3-rasm.



3.4-rasm.

Silindr sirtida simmetrik joylashgan (0,h-r) va $(0,-(h-r))$ nuqtalarni (3.28) formulaga qo`yib, k uchun tenglama hosil qilamiz:

$$k\ln\left|\frac{i(h-r)+i\sqrt{h^2-r^2}}{-i(h-r)-i\sqrt{h^2-r^2}}\right|-k\ln\left|\frac{-i(h-r)+i\sqrt{h^2-r^2}}{-i(h-r)-i\sqrt{h^2-r^2}}\right|=2k\ln\frac{h+\sqrt{h^2-r^2}}{r}=U_0.$$

bundan

$$k=\frac{U_0}{2\ln\dfrac{h+\sqrt{h^2-r^2}}{r}}.$$

ega bo`lamiz.

Agar $\frac{r}{h}\ll 1$, u holda $k\approx\dfrac{U_0}{2\ln\dfrac{2h}{r}}$ va

$$F(z)\approx\frac{U_0}{2\ln\dfrac{2h}{r}}Log\frac{z+i\sqrt{h^2-r^2}}{z-i\sqrt{h^2-r^2}}.$$

JAVOBLAR

1. $2e^{z}\cos\left(2e^{z}\right)$**;** **2.** $2shze^{chz}$**;** **3.** $(1-z)e^{-z}$ **4.** $\left(\frac{1}{z}-\frac{1}{z^{2}}\right)e^{z},\ z\neq 0$**;** **5.** $\frac{\left(1+z^{2}\right)\left(\cos z-z\sin z\right)-2z^{2}\cos z}{\left(1+z^{2}\right)^{2}},\ z\neq\pm i$**;** **6.** $\frac{2e^{z}}{\left(e^{z}-1\right)^{2}}$**;** **7.** $\cos 2z$**;** **8.** $-2\frac{e^{z}+e^{-z}}{\left(e^{z}+e^{-z}\right)^{3}}$**;** **20.** $u=C_{1}x+C_{2}$**;** **21.** $u=C_{1}\left(ax+by\right)+C_{2}$**;** **22.** $u=C_{1}y+C_{2}$ **23.** $C_{1}arctg\frac{y}{x}+C_{2}$**;** **24.** $u=C_{1}\ln\left(x^{2}+y^{2}\right)+C_{2}$**;** **25.** $u=\frac{C_{1}x}{x^{2}+y^{2}}+C_{2}$**;** **26.** $u=C_{1}\sqrt{x+\sqrt{x^{2}+y^{2}}}+C_{2}$**;** **27.** Yechimi yo’q; **28.** $u=C_{1}\left(x^{2}-y^{2}\right)+C_{2}$**;** **29.** $f\left(z\right)=z^{2}+2z$**;** **30.** $f\left(z\right)=2shz-z^{2}$**;** **31.** $f\left(z\right)=2\cos 2z+z$**;** **32.** $f\left(z\right)=2i\left(\cos z-1\right)-iz^{2}+2$**;** **33.** $f\left(z\right)=z^{2}+\left(5-i\right)z-\frac{1}{z}+iC$**;** **34.** $f\left(z\right)=\frac{1}{2z}+iz^{2}+3i+C$**;** **35.** $f\left(z\right)=ze^{z}+2i\cos z+z^{3}-iz+iC$**.**

IV BOB

KOMPLEKS SOHADA INTEGRALLAR

Bu bobda $\omega = f(z)$ kompleks argumentli funksiyalardan tekislikdagi biror Γ chiziq buylab olingan egri chiziqli integral bilan tanishamiz. Matematik analiz kursidagi aniqmas va aniq integrallar bilan tanish bo'lgan o'quvchi uchun bu bobning materialini o'zlashtirish qiyinchilik tug'dirmaydi. Bundan keyingi bazi boblarda ham integrallar haqida so'zlashga to'g'ri keladi. Integral nazariyasiga oid tushunchalarni (1) dan topish mumkin.

4.1-§. INTEGRALNING TARIFI

Tekislikning biror qismidan iborat bulgan bir bog'lamli G sohada kompleks argumentli

$$\omega = f(z) = u(x, y) + iv(x, y) \tag{4.1}$$

bir qiymatli va uzluksiz funksiya aniqlangan bo'lsin. Shu soha ichidagi ixtiyoriy Γ silliq chiziqni olamiz. Usha chiziqni ixtiyoriy ravishda

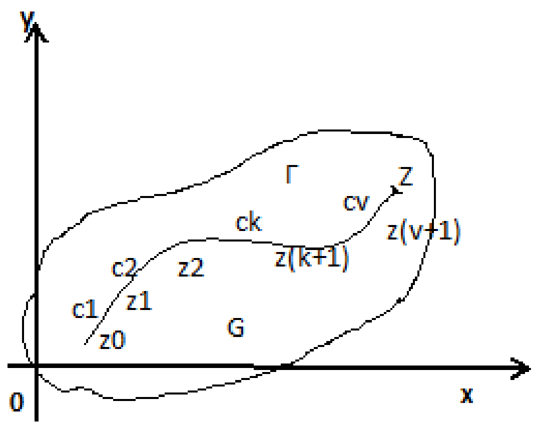
$$z_0, z_1, z_2, \dots, z_{k-1}, z_k, \dots, z_{n-1}, z_n = z$$

nuqtalar vositasi bilan n ta bo'lakka bo'lib, har biri orasidan ixtiyoriy ravishda bittadan

$$z_1, z_2, \dots z_k, \dots, z_n$$

nuqta olamiz (4.1-rasm). Quyidagi belgilarni kiritaylik:

$$\triangle z_1 = z_1 - z_0,\ \triangle z_2 = z_2 - z_1, \dots,\ \triangle z_k = z_k - z_{k-1},\ \dots,\ \triangle z_n = z - z_{n-1}$$



4.1-rasm

Shularga asoslanib ushbu integral yig'indini tuzib olamiz:

$$S_n = f(\zeta_1)\triangle z_1 + f(\zeta_2)\triangle z_2 + \dots + f(z_k)\triangle z_k + \dots + f(\zeta_n)\triangle z_n = \sum_{k=1}^{n} f(\zeta_k)\triangle z_k \tag{4.2}$$

Ma’lumki, $|\triangle z_k| = |z_k - z_{k-1}|$ son z_{k-1} bilan z_k nuqtalar orasdagi masofani bildiradi. Mana shu masofaning eng kattasini δ orqali belgilab olaylik:

$$|\triangle z_1| \le \delta,\ |\triangle z_2| \le \delta,\ ...,\ |\triangle z_n| \le \delta .$$

Endi integralga tarif berish mumkin.

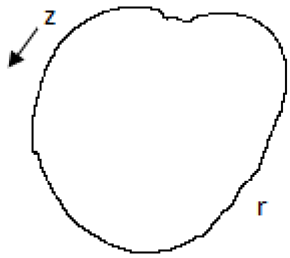
Ta’rif. Agar δ nolga intilganda s_n integral yig'indi $z_1, z_2, ..., z_k, ..., z_{n-1}$ va $\zeta_1, \zeta_2, ..., \zeta_n$ nuqtalar chiziqning qaysi joyidan olinishidan qatiy nazar aniq birgina chekli limitga intilsa, o‘sha limit $f(z)$ funksiyadan Γ chiziq buylab olingan _integral_ deyiladi va quyidagicha yoziladi:

$$\lim_{\delta \to 0} \sum_{k=1}^{n} f(\zeta_k) \triangle z_k = \int_{\Gamma} f(z)\, dz \qquad (4.3)$$

Ma‘lumki, δ nolga intilganda n cheksizlikka intiladi, shu sababli ba‘zan (4.3) dagi limit belgisi ostiga $n \to \infty$ deb yoziladi. Agar Γ chiziq yopiq bo’lsa, ya’ni uning z_0 bosh nuqtasi bilan oxirgi z nuqtasi ustma ust tushsa, (4.3) integral ko'pincha

$$\oint_{\Gamma} f(z)\, dz \qquad (4.4)$$

ko'rinishida yozilib, Γ chiziq integrallash konturi deyiladi (4.2-rasm).



4.2-rasm

4.2-§. INTEGRALNI HISOBLASH

Ba’zan (4.1) ga asoslanib (4.3) integral quyidagicha yoziladi:

$$\int_{\Gamma} f(z)\, dz = \int_{\Gamma} (u + iv)(dx + idy) = \int_{\Gamma} udx - vdy + i \int_{\Gamma} vdx + udy \qquad (4.5)$$

Buning o'ng tomonidagi har bir had egri chiziqli integraldan iborat. Integralni hisoblashning turli usullari mavjud. Agar Γ chiziqning tenglamasi Dekart kordinatalari sestemasida berilgan bo'lsa:

$$y = f(x), \quad a \le x \le b \qquad (4.5)$$

(4.4) dagi y va dy o'rniga (4.5) dan qiymatlar qo'yilib aniq integralga aylantiriladi. Agar Γ chiziqning

$$x = x(t), \quad y = y(t), \quad t_0 \le t \le T \qquad (4.6)$$

parametrik tenglamalari, yani z=z(t) berilgan bo'lsa, uni (3) ga qo'yib yana aniq integral hosil qilinadi:

$$\int_{\Gamma} f(z)\, dz = \int_{t_0}^{T} f[(z(t)]z'(t) dt = \int_{t_0}^{T} R(t)\, dt + i \int_{t_0}^{T} I(t)\, dt \qquad (4.7)$$

Chap tomondagi integral belgisi ostidagi funksiyaning haqiqiy qismini $R(t)$ bilan, mavhum qismini $I(t)$ belgiladik. Integrallashga doir ba'zi tushunchalarni esga olib o'tamiz.

a) Agar $f(z)$ va $\varphi(z)$ funksiyalar bir bog'lamli G sohada analitik, yani hosilaga ega bo'lsa, u holda quyidagi bulaklab integrallash formulasi urinli bo'ladi:

$$\int_{z_0}^{z} f(z)\varphi'(z)dz = f(z)\varphi(z)\Big|_{z_0}^{z} - \int_{z_0}^{z} \varphi(z)f'(z)dz \qquad (4.8)$$

b) Integralni soddalashtirish uchun ba'zan z o'zgaruvchini boshqa bir ω o'zgaruvchiga $z=\varphi(\omega)$ orqali almashtirishga to'g'ri keladi:

$$\int_{\Gamma} f(z)dz = \int_{\Gamma} f(\psi,\omega)\psi'(\omega)d\omega \qquad (4.9)$$

bundagi Γ_1 chiziq Γ ning *Ouv* tekislikdagi aksidan iborat.

v) Agar Γ chiziq markazi $\alpha = a+ib$ nuqtaga joylashgan aylanadan iborat bo'lsa, integralni osonroq hisoblash uchun ushbu aylana tenglamasidan foydalnamiz:

$$z-a=re^{i\varphi} \qquad (4.10)$$

g) Agar Γ chiziq 0 nuqtadan chiquvchi to'g'ri chiziq-nurdan iborat bo`lsa ham (4.10) dan foydalanish tavsiya etiladi, bunda φ o'zgarmas bo'ladi$(0\le r\le\infty)$.

1-misol. Ushbu integralni hisoblang:

$$I=\int_{\Gamma}\left(z\overline{z}+z^2\right)dz$$

bu yerda Γ chiziq $|z|=1$ aylananing yuqori yarmi, yani $0\le \arg z\le\pi$.

Yechilishi: I ni hisoblash uchun (4.10) dan foydalanamiz, bizning misolda

$$x=0,\ r=1;\ dz=e^{i\varphi}id\varphi,\ \ 0\le\varphi\le\pi,\ \ z\overline{z}=e^{i\varphi}\cdot e^{-i\varphi}=1.$$

Shuning uchun

$$I=i\int_0^{\pi}(1+e^{2i\varphi})\cdot e^{i\varphi}d\varphi=\int_0^{\pi}(e^{i\varphi}+e^{3i\varphi})d(i\varphi)=(e^{i\varphi}+\frac{1}{3}e^{3i\varphi})\Big|_0^{\pi}=(e^{\pi i}-e^0)+\frac{1}{3}(e^{3\pi i}-e^0)$$

Bu yerda $e^{\pi i}=\cos\pi+i\sin\pi=-1,\quad e^{3\pi i}=\cos3\pi+i\sin3\pi=-1$ ekanligini hisobga olsak:

$$I=-\frac{8}{3}$$

2-misol. Ushbu integralni hisoblang:

$$I=\int_{\Gamma}(1+i-2\overline{z})dz,$$

Bu yerda Γ chiziq $z_0 = 0,\ z = 1+i$ nuqtalardan utadigan $y = x^2$ paraboladan iborat.

Yechilishi: $y = x^2, \quad x = 0, \quad x = 1 \quad dy = 2dx, \quad z = x+iy, \quad \vec{z} = x - iy.$

$$1+i-2\overline{z} = 1+i-2(x-iy) = (1-2x)+i(1+2y).$$

Shunday qilib,

$$(1+i-2\overline{z})dz = \left[(1-2x)+i(1+2y)\right](dx+idy) = \left[(1-2x)dx-(1+2y)dy\right]+i\left[(1+2y)dx+(1-2x)dy\right],$$

ko'rinishga keladi. Endi y va dy o'rniga chiziqning tenglamasidan keladigan qiymatlarni qo'yib $0 \le x \le 1$ oraliqda aniq integral olish kerak. Natijada

$$I = \int_0^1 \{(1-2x)-(1+2x^2)2x\}dx + i\int_0^1 \{(1+2x^2)+(1-2x)2x\}dx = -2+\frac{4}{3}i.$$

$$I = -2+\frac{4}{3}i;$$

3-misol. Ushbu integralni hisoblang.

$$I = \int_T e^z dz$$

Bu yerda T chiziq $z_0 = 0$ va $z = \pi - i\pi$ nuqtalarni tutashtiruvchi $y = -x$ to'g'ri chiziq kesmasidan iborat.

Yechilishi: Berilishiga ko'ra

$$x = 0; \quad x = \pi; \quad y = 0; \quad y = -\pi$$

T ning parametrik tenglamalarini tuzib olaylik

$$x = t, \quad y = -x = -t, \quad 0 \le t \le \pi;$$

$$z = x+iy = t-it = (1-\mathrm{i})\,\mathrm{t}; \ \overline{z} = (1+i)t,$$

u holda $e^{\bar{z}} = e^{(1+i)t}; \quad \mathrm{dz} = (1-i)dt.$

$$I = (1-i)\int_0^x e^{(1+i)t}dt = \frac{1-i}{1+i}\int_0^\pi e^{(1+i)t}d\left[(1+i)t\right] = \frac{1-i}{1+i}e^{(1+i)t}\Big|_0^\pi = (1+e^\pi)i\ .$$

4-misol. Ushbu integralni hisoblang.

$$I = \int_0^i z\sin z dz$$

Yechilishi: I ni hisoblab integrallaymiz:

$$\int z\sin z dz = -z\cos z + \int \cos z dz = -z\cos z + \sin z,$$

$$I = (-z\cos z + \sin z)\Big|_0^i = -i\cos i + \sin i = -\frac{i}{2}.$$

Bu yerda

$$\cos z = \frac{1}{2}(e^{iz}+e^{-iz}) \ \ va \ \ \sin z = \frac{1}{2i}(e^{iz}-e^{-iz})$$

formuladan foydalandik.

5-misol. Ushbu integralni hisoblang $\mathrm{I}=\int_{\gamma}\operatorname{Re} z dz,$

γ chiziq $z=z(t)=(2+i)t$ parametrik tenglama bilan aniqlangan, $(0\le t\le 1)$.

Yechilishi: Ma'lumki $Z=x+iy,\quad \operatorname{Re} z=x,\quad dz=dx+idy.$

Parametrik tenglamadan esa, $x=2t$, $y=t$, ya'ni $y=\frac{x}{2}$ kelib chiqadi. Bulardan $dx=2dt,\quad dy=dt$ va $\operatorname{Re} z\, dz=x(dx+i\,dy)=4tdt+i\,2tdt$ ekan. Demak, biz berilgan integralni t parametr orqali aniq integral shaklida yozib oldik.

$$\mathrm{I}=4\int_0^1 t\,dt+2i\int_0^1 tdt=2+i$$

6-misol. Ushbu integralni hisoblang;

$$I=\oint_{\Gamma} z\bar{z}dz,$$

bu yerda Γ chiziq $|z|=1$ aylanadan iborat

Yechilishi: I ni (4.10) ga asoslanib hisoblaymiz:

$$z=e^{i\varphi},\quad z\bar{z}=e^{i\varphi}\cdot e^{-i\varphi}=1;$$

$$dz=ie^{i\varphi}d\varphi,\quad 0\le\varphi\le 2\pi,$$

Bularga asosan

$$I=\int_0^{2\pi} e^{i\varphi}d(i\varphi)=e^{2\pi i}-e^0=\cos 2\pi+i\sin 2\pi-1=1-1=0.$$

Demak, I=0.

7-misol. Ushbu integralni hisoblang;

$$I=\int_{\Gamma} e^{|z|}\operatorname{Re} z dz,$$

Bu yerda Γ chiziq $z=0$ va $z=1+i$ nuqtalarni tutashtiruvchi to`g'ri chiziq kesmasidan iborat.

Yechilishi: Ma'lumki,

$$z=x+iy,\ dz=dx+idy,\ |z|^2=|x+iy|^2=x^2+y^2,$$

$$\operatorname{Re} z=x,\ z_0=0,\ z=1+i,\ 0\le x\le 1,$$

Г chiziqning tenglamasi y=x bulgani uchun

$$dy=dx,\ e^{|z|^2}\operatorname{Re} zdz=e^{x^2+y^2}x(dx+idy)=e^{2x^2}x(dx+idx)=e^{2x^2}xdx+ie^{2x^2}xdx;$$

bularga binoan

$$I=\frac{1}{4}\int_0^1 e^{2x^2}d(2x^2)+\frac{1}{4}i\int_0^1 e^{2x^2}d(2x^2)=\frac{1}{4}(1+i)e^{2x^2}\Big|_0^1=\frac{1}{4}(1+i)(\mathrm{e}^2-1).$$

8-misol. Ushbu integralni hisoblang:

$$I=\int_{\Gamma} z\,\mathrm{lm}\,z^2 dz,$$

Bu yerda Γ tug'ri chiziq $|z|=1$ aylananing pastki yarmi $(-\pi\le\varphi\le 0)$.

Yechilishi: (4.10) dan foydalanamiz:

$$z=e^{i\varphi},\ dz=ie^{i\varphi}d\varphi,\ z^2=e^{2i\varphi}=\cos 2\varphi+i\sin 2\varphi,$$

$$\mathrm{lm}\,z^2=\sin 2\varphi;\ zdz=ie^{2i\varphi}d\varphi;\ z\,\mathrm{lm}\,z^2dz=i\sin 2\varphi\cdot e^{i\varphi}d\varphi=$$

$$=i\sin 2\varphi(\cos 2\varphi+i\sin 2\varphi)d\varphi=i(\sin 2\varphi\cos 2\varphi+i\sin^2 2\varphi)d\varphi$$

Demak,

$$I=i\int_{-\pi}^{0}\sin 2\varphi\cos 2\varphi d\varphi-\int_{-\pi}^{0}\sin^2 2\varphi d\varphi=\frac{i}{2}\int_{-\pi}^{0}\sin 2\varphi d(sin2\varphi)-\frac{1}{2}\int_{-\pi}^{0}(1-cos4\varphi)d\varphi,$$

bularni hisoblab chiqsak:

$$I=\frac{\pi}{2}$$

9-misol. Ushbu integralni hisoblang:

$$I=\oint_{\Gamma}\ln zdz,$$

bu yerda Γ $|z|=1$ aylana bo'ylab $z_0=1$ nuqtadan chiquvchi soat mili yo'nalishiga teskari yo'nalishli chiziq.

Yechilishi: (4.10)ga muvofiq quyidagicha belgilab olamiz:

$$z=e^{i\varphi},\ \ \ln z=\ln z^{i\varphi}=i\varphi,\ \ dz=ie^{i\varphi}d\varphi,\ \ln zdz=-\varphi e^{i\varphi}d\varphi$$

$$\ln zdz=-\varphi e^{i\varphi}d\varphi,\quad 0\le\varphi\le 2\pi.$$

U holda bo`laklab integrallash natijasida quyidagiga ega bo`lamiz:

$$I=-\int_{0}^{2\pi}\varphi e^{i\varphi}d\varphi=2\pi i.$$

10-misol. Ushbu integralni hisoblang;

$$I=\int_{\Gamma}[(y+1)-ix]dz,$$

Bu yerda Γ chiziq $z_0=1$ va $z=-i$ nuqtalarni tutashtiruvchi to'g'ri chiziq kesmasidan iborat.

Yechilishi: Berilgan ikki nuqtadan utadigan tug'ri chiziqning tenglamasi $y=x-1$ bo'lganligi sababli:

$$dy=dx,\ \ 0\le x\le 1;\ \ y+1=x,$$

$(y+1-ix)dz = x(1-i)(dx+idy) = (1-i)x(dx+idx) = (1-i)(1+i)xdx = 2xdx,$
bundan integral olsak

$$I = 2\int_1^0 xdx = -1$$

kelib chiqadi.

11-misol. Ushbu integralni hisoblang;

$$I = \oint_\Gamma \frac{dz}{z-(1+i)},$$

bu yerda Γ chiziq $|z-(1+i)| = 1$ aylanadan iborat.

Yechilishi: Berilgan aylananing parametrik tenglamalarini yozib olamiz:

$$x-1 = \cos\varphi, \quad y-1 = \sin\varphi; \quad x = 1+\cos\varphi, \quad y = 1+\sin\varphi;$$

$$z = x+iy = (1+\cos\varphi)+i(1+\sin\varphi) = (1+i)+(\cos\varphi+i\sin\varphi) = (1+i)+e^{i\varphi}$$

$$0 \le \varphi \le 2\pi, \qquad zdz = ie^{i\varphi}d\varphi;$$

$$\frac{dz}{z-(1+i)} = id\varphi.$$

U holda

$$I = i\int_0^{2\pi} d\varphi = 2\pi i.$$

12-misol. Quyidagi integralni hisoblang.

$$I = \oint_\gamma \ln zdz$$

Bu yerda γ chiziq $|z| = 1$ aylana bo’ylab $z_0 = 1$ nuqtadan chiquvchi soat strelkasiga teskari yo’nalishli chiziqdir.

Yechilishi: Integral aylana bo’ylab bo’lgani uchun quyidagi almashtirishlarni bajaramiz:

$$z = e^{i\varphi}, \quad \ln z = \ln e^{i\varphi} = i\varphi, \quad dz = ie^{i\varphi}d\varphi,$$

$$\ln zdz = -\varphi e^{i\varphi}dy, \quad 0 \le \varphi \le 2\pi.$$

U holda bo’laklab integrallash natijasida quyidagiga ega bo’lamiz**.**

$$I = -\int_0^{2\pi} \varphi e^{i\varphi}d\varphi = 2\pi i$$

4.3- §. NYUTON- LEYBNIS FORMULASI

Ba'zi integrallarni osonroq yo'l bilan hisoblash mumkin. Agar $f(z)$ funksiya z_0 va z nuqtalarni o'z ichiga oluvchi bir bog'lamli G sohada analitik, yani, hosilaga ega bo'lsa u holda ushbu

$$\int_{z_0}^{z} f(z)dz = \Phi(z)\big|_{z_0}^{z} = \Phi(z) - \Phi(z_0) \qquad (4.11)$$

Nyuton-Leybnis formulasi o'rinli bo'lib, bu yerda $\Phi(z)$ funksiya $f(z)$ ning biror funksiyasidan iborat, yani $\Phi'(z) = f(z),\ z \in G.$

Misollar ko'raylik;

1-misol. $\int_0^{1+i} z^3 dz = \frac{1}{4}z^4\big|_0^{1+i} = \frac{1}{4}(1+i)^4 = -1,$ bu yerda $f(z) = z^3$- analitik funksiya.

2-misol. $I = \int_1^i (3z^4 - 2z^3)dz = (\frac{3}{5}z^5 - \frac{1}{2}z^4)\big|_1^i = \frac{3}{5}(i^5 - 1) - \frac{1}{2}(i^4 - 1) =$

$= \frac{3}{5}(i-1),$ chunki $i^4 = 1,\ i^5 = i.$

3-misol. $I = \int_1^i ze^z dz = \left(ze^z - e^z\right)\big|_1^i = \left(i-1\right)e^i.$

4-misol. $I = \int_0^i \left(z - i\right)e^{-z} dz$ integralni bo`laklab integrallasak,

$$I = -e^{-z}\left(1 - i + z\right)\big|_0^i = 1 - i - e^{-i} = \left(1 - \cos 1\right) - i\left(1 - \sin 1\right).$$

5-misol. $I = \int_1^i \frac{\ln z}{z} dz$ integral $z_0 = 1$ va $z = i$ nuqtalarni tutashtiruvchi tug'ri chiziq kesmasi bo`ylab olingan.

Nyuton-Leybnis formulasiga ko'ra:

$$I = \int_1^i \frac{\ln z}{z} dz = \int_1^i \ln z d\left(\ln z\right) = \frac{1}{2}\ln^2 z\big|_1^i = \frac{1}{2}\left[\ln i\right].$$

6-misol. $I = \int_0^{1+i} \sin z \cos z dz = \frac{1}{2}\int_0^{1+i} \sin 2z dz = \frac{1}{4}\left(-\cos 2z\right)\big|_0^{1+i} =$

$$=\frac{1}{4}\left[1-\cos\left(2+2z\right)\right].$$

7-misol. $I=\int_{1}^{i}\frac{\ln(1+z)}{1+z}dz$ integral $|z|=1$ aylananing birinchi chorakdagi qismi bo'ylab olingan. Uni hisoblaylik

$$I=\int_{1}^{i}\ln\left(1+z\right)d\left[\ln\left(1+z\right)\right]=\frac{1}{2}\ln^{2}\left(1+z\right)|_{1}^{i}=\frac{1}{2}\ln^{2}\left(1+i\right)-\frac{1}{2}\ln^{2}2;$$

$$\ln\left(1+i\right)=\ln\sqrt{2}+\frac{\pi}{4}i;\qquad \ln^{2}\left(1+i\right)=\left(\ln\sqrt{2}+\frac{\pi}{4}i\right)^{2}=$$

$$=\ln^{2}\sqrt{2}+i\frac{\pi}{2}\ln\sqrt{2}+\frac{\pi^{2}}{16}i^{2}=\left(\frac{1}{\sqrt{2}}\ln 2\right)^{2}+\frac{1}{4}\pi i\ln 2-\frac{\pi^{2}}{16};$$

bularni yuqoriga keltib, ixchamlashtirilsa ushbu natija kelib chiqadi:

$$I=-\frac{1}{8}\left(\frac{\pi^{2}}{4}+3\ln^{2}2\right)+\frac{\pi i}{8}\ln 2.$$

4.4- §. KO'P QIYMATLI FUNKSIYALAR. TARMOQLANISH NUQTASI VA QIRQIMLAR

Ma'lumki, agar $z=x+iy$ ga bitta qiymat berganda $w=f\left(z\right)$ funksiya ham birgina qiymatni qabul qilsa, funksiya ***bir qiymatli***, aks holda ***ko'p qiymatli*** deyiladi. Har bir qiymati o'sha funksiyaning ***tarmog'i*** deyiladi.

Masalan, bizga ma'lum logarifimli funksiyaning cheksiz ko'p qiymati mavjud edi.

$$f\left(z\right)=\ln z=\ln|z|+i\arg z+2k\pi\, i,\qquad \left(z=0,\pm1,\pm2.....\right).\qquad (4.12)$$

Ixtiyoriy kompleks sonining logarifimi deganda yuqoridagi ifodaning qabul qilishi mumkun bo'lgan biror qiymati nazarda tutiladi. Bunday ko'p qiymatli funksiyalar ustida amallar bajarishda ***Riman sirtlari*** tushunchasi juda katta qulayliklar yaratadi. Logarifmning ushbu

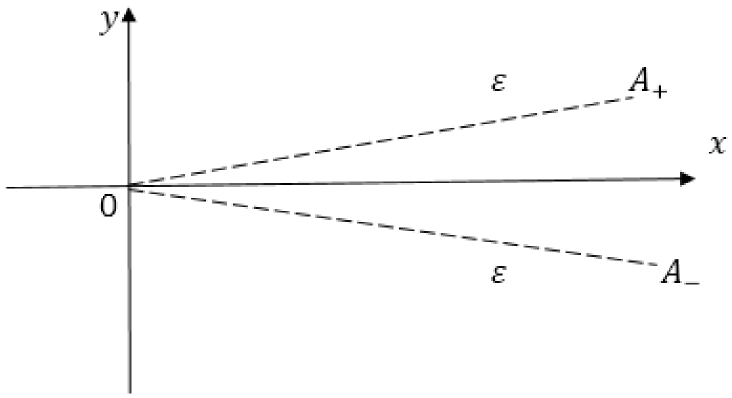
$$\ln z=\ln|z|+i\arg z,(0\le\arg z\le 2\pi).$$

qiymatini, odatda bosh qiymat deyiladi. Logarifmning boshqa qiymatlarini hosil qilish uchun, kompleks tekislikda haqiqiy o'qning musbat qismini kompleks tekislikdan "ajratib" tashlaymiz (musbat yo'nalishda "qirqim" hosil qilamiz). Qirqimning qarama-qarshi "qirg'oqlarrida"gi nuqtadagi logarifmning qiymatlari uchun

$$f\left(A_{+}\right)=\ln|z|+i\left(\arg z+\varepsilon\right),\qquad \varepsilon\to 0_{+}$$

$$f\left(A_-\right)=\ln|z|+i\left(\arg z+2\pi-\varepsilon\right),\ \ \varepsilon\to 0_- \qquad (4.3\text{-rasm})$$

larni hosil qilamiz.



4.3-rasm.

Qirqimlardan o'tmasdan $\arg z$ ni 2π ga o'zgartirib bo'lmaydi. Demak, kompleks tekislikda "qirqim" hosil qilish oqibatida bir qiymatli funksiya hosil qilamiz, ammo bu funksiya uzliksiz emas, chunki.

$$\lim_{\varepsilon\to 0}\left|f\left(A_+\right)-f\left(A_-\right)\right|=2\pi i\neq 0 \qquad (4.13)$$

Shunday qilib, cheksiz ko'p qiymatli funksiyadan cheksiz ko'p bir qiymatli funksiya hosil qildik. Endi kitob varaqlari kabi taxlangan kompleks tekisliklar sistemasini ko'raylik, har bir varaq k ning qiymatlariga mos keladi, har bir tekislik haqiqiy o'qning musbat yo'nalishi bo'yicha qirqilgan. Hamma joyda o'zaro bir qiymatli moslik o'rnatish uchun $k-$ varaqning yuqori "qirg'og'i" ni $(k-1)$ - varaqning quyi "qirg'og'i " bilan yelimlaymiz. Bu ish cheksiz davom etadi. Shunday usulda yasalgan sirt ***Riman sirti*** deb ataladi. Ya'ni, $z=0$ nuqta atrofida aylanishning yo'nalishiga qarab u tekislikdan bu tekislikka qirqim orqali yo yuqoriga ko'tariladi, yoki pastga tushiladi (xuddi vintga o'xshagan zinapoyadandek).

$z=0$ va $z=\infty$ nuqtalar Riman sirtidagi barcha varaqlar uchun, umumiydir. Har bir "varaq" ***Riman sirti varag'i*** deyiladi va ularda $f(z)$ funksiya bir qiymatlidir.

Funksiyaning har bir varaqdagi qiymatini - ***funksiyaning tarmog'i*** deyiladi. Hamma varaq uchun bir xil bo'lib, ularni atrofida aylanish natijasida ko'p qiymatlilik hosil bo'lgan nuqtalar ($z=0$ va $z=\infty$) ***tarmoqlanish nuqtalari*** deb nomlanadi.

1-misol. $z=\sqrt[n]{w}$ ning Riman sirtini tuzing.

$$z=re^{i\varphi},\quad w=\rho e^{\theta}$$

deylik, unda

$$r=\sqrt[n]{\rho},\quad \varphi=\frac{\theta+2k\pi}{n}\quad,\quad \kappa=0,1,....(n-1).$$

ga tengdir.

$$z_0,\ z_1,\ z_2,\ \dots\ z_{n-1}$$

nuqtalarning moduli $r=\sqrt[n]{\rho}$ ga teng , argumentlari esa mos ravishda

$$\varphi_0=\frac{\theta}{n}\quad,\quad \varphi_1=\frac{\theta+2\pi}{n}\quad,\quad \varphi_{n-1}=\frac{\varphi+2\pi(n-1)}{n}$$

bo'ladi. Agar biz w ning har bir qiymatiga z ning faqat bitta qiymatini mos qilib qo'ymoqchi bo'lsak, z ga tegishli nuqtalarni quyidagi

$$0<\varphi<\frac{2\pi}{n},\ \frac{2\pi}{n}<\varphi<\frac{4\pi}{n},\ \dots\ \frac{2(n-2)\pi}{n}<\varphi<\frac{2(n-1)\pi}{n}\ .$$

burchaklarning faqat bittasini ichida olishimiz kerak. Bularni quyidagicha belgilaylik (tarmoqlar):

$$z_0=(\sqrt[n]{w})_0,\ z_1=(\sqrt[n]{w})_1,\ \dots\ z_{n-1}=(\sqrt[n]{w})_{n-1}.$$

Haqiqiy o'qning musbat yo'nalishi bo'yicha qirqim hosil qilib, 1-varaqning pastki qirg'og'ini 2-varaqning yuqori "qirg'og'iga" yopishtiramiz, v.h. Agar z nuqta $z=0$ aylana bo'yicha $z=0$ nuqtani aylanib chiqsa, w nuqta $w=0$ nuqtani $|w|=\rho$ aylana bo'yicha n marta aylanib chiqadi, ya'ni Riman sirtida w nuqta bir varaqdan ikkinchiga, v.h. o'tadi. Bu funksiyaning logarifmdan farqi shundaki , $k=n$ bo'ladi. Chunki,

$$z_n=(\sqrt[n]{w})_n=z_0$$

ga teng bo'ladi, ya'ni n- qiymat yana birinchi varaqqa qaytib keladi. Shuning uchun , $w=0$ $(w\to\infty)$ nuqta $(n-1)$- tartibli ***tarmoqlanish nuqtalari*** deyiladi. Logarifimik funksiya holida esa , hech qachon boshlang'ich tarmoqqa qaytib kelinmaydi. Bunday xossaga ega bo'lgan tarmoqlanish nuqtasini ***cheksiz tartibli,*** yoki ***logarifmik tarmoqlanish*** nuqtasi deyiladi.

2-misol. $f(z)=(z-1)^{1/2}$ funksianing Riman sirti tuzilsin .

$$z=1+re^{i\varphi}$$

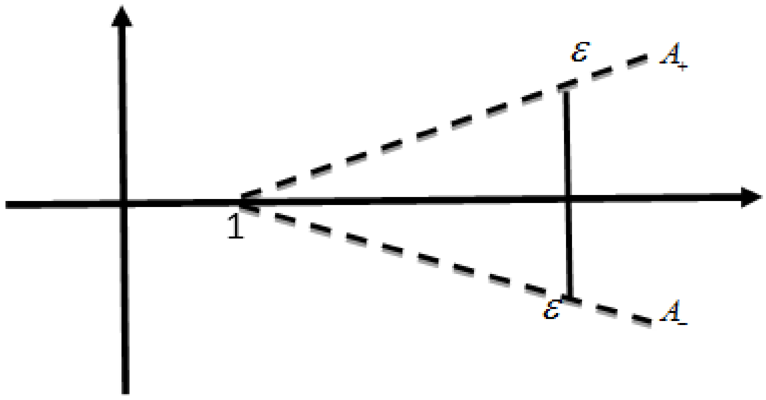
deb $f(z)=\sqrt{r}e^{i\varphi/2}$ ni olamiz. Ildizning qiymatini doim musbat tanlab olamiz. φ ni $\varphi+2\pi$ ga o'zgartirsak

$$f(z)=\sqrt{r}e^{i(\varphi+2\pi)/2}=-\sqrt{r}e^{i\varphi/2}=-f(z)\ .$$

Ya'ni, bu funksiya ikki qiymatli funksiyadir. Tarmoqlanish nuqtasi esa, $z=1$ dir,

$$f(z)=(z-1)^{1/2}$$

ikki tarmoqqa ega. Riman sirti varag'i $\varphi_1 < \varphi < \varphi_1 + 2\pi$ bilan aniqlanadi.(4.4-rasm)



4.4-rasm.

Mashqlar.

Quyidagi funksiyalarning Riman sirti tuzilsin.

1. $w=e^z$
2. $z=Arg\sin w$
3. $z=Arctgw$
4. $w=\cos z$
5. $w=\sin z$

Misollar

1-misol. $I=\int\limits_\gamma \frac{dz}{\sqrt{z}}$ ni hisoblang. Bu yerda γ chiziq $|z|=1$ aylananing yuqori qismi. $w=\frac{1}{\sqrt{z}}$ funksiyaning shunday tarmog'i olinsinki, natijada $\sqrt{1}=-1$ bo'lsin.

Yechilishi: Bu misolni 2 xil usul bilan yechaylik:

1-usul. $z=r(\cos\varphi+i\sin\varphi);\quad r=|z|,\quad \varphi=\arg z.$

$$w_k=\sqrt{z}=\sqrt{r}(\cos\frac{\varphi+2k\pi}{2}+i\sin\frac{\varphi+2k\pi}{2}),\quad k=0,1;\quad r=1.$$

$$W_O=\cos\frac{\varphi}{2}+i\sin\frac{\varphi}{2},\quad W_1=\cos\left(\frac{\varphi}{2}+\pi\right)+i\sin\left(\frac{\varphi}{2}+\pi\right)=-\left(\cos\frac{\varphi}{2}+i\sin\frac{\varphi}{2}\right),$$

Berilgan $\sqrt{1}=-1$ shartni w_1 ildiz qanoatlantiradi, chunki $z=1$ bo'lganda $\varphi=0$ bo'lib, $w_1=-1$ ga teng bo'ladi: $W_1=\sqrt{z}=-1.$

Niyuton –Leybnis formulasiga ko'ra

$$\mathrm{I}=\int_1^{-1} z^{-\frac{1}{2}}dz=2\sqrt{z}\Big|_1^{-1}=2\left(\sqrt{-1}-\sqrt{1}\right)$$

Ma'lumki, $z=-1$ bo'lganda $\varphi=\pi$ bo'lib, w_1 ning ifodasiga muvofiq

$$\sqrt{-1}=-\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=-i$$

demak,

$$\mathrm{I}=2(-i+1)=2(1-i)$$

2-usul.

$$z=re^{i\varphi},\quad r=|z|=1,\quad z=e^{i\varphi},$$

$$\sqrt{z}=\sqrt{e^{i\varphi}}=\sqrt{\cos\varphi+i\sin\varphi}=\cos\frac{\varphi+2k\pi}{2}+i\sin\frac{\varphi+2k\pi}{2}=W_k,\quad \mathrm{k}=0,1;$$

$$W_o=\cos\frac{\varphi}{2}+i\sin\frac{\varphi}{2}=e^{\frac{i\varphi}{2}};\qquad W_1=\cos\frac{\varphi+2\pi}{2}+i\sin\frac{\varphi+2\pi}{2}=e^{i\left(\frac{\varphi}{2}+\pi\right)}$$

Endi $\sqrt{1}=1$ shartga ko'ra w_1 ildizni olamiz, chunki z=1 bo'lganda $\varphi=0$ bo'lib

$$W_1=e^{\pi i}=\cos\pi+i\sin\pi=-1,\qquad dz=ie^{i\varphi}d\varphi.$$

Aylananing ustki yarimida $0\le\varphi\le\pi.$.

Shu sababli berilgan integralni aniq integralga aylantiramiz.

$$\mathrm{I}=i\int_o^{\pi}\frac{e^{i\varphi}d\varphi}{e^{i\left(\frac{\varphi}{2}+\pi\right)}}=i\int_o^{\pi}e^{i\left(\frac{\varphi}{2}-\pi\right)}d\varphi=2\int_o^{\pi}e^{i\left(\frac{\varphi}{2}-\pi\right)}d\left[i\left(\frac{\varphi}{2}-\pi\right)\right]=2e^{i\left(\frac{\varphi}{2}-\pi\right)}\Bigg|_0^{\pi}=2\left(e^{-\frac{\pi i}{2}}-e^{-\pi i}\right)=2(1-i).$$

Demak, $\mathrm{I}=2(1-i)$

2-misol. $I=\int_{\Gamma}\frac{dz}{\sqrt[4]{z^3}}$ ni hisoblab, bu yerda Γ chiziq $|z|=1$ aylananing ustki yarmi. $\varpi=\dfrac{1}{\sqrt[4]{z^3}}$ funksiyaning to`rtta tarmog`idan shunday bittasi tanlab olinsinki, natijada $\sqrt[4]{1}=1$ bo`lsin.

Yechilishi: Nyuton-Leybnis formulasiga muvofiq

$$I=\int_1^{-1}z^{-\frac{3}{4}}dz=4z^{\frac{1}{4}}\Bigg|_1^{-1}=4(\sqrt[4]{-1}-\sqrt[4]{1})=4(\sqrt[4]{-1}-1),$$

bu yerda $a_k=\sqrt[4]{-1}=\sqrt[4]{\cos\pi+i\sin\pi}=\cos\dfrac{\pi+2k\pi}{4}+i\sin\dfrac{\varphi+2k\pi}{4}.$

$k=0,1,2,3.$

Endi mana shu to`rtta ildizdan qaysi birini olish kerak degan savol tug`iladi. Uning uchun misolda berilgan shartdan foydalanamiz, ya`ni

$$\beta_k = \sqrt[4]{1} = \sqrt[4]{\cos(0+2k\pi) + i\sin(0+2k\pi)} = \cos\frac{2k\pi}{4} + i\sin\frac{2k\pi}{4} = 1$$

bo`lishi uchun *k=0* deb olishga majburmiz. Shu sababli to`rtta ildizdan faqat $a_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\cdot$ ildizni olishga to`g`ri keladi.

U holda $I = 4(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} - 1) = 2\sqrt{2} - 4 + i2\sqrt{2}.$

3. $I = \int\limits_{-1}^{i} \frac{\cos z dz}{\sqrt{\sin z}}$ ni hisoblang. Bu yerda chiziq $z_0 = 1$ va $z = i$ nuqtalar orasidagi to`g`ri chiziq kesmasi bo`lib, ikki qiymatli $\frac{\cos z}{\sqrt{\sin z}}$ funksiyaning shunday tarmog`i olinsinki, natijada $\sqrt{\sin(-1)} = i\sqrt{\sin 1}$ bo`lsin.

Yechilishi: Nyuton-Leybnis formulasidan foydalanamiz:

$$I = \int\limits_{-1}^{i} \sin^{-\frac{1}{2}} z d(\sin z) = 2\sin^{\frac{1}{2}} Z\Big|_{-1}^{i} = 2(\sqrt{\sin i} - \sqrt{\sin(-1)} = 2(\sqrt{\sin i} - i\sqrt{\sin 1});$$

$$\sqrt{\sin i} = \sqrt{\frac{e^{-1} - e}{2i}} = \sqrt{i}\cdot\sqrt{sh1};$$

$$a_k = \sqrt{i} = \sqrt{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}} = \cos\frac{\frac{\pi}{2} + 2k\pi}{2} + i\sin\frac{\frac{\pi}{2} + 2k\pi}{2}, \qquad k = 0,1$$

bundan biz faqat birinchi $a_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}.$

Ildizni $(\mathrm{k} = 0)$ olamiz, chunki boshlang`ich shartga muvofiq

$\sqrt{\sin(-1)} = \sqrt{-1\cdot\sin 1} = i\sqrt{\sin 1},$ ya`ni $\sqrt{-1} = i$

deb olishimizga sabab:

$$\beta_k = \sqrt{-1} = \sqrt{\cos\pi + i\sin\pi} = \cos\frac{\pi + 2k\pi}{2} + i\sin\frac{\pi + 2k\pi}{2}, \qquad k = 0,1$$

$$\beta_0 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i, \quad \beta_1 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i.$$

Demak, $I = 2\left[(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})\sqrt{sh1} - i\sqrt{\sin 1}\right] = \sqrt{2sh1} + i(\sqrt{2sh1} - 2\sqrt{\sin 1}).$

Mashqlar

Kompleks sohada berilgan quyidagi integrallarni hisoblang.

6. $\int_{\Gamma}\left(x^2+iy^2\right)dz,$ Γ chiziq $z_0=1+i$ va $z=2+3i$ nuqtalarni tutashtiruvchi to`g`ri chiziq;

7. $\int_{i}^{1+i} zdz$;

8. $\oint_{i} \bar{z}dz$ bu yerda Γ chiziq $x=\cos t, \quad y=\sin t$, $(0\le t\le 2\pi)$;

9. $\oint \frac{dz}{z-4}$, bu yerda Γ chiziq $x=3\cos t, \quad y=2\sin t$ ellipsdan iborat;

10. $\int_{\Gamma} z^2dz$ bu yerda Γ chiziq $z_0=1$ va $z=i$ nuqtalarni tutashtiruvchi to`g`ri chiziq kesmasi;

11. $\oint \frac{dz}{z}$ bu yerda Γ chiziq $x=\cos t, \quad y=\sin t$ aylanadan iborat;

12. $\int_{1+i}^{-1-i}(2z+1)dz$ **;**

13. $\int_{\Gamma}^{1+i} e^z dz$, bu yerda Γ chiziq $z_0=0$ va $z=1+i$ nuqtalardan o`tadigan $y=x^2$ parabolaning qismidir;

14. $\int_{\Gamma}\cos zdz,$ bu yerda Γ chiziq $z_0=\frac{\pi}{2}$ va $z=\pi+i$ nuqtalarni tutashtiruvchi to`g`ri chiziq kesmasi;

15. $\int_{1+i}^{2i}(z^3-z)e^{\frac{z^2}{2}}dz$;

16. $\int_{0}^{i} \mathrm{z}\cos zdz$;

17. $\int_{1}^{i} \mathrm{z}\sin zdz$;

18. $\int_{\Gamma}\mathrm{Re}(\sin z)\cos zdz,$ bu yerda Γ chiziq $|\mathrm{Im}\, z|\le 1, \quad \mathrm{Re}\, z=\frac{\pi}{4}$;

19. $\int_{\Gamma} z\,\mathrm{Im}(z^3)dz$ bu yerda Γ chiziq: $|\mathrm{Im}\, z|\le 1 \quad \mathrm{Re}\, z=1$;

20. $\int_{-i}^{i} ze^{z^2}dz$;

21. $\int_{0}^{1}(1+it)^2\,dt,$ t - haqiqiy o`zgaruvchi;

22. $\int_0^1 \frac{dt}{1+it}$, t- haqiqiy o`zgaruvchi;

24. $\int_0^1 \frac{1+it}{1-it}dt$, t- haqiqiy o`zgaruvchi;

25. $\int_0^{\pi} b^{it}dt$, t- haqiqiy o`zgaruvchi.

Javoblar

6. $-\frac{19}{3}+9i$; 7. $\frac{1}{2}+i$; 8. $2\pi i$; 9. 0; 10. $-\frac{1}{3}(1+i)$; 11. $2\pi i$; 12. $-2(1+i)$; 13. $-1+e\cos1+ie\sin1$; 14. $-(1+ish1)$; 15. $-7e^{-2}+(3-2i)\mathrm{e}$; 16. $e^{-1}-1$; 17. $\cos1-\sin1-ie^{-1}$; 18. $\frac{1}{4}sh2+\frac{1}{2}i$; 19. $-\frac{4}{3}$; 20. 0; 21. $\frac{2}{3}+i$; 22. $\frac{\pi}{4}-\frac{i\ln2}{2}$; 23. $\frac{\pi}{2}-1+i\ln2$; 24. $-2i$;

V BOB.

KOSHI FORMULASIDAN FOYDALANIB HISOBLANADIGAN INTEGRALLAR

Ba`zi kompleks argumentli funksiyalardan olingan egri chiziqli integrallarni Koshi formulasidan foydalanib hisoblash qulaydir. Bu bob ana shunday integrallarni hisoblashga doir misollar yechishga bag`ishlanadi.

5.1-§. KOSHI TEOREMALARI HAQIDA

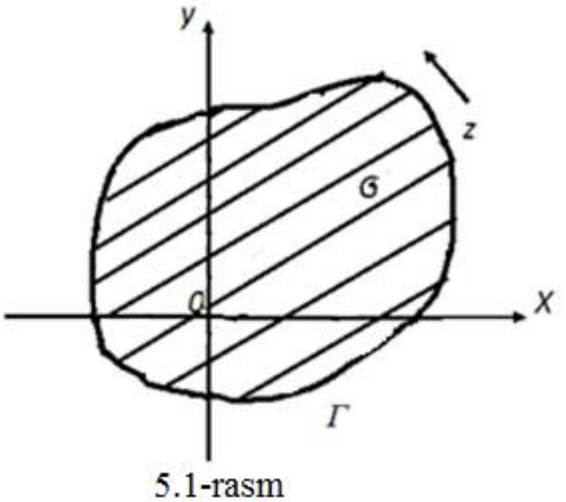
Ba`zi integrallarning javobini Koshi teoremasiga asoslanib aytish mumkin. Bunday integrallarni hisoblab o`tirishning mutlaqo hojati yo`q. Shu teoremani esga olib o`tamiz:

1-teorema. Agar bir bog`lamli G sohada $f(z)$ funksiya analitik bo`lsa, u holda G sohada yotuvchi har bir G yopiq kontur bo`ylab $f(z)$ funksiyadan olingan integral nolga teng bo`ladi:

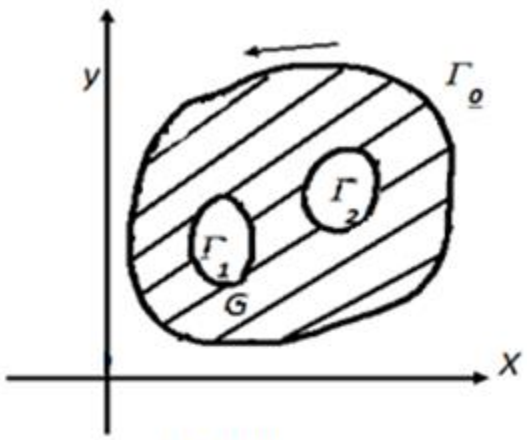
$$\oint_{\Gamma} f(z)dz = 0 \tag{5.1}$$

Xususiy holda G soha yopiq va uning konturi o`sha $\Gamma$ chiziqdan iborat (5.1-rasm) bo`lib qolishi ham mumkin.

Demak, (5.1) integralning nolga tengligini bilish uchun dastlab



5.1-rasm



5.2-rasm

integral belgisi ostidagi $f(z)$ funksiyaning G sohada analitik, ya`ni hosilaga ega ekanligini tekshirish zarur. Hozircha zarurat bo`lmasa-da, Koshining yana bitta teoremasini isbotsiz keltiramiz:

2-teorema. Agar ko`p bog`lamli yopiq G sohada $f(z)$ funksiya analitik bo`lsa, u holda funksiyadan tashqi kontur bo`ylab olingan integral ichki konturlar bo`ylab olingan integrallar yig`indisiga teng bo`lib, bunda barcha konturlar bo`ylab yo`nalishi bir xilda olinadi:

$$\oint_{\Gamma} f(z)\,dz = \oint_{\Gamma_1} f(z)\,dz + \oint_{\Gamma_2} f(z)\,dz + \ldots\ldots \oint_{\Gamma_n} f(z)\,dz \qquad (5.2)$$

Ko`pincha barcha konturlar bo`ylab yo`nalish soat strelkasi yo`nalishiga qarshi qilib olinadi (5.2-rasm).

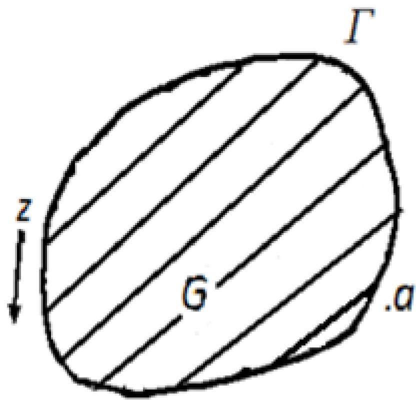
Endi Koshining birinchi teoremasi yordamida yechiladigan ba`zi misollarni keltiramiz:

1-misol. $f(z)=(z-a)^n$, $(n=0,1,2,....)$ funksiya har qanday chekli sohada analitik, ya`ni hosilaga ega : $f''(z)=n(z-a)^{n-1}$.

Shu sababli, (5.1) ga muvofiq $\oint_{\Gamma}(z-a)^n\,dz=0.$

Xususiy holda $a=0$ bo`lsa, $\oint_{\Gamma} z^n\,dz=0.$

2-misol. $f(z)=\dfrac{1}{(z-a)^n}$, $(n=1,2,......)$



5.3-rasm

funksiya a nuqtani o`z ichiga olmaydigan har qanday chekli Γ sohada analitik (5.3-rasm)

chunki $f'(z)=-\dfrac{n}{(z-a)^{n+1}}$, $z\neq a,$

demak,

$$\oint_{\Gamma}\frac{dz}{(z-a)^n}=0 \quad (5.3)$$

Xususiy holda a=0 bo`lsa, $\oint_{\Gamma}\dfrac{dz}{z^n}=0$, n=1,2,3,

Agar $n=1$ bo`lsa, (5.3) dan: $\oint_{\Gamma}\dfrac{dz}{(z-a)}=0$.

3-misol. $I=\oint_{\Gamma}\dfrac{z^2dz}{z-2i}$, bu yerda Γ aylana bo`lib, markazi 0 nuqtada va radiusi $R=1$ ga teng. Demak, $\overline{G}$ soha birlik doiradan iborat bo`lib, $z=2i$ nuqta uning ichida emas, shu sababli

$$f(z)=\frac{z^2}{z-2i}$$

funksiya o`sha doirada hosilaga ega. (5.1) ga asosan $I=0$.

4-misol. Agar Γ chiziq $|z|=2$ aylanadan iborat bo`lsa, uning ichidagi G doirada $f(z)=\frac{1}{z^2+9}$ funksiya analitik bo`ladi, shu sababli, $\int\limits_{\Gamma}\frac{dz}{z^2+9}=0.$

5-misol. Agar Γ chiziq $|z|=\frac{1}{2}$ aylanadan iborat bo`lsa, u bilan chegaralangan G doira ichida $\int\limits_{\Gamma}\frac{\sin z dz}{z+i}=0.$ bo`ladi.

6-misol. Agar Γ chiziq $|z-i|=1$ aylanadan iborat bo`lsa, u bilan chegaralangan G doira ichida $f(z)=\frac{e^z}{(z+4)^3}$ funksiya analitik. (1) ga muvofiq $\oint\limits_{\Gamma}\frac{e^z dz}{(z+4)^3}=0.$

Shunday qilib, umuman, $f(z)$ funksiya hosilasi mavjud bo`lmagan nuqtalar $\overline{G}$ sohaning tashqarisida qolsa, $\overline{G}$ ning $\Gamma$ chegarasi bo`ylab olingan integral nolga teng bo`laveradi.

7-misol. $1+i$ nuqtani o`z ichiga olmaydigan va Γ bilan chegaralangan har qanday $\overline{G}$ sohada $\oint\limits_{\Gamma}\frac{tgzdz}{z-1-i}=0.$

5.2-§. KOSHINING INTEGRAL FORMULASI

Nolga teng bo`lmagan integrallarni Koshining integral formulasi deb ataluvchi

$$f(a)=\frac{1}{2zi}\oint\limits_{\Gamma}\frac{f(z)\,dz}{z-a} \qquad (5.4)$$

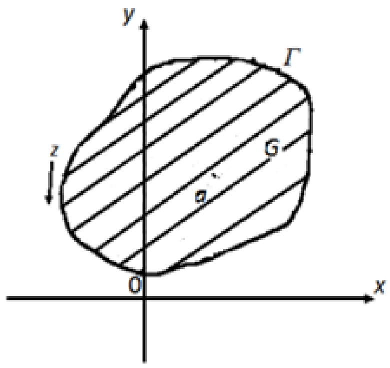
Integral formula asosida topish mumkin. Bu yerda a nuqta Γ soha ichida, z nuqta esa Γ chegarada yotadi va $f(z)$ Γ da analitikdir. (5.4-rasm)

Kompleks argumentli funksiyalar nazarayasidan ma`lumki, (5.4) dan quyidagi formulani keltirib chiqarish mumkin:

$$f^{(n)}(a)=\frac{n!}{2\pi i}\oint\limits_{\Gamma}\frac{f(z)\,dz}{(z-a)^{n+1}},\quad n=1,2,3,... \qquad (5.5)$$

$i=\sqrt{-1},\quad i^2=-1 \qquad \pi=3{,}141592.....$

$\int\limits_{\Gamma}\frac{f(z)dz}{z-a}=2\pi i\cdot f(a)$ va



5.4-rasm

$$\int_{\Gamma}\frac{f(z)dz}{(z-a)^{n+2}}=\frac{2\pi i}{n!}f^{(n)}(a),$$

Bu yerda $f^{(n)}(a)$ ifoda $f(z)$ funksiyadan *n*-tartibli hosila olib, so`ngra *z* o`rniga *a* qo`yishni bildiradi. Misol yechishda *a* nuqtaning Γ soha ichida ekanligiga e`tibor berishimiz lozim, aks holda Koshining birinchi teoremasiga asosan integral nolga teng bo`lib qoladi. *a* ni o`zgarmas kompleks son deb faraz qilamiz, xususiy holda haqiqiy sondan iborat bo`lishi ham mumkin.

1-misol. $I=\int_{\Gamma}\frac{dz}{z-a}$ ni hisoblang, bu yerda *Γ* chiziq markazi *a* nuqtada joylashgan ixtiyoriy aylana. (5.5-rasm).

Yechilishi: Bu misolda $f(z)=1$ bo`lgani uchun (5.4) formulasiga asosan

$$\int_{\Gamma}\frac{dz}{z-a}=2\pi i. \tag{5.6}$$

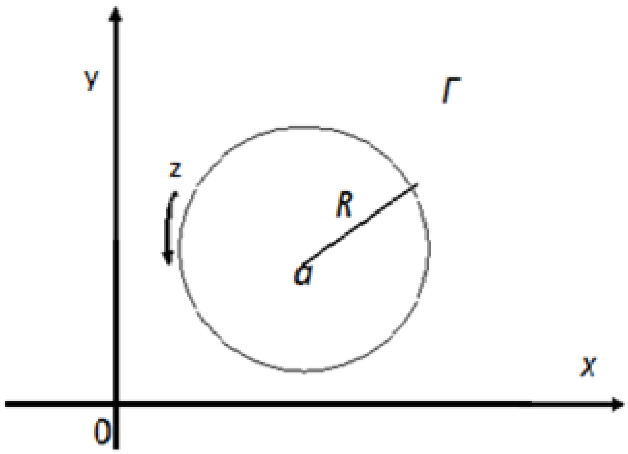
Xususiy holda $a=0$ bo`lib aylana markazi 0 nuqtada bo`ladi, u holda $\int_{\Gamma}\frac{dz}{z}=2\pi i$

2-misol. $I=\int_{\Gamma}\frac{z^2dz}{z-2i}$ ni hisoblang, bu yerda *Γ* chiziq $|z|=3$ aylanadan iborat.

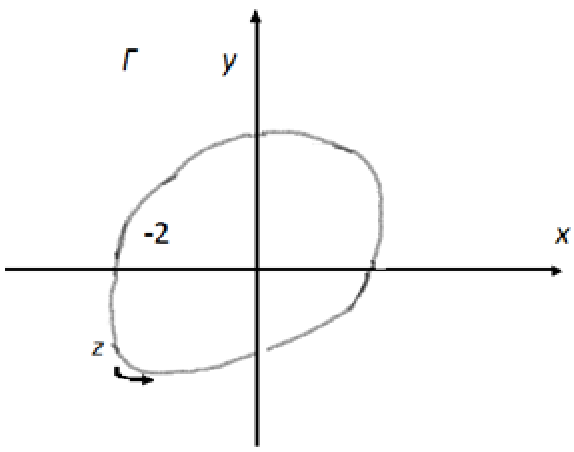
Yechilishi: $a=2i$ nuqta aylana ichida bo`lgani uchun (5.4) formulaga asosan:

$$f(z)=z^2\ ,$$

$$f(a)=f(2i)=(2i)^2=-4;\ I=2\pi i f(a)=-8\pi i.$$



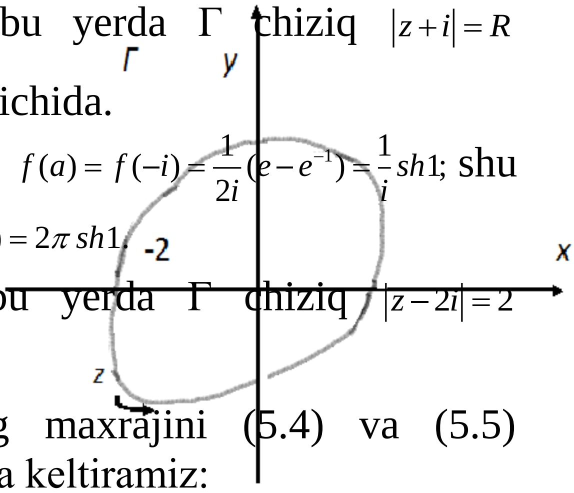
5.5-rasm



5.6-rasm

3-misol. $I=\oint\limits_{\Gamma}\frac{\sin z}{z+i}dz$ ni hisoblang, bu yerda Γ chiziq $|z+i|=R$ aylanadan iborat bo'lib, $a=-i$ nuqta uning ichida.

Yechilishi: $f(z)=\sin z=\frac{1}{2i}(e^{iz}-e^{iz}),\qquad f(a)=f(-i)=\frac{1}{2i}(e-e^{-1})=\frac{1}{i}sh1;$ shu sababli (5.4) formulaga muofiq $I=2\pi i\cdot f(a)=2\pi\, sh1.$



4-misol. $I=\oint\limits_{\Gamma}\frac{dz}{z^2+9}$ ni hisoblang, bu yerda Γ chiziq $|z-2i|=2$ aylanadan iborat.

Yechilishi: Bu misoldagi kasrning maxrajini (5.4) va (5.5) formuladagi maxrajga o'xshash ko'rinishga keltiramiz:

$$z^2+9=z^2-i^2 9=z^2-(3i)^2=(z-3i)(z+3i)$$

Bu yerda ikkita nuqta bor bo'lib, shulardan $a=3i$ yuqoridagi aylana ichidadir. Shu sababli berilgan integralni quyidagicha yozib olamiz:

$$I=\oint\limits_{\Gamma}\frac{\frac{dz}{z+3i}}{z-3i};\quad f(z)=\frac{1}{z+3i};\quad f(a)=f(3i)=\frac{1}{3i+3i}=\frac{1}{6i}.$$

Demak, (5.4) ga muvofiq

$$I=2\pi i\cdot f(a)=\frac{\pi}{3}$$

5-misol. $I=\oint\limits_{\Gamma}\frac{e^z dz}{(z+2)^2}$ ni hisoblang. Bu yerda Γ chiziq $a=-2$ nuqtani o'z ichiga olgan yopiq kontur (5.6-rasm)

Yechilishi: (5.5) formulaga asosan:

$$f(z)=e^z;\ n=3;\ f'''(z)=e^z;\ a=-2;\ f'''(-2)=e^{-2};$$

$$I=\frac{2\pi i}{n!}f'''(a)=\frac{2\pi i}{3!}\cdot e^{-2}=\frac{\pi i}{3e^2}.$$

6-misol. $I=\oint\limits_{\Gamma}\frac{dz}{(z-1)^3(z+1)^3}$ ni hisoblang bu yerda Γchiziq $|z-1|=R$ aylanadan iborat bo'lib, $R<2$.

Yechilishi: Maxrajdagi ikkita 1 va -1 nuqtadan birinchisi naylana ichida yotgani uchun I ni $I=\oint\frac{\frac{dz}{(z+1)^3}}{(z-1)^3}$ ko'rinishda yozib olib, yechishni davo ettiramiz:

$$f(z)=\frac{1}{(z+1)^3}=(z+1)^{-3};\ n=2;\ f'(z)=-3(z+1)^{-4};$$

$$f''(z)=12(z+1)^{-5},\ f''(1)=12\cdot 2^{-5}=\frac{3}{8};$$

(5.5) ga asosan

$$I=\frac{2\pi i}{2!}\cdot f^6(a)=\pi i\cdot\frac{3}{8}=\frac{3\pi i}{8}$$

7-misol. $I=\oint\limits_{|z|=2}\frac{dz}{z^2+1}$ ni hisoblang.

Yechilishi: I ni (5.4) va (5.5) larning biriga keltiramiz:

$$z^2+1=z^2-i^2=(z-1)(z+1)$$

Bu yerda i va $-i$ nuqtalar $|z|=2$ ayalana ichida bo'lganligi uchun integralni shunday ikkiga ajratimizki, natijada aylana ichida o'sha nuqtalardan faqat bittasi joylashgan bo'lsin, xolos. Buning uchun quyidagicha ish ko'ramiz:

$$\frac{1}{z^2+1}=\frac{1}{(z-i)(z+i)}=\frac{A}{z-i}+\frac{B}{z+i};\ 1=A(z+i)+B(z-i);$$

$$A=\frac{1}{2i};\ \ B=-\frac{1}{2i};\ \ \frac{1}{z^2+1}=\frac{\frac{1}{2i}}{z-1}-\frac{\frac{1}{2i}}{z+i}.$$

u holda

$$I=I_1+I_2=\frac{1}{2i}\oint\limits_{|z|=2}\frac{dz}{z-i}-\frac{1}{2i}\oint\limits_{|z|=2}\frac{dz}{z+i};$$

Bulardan ikkalasi ham $f(z)=1$ bo'lib, birinchisi $a=i$, ikkinchisida esa $a=-i$. Demak, (5.4) ga ko'ra $I=I_1+I_2=\frac{1}{2i}\cdot 2\pi i-\frac{1}{2i}\cdot 2\pi i=\pi-\pi=0$.

8-misol. $I=\oint\limits_{|z|=2}\frac{e^z dz}{z^2-1}$ ni hisoblang.

Yechilishi: Bu yerda ikkala $z=\pm 1$ nuqta ham $|z|=2$ aylana ichida joylashgani sababli integralni, yuqoridagi misolga o'xshash ikkiga ajratsamiz,

$$z^2-1=(z-1)(z+1);\ \ \frac{1}{z^2-1}=\frac{A}{z-1}+\frac{B}{z+1};\quad 1=A(z+1)+B(z-1);$$

$$A=\frac{1}{2};\ \ B=-\frac{1}{2};\ \ f(z)=e^z;\ \ f(1)=e,\ \ f(-1)=e^{-1}=\frac{1}{e}.$$

Bularga asosan

$$I=I_1+I_2=\frac{1}{2}\oint\limits_{|z|=2}\frac{e^z dz}{z-1}-\frac{1}{2}\oint\limits_{|z|=2}\frac{e^z dz}{z+1}=\frac{1}{2}\cdot 2\pi i\cdot e-\frac{1}{2}2\pi i\cdot e^{-1}=\frac{1}{2}\cdot(e-e^{-1})2\pi i=2\pi i\cdot sh1.$$

9-misol. $I=\oint\limits_{|z+1|=1}\frac{dz}{(z+1)(z-1)^3}$ ni hisoblang.

Yechilishi: $I=\oint\limits_{|z+1|=1}\frac{\frac{dz}{(z-1)^3}}{z+1}$, chunki $a=-1$ nuqta $|z+1|=1$ aylana ichida

demak,

$$f(z)=\frac{1}{(z-1)^3},\quad f(a)=f(-1)=\frac{1}{(-2)^3}=-\frac{1}{8},$$

(5.4) formulaga binoan

$$I = 2\pi i \cdot \left(-\frac{1}{8} \right) = -\frac{\pi i}{4}.$$

10-misol. $I = \oint\limits_{\Gamma} \frac{e^z dz}{z(1-z)^3}$ ni hisoblang, bu yerda Γ chiziq $|z| = \frac{1}{2}$ doiraning chegarasidadir.

Yechilishi: $a = 0$ nuqta o'sha chegara ichida joylashganligi uchun $f(z)$ deb ushbu funksiyani qabul qilamiz:

$$f(z) = \frac{e^z}{(1-z)^3}, \qquad f(0) = \frac{e^0}{1^3} = 1,$$

u holda (5.4) ga ko'ra

$$I = 2\pi i.$$

11-misol. Oldingi misolda $|z| < \frac{3}{2}$ deb I ni hisoblang.

Yechilishi: Bu doira ichida ikkita $a = 0$ va $a = 1$ nuqtalar joylashadi. Shu sababli integralni bir necha integrallarga ajratamiz. Buning uchun matematik analiz kursidagi ushbu qoidadan foydalanamiz:

$$\frac{1}{z(1-z)^3} = \frac{A}{z} + \frac{B}{(1-z)^3} + \frac{C}{(1-z)^2} + \frac{D}{1-z},\ 1 = A(1-z)^3 + Bz + Cz(1-z) + Dz(1-z)^2$$

bu yerda z ga $0, \pm 1, \pm 2$, qiymatlar berilsa,

$$A = B = C = D = 1$$

hosil bo'ladi. Demak,

$$I = I_1 + I_2 + I_3 + I_4 = \oint\limits_{\Gamma} \frac{e^z dz}{z} + \oint\limits_{\Gamma} \frac{e^z dz}{(1-z)^3} + \oint\limits_{\Gamma} \frac{e^z dz}{(1-z)^2} + \oint\limits_{\Gamma} \frac{e^z dz}{1-z} = \oint\limits_{\Gamma} \frac{e^z dz}{z} - \oint\limits_{\Gamma} \frac{e^z dz}{(z-1)^2} + \oint\limits_{\Gamma} \frac{e^z dz}{(z-1)^2} - \oint\limits_{\Gamma} \frac{e^z dz}{z-1}$$

$$f(z) = e^z,\ \ f'(z) = e^z.\ \ f''(z) = e^z;\ \ f(0) = e^0 = 1,\ \ f'(1) = e,\ \ f''(1) = e;$$

(5.4) va (5.5) formulalarga muofiq

$$I = 2\pi i \cdot 1 - \frac{2\pi i}{2!} e + 2\pi i e - 2\pi i \cdot e = (2-e)\pi i$$

Shunday qilib, $|z| < \frac{3}{2}$ bo'lganda

$$I = (2-e)\pi i$$

bo'lar ekan.

12-misol. $I = \oint\limits_{|z|=2} \frac{ch\, i\, z\, d\, z}{z^2 + 4z + 3}$ ni hisoblang.

Yechilishi: I ning maxraji (5.4) va (5.5) formulalarga to'g'ri kelmaydi, shu sababli uni ko'paytuvchilarga ajratib olamiz:

$$z^2 + 4z + 3 = (z+1)(z+3)$$

$a = -1$ nuqta $|z| = 2$ aylana ichida bo'lib,

$$f(z) = \frac{ch\, i\, z}{z+3}$$

funksiya doira ichida analitik bo'lgani uchun

$$I=\oint_{|z|=2}\frac{\frac{ch\,i\,z}{z+3}}{z+1}dz=2\pi i\cdot f(a)=2\pi i\cdot\frac{ch(-i)}{2}=\pi i\cdot\cos 1.$$

Demak, $I=\pi i\cdot\cos 1$.

13-misol. $I=\oint\limits_{|z-2|=3}\frac{e^{z^2}dz}{z^2-6}=\oint\limits_{|z-2|=3}\frac{e^{z^2}dz}{z(z-6)}$ ni hisoblang.

Yechilishi: Bu yerda $a=0$ nuqta $|z-2|=3$ aylana ichida bo'lgani uchun

$$f(z)=\frac{e^2}{z-6},\quad f(0)=\frac{e^0}{-6}=-\frac{1}{6};$$

$$I=2\pi i\left(-\frac{1}{6}\right)=-\frac{\pi i}{3}.$$

14-misol. Bundan oldingi misolda integrallash konturi $|z-2|=5$ aylanadan iborat bo'lsin. U holda bu misolni boshqa usul bilan yechish ham mumkin.

Yechilishi: $a=0$ nuqtani kichik C_1 aylana bilan va $a=6$ nuqtani C_2 aylana bilan o'rab olamiz. U holda o'sha aylana bilan chegaralangan uch bog'lamli sohada

$$f(z)=\frac{e^{z^2}}{z(z-6)}$$

funksiya analitik bo'ladi, chunki, o'sha sohada $z\neq 0$, $z\neq 6$. Shu sababli Koshining 2- teoremasiga binoan tashqi $|z-2|=5$ kontur bo'ylab olingan integral ichki C_1 va C_2 konturlar bo'ylab olingan integrallar yig'indisiga teng bo'ladi:

$$I=\oint_{|z-2|=5}\frac{e^{z^2}dz}{z^2-6z}=\oint_{C_1}\frac{e^{z^2}dz}{z(z-6)}=I_1+I_2,$$

bu yerda I_1 uchun:

$$f(z)=\frac{e^{z^2}}{z-6},\quad f(0)=\frac{e^0}{-6}=-\frac{1}{6}.$$

I_2 uchun:

$$f(z)=\frac{e^{z^2}}{z},\qquad f(6)=\frac{e^{6^2}}{6}.$$

Demak, (5.4) formulaga binoan

$$I=2\pi i\cdot\left(-\frac{1}{6}\right)+2\pi i\cdot\frac{e^{36}}{6}=\frac{e^{36}-1}{3}\pi i.$$

15-misol. $I=\oint\limits_{|z-1|=1}\frac{\sin\pi z dz}{(z^2-1)^2}$ ni hisoblang.

Yechilishi: Bu misolda $a=1$ nuqta $|z-1|=1$ aylana ichida bo'lgani uchun:

$$f(z)=\frac{\sin\pi z}{(z+1)^2},\quad n=1;\quad f'(z)=\frac{\pi\cos\pi z(z+1)^2-2(z+1)\sin\pi z}{(z+1)^4}=\frac{\pi\cos\pi z(z+1)-2\sin\pi z}{(z+1)^3},$$

$$f'(a)=f'(1)=-\frac{\pi}{4},$$

(5.5) formulaga binoan

$$I=2\pi i\cdot\left(-\frac{\pi}{4}\right)=-\frac{\pi^2 i}{2}.$$

16-misol. $I=\oint\limits_{|z|=2}\frac{ch\,z\,d\,z}{(z+1)^3(z-1)}$ ni hisoblang.

Yechilishi: Bu misolda aylana ichiga ikkita nuqta $a=1$ va $a=-1$ joylashgan. O'sha nuqtalarni mos ravishda C_1 va C_2 aylanalar bilan o'rab olaylik, natijada uch bog'lamli soha hosil bo'ladi. Koshining 2-teoremasiga binoan

$$I=I_1+I_2=\oint\limits_{C_1}\frac{ch\,zdz}{(z+1)^3(z-1)}+\oint\limits_{C_2}\frac{ch\,zdz}{(z+1)^3(z-1)}$$

bo'lib I_1 uchun:

$$f(z)=\frac{ch\,z}{(z+1)^3},\quad f(1)=\frac{ch1}{8};.$$

I_2 uchun: $f(z)=\frac{ch\,z}{z-1},\quad n=2;\ f'(z)=\frac{(z-1)sh\,z-ch\,z}{(z-1)^2},$

$$f''(z)=\frac{(ch\,z\,(z-1)+sh\,z)(z-1)^2-sh\,z\,(z-1)^2-2(z-1)\left[(z-1)\,shz-ch\,z\right]}{(z-1)^4}=$$

$$=\frac{((z-1)^2+2)ch\,z-2(z-1)sh\,z}{(z-1)^3},$$

$$f''(-1)=\frac{3ch1-2sh1}{-4},\ \text{demak,}$$

$$I=I_1+I_2=2\pi i\cdot\frac{ch1}{1!}=\frac{2\pi i}{1!}\cdot\frac{ch1}{8}-\frac{2\pi i}{2!}\cdot\frac{3ch1-2sh1}{4}=\frac{\pi i}{2}(sh1-ch1),$$

$$sh1-ch1=\frac{e-e^{-1}}{2}-\frac{e+e^{-1}}{2}=-\frac{1}{e}.$$

Shunday qilib, $I=-\frac{\pi i}{2e}$.

Mashqlar

Integrakni hisoblang. (Bu misollarni ko'pchiligi I integrallash konturi $|z-a|=R$ ko'rishdagi aylanadan iborat bo'lib, uning , markazi a nuqtada radiusi esa R ga teng.)

1. $\oint\limits_{|z-1|=\frac{1}{2}} \frac{e^{\frac{1}{z}}dz}{z^2+z}$; 2. $\oint\limits_{|z|=1} \frac{e^z \cos\pi z}{z^2+2z}dz$; 3. $\oint\limits_{|z-2|=2} \frac{ch\, z\, d\, z}{z^2-1}$; 4. $\oint\limits_{|z-1-i|=1} \frac{\sin\pi(z-1)}{z^2-2z+2}dz$; 5. $\oint\limits_{|z|=1} \frac{tg\, zdz}{\frac{1}{z+2}}$; 6. $\oint\limits_{|z|=3} \frac{\cos(z+\pi i)}{z(e^z+2)}dz$; 7. $\oint\limits_{|z|=5} \frac{dz}{z^2+16}$; 8. $\oint\limits_{|z|=4} \frac{dz}{\left(z^2+9\right)\left(z+9\right)}$; 9. $\oint\limits_{\Gamma} \frac{sh(z+1)}{z^2+1}dz$,

bu yerda Γ astroidadan iborat: $x^{\frac{2}{3}}+y^{\frac{2}{3}}=3^{\frac{2}{3}}$;

10. $\oint\limits_{|z|=2} \frac{\sin z\cdot\sin(z-1)}{z^2-z}dz$; 11. $\oint\limits_{|z|=1} \frac{\cos zdz}{z^3}$; 12. $\oint\limits_{|z|=1} \frac{sh^2 zdz}{z^3}$; 13. $\oint\limits_{|z-1|=1} \frac{\sin\frac{\pi z}{4}dz}{(z-1)^2\cdot(z-3)}$; 14. $\oint\limits_{|z|=2} \frac{z\, sh\, zdz}{(z^2-1)^2}$;

15. $\oint\limits_{|z-3|=6} \frac{zdz}{(z-2)^3(z+4)}$; 16. $\oint\limits_{|z-2|=3} \frac{ch\, e^{i\pi z}dz}{z^3-4z^2}$; 17. $\oint\limits_{|z|=\frac{1}{2}} \frac{\cos\frac{\pi}{z+1}}{z^3}dz$; 18. $\oint\limits_{|z-2|=1} \frac{e^{\frac{1}{z}}dz}{(z^2+4)^2}$; 19. $\oint\limits_{|z|=\frac{1}{2}} \frac{1-\sin z}{z^2}dz$;

20. $\oint\limits_{|z-1|=\frac{1}{2}} \frac{e^{iz}dz}{(z^2-1)^2}$; 21. $\oint\limits_{|z|=2} \frac{dz}{(z-a)^n(z-b)}$ $(|a|<z<|b|$, n = 1, 2, 3, ...); 22. $\oint\limits_{|z-i|=1} \frac{\cos z\, d\, z}{(z-i)^3}$.

Javoblar

1. 0; 2. πi; 3. $\frac{1}{2}\pi ch1i$; 4. $i\pi sh\pi$; 5. 0; 6. $\frac{3}{2}i\pi ch\pi$; 7. 0; 8. $-\frac{\pi i}{45}$; 9. $i2\pi\sin1ch1$; 10. 0; 11. $-\pi i$; 12. πi; 13. $-\frac{\pi(\pi+2)\sqrt{2}}{8}i$; 14. 0; 15. $-\frac{\pi i}{27}$; 16. $-\frac{\pi^2}{2}sh1$; 17. $\pi 3$; 18. $-\frac{3\pi\sqrt{e}}{32}i$; 19. $-2\pi i$; 20. $-\frac{1+i}{2}e^2$; 21. $-2\pi i(b-a)^{-n}$; 22. $-\pi i\, ch1$.

VI BOB

KOMPLEKS HADLI QATORLAR

Bu bobda hadlari kompleks sonlardan yoki kompleks argumentli funsiyalardan iborat bo‘lgan qatorlar jumladan, amaliyotda ko‘p ishlatiladigan darajali qatorlarga doir misollar bilan shug‘ullanamiz.

6.1-§. SONLI QATORLAR

Kompleks sonlarning ushbu cheksiz

$$z_1, z_2, z_3, ..., z_n, ... \qquad (6.1)$$

ketma-ketligi berilgan bo‘lsin, bunda $z_n = x_n + iy_n, \mathrm{n} = 1,2,3,...$ (6.1) vositasida

$$z_1 + z_2 + z_3 + ... + z_{n-1} + z_n + ... \qquad (6.2)$$

sonli qator tuziladi. Uni qisqacha

$$\sum_{n=1}^{\infty} z_n \qquad (6.2')$$

ko‘rinishda ham yoziladi. Odatda qatorning yaqinlashish yoki uzoqlashish masalasi muhim bo‘lib, ularning ta’riflarini berish uchun (2) ning hadlaridan ushbu xususiy yoig‘indilarni tuzamiz:

$$\begin{aligned} &S_1 = z_1, \\ &S_2 = z_1 + z_2, \\ &S_3 = z_1 + z_2 + z_3, \\ &\text{.........................} \\ &S_n = z_1 + z_2 + ... + z_n, \\ &\text{..........................} \end{aligned} \qquad (6.3)$$

Ta’rif. Agar $n \to \infty$ da S_n xususiy yig‘indi birgina chekli songa intilsa, u holda (6.2) qator yaqinlashuvchi deyiladi.

Demak bu ta’rifga muvofiq, agar (6.2) qator yaqinlashuvchi bo‘lsa,

$$S = \lim_{n \to \infty} S_n \qquad (6.4)$$

Chekli bo‘lib, S son (6.2) qatorning yig‘indisi deyiladi.

$$S = z_1 + z_2 + ... + z_n + ... = \sum_{n=1}^{\infty} z_n. \qquad (6.5)$$

Ta’rifga ko‘ra ,agar qator uzoqlashuvchi bo‘lsa, u holda S_n xususiy yig‘indi +∞ yoki -∞ ga intiladi yoki hech qanday songa intilmaydi.

Yuqoridagi (2) qatorni ushbu

$$(x_1 + x_2 + ... + x_n + ...) + i(\mathrm{y}_1 + \mathrm{y}_2 + ... + \mathrm{y}_n + ...)$$

ko‘rinishda yozib, bundan

$$(x_1+x_2+...+x_n+...) \quad (6.6) \text{ va } (y_1+y_2+...+y_n+...) \quad (6.7)$$

qatorlarni yozish ham mumkin. Agar (6.2) qator yaqinlashuvchi bo‘lsa, u holda (6.6) va (6.7) qatorlar ham yaqinlvshvdi va aksincha, chunki (6.4) ga binoan

$$S=\lim_{n\to\infty} S_n=\lim_{n\to\infty}(\sigma_n+\mathrm{i}\,\tau_n)=\sigma+\mathrm{i}\,\tau \qquad (6.8)$$

Bunda $S=\sigma+i\tau$ chekli son va

$$\sigma_n=x_1+x_2+...+x_n \quad \text{va} \quad \tau_n=y_1+y_2+...+y_n,$$

yuqoridagi (6.6) va (6.7) qatorlarning hadlari haqiqiy sonlardan iborat bo‘lgani uchun matematik analiz kursidagi yaqinlashish alomatlarini (6.2) qatorga ham tatbiq etish mumkin. Jumladan, biz Dalamber va Koshi alomatlaridan keng miqyosda foydalanamiz.

Endi (6.2') qator hadlarining modullaridan ushbu qatorni tuzamiz:

$$|z_1|+|z_2|+...+|z_n|+...=\sum_{n=1}^{\infty}|z_n| \qquad (6.9)$$

Agar bu qator yaqinlashuvchi bo‘lsa, u holda (6.2) ni absolyut yaqinlashuvchi qator deyiladi. Bu holda, qatorlar nazariyasidan ma’lumki, (6.6) va (6.7) ham absolyut yaqinlashadi.

1. *Dalamber alomati.* Agar

$$\lim_{n\to\infty}\frac{|z_n|}{|z_{n-1}|}=\lambda \qquad (6.10)$$

bo‘lib,

a) $\lambda<1$ bo‘lsa (6.9) qator yaqinlashadi;

b) $\lambda>1$ bo‘lsa, (6.9) qator uzoqlashadi;

v) $\lambda=1$ bo‘lsa, (6.9) qatorning yaqinlashish masalasi ochik qoladi, ya’ni uning yaqinlashuvchi yoki uzoqlashuvchi ekanligini ayta olmaymiz. Bu holda (6.9) qatorni boshqa usullar bilan tekshirishga to‘g‘ri keladi.

2. *Koshi alomati.* Agar

$$\lim_{n\to\infty}\sqrt[n]{|z_n|}=\lambda \qquad (6.11)$$

bo‘lib,

a) $\lambda<1$ bo‘lsa, (6.9) qator yaqinlashadi;

b) $\lambda>1$ bo‘lsa, (6.9) qator uzoqlashadi;

v) $\lambda=1$ bo‘lsa, (6.9) qatorning yaqinlashish masalasi ochiq qoladi.

Ma’lumki, yuqoridagi alomatlardan tashqari ikkita qatorni o‘zaro taqqoslab ko‘rish usullari ham mavjud.

Ba’zi murakkabroq qatorlarni tekshirishda quyidagi alomatlardan foydalanishga to‘g‘ri keladi. Faraz qilaylik, ushbu

$$a_1+a_2+a_3+...+a_n+a_{n+1}+...=\sum_{n=1}^{\infty}a_n \qquad (6.12)$$

qator musbat hadli hamda

$$B_n=n(1-\frac{a_{n+1}}{a_n}) \qquad (6.13)$$

bo‘lsin.

3. *Raabe alomati.* Agar etarli darajada katta p sonlar uchun

$B_n >> r$ (r — o‘zgarmas son)

bo‘lib, $r>1$ bo‘lsa, u holda (6.12) qator yaqinlashuvchi, agar biror p dan boshlab

$$B_n << 1$$

bo‘lsa, (6.12) qator uzoqlashuvchi bo‘ladi.

Endi quyidagi

$$a_1b_1+a_2b_2+...+a_nb_n+...=\sum_{n=1}^{\infty}a_nb_n \qquad (6.14)$$

qator berilgan bo‘lib (a_n, b_n kompleks sonlar bo‘lishi ham mumkin),

$$B_1=b_1, B_2=b_1+b_2,..., B_n=b_1+b_2+...+b_n,$$

$p=1,2,3,...$ bo‘lsin.

*4. Dirixle alomati***.** Agar ushbu

$$a_1, a_2, a_3,..., a_n,...,$$

ketma-ketlik monoton ravishda nolga intilib, ushbu

$$b_1+b_2+...+b_n+... \qquad (6.15)$$

qatorning xususiy yig‘indilaridan tuzilgan

$$B_1, B_2,..., B_n,... \qquad (6.16)$$

ketma-ketlik chegaralangan bo‘lsa, u holda (6.14) qator yaqinlashuvchi bo‘ladi.

5. *Abel alomati.* Agar ushbu

$$a_1, a_2, a_3,..., a_n,...$$

ketma-ketlik monoton va chegaralangan bo‘lib,

$$b_1+b_2+...+b_n+...$$

qator yaqinlashuvchi bo‘lsa, u holda (6.14) qator ham yaqinlashuvchi bo‘ladi.

Izox. Yuqoridagi ketma-kegliklarni qisqacha $\{a_n\}$ va $\{b_n\}$ ko‘rinishda yozish ham mumkin**.**

1-misol. $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^2}+\frac{4}{2^2}+...+\frac{n}{2^n}+...$ qatorni tekshiring.

YYechilishi: Bu qator (6.6) yoki (6.7) qator ko‘rinishda (hadlari haqiqiy sonlardan iborat). Dalamberning (6.10) alomatini qo‘llaymiz:

$$\lim_{n\to\infty}\left(\frac{n}{2^n}:\frac{n-1}{2^{n-1}}\right)=\frac{1}{2}\lim_{n\to\infty}\frac{n}{n-1}=\frac{1}{2}\lim_{n\to\infty}\frac{1}{1-\frac{1}{n}}=\frac{1}{2}$$

demak, $\lambda=\frac{1}{2}<1$ bo‘lgani sababli qator yaqinlashuvchi ekan.

2-misol. $\sum_{n=1}^{\infty} e^{in}$ qatorni tekshiring.

YYechilishi: Koshining (6.11) alomatini qo‘llasak bo‘ladi;

$$e^{in}=\cos n+i\sin n, n=1,2,3,..., \left|e^{in}\right|=\left|\cos n+i\sin n\right|=\sqrt{\cos^2 n+\sin^2 n}=1;\ \lim_{n\to\infty}\sqrt[n]{|z_n|}=1,\ \lambda=1.$$

Qatorning yaqinlashish masalasi ochiq qoladi. Shu sababli

$\cos n$ va $\sin n$.

qatorlarni alohida tekshirib ko‘ramiz. Ma’lumki,

$$1+\cos x+\cos 2x+...+\cos nx=\frac{\sin\frac{n+1}{2}\cos\frac{nx}{2}}{\sin\frac{x}{2}}$$

bunda $x=1$ deb olsak,

$$1+\cos 1+\cos 2+...+\cos n=\frac{\sin\frac{n+1}{2}\cos\frac{n}{2}}{\sin\frac{1}{2}}$$

Bundan ko‘rinadiki, $p\to\infty$ da kasrning suratidagi sinus va kosinuslar aniq bir songa intilmaydi. Shu sababli $\sum_{n=1}^{\infty} cosn$ qator, va demak, berilgan qator ham uzoqlashuvchi bo‘ladi.

3-misol. $\sum_{n=1}^{\infty}\frac{e^{in}}{n}$ qatorni tekshiring.

Yechilishi: Ma’lumki,

$$\left|e^{in}\right|=\left|\cos n+i\sin n\right|=1,\quad \left|\frac{e^{in}}{n}\right|=\frac{1}{n};$$

buning o‘ng tomoni uzoqlashuvchi garmonik qatorning umumiy hadi bo‘lgani uchun berilgan qator absolyut yaqinlashuvchi emas. Endi uning shartli yaqinlashuvchi ekanligini Dirixle alomati yordamida tekshirib ko‘raylik;

$$\frac{e^{in}}{n}=\frac{1}{n}(\cos n+i\sin n)=\frac{\cos n}{n}+i\frac{\sin n}{n};$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\cos n \quad \text{va} \quad \sum_{n=1}^{\infty}\frac{1}{n}\sin n$$

Dastlab bu qatorlarning birinchisini tekshirib ko‘raylik, buning uchun $a_n = \frac{1}{n}$ va $b_n = \cos n$ deb belgilaymiz. So‘ngra

$$B_n = \cos 1 + \cos 2 + \cos 3 + ... + \cos n = \frac{\cos\frac{n+1}{2}\sin\frac{n}{2}}{\sin\frac{1}{2}},$$

$$\left|\cos\frac{n+1}{2}\right| \le 1 \quad \text{va} \quad \left|\sin\frac{n}{2}\right| \le 1$$

bo‘lgani sababli

$$|B_n| \le \frac{1}{\sin\frac{1}{2}}. \quad \{a_n\} = \left\{\frac{1}{n}\right\}$$

ketma-ketlik monoton kamayib, nolga intiladi, demak, Dirixle alomatiga muvofiq, $\sum_{n=1}^{n}\frac{1}{n}\cos n$ qator yaqinlashuvchi. Xuddi shu usulda ikkinchi qatorning ham yaqinlashuvchi ekanligini ko‘rsatish mumkin.

Shunday qilib, berilgan qator shartli yaqinlashuvchi, lekin absolyut yaqinlashuvchi emas.

4-misol. $\sum_{n=1}^{\infty}\frac{n(2i-1)}{3^n}$ qatorni tekshiring.

Yechilishi: Dastlab uning absolyut yaqinlashish masalasini tekshirib ko’raylik. Bu misolga Koshi alomatini qo‘llash qulay:

$\sqrt[n]{|c_n|} = \sqrt[n]{n\left|\frac{2i-1}{3}\right|^n} = \sqrt[n]{n}\,\frac{|2i-1|}{3} = \frac{1}{3}\sqrt{5}\sqrt[n]{n},$ chunki $|2i-1| = \sqrt{2^2+(-1)^2} = \sqrt{5}; \sqrt[n]{n} = n^{\frac{1}{n}} = p;$

$$\frac{1}{n}\ln n = \ln p, \quad \lim_{n\to\infty}(\ln p) = \lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{\frac{1}{n}}{1} = 0; \quad \lim_{n\to\infty} \mathrm{n}^{\frac{1}{n}} = \lim_{n\to\infty} p = 1;$$

biz bu o‘rinda Lapital qoidasini tatbiq etdik, demak,

$$\lim_{n\to\infty}\sqrt[n]{|c_n|} = \frac{1}{3}\sqrt{n}\lim_{n\to\infty}\sqrt[n]{n} = \frac{\sqrt{5}}{3} < 1.$$

Shunday qilib, Koshi alomatiga muvofiq, berilgan qator absolyut yaqinlashar ekan.

5-misol. $\sum_{n=1}^{\infty}\frac{1}{(n+i)\sqrt{n}}$ qatorni tekshiring.

Yechilishi: Bu qatorni quyidagicha tekshirish qulayroq:

$$|c_n| = \frac{1}{\sqrt{n}|n+i|} = \frac{1}{\sqrt{n}\bullet\sqrt{n^2+1}} < \frac{1}{\sqrt{n}\bullet\sqrt{n^2}} = \frac{1}{n^{\frac{3}{2}}}$$

o‘ng tomondagi had ushbu

$$1+\frac{1}{2^p}+\frac{1}{3^p}+...+\frac{1}{n^p}+... \qquad (6.17)$$

Dirixle qatorining umumiy hadidan iborat bo‘lib, $p=\frac{3}{2}>1$ bulgani uchun (6.17) qator yaqinlashuvchi, $|c_n|$ esa uning umumiy hadidan kichik. Demak, berilgan qator absolyut yaqinlashuvchi ekan.

6-misol. $\sum\limits_{n=1}^{8}\frac{1}{\sqrt{n}+i}$ qatorni tekshiring.

Yechilishi: Dastlab absolyut yaqinlashish masalasini tekshirib ko‘raylik:

$$|c_n|=\frac{1}{\left|\sqrt{n}+i\right|}=\frac{1}{\sqrt{n+1}}\approx\frac{1}{n^{\frac{1}{2}}},$$

buni (6.17) Dirixle qatori bilan solishtirib ko‘rsak, $p=\frac{1}{2}<1.$ Demak, berilgan qator absolyut yaqinlashuvchi emas. Endi uning shartli yaqinlashishini tekshirish uchun haqiqiy va mavhum qismlarini ajratamiz:

$$\frac{1}{\sqrt{n}+i}=\frac{\sqrt{n}-i}{\left(\sqrt{n}\right)^2-i^2}=\frac{\sqrt{n}}{n+1}-\frac{i}{n+1}\approx\frac{\sqrt{n}}{n}-\frac{i}{n}=\frac{1}{\sqrt{n}}-\frac{i}{n}$$

(6.17) ga asosan $p=\frac{1}{2}<1$ va ikkinchisi garmonik qator hadi bo‘lgani uchun ushbu $\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ va $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ qatorlar uzoqlashuvchidir. Demak, berilgan qator shartli yaqinlashuvchi emas, ya’ni uzoqlashuvchidir.

7-misol. $\sum\limits_{n=1}^{\infty}\frac{i^n}{n}$ qatorni tekshiring.

Yechilishi: Koshi va Dalamber alomatlari bilan tekshiradigan bo‘lsak, $\lambda=1$ bo‘lib, qatorning yaqinlashish masalasi ochiq qoladi. Shu sababli Raabe alomatiga murojaat qilamiz;

$$\left|\frac{c_{n+1}}{c_n}\right|=\left|\frac{i^n}{n}:\frac{i^{n-1}}{n-1}\right|=\left|\frac{n-1}{n}i\right|=\frac{n-1}{n},\ |i|=1,$$

$$B_n=n\left(1-\frac{n-1}{n}\right)=n-(n-1),\quad \lim_{n\to\infty}\mathrm{B}_n=1$$

Demak, qator uzoqlashuvchi. Endi uning shartli yaqinlashishini tekshiraylik:

$$\frac{i^n}{n}=\frac{1}{n}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)^n=\frac{\cos\frac{\pi n}{2}}{n}+i\frac{\sin\frac{\pi n}{2}}{n}.$$

Oldingi 3-misolga o‘xshash, Dirixle alomatiga asosan ushbu qatorlar yaqinlashuvchidir:

$$\sum_{n=1}^{\infty}\frac{1}{n}\cos\frac{\pi n}{2} \quad \text{va} \quad \sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{\pi n}{2}$$

Demak, berilgan qator shartli yaqinlashuvchi ekan.

8-misol. $\sum_{n=1}^{\infty} n^{-\alpha} e^{in}$ qator α haqiqiy parametrning qaysi qiymatlarida yaqinlashuvchi bo‘ladi?

Yechilishi: Dalamber alomati yordamida tekshiramiz:

$$\left|\frac{c_n}{c_{n-1}}\right| = \left|\frac{n^{-\alpha}e^{in}}{(n-1)^{-\alpha}e^{i(n-1)}}\right| = \left(\frac{n-1}{n}\right)^{\alpha}\left|e^{i}\right| = \left(1-\frac{1}{n}\right)^{\alpha} < 1$$

bo‘lishi uchun $\alpha > 0$ bo‘lishi kifoya. Demak, berilgan qator $\alpha > 0$ da yaqinlashuvchi bo‘ladi.

9-misol. $\sum_{n=1}^{\infty}\frac{(a+1)(a+2)...(a+n)}{n!}e^{in}$ qator a ning qaysi qiymatlarida yaqinlashuvchi bo‘ladi?

Yechilishi: Dalamber alomati bo‘yicha

$$\left|\frac{c_n}{c_{n-1}}\right| = \left|\frac{(a+1)(a+2)...(a+n)(n-1)!}{(a+1)(a+2)...(a+n-1)\text{n}!}e^{i}\right| = \left|\frac{a+n}{n}\right|\left|e^{i}\right| = 1+\frac{a}{n}$$

Demak, $1+\frac{a}{n} < 1$ bo‘lishi uchun $a < 0$ bo‘lishi shart. a ning shu qiymatlarida berilgan qator yaqinlashadi,

10-misol. $\sum_{n=1}^{\infty}\frac{\cos in}{2^n}$ qatorni tekshiring.

Yechilishi: Dalamber alomatidan foydalanamiz:

$$\frac{c_n}{c_{n-1}} = \frac{\cos in}{2^n} : \frac{\cos i(n-1)}{2^{n-1}} = \frac{1}{2}\frac{\cos in}{\cos i(n-1)}$$

Ma’lumki, $\cos z = \frac{1}{2}\left(e^{iz}+e^{-iz}\right)$, $\cos in = \frac{1}{2}\left(e^{-n}+e^{n}\right)$, $\cos i(n-1) = \frac{1}{2}\left(e^{-(n-1)}+e^{(n-1)}\right)$,

$$\frac{c_n}{c_{n-1}} = \frac{1}{2}\frac{e^{-n}+e^{n}}{e^{-n}\cdot e+e^{n}\cdot e^{-1}} = \frac{1}{2}\frac{e^{n}+1}{e+e^{-1}\cdot e^{2n}} = \frac{1}{2}\frac{1+\frac{1}{e^{2n}}}{\frac{e}{e^{2n}}+e^{-1}}$$

bundan

$$\lambda = \lim_{n\to\infty}\left|\frac{c_n}{c_{n-1}}\right| = \frac{1}{2e^{-1}} = \frac{e}{2} > 1, \quad \lim_{n\to\infty}\frac{1}{e^{2n}} = 0, \quad e = 2,71828...$$

Demak, $\lambda > 1$ bulgani sababli qator uzoqlashuvchi.

Mashqlar

Quyidagi kompleks hadli qatorlarning yaqinlashuvchi yoki uzoqlashuvchi ekanligini aniqlang.

$$1.\sum_{n=1}^{\infty}\frac{n\sin in}{3^n};\ 2.\sum_{n=1}^{\infty}\frac{e^{2in}}{n\sqrt{n}};\ 3.\sum_{n=1}^{\infty}\frac{e^{i\frac{\pi}{n}}}{\sqrt{n}};\ 4.\sum_{n=1}^{\infty}\frac{(1+i)^n}{2^{\frac{n}{2}}\cos in};\ 5.\sum_{n=1}^{\infty}\frac{shi\sqrt{n}}{\sin in};\ 6.\sum_{n=1}^{\infty}\frac{\ln n}{shin};\ 7.\sum_{n=1}^{\infty}\frac{ch\frac{\pi i}{n}}{n^{\ln n}};$$

$$8.\sum_{n=1}^{\infty}\frac{n}{tgin\pi};\ 9.\sum_{n=1}^{\infty}\frac{e^{in}}{n^2};\ 10.\sum_{n=1}^{\infty}\frac{1}{\left[n+(2n-i)\right]^2};\ 11.\sum_{n=1}^{\infty}\frac{n(2+i)^n}{2^n};\ 12.\sum_{n=1}^{\infty}\left[\frac{n(2-i)+1}{n(3-2i)-3i}\right]^n.$$

Quyidagi qatorlar α haqiqiy parametrning qanday qiymatlarida yaqinlashuvchi bo‘ladi:

$$13.\sum_{n=1}^{\infty}n^{-\alpha}e^{\frac{\pi i}{n}};\qquad 14.\sum_{n=1}^{\infty}\left(n^2+1\right)^{-\alpha}\left(e^{\frac{\pi i}{n}}-1\right);\qquad 15.\sum_{n=1}^{\infty}i^n\frac{\left[\ln\left(n^2+1\right)\right]\alpha}{n};\qquad 16.$$

$$\sum_{n=1}^{\infty}2^{-\frac{n}{2}}(1+i)^n\left(\ln ctg\frac{\pi}{4n}\right)^{\alpha};$$

6.2-§. DARAJALI QATORLAR

Kompleks sohadagi darajali qatorning umumiy ko‘rinishi quyidagicha:

$$c_0+c_1z+c_2z^2+...+c_nz^n+...=\sum_{n=1}^{\infty}c_nz^n \qquad (6.18)$$

bunda $c_0, c_1, c_2...$ - o‘zgarmas kompleks sonlar. $z=x+iy$ kompleks o‘zgaruvchi – argument.

Agar (6.18) da z o‘rniga biror o‘zgarmas $z_0=x_0+iy_0$ son qo‘yish natijasida hosil bo‘lgan sonli qator yaqinlashuvchi bo‘lsa, u holda (6.18) qator z_0 *nuqtada yaqinlashadi* deyiladi. Barcha yaqinlashish nuqtalari to‘plami (6.18) ning *yaqinlashish sohasi* deyilib, u biror *doiradan* iborat bo‘ladi. Yaqinlashish doirasini ikki usulda topishni eslatib o‘tamiz:

a) yaqinlashish doirasini Dalamber alomati yordamida topish:

$$\lim_{n\to\infty}\left|\frac{c_nz^n}{c_{n-1}z^{n-1}}\right|=\lim_{n\to\infty}\left|\frac{c_n}{c_{n-1}}\right||z|<1.$$

Agar

$$L=\lim_{n\to\infty}\left|\frac{c_n}{c_{n-1}}\right|,\quad \frac{1}{L}=R \qquad (6.19)$$

deb belgilansa, izlanayotgan doira $|z|<R$ bo‘ladi;

b) yaqinlashish doirasini Koshi alomati yordamida topish.

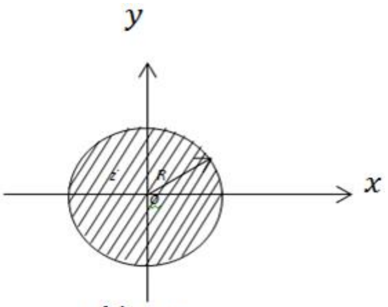
$$\lim_{n\to\infty}\sqrt[n]{|c_n|}\,|z|<1.$$

Agar

$$L=\varlimsup_{n\to\infty}\sqrt[n]{|c_n|},\quad \frac{1}{L}=R \tag{6.20}$$

deb belgilansa, izlanayotgan doira $|z|<R$ bo‘ladi (6.1-rasm). R darajali qatorning *yaqinlashish radiusi* deyiladi. Yaqinlashish radiusini topish uchun R ni topish kifoya, markaz esa nol nuqtada joylashgan.

Umuman darajali qator yaqinlashish doirasining radiusi R quyidagi Koshi-Adamar formulasi bilan hisoblanadi:



6.1-rasm

$$\rho=\frac{1}{R}=\varlimsup_{n\to\infty}\sqrt[n]{|c_n|}$$

Bu erdagi limit quyidagicha ta’riflanadi. Agar $[\alpha_n]$ sonli ketma-ketlik uchun

1) $[\alpha_n]$ dan limiti A songa teng bo‘lgan α_{n_k} qismiy ketma-ketlik ajratib olish mumkin, ya’ni

$\lim\limits_{k\to\infty}\alpha_{n_k}$ bo‘lsa;

2) har qanday $\alpha>0$ son uchun shunday N_ε butun musbat sonni ko‘rsatish mumkin bo‘lsaki, barcha $n>N_\varepsilon$ lar uchun $\alpha_n<A+\varepsilon$ bo‘lsa , u holda A son $[\alpha_n]$ ketma-ketlikning yuqori limiti deyiladi va $\varlimsup\limits_{n\to\infty}\alpha_n=A$ deb yoziladi.

Masalan,

$$0{,}1,\ 0{,}1+\frac{1}{2},\ 0{,}1+\frac{1}{3},\ldots,\ 0{,}1+\frac{1}{n},\ldots,$$

ketma-ketlik uchun $A=1$ son yuqoridagi ta’rifning ikkala shartini qanoatlantiradi. Shu sababli bu ketma-ketlik uchun 1 soni yuqori limitdir.

Agar $\lim\limits_{n\to\infty}|c_n|$ mavjud bo‘lsa, Koshi va Dalamber alomatlaridan bevosita foydalanish ham mumkin.

1-misol. $\sum\limits_{n=0}^{\infty}e^{in}z^n$ qatorning yaqinlashish doirasini toping.

Yichilishi. $c_n=e^{in},\ \sqrt[n]{|c_n|}=\sqrt[n]{|e^i|^n}=|e^i|=1,\quad L=\lim\limits_{n\to\infty}\sqrt[n]{|c_n|}=1,\ R=\frac{1}{L}=1.$

demak, qatorning yaqinlashish doirasi $|z|<1.$

2-misol.. $\sum\limits_{n-1}^{\infty}\left(\frac{z}{\text{in}}\right)^n$ qatorning yakinlashish doirasini toping

Yechilishi: $c_n=\frac{1}{(\text{in})^n},\ \varlimsup\limits_{n\to\infty}\sqrt[n]{|c_n|}=\varlimsup\limits_{n\to\infty}\frac{1}{|\text{in}|}=\varlimsup\frac{1}{n}=0=L,\ \ \frac{1}{L}=R=+\infty.$

Demak, qator butun kompleks tekislikda yaqinlashadi.

3-misol.. $\sum_{n=1}^{\infty} z^n ch\frac{i}{n}$ qatorning yaqinlashish radiusi va doirasini toping.

Yechilishi: Dalamber alomatidan foydalanamiz:

$$chz=\frac{e^z+e^{-z}}{2};\ c_n=ch\frac{i}{n}=\frac{1}{2}\left(e^{\frac{i}{n}}+e^{-\frac{i}{n}}\right)=\cos\frac{1}{n},\ \ c_{n-1}=ch\frac{i}{n-1}=\cos\frac{1}{n-1};$$

$$\overline{\lim_{n\to\infty}}\left|\frac{c_n}{c_n-1}\right|=\frac{\lim\limits_{n\to\infty}\cos\frac{1}{n}}{\lim\limits_{n\to\infty}\cos\frac{1}{n-1}}-\frac{\cos 0}{\cos 0}=1,$$

Demak, $R=1$.

4-misol.. $\sum_{n=1}^{\infty}\left(\frac{z}{\ln in}\right)^n$ qatorni tekshiring.

Yechilishi: Buni Koshi alomati yordamida tekshiramiz:

$$c_n=\frac{1}{(\ln in)^n},\ \ \sqrt[n]{|c_n|}=\frac{1}{\left|\ln n+\frac{\pi}{2}i\right|}=\frac{1}{\sqrt{ln^2 n+\frac{\pi}{4}^2}};\ L=\overline{\lim_{n\to\infty}}\sqrt[n]{|c_n|}=0,$$

Demak, $L=0,\quad R=\frac{1}{L}=+\infty.$

5-misol. $\sum_{n=1}^{\infty}(\mathrm{n+i})z^n$ qatorni tekshiring.

Yechilishi: Koshi alomatiga ko'ra

$$|c_n|=|n+i|=\sqrt{n^2+1};\quad \sqrt[n]{|c_n|}=\sqrt[n]{n^2+1}=(n^2+1)^{\frac{1}{n}}=w;$$

$$\lim_{n\to\infty}w=?\qquad \ln w=\frac{1}{n}\ln(n^2+1);$$

$$\lim_{n\to\infty}(\ln w)=\lim_{n\to\infty}\frac{1}{n}\ln(n^2+1)=\lim_{n\to\infty}\frac{(\ln(n^2+1))_n'}{(n)'}=\lim_{n\to\infty}\frac{\frac{2n}{n^2+1}}{1}=\lim_{n\to\infty}\frac{\frac{2}{n}}{1+\frac{1}{n^2}}=0;$$

demak,

$$\lim_{n\to\infty}(\mathrm{lnw})=0;\ \lim_{n\to\infty}w=1;\ \ L=\lim_{n\to\infty}\sqrt[n]{|c_n|}=1;\ R=1.$$

Shunday qilib, bu qatorning yaqilashish sohasi birlik doira ekan, $|z|<1$

6-misol. $\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2}z^n$ qatorni tekshiring.

Yechilishi: Dalamber alomatiga muvofiq,

$$|c_n|=\frac{(2n)!}{(n!)^2},\ |c_{n+1}|=\frac{(2n+2)!}{[(n+1)!]^2},\ \left|\frac{c_{n+1}}{c_n}\right|=\frac{(2n+1)(2n+2)}{(n+1)^2}=\frac{2\left(2+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^2},$$

$$L=\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|=4, \qquad R=\frac{1}{4}.\ \text{Demak, } |z|<\frac{1}{4}.$$

Mashiqlar

Qo'yidagi darajali qatorlarning yaqinlashish radiuslari va yaqinlashish doiralarini toping.

17. $\sum_{n=1}^{\infty}\frac{z^n}{n}$. 18. $\sum_{n=0}^{\infty}\frac{z^n}{n!}$. 19. $\sum_{n=1}^{\infty}n^n z^n$. 20. $\sum_{n=1}^{\infty}\frac{n}{2^n}z^n$. 21. $\sum_{n=1}^{\infty}\frac{n!}{n^n}z^n$. 22. $\sum_{n=1}^{\infty}2^n z^{n!}$. 23. $\sum_{n=1}^{\infty}z^{2^n}$.

24. $\sum_{n=1}^{\infty}\cos in\, z^n$. 25. $1+\sum_{n=1}^{\infty}\frac{a(a+1)(a+2)..,(a+n-1)\cdot\beta(\beta+1)(\beta+2)..,(\beta+\mathrm{n}-1)}{n!\tau(\tau+1)(\tau+2)...(\tau+\mathrm{n}-1)}2n$.

26. $\sum_{n=1}^{\infty}n^k z^n$. 27. $\sum_{n=0}^{\infty}\left[\ln\left(n+2j\right)\right]^k z^n$. 28. $\sum_{n=0}^{\infty}\frac{z^n}{(\mathrm{n}!)^a}, a>0$. 29. $\sum_{n=0}^{\infty}e^{-\sqrt{n}}z^n$.

30. $\sum_{n=1}^{\infty}\frac{(kn)!\mathrm{z}^n}{\mathrm{n}!(n+1)!...(\mathrm{n}+\mathrm{k}-1)!}$. 31. $\sum_{n=1}^{\infty}\frac{n(2+i)^n}{2^n}z^n$. 32. $\sum_{n=1}^{\infty}\frac{(z-2i)^n}{n3^n}$. 33. $\sum_{n=0}^{\infty}\frac{z^{2n}}{2^n}$. 34. $\sum_{n=1}^{\infty}e^{\frac{xi}{n}}z^n$.

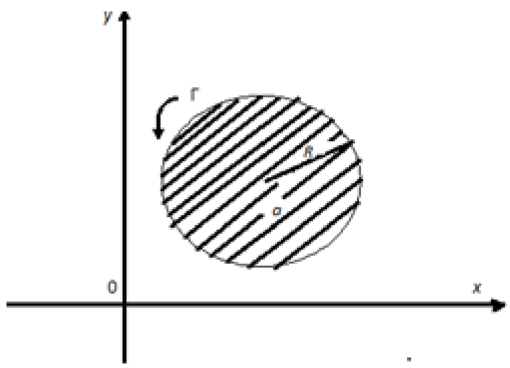
35. $\sum_{n=0}^{\infty}i^n z^n$. 36. $\sum_{n=1}^{\infty}\sin\frac{\pi!}{n}z^n$. 37. $\sum_{n=1}^{\infty}\cos^n\frac{\pi!}{\sqrt{n}}$. 38. $\sum_{n=0}^{\infty}\frac{z^n}{sh^a(1+in)}$.

6.3-§ TEYLOR VA MAKLOREN QATORLARI

Kompleks argumentli funksiyalar nazariyasidan ma'lumki, funksiya α nuqtaning biror atrofida analitik, ya'ni hosilaga ega bo'lsa, uni $(z-a)$ ga nisbatan musbat darajali qatorga yoyish mumkin. Ma'lumki, markazi α nuqta bo'lgan har qanday doira o'sha τ nuqtaning atrofi deyiladi. Shunday qilib, agar $f(z)$ funksiyani quyidagi

$$f(z)=A_0+A_1(z-a)+A_2(z-a)^2+\cdots+A_n(z-a)^n+\cdots=\sum_{n=0}^{\infty}A_n(z-a)^n \qquad (6.21)$$

ko'rinishida yozish mumkin bo'lsa u holda $f(z)$ funksiya a nuqta atrofida musbat darajali qatorga yoyilgan deyiladi. $A_0, A_1, A_2,...$ koeffitsientlarni $f(z)$ funksiya orqali ifodalash mumkin. Buning uchun $f(z)$ni $|z-a|<R$ doirada (6.2-rasm) analitik deb hisoblab, ketma-ket hosila olgandan so'ng, z ning 6.2-rasm o'rniga a qo'yiladi.



6.2-rasm

$$f'(z)=A_1+2A_2(z-a)+3A_3(z-a)^2+...+nA_n(z-a)^{n-1}+$$

$$f''(z)=2!A_2+2\cdot3A_3(z-a)+...+(n-1)nA_n(z-a)^{n-2}+.$$

$$f'''(z)=3!A_3+2\cdot3\cdot4A_4(z-a)+...+(n-2)(n-1)nA_n(z-a)^{n-3}+...,$$

....

Bulardan quyidagi munosabatlar kelib chiqadi :

$$A_0=f(a),\ A_1=f'(a),\ A_2=\frac{f''(a)}{z!},\ A_3=\frac{f'''(a)}{3!},...,A_n=\frac{f^n(a)}{n!},.... \qquad (6.22)$$

Bularni (6.21) ga qo'yish natijasida Teylorning mashhur

$$f(z)=f(a)+\frac{f(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+...+\frac{f^{(n)}(a)}{n!}(z-a)^n+...=\sum_{n=0}^{\infty}\frac{t^{(n)}(a)}{n!}(z-a)^n$$

(6.23)

qatori hosil bo'ladi. Koshinig oldingi boblarda qayd etilgan integral formulalariga muvofiq,

$$A_n=c_n=\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(z)dz}{(z-a)^{n+1}}-\frac{f^{(n)}(a)}{n!},\ n=0,1,2,... \qquad (6.24)$$

Bu formuladagi Γ chiziq $|z-a|<R$ doiraning chegarasi bo'lgan aylanadir.

Berilgan $f(z)$ funksiyani a nuqta atrofida Teylor qatoriga yoyish uchun uning c_n koeffitsientlarini (6.24) formuladan topish talab qilinadi. Lekin a nuqtaning atrofida $f(z)$ funksiya berilgan bo'lsa, (6.24) formula bo'yicha emas, balki ketma-ket $f(z)$ funksiyaning a dagi hosilalarini hisoblash mumkin.

Agar $a=0$ deb faraz qilinsa, (6.23) dan ushbu Makloren qatori kelib chiqadi:

$$f(z)=f(0)+f'(0)z+\frac{f''(0)}{2!}z^2+...+\frac{t^{(n)}(0)}{n!}z^n+..., \qquad (6.25)$$

Bu esa (z) funksiyaning $z=0$ nuqta atrofidagi yoyilmasi bo'lib, $|z|<R$ doirada absolut yaqinlashuvchidir. Ma'lumki, har bir qatorning o'ziga xos R yaqinlashish radiusi mavjud bo'lib, umuman, $0\le R<+\infty$.

Ba'zi soddaroq funksiyalarni qatorga yoyishda cheksiz kamayuvchi

$$a+aq+aq^2+...+aq^{n-1}+...=\frac{a}{1-q} \qquad (6.26)$$

geometrik progressiyadan foydalanish mumkin, bu yerda $|q|<1$.

Oldingi boblarning birida $e^z,\ \cos z,\ \sin z$ ning darajali qatorga yoyilmalarini ta'rif orqali kiritgan edik. Endi esa (6.25) ga asoslanib, ularning kelib chiqishini isbot qilish mumkin.

a) $f(z)=e^z$ bo'lsin. U holda

$$f'(z)=(e^z)'=e^z,\quad f''(z)=(e^z)'=e^z,\ ...,\quad f^{(n)}(z)=(e^z)'=e^z,\ \dots\quad z=0 \text{ da}$$

$$f(0)=e^0=1,\quad f'(0)=e^0=1,\quad f''(0)=e^0=1,\dots,$$

Bularni (6.25) ga qo'yamiz:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ... + \frac{z^n}{n} + ... = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad (6.27)$$

$0! = 1! = 1$ ekanligi ma'lum. Bunda $L = 0,\ R = +\infty.$

b) $f(z) = \cos z$ bo'lsin. U holda

$$f'(z) = -\sin z,\ f''(z) = -\cos z,\ f'''(z) = \sin z,\ f^{IV}(z) = cosz;\ \ldots$$

$z = 0$ da $f(0) = \cos 0 = 1,\quad f'(0) = -\sin 0 = 0,\quad f''(0) = -1,\quad f'''(0) = 0,\ f^{IV}(0) = 1;$

Bularni (6.25) ga qoyamiz :

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ... + (-1)^n \frac{z^{2n}}{(2n)!} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}. \qquad (6.28)$$

Bu qatorning yaqinlashish radiusi $R = +\infty.$

v) $f(z) = \sin z$ bo'lsin. U holda

$$f'(z) = \cos z,\quad f''(z) = -\sin z,\quad f'''(z) = -\cos z,\ f^{(4)}(z) = \sin z,\ \ldots$$

$$z = 0,\ f(0) = \sin 0 = 0,\ f'(0) = \cos 0 = 1,\ f''(0) = 0,\ f'''(0) = -1,\ f^{IV}(0) = 0, \ldots$$

Bularni (6.25) ga qo'yilsa,

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ... + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad (6.29)$$

Bu yerda ham $R = +\infty.$

Shunday qilib, uchala qator ham har qanday katta radiusli doirada absolut yaqinlashadi. (6.27) dagi z o'rniga iz qo'yib, ushbu *Eyler formulasini* keltirib chiqargan edik:

$$e^{iz} = \cos z + i \sin z \qquad (6.30)$$

Bunda $z = x + iy,\ i = \sqrt{-1},\quad i^2 = -1.$

Makloren qatoridan foydalanib, ko'p uchraydigan ushbu funksiyani $z = 0$ nuqta atrofida musbat darajali qatorga yoyish mumkin: $f(z) = (1+z)^a$, a-haqiqiy son.

$$f'(z) = a(1+z)^{a-1},\ f''(z) = a(a-1)(1+z)^{a-2},\ f'''(z) = a(a-1)(a-2)(1+z)^{a-3}, \ldots.$$

$$z = 0;\ f(0) = 1,\ f'(0) = a,\ f''(0) = a(a-1),\ f'''(0) = a(a-1)(a-2), \ldots$$

Bularni (6.25) ga qo'ysak ushbu

$$(1+z)^a = 1 + az + \frac{1}{2!} a(a-1) z^2 + ... + \frac{1}{n!} a(a-1)(a-2)...(a+n-1) z^n + ... \qquad (6.31)$$

qator hosil bo'lib, uning yaqinlashish radiusi $R = 1$.

Xususiy holda $a = -1$ bo'lsa, (6.31) dan quyidagi

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + ... + (-1)^n z^n + ... = \sum_{n=0}^{\infty} (-1)^n z^n,\ R = 1 \qquad (6.32)$$

qator kelib chiqadi.

Shuningdek, $f(z)=\ln(1+z)$ logarifmik funksiyani Makloren qatoriga yoyish mumkin:

$$f(z)=\ln(1+z),\ f'(z)=\frac{1}{1+z}=(1+z)^{-1},\ f''(z)=-(1+z)^{-2},\ f'''(z)=2!(1+z)^{-3},\ \ldots$$

$$z=0;\ f(0)=\ln 1=0,\ f'(0)=1,\ f''(0)=-1,\ f'''(0)=2,\ \ldots,$$

Bularni (25) ga qo'ysak,

$$\ln(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\ldots+(-1)^{n-1}\frac{z^n}{n}+\ldots=\sum_{n=0}^{\infty}(-1)^{n-1}\frac{z^n}{n}, \qquad (6.33)$$

bunda $R=1$.

Quyidagi funksiyalarni $z=0$ nuqta atrofida darajali qatorga yoying.

1-misol. $f(z)=\cos az.$

Yechilishi: $f(z)$ ning $z=0$ nuqtadagi ketma-ket hosilalarni hisoblaymiz:

$$f(z)=\cos az,\ f'(z)=(\cos az)'=-a\sin az,\ f''(z)=-a^2\cos az,$$

$$f'''(z)=a^3\sin az,\ f^{IV}(z)=a^4\cos az,$$

$$z=0 \text{ da } f(0)=1,\ f'(0)=0,\ f''(0)=-a^2,\ f'''(0)=0,\ f^{(IV)}(0)=a^4,\ldots$$

Bularni (6.25) Makloren qatoriga qo'yamiz:

$$\cos az=1-\frac{(az)^2}{2!}+\frac{(az)^4}{4!}-\frac{(az)^6}{6!}+\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{(az)^{2n}}{(2n)!}. \qquad (6.34)$$

2-misol: $f(z)=e^{az}.$

Yechilishi: $f(z)=e^{az}.\ f'(z)=(e^{az})'=ae^{az},\ f''(z)=a^2e^{az},\ f'''(z)=a^3e^{az},\ldots;z=0$ da

$$f(0)=e^0=1,\ f'(0)=a,\ f''(0)=a^2,\ f'''(0)=a^3,\ldots$$

Bularni (6.25) ga qo'yamiz:

$$e^{az}=1+az+\frac{az^2}{2!}+\frac{az^3}{3!}+\ldots+\frac{(az)^n}{n!}+\ldots=\sum_{n=0}^{\infty}\frac{(az)^n}{n!}. \qquad (6.35)$$

3-misol. $f(z)=e^{-az}.$

Yechilishi: Bundan oldingi misolda a o'rniga $(-a)$ qo'yish kifoya:

$$e^{-az}=\sum_{n=o}^{\infty}(-1)^n\frac{(az)^{2n}}{(2n)!}. \qquad (6.36)$$

4-misol. $f(z)=chaz$.

Yechilishi: $f(z)$ ning yoyilmasi giperbolik kosinusning ta'rifiga asosan oldingi ikki misoldan kelib chiqadi:

$$chaz=\frac{1}{2}\left(e^{az}+e^{-az}\right)=1+\frac{(az)^2}{2!}+\frac{(az)^4}{4!}+\ldots+\frac{(az)^{2n}}{(2n)!}+\ldots=\sum_{n=0}^{\infty}\frac{(az)^{2n}}{(2n)!}. \qquad (6.37)$$

5-misol. $f(z)=shaz.$

Yechilishi: $f(z)$ning yoyilmasini topish uchun (6.35) dan (6.36) ni ayirish kifoya

$$shaz = \frac{1}{2}\left(e^{az} - e^{-az}\right) = \sum_{n=0}^{\infty} \frac{(az)^{2n+1}}{(2n+1)!}. \tag{6.38}$$

6-misol. $f(z) = \frac{1}{1-z}$.

Yechilishi: Yuqoridagi (6.31) ga muvofiq.

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots + z^n + \ldots = \sum_{n=0}^{\infty} z^n \tag{6.39}$$

bunda $R = 1$, yaqinlashish doirasi $|z| < 1$.

7-misol. $f(z) = \frac{1}{(1-z)^2}$.

Yechilishi: Ma'lumki, (6.39) qatorni o'zining yaqinlashish doirasi ichida hama-had differensiallash mumkin, u holda

$$\left(\frac{1}{1-z}\right)' = \frac{1}{(1-z)^2} = \left(1 + z + z^2 + \ldots + z^n + \ldots\right)' = 1 + 2z + 3z^2 + \ldots + nz^{n-1} + \ldots.$$

Demak,

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n.$$

8-misol. $f(z) = \frac{2}{(1+z)^3}$.

Yechilishi: Yuqoridagi (6.31) formulada a o'rniga (-3) qo'yish kifoya:

$$\frac{2}{(1+z)^3} = \sum_{n=0}^{\infty} (-1)^n (n+1)(n+2) z^n, \mathrm{R} = 1, |z| < 1$$

9-misol. $f(z) = \frac{1}{a^2 + z^2}$, $a -$ o'zgarmas son.

Yechilishi: Buni (6.32) ga asosan quyidagicha hisoblash ham mumkin:

$$\frac{1}{a^2+z^2} = \frac{1}{a^2} \cdot \frac{1}{1+\left(\frac{z}{a}\right)^2}, \left(\frac{z}{a}\right) = t; \ \frac{1}{a^2+z^2} = a^{-2} \cdot \frac{1}{1+t^2} = a^{-2}\left(1 - t^2 + t^4 - t^6 - \ldots + (-1)^n t^{2n} + \ldots\right)$$

Demak, $\frac{1}{a^2+z^2} = \sum_{n=0}^{\infty} (-1)^n a^{-2} \left(\frac{z}{a}\right)^{2n}, \left|\frac{z}{a}\right| < 1,$ ya'ni $|z| < |a| \neq 0.$

10-misol. $f(z) = \frac{1}{\left(1+z^2\right)^2}$.

Yechilishi: Bu misolni hal qilish uchun oldingi qatorda $a = 1$ deb faraz qilib, so'ngra kvadratga ko'tarish kifoya:

$$\frac{1}{1+z^2}=1-z^2+z^4-z^6+\ldots;$$

$$\frac{1}{\left(1+z^2\right)^2}=\left(1-z^2+z^4-z^6+\ldots\right)\left(1-z^2+z^4-z^6+\ldots\right)=\sum_{n=0}^{\infty}(-1)^n(n+1)z^{2n}, |z|<1, R=1$$

11-misol. $f(z)=\dfrac{1}{(z+1)(z-2)}$.

Yechilishi: Dastlab $f(z)$ ni eng sodda kasrlarga ajratib olamiz:

$$\frac{1}{(z+1)(z-2)}=\frac{A}{z+1}+\frac{B}{z-2}, \qquad\qquad 1=A(z-2)+B(z+1),$$

$z=-1$ bo'lganda $A=-\dfrac{1}{3}, z=2$ bo'lganda $B=\dfrac{1}{3};$ endi (6.32) va (6.39) ga muvofiq, bu ikki kasrni qatorga yoyish mumkin:

$$-\frac{1}{3}\cdot\frac{1}{1+z}+\frac{1}{3}\cdot\frac{1}{z-2}=-\frac{1}{3}\cdot\frac{1}{1+z}-\frac{1}{6}\cdot\frac{1}{1-\frac{z}{2}}=-\frac{1}{3}\left(1-z+z^2-z^3+\ldots\right)-\frac{1}{6}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\ldots\right]=$$

$$=\frac{1}{3}\left(-1+z-z^2+z^3-z^4+\ldots\right)-\frac{1}{3}\left(\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\frac{z^3}{2^4}+\ldots\right)=$$

$$=\frac{1}{3}\left[\left(-1-\frac{1}{2}\right)+\left(1-\frac{1}{2^2}\right)z+\left(-1-\frac{1}{2^3}\right)z^2+\left(1-\frac{1}{2^4}\right)z^3+\ldots\right],$$

Shunday qilib, $\dfrac{1}{(z+1)(z-2)}=\dfrac{1}{3}\sum\limits_{n=0}^{\infty}\left[(-1)^{n+1}-\dfrac{1}{2^{n+1}}\right]z^n.$

12. $f(z)=\dfrac{1}{1+z+z^2}$

Yechilishi: Tekshirishlar ko'rsatadiki, bu misolni yuqoridagi usul bilan yechish maqsadga muvofiq emas. Shu sababli uni sun'iy usul bilan hal qilamiz. Kasrning suratini maxrajiga bo'lamiz, u holda

$$
\begin{array}{r|l}
-\;1 & 1+z+z^2 \\ \cline{2-2}
\underline{1+z+z^2} & 1-z+z^3-z^4+z^6-z^7+z^9 \\
-z-z^2 & \\
-\;\underline{-z-z^2-z^3} & \\
-\;z^3 & \\
\underline{z^3+z^4+z^5} & \\
-z^4-z^5 & \\
-\;\underline{-z^4-z^5-z^6} & \\
z^6 &
\end{array}
$$

—

$z^6+z^7+z^8$

$-z^7-z^8$

—

$-z^7-z^8-z^9$

— z^9

....

$$\frac{1}{1+z+z^2}=(1+z^3+z^6+z^9+\cdots)-(z+z^4+z^7+z^{10}+\cdots)=\sum_{n=0}^{\infty}(z^{3n}-z^{3n+1}).$$

13-misol. $f(z)=\sin^2 z.$

Yechilishi:

$$f'(z)=2\sin z\cos z=\sin 2z,\ f''(z)=2\cos 2z,$$

$$f'''(z)=-2^2\sin 2z,\ f^{IV}(z)=-2^3\cos 2z,...;$$

$z=0$ da

$$f(0)=0,\ f'(0)=0,\ f''(0)=2,\ f'''(0)=0,\ f^{IV}(0)=-2^3,\ f^{(V)}(0)=0,\ f^{(VI)}(0)=2^6,\ldots.$$

Bularni Makloren qatoriga qo'yish natijasida

$$\sin^2 z=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^{2n-1}}{(2n)!}z^{2n}$$

hosil bo'ladi.

Bu misolni boshqa yo'llar bilan hal qilish ham mumkin, masalan, $\sin z$ ni qatorga yoyib, so'ngra kvadratga ham ko'tarilsa bo'ladi.

14-misol. $f(z)=ch z\cos z$.

Yechilishi: Ma'lumki,

$$chz=\cos iz;\cos\alpha\cos\beta=\frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right],$$

$$f(z)=chz\cos z=\cos iz\cos z=\frac{1}{2}\left[\cos(1+i)z-\cos(1-z)z\right].$$

Bundan ketma-ket hosilalar olib, $z=0$ da hisoblaymiz.

$$f'(z)=-\frac{1}{2}\left[(1+i)\sin(1+i)z+(1-i)\sin(1-i)z\right],$$

$$f''(z)=-\frac{1}{2}\left[(1+i)^2\cos(1+i)z+(1-i)^2\cos(1-i)z\right],$$

$$f'''(z)=\frac{1}{2}\left[(1+i)^3\sin(1+i)z+(1-i)^3\sin(1-i)z\right],$$

$$f^{(4)}(z)=\frac{1}{2}\left[(1+i)^4\cos(1+i)z+(1-i)^4\cos(1-i)z\right],\ldots$$

$z=0$ da $f(0)=\frac{1}{2}[\cos 0+\cos 0]=1;$

$$f'(0)=0;\ f''(0)=-\frac{1}{2}\left[(1+i)^2+(1-i)^2\right];\ f'''(0)=0;\ f^{(4)}(0)=\frac{1}{2}\left[(1+i)^4+(1-i)^4\right],$$

bularga o'xshash yig'indilarni hisoblab chiqish uchun kompleks sonning trigonometrik formasidan va Muavr formulasidan foydalanamiz, u holda

$$1+i=\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}), \qquad 1-\mathrm{i}=\sqrt{2}(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}),$$

$$(1+\mathrm{i})^4+(1+i)^4=(\sqrt{2})^4\left(\cos\frac{4\pi}{4}+i\sin\frac{4\pi}{4}\right)+(\sqrt{2})^4\left(\cos\frac{4\pi}{4}-i\sin\frac{4\pi}{4}\right)=$$

$$=2^2\cdot 2\cos\pi=-2^3, \qquad \cos\pi=-1, \qquad f''''(0)=-2^2$$

va hokazo. Bularning hammasini Makleron qatoriga qo'yish natijasida ushbu qator hosil bo'ladi:

$$ch\cos z=\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}z^{4n}}{(4n)!}$$

15-misol. $f(z)=Arctgz, \qquad Arctg0=0.$

Yechilishi: Ma'lumki,

$$\frac{1}{1+z^2}=1-z^2+z^4-z^6+z^8-z^{10}+\ldots, \qquad \left|z^2\right|=\left|z\right|^2<1, \quad R=1;$$

buning ikki tomonidan, o'sha doira ichida faraz etib, integral olishga haqlimiz, ya'ni

$$\int\frac{dz}{1+z^2}=Arctgz+C=\int(1-z^2+z^4-z^6+\ldots)dz=z-\frac{z^3}{3}+\frac{z^5}{5}-\frac{z^7}{7}+\ldots;$$

$$z=0 \quad da \qquad Arctg0+C=0, \qquad C=0,$$

demak,

$$Arctgz=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{2n+1}.$$

Quyidagi funksiyalarn $z-a$ ga nisbatan musbat darajali qatorga yoying.

16-misol. $f(z)=\dfrac{z}{z+2}$ funksiyani $z-1$ ga nisbatan, ya'ni $z=a=1$ nuqta atrofida darajali qatorga yoying.

$$f(\mathrm{z})=\frac{z}{z+2}=\frac{(z+2)-2}{z+2}=1-\frac{2}{z+2}=1-2(z+2)^{-1}, \qquad f'(z)=2(z+2)^2,$$

$$f''(z)=-2^2(z+2)^{-3}, \qquad \mathrm{f}'''(\mathrm{z})=3\cdot 4(z+2)^{-4}, \qquad f''''(\mathrm{z})=-3\cdot 4^2(z+2)^{-5},$$

$$f'''''(z)=3\cdot 4^2\cdot 5(z+2)^{-6},\ldots$$

Endi yuqoridagilarni $z=1$ da hisoblab, topilgan qiymatlarni (6.23) Teylor qatorlariga qo'yish natijasida ushbu

$$\frac{z}{z+2}=\frac{1}{3}+2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(z-1)^n}{3^{n+1}}, \qquad R=3$$

qator hosil bo'ladi.

17-misol. $f(z)=\ln z$ funksiyani $z-1$ ning darajalari bo'yicha qatorga yoying.

Yechilishi:

$$f(z)=\ln z, \qquad f'(z)=\frac{1}{z}=z^{-1}, \qquad f''(z)=-z^{-2}, \qquad f'''(z)=2!z^{-3},$$

$$f^{iv}(z)=-3!z^{-4}, \qquad f^{v}(z)=4!z^{-5}, \ \dots \ ;$$

Natijada ushbu hosil bo'ladi: $\ln z=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(z-1)^n}{n}.$

18-msiol. $f(z)=\frac{1}{1+3z}$ funksiyani $z+2$ ga nisbatan darajali qatorga yoying.

Yechilishi: (6.39) qatordan foydalanamiz:

$$\frac{1}{1+3z}=\frac{1}{1+3(z+2-2)}=\frac{1}{-5+3(z+2)}=-\frac{1}{5}\cdot\frac{1}{1-\frac{3}{5}(z+2)}=$$

$$=-\frac{1}{5}\left[1+\frac{3}{5}(z+2)+\frac{3^2}{5^2}(z+2)^2+\frac{3^3}{5^3}(z+2)^3+\dots\right]=-\frac{1}{5}\sum_{n=0}^{\infty}\left[\frac{3}{5}(z+2)\right]^n;$$

$$R=\frac{5}{3}$$

ya'ni, bu qatorning yaqinlashish sohasi $|z+2|<\frac{5}{3}.$

19-misol. $f(z)=\sin(2z+1)$ funksiyani $z+1$ ga nisbatan Teylor qatoriga yoying.

Yechilishi:

$$f(z)=\sin(2z+1), \qquad f'(z)=2\cos(2z+1), \qquad f''=-2^2\sin(2z+1),$$

$$f'''(z)=-2^3\cos(2z+1), \qquad f''''(z)=2^4\sin(2z+1), \quad \dots \ .$$

$z=a=-1$ da $f(-1)=\sin(-1)=-\sin 1, \qquad f'(-1)=2\cos(-1)=2\cos 1,$

$$f''(-1)=2^2\sin 1, \qquad f'''(-1)=-2^3\cos 1, \qquad f''''(-1)=-2^4\sin 1, \ \dots .$$

Bularni Teylor qatoriga qo'yamiz:

$$f(z)=\sin(2z+1)=-\sin 1+2(z+1)\cos 1+\frac{2}{2!}(z+1)^2\sin 1-\frac{2^3}{3!}(z+1)^3\cos 1-$$

$$-\frac{2^4}{4!}(z+1)^4\sin 1+\dots,$$

bunda $\sin 1 \approx \sin 57°17'45''$ va $\cos 1 \approx \cos 57°17'45''$.

Mashqlar

Quyidagi funksiyalarni Makleron qatoriga, ya'ni $z=0$ nuqta atrofida darajali qatorga yoying.

39. $\frac{1}{az+b}$, $b \neq 0$. **40.** chz. **41.** shz. **42.** $ch^2 z$. **43.** $\ln\frac{1+z}{1-z}$. **44.** $\ln(z^2-3z+2)$. **45.** $\frac{z(z+a)}{(a-z)^3}$. $|z|<a$, $a \neq 0$. **46.** $\cos\sqrt{z}$. **47.** $\frac{z}{z^2+i}$. **48.** $sh^2\frac{z}{2}$. **49.** $\ln(2+z)$.

Quyidagi funksiyalarni Teylor qatoriga, ya'ni $z=a$ nuqta atrofida darajali qatorga yoying.

50. $ch(1-z)$ funksiyani $z-\left(1-\frac{z_i}{2}\right)$ ga nisbatan. **51.** $\cos z$ funksiyani $z+\frac{\pi}{4}$ ga nisbatan. **52.** $\frac{1}{3-2z}$ funksiyani $z-3$ ga nisbatan. **53.** $\frac{z^2-5}{z^2-4z+3}$ funksiyani $z-2$ ga nisbatan. **54.** $\frac{z}{(z+1)(2-z)}$ funksiyani $z-1$ ga nisbatan.

6.4-§. MANFIY DARAJALI QATORLAR

Yuqorida qaralgan Teylor qatori $z-a$ ga nisbatan musbat darajali qator bo'lib, uning yaqinlashish sohasi R radiusli biror doiradan iborat o'sha $z-a$ bo'yicha *manfiy darajali* qatorlar ham ko'p uchrab turadi. Uni quyidagicha yozish mumkin:

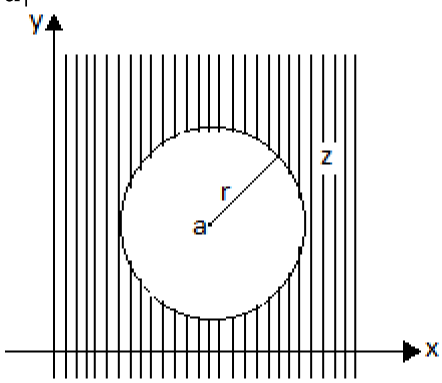
$$\frac{b_1}{z-a}+\frac{b_2}{(z-a)^2}+\frac{b_3}{(z-a)^3}+\ldots+\frac{b_n}{(z-a)^n}+\ldots=\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n} \qquad (6.40)$$

Bunda $a,\ b_1,\ b_2,\ \ldots$ o'zgarmas sonlar bo'lib, xususiy holda haqiqiy sonlar bo'lishi ham mumkin:

$$\overline{\lim_{n\to\infty}}\left|\frac{b_n}{(z-a)^n}:\frac{b_{n-1}}{(z-a)^{n-1}}\right|=\overline{\lim_{n\to\infty}}\left|\frac{b_n}{b_{n-1}}\right|\cdot\frac{1}{|z-a|}<1,$$

ya'ni

$$\overline{\lim_{n\to\infty}}\left|\frac{b_n}{b_{n-1}}\right|=r, \qquad |z-a|>r \qquad (6.41)$$



6.3-rasm

Demak, manfiy darajali (6.40) qatorning yaqinlashish sohasi $|z-a|>r$ *doira tashqarisidan* iborat ekan (6.3-rasm). Umuman olganda $r>0$ bo'ladi. Odatda $b_n = c_{-n}$ deb, (6.40) qator quyidagicha yoziladi:

$$\ldots + c_{-n}(z-a)^{-n} + c_{-(n-1)}(z-a)^{-(n-1)} + \ldots + c_{-2}(z-a)^{-2} + c_{-1}(z-a)^{-1} = \sum_{n=-\infty}^{-1} c_n (z-a)^n. \quad (6.40')$$

Xususiy holda $a=0$ bo'lsa, $(6.40')$ dan quyidagi qator kelib chiqadi:

$$\sum_{n=-\infty}^{-1} c_n z^n \qquad (6.42)$$

u holda (6.41) doira tashqarisida $|z|>r$ ko'rinishiga ega bo'ladi.

Quyidagi manfiy darajali qatorlarning yaqinlashish sohalarini toping.

1-misol. $\sum_{n=1}^{\infty} e^n (iz)^{-z}$ **.**

Yechilishi:

$$c_{-n} = e^n i^{-n} = \frac{e^n}{i^n}, \qquad c_{-(n-1)} = \frac{e^{n-1}}{i^{n-1}}, \quad \mathrm{r} = \lim_{n\to\infty}\left|\frac{c_{-n}}{c_{-(n-1)}}\right| = \lim \frac{e}{|i|} = e, \quad r = e,$$

Demak, qatorning yaqinlashish sohasi $|z|>e$ doira tashqarisidan iborat ekan.

2-misol. $\sum_{n=1}^{\infty} \frac{z^{-n}}{\cos i n}.$

Yechilishi:

$$c_{-n} = \frac{1}{\cos i n} = \frac{1}{chn}, \qquad c_{-(n-1)} = \frac{1}{ch(n-1)},$$

$$r = \lim_{n\to\infty}\left|\frac{c_{-n}}{c_{-(n-1)}}\right| = \lim \frac{ch_{(n-1)}}{ch_n} = \lim_{n\to\infty} \frac{e^{n-1} + e^{-(n-1)}}{e^n + e^{-n}} = \lim_{n\to\infty} \frac{e^{-1} + e\cdot e^{-2n}}{1 + e^{-2n}} = e^{-1},$$

chunki $\lim_{n\to\infty} e^{-2n} = \lim_{n\to\infty} \frac{1}{e^{2n}} = 0, \quad r = e^{-1}, \quad |z| > e^{-1}.$

3-misol. $\sum_{n=1}^{n} \frac{3^n + 1}{(z+2i)^n}.$

Yechilishi:

$$a = -2i, \quad c_{-n} = 3^n + 1, \quad c_{-(n-1)} = 3^{n-1} + 1, \qquad \left|\frac{c_{-n}}{c_{-(n-1)}}\right| = \frac{3^n + 1}{3^{n-1} + 1} = \frac{3 + \frac{1}{3^{n-1}}}{1 + \frac{1}{3^{n-1}}},$$

$$r=\lim_{n\to\infty}\left|\frac{c_{-n}}{c_{-(n-1)}}\right|=3,\quad \lim_{n\to\infty}\frac{1}{3^n}\ \text{demak},\qquad |z+2i|>3.$$

4-misol. $\sum_{n=1}^{\infty}\frac{n2^{-n}}{(z-2-i)^n}.$

Yechilishi: $\quad z-2-i=z-(2+i),\qquad a=2+i,\qquad c_{-n}=n2^{-n},$

$$c_{-(n-1)}=(n-1)\,2^{-(n-1)},\quad r=\lim_{n\to\infty}\left|\frac{c_{-n}}{c_{-(n-1)}}\right|=\frac{1}{2},\qquad |z-2-i|>\frac{1}{2}.$$

Quyidagi funksiyalarni manfiy darajali qatorlarga yoying.

5-misol. $\frac{1}{z-b}$ funksiyani $\frac{b}{z}$ ga nisbatan manfiy darajali qatorga yoying.

Yechilishi: Ma’lumki $|q|<1$ bo’lsa,

$$a+aq+aq^2+\ldots+aq^{n-1}+\ldots=\frac{a}{1-q}$$

bo’ladi. Shunga asosan

$$\frac{1}{z-b}=\frac{1}{z\left(1-\frac{b}{z}\right)}=\frac{1}{z}\cdot\left[1+\frac{b}{z}+\left(\frac{b}{z}\right)^2+\left(\frac{b}{z}\right)^3+\ldots+\left(\frac{b}{z}\right)^n+\ldots\right]=\sum_{n=0}^{\infty}\frac{b^n}{z^{n+1}},$$

$$\left|\frac{b}{z}\right|<1,\quad |z|>|b|.$$

Xususiy holda, agar $b=1$ bo’lsa, quyidagi qatorga ega bo’lamiz.

$$\frac{1}{z-1}=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}},\qquad |z|>1.$$

6-misol. $\frac{1}{(z-b)^2}$ funksiyani $\frac{b}{z}$ ga nisbatan qatorga yoying.

Yechilishi: Oldingi qatordan o’sha yaqinlashiah sohasida z bo’yicha hosila olish kifoya:

$$\left(\frac{1}{z-b}\right)'=\sum_{n=0}^{\infty}b^n(z^{-n-1})',$$

bundan $\quad \frac{1}{(z-b)^2}=\sum_{n=0}^{\infty}\frac{(n+1)b^n}{z^{n+2}},\qquad |z|>|b|.$

7-misol. $\frac{z^2}{z^2+b^2}$ funksiyani $\frac{b}{z}$ ga nisbatan darajali qatorga yoying.

Yechilishi: Buni suniy usulda bajaramiz:

$$\frac{z^2}{z^2+b^2}=\frac{(z^2+b^2)-b^2}{z^2+b^2}=1-\frac{b^2}{z^2+b^2}=1-\frac{\left(\frac{b}{z}\right)^2}{1+\left(\frac{b}{z}\right)^2}=1-\left(\frac{b}{z}\right)^2+\left(\frac{b}{z}\right)^4-\left(\frac{b}{z}\right)^6+$$

$$+\ldots=\sum_{n=0}^{\infty}\left(-1\right)^n\left(\frac{b}{z}\right)^{2n}.$$

Bu ba'zan quyidagicha ham yoziladi:

$$\frac{z^2}{z^2+b^2}=\sum_{n=-\infty}^{0}\left(-1\right)^n b^{-2n}z^{2n},\ |z|>|b|.$$

Mashqlar

Quyidagi qatorlarning yaqinlashish sohalarini aniqlang:

55. $\sum_{n=1}^{\infty}\frac{1}{(1-i)^n z^n}$. **56.** $\sum_{n=1}^{\infty}\frac{1}{4^n(1+z)^n}$. **57.** $\sum_{n=1}^{\infty}\frac{(z+1-i)^{-n}}{n+i}$. **58.** $\sum_{n=1}^{\infty}\frac{(\sqrt{2}+i\sqrt{2})^n}{z^n}$. **59.** $\sum_{n=-\infty}^{-1}b^{-2(n+1)}z^{2n}$. **60.** $\sum_{n=-\infty}^{-1}(b-a)^{-n-1}(z-a)^n,\ a\neq b.$ **61.** $-\sum_{n=-\infty}^{-1}\frac{1}{2n+1}a^{-2n-1}z^{2n+1}$.

Quyidagi funksiyalarni manfiy darajali qatorlarga yoying:

62. $\frac{1}{2z-5}$ funksiyani z ga nisbatan, ya'ni z - nuqta atrofida manfiy darajali qatorga yoying.

63. $z^2\cos\frac{1}{z}$ funksiyani $z=0$ nuqta atrofida yoying.

64. $\frac{1}{z^2-a^2}$ funksiyani $z=0$ nuqta atrofida yoying.

65. $\frac{1}{z-a}$ funksiyani $z=0$ ga nisbatan yoying.

6.5-§. LORAN QATORI.

Loran darajali qatorlarining umumiy ko'rinishi:

$$\begin{aligned}&\ldots+\frac{c_{-n}}{\left(z-a\right)^n}+\frac{c_{-(n-1)}}{(z-a)^{n-1}}+\ldots+\frac{c_{-2}}{\left(z-a\right)^2}+\frac{c_{-1}}{z-a}+\\&+c_0+c_1\left(z-a\right)+c_2\left(z-a\right)^2+\ldots+c_n\left(z-a\right)^n+\ldots=\sum_{n=-\infty}^{\infty}c_n\left(z-a\right)^n\end{aligned}\qquad(6.43)$$

bo'lib, u ikki qismdan iborat:

a) To’g’ri qismi:

$$P = c_0 + c_1(z-a) + c_2(z-a)^2 + ... + c_n(z-a)^n + ... \qquad (6.44)$$

b) Bosh qismi:

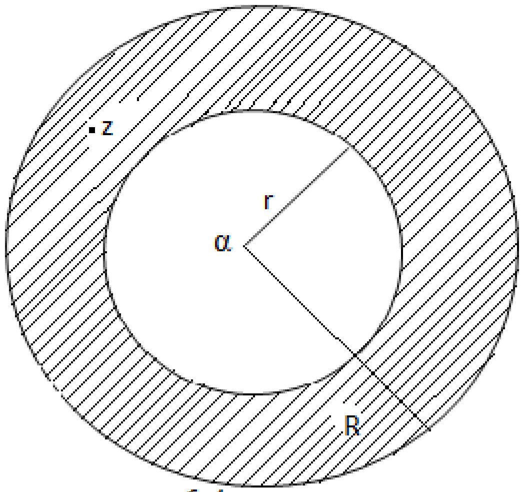
$$Q = ... + c_{-n}(z-a)^{-n} + c_{-(n-1)}(z-a)^{-(n-1)} + \ldots + c_{-2}(z-a)^{-2} + c_{-1}(z-a)^{-1} =$$

$$= \sum_{n=-\infty}^{-1} c_n (z-a)^n \qquad (6.45)$$

Ma’lumki, bu qatorning yaqinlashish sohasi $|z-a| < R$ doiradan iborat;

Buning yaqinlashish sohasi $|z-a| > r,$ ya’ni doira tashqarisidan iborat. Shunday qilib, Loran umumiy qatorining yaqinlashish sohasi markazi a nuqta bo’lgan r va R radiusli aylanalar orasidagi halqadan iborat bo’ladi (6.4-rasm):

$$r < |z-a| < R$$



6.4-rasm

Bu yerda uch hol bo’lishi mumkin:

1. Agar $r < R$ bo’lsa, halqa mavjud bo’lib, Loran qatori o’sha halqa ichida absolyut yaqinlashadi.

2. Agar $r = R$ bo’lsa, halqa faqat aylanadan iborat bo’lib, Loran qatori uning ba’zi nuqtalaridagina yaqinlashuvchi bo’lishi mumkin.

3. Agar $r > R$ bo’lsa, hech qanday halqa yo’q. Shu sababli Loren qatori uzoqlashuvchi bo’ladi.

Agar Loran qatori berilgan bo’lsa, uning ikkala qismini alohida tekshirish halqani topish lozim bo’ladi. Aksincha, agar funksiya berilgan bo’lsa, uni Loran qatoriga yoyish uchun oldindan berilgan halqaga qarab ish ko’rish talab qilinadi.

Kompleks argumentli funksiyalar nazariyasidan ma’lumki, Loran qatorining c_n koeffitsienti ushbu formula orqali ifodalanadi:

$$c_n = \frac{1}{2\pi l}\oint_{\Gamma} \frac{f(z)dz}{(z-a)^{n+1}}, \quad n = 0, \pm1, \pm2, \pm3, ... \qquad (6.46)$$

Bunda a son ikkala aylana markazi, $f(z)$ - halqa ichida analitik funksiya; Γ chiziq – halqa ichidagi a markazli ixtiyoriy aylanadan iborat. Odatda c_n koeffitsienti (6.46) formula orqali topish qiyin bo'lganda, ba'zan sun'iy yo'l bilan ish ko'rishga to'g'ri keladi.

1-misol. Quyidagi Loran qatorlarini yaqinlashish sohalarini aniqlang.

$$\sum_{n=1}^{\infty}\left(\frac{2}{z}\right)^n+\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n.$$

Yechilishi: Dastlab, qatorning to'gri qismini tekshirib ko'raylik:

$$P=\sum_{n=0}^{\infty}\frac{z^n}{4^n};\qquad c_n=\frac{1}{4^n},\qquad c_{n-1}=\frac{1}{4^{n-1}},\qquad L=\lim_{n\to\infty}\left|\frac{c_n}{c_{n-1}}\right|=\lim_{n\to\infty}\frac{4^{n-1}}{4^n}=\frac{1}{4},$$

$$L=\frac{1}{4},\qquad R=\frac{1}{L}=4,\qquad |z|<4.$$

Endi qatorning bosh qismini tekshiramiz:

$$Q=\sum_{n=1}^{\infty}\frac{2^n}{z^n}=\sum_{n=-\infty}^{-1}2^{-1}z^n;\ c_{-n}=2^n,\ c_{-(n-1)}=2^{n-1}.$$

$$r=\lim_{n\to\infty}\left|\frac{c_{-n}}{c_{-(n-1)}}\right|=\lim_{n\to\infty}\frac{2^n}{2^{n-1}}=2,\ \ |2|>2.$$

Shunday qilib, biz izlayotgan soxa $2<|z|<4$ halqadan iborat ekan.

2-misol. $\sum_{n=1}^{\infty}\frac{\sin in}{(z-i)^n}+\sum_{n=0}^{\infty}\frac{(z-i)^n}{n!}.$

Yechilishi: Qatorning to'g'ri qismi:

$$P=\sum_{n=0}^{\infty}\frac{(z-i)^n}{n!},\ a=i\ c_n=\frac{1}{n!},\ c_{n-1}=\frac{1}{(n-1)!}$$

$$L=\lim_{n\to\infty}\frac{(\mathrm{n}-1)!}{n!}=\lim_{n\to\infty}\frac{1}{n}=0,\ R=\frac{1}{L}=+\infty,\ \ |z-i|<+\infty$$

Qatorning bosh qismi:

$$Q=\sum_{n=1}^{\infty}\frac{\sin in}{(z-i)^n};\ c_{-n}=\sin in=\frac{1}{2i}(\mathrm{e}^{-n}-\mathrm{e}^n);$$

$$c_{-(n-1)}=\frac{\mathrm{e}^{-(n-1)}-\mathrm{e}^{n-1}}{2i};\qquad r=\lim_{n\to\infty}\left|\frac{c_{-n}}{c_{-(n-1)}}\right|=\lim_{n\to\infty}\frac{e^{-2n}-1}{e\cdot e^{-2n}-e^{-1}}=\frac{1}{e^{-1}}=e,$$

$$r=e;\ |z-i|>e.$$

Demak, soha: $e<|z-i|<\infty.$

3-misol. $\sum_{n=1}^{\infty}\frac{2^n-1}{(z+1)^n}+\sum_{n=0}^{\infty}\frac{(z+1)^n}{(n+i)^n}.$

Yechilishi: qatorning to'g'ri qismi:

$$P=\sum_{n=0}^{\infty}\frac{(z+1)^n}{(n+i)^n},\ a=-1;\ c_n=\frac{1}{(n+i)^n}:\ \sqrt[n]{|c_n|}=\frac{1}{\sqrt{n^2+1}}.$$

$$L=\lim_{n\to\infty}\sqrt[n]{|c_n|}=0,\ \ R=\frac{1}{L}=+\infty,\ \ |z+1|<\infty$$

Qatorning bosh qismi:

$$Q=\sum_{n=1}^{\infty}\frac{2^n-1}{(z+1)^n},\ c_{-n}=2^n-1,\ r=\lim_{n\to\infty}\left|\frac{c_{-n}}{c_{-(n-1)}}\right|=\lim_{n\to\infty}\frac{2^n-1}{2^{n-1}-1}=\lim_{n\to\infty}\frac{2-\frac{1}{2^{n-1}}}{1-\frac{1}{2^{n-1}}}=2,$$

$$|z+1|>2.$$

Demak, $2<|z+1|<\infty$, yani markazi a=-1 nuqtadan iborat bo'lgan doira tashqarisi.

4-misol. $\sum_{n=1}^{\infty}\frac{n}{(z+1-i)^n}+\sum_{n=0}^{\infty}n(z+1-i)^{n.}$

Yechilishi:.Qatorning to'gri qismi P da

$$c_n=n,\ c_{n-1}=n-1,\ L=\lim_{n\to\infty}\frac{n}{n-1},\ \lim_{n\to\infty}\frac{1}{1-\frac{1}{n}}=1,\ R=1.$$

Qatorning bosh qismi Q da

$$c_{-n}=n,\ c_{-(n-1)}=n-1,\ r=\lim_{n\to\infty}\frac{n}{n-1}=1,\ r=1.$$

Shunday qilib, r=R=1, yani halqa aylanadan iborat bo'lib, qator uzoqlashuvchi.,

5-misol. $\sum_{n=1}^{\infty}\frac{a^n}{z^n}+\sum_{n=0}^{\infty}\frac{z^n}{b^n},\quad (b\neq 0).$

Yechilishi: Qatorning to\`g\`ri qismi P da

$$\left|\frac{c_n}{c_n-1}\right|=\left|\frac{1}{b_n}\bullet\frac{1}{b^{n-1}}\right|=\frac{1}{|b|},\ L=\frac{1}{|b|},\ R=\frac{1}{L}=|b|,\ |z|<|b|.$$

Qatorning bosh qismi Q da $\left|\frac{c_{-n}}{c_{-(n-1)}}\right|=\left|\frac{a^n}{a^{n-1}}\right|=|a|,\ r=|a|,\quad |z|>|a|.$

Demak, $|a|<|b|$. bo\`lsa, yaqinlashish halqasi: $|a|<|z|<|b|$. Agar $|a|>|b|$. bo\`lsa, qator uzoqlashuvchidir.

6-misol. $-\dfrac{i}{2(z-i)}+\dfrac{1}{4}\sum_{n=0}^{\infty}(-1)^n\dfrac{(z-i)^n}{(2i)^n}$

Yechilishi: Bu misolning bosh qismi birgina haddan iborat bo\`lgani uchun uning faqat to\`g\`ri qismining yaqinlashish sohasini tekshiramiz:

$$a=i; \quad |c_n|=\frac{1}{4}\left|\frac{1}{(2i)^n}\right|=\frac{1}{4}\cdot\frac{1}{2^n}, \quad \left|\frac{c_n}{c_{n-1}}\right|=\frac{1}{4}\cdot\frac{1}{2^n}:\left(\frac{1}{4}\cdot\frac{1}{2^{n-1}}\right)=\frac{1}{2} \qquad |i|=1;$$

$$L=\frac{1}{2}, \quad R=\frac{1}{L}=2,$$

$$0<|z-i|<2.$$

7-misol. $f(z)=\dfrac{\sin z}{z}$ ni $z=0$ atrofida Loran qatoriga yoying.

Yechilishi: Ma\`lumki, $\sin z=z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\dfrac{z^7}{7!}+......\mathrm{R}=\infty, |z|<\infty.$

Shu sabali: $\dfrac{\sin z}{z}=1-\dfrac{z^2}{3!}+\dfrac{z^4}{5!}-\dfrac{z^6}{7!}+......$

8-misol. $f(z)=z^4\cos\dfrac{1}{z},\ z=0$ nuqta atrofida Loran qatoriga yoying.

Yechilishi: Ma'lumki,

$$\cos\frac{1}{z}=1-\frac{1}{2!}\left(\frac{1}{z}\right)^2+\frac{1}{4!}\left(\frac{1}{z}\right)^4-\frac{1}{6!}\left(\frac{1}{z}\right)^6+.....$$

Shu sababli

$$z^4\cos\frac{1}{z}=z^4-\frac{z^2}{2!}+\frac{1}{4!}-\frac{1}{6!z^2}+\frac{1}{8!z^4}-\frac{1}{10!z^6}+...,\mathrm{R}=+\infty.$$

9-misol. $\dfrac{e^z-1}{z}$ ni $z=0$ nuqta atrofida Loran qatoriga yoying.

Yechilishi: Ma'lumki,

$$e^z=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+...+\frac{z^n}{n!}+...=\sum_{n=0}^{\infty}\frac{z^n}{n!}.$$

Shu sababli

$$\frac{e^z-1}{z}=1+\frac{z}{2!}+\frac{z^2}{3!}+...+\frac{z^{n-1}}{n!}+...=\sum_{n=0}^{\infty}\frac{z^{n-1}}{n!},\ R=+\infty.$$

10-misol. $f(z)=\frac{1+\cos z}{z^4}$ ni z=0 nuqta atrofida Loran qatoriga yoying.

Yechilishi: Ma'lumki,

$$\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\dots\ .$$

Buning ikkala tomoniga 1 ni qo'shib, so'ngra z^4 ga bo'lamiz:

$$\frac{1+\cos z}{z^4}=\frac{2}{z^4}-\frac{1}{2!z^2}+\frac{1}{4!}-\frac{z^2}{6!}+\dots,\ R=+\infty.$$

11-misol. $f(z)=\frac{\sin^2 z}{z}$ ni $z=0$ nuqta atrofida Loran qatoriga yoying.

Yechilishi: Ma`lumki,

$$\sin^2 z=\frac{1-\cos 2z}{2}=\frac{1}{2}-\frac{1}{2}\left[1-\frac{(2z)^2}{2!}+\frac{(2z)^4}{4!}-\frac{(2z)^6}{6!}+\dots\right]=$$

$$=\left[\frac{(2z)^2}{2!}-\frac{(2z)^4}{4!}+\frac{(2z)^6}{6!}-\frac{(2z)^8}{8!}+\dots\right]\frac{1}{2}.$$

Shunga asosan

$$\frac{\sin^2 z}{z}=\left[\frac{2^2 z}{2!}-\frac{2^4 z^3}{4!}+\frac{2^6 z^5}{6!}-\frac{2^8 z^7}{8!}+\dots.\right]\frac{1}{2},\ R=+\infty.$$

12-misol. $f(z)=\frac{1}{(z-2)(z-3)}$ funksiyani 2<$|z|$<3 halqada Loran qatoriga yoying.

Yechilishi: Buniung uchun berilgan funksiyani eng sodda kasirga ajratib olamiz:

$$\frac{1}{(z-2)(z-3)}=\frac{A}{(z-2)}+\frac{B}{(z-3)},1=A(z-3)+B(z-2),$$

$z=2$ bo'lganda A=-1; $A=1,\ z=3$ bo'lganda $B=1$. Berilgan halqaga binoan ish ko'ramiz:

a) $|z|>2$, yani $\frac{2}{|z|}<1$ bo'lishi uchun quyidagi ko'rinishda yozamiz:

$$-\frac{1}{z-2}=-\frac{1}{z}\cdot\frac{1}{1-\frac{2}{z}}=-\frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+...\right]=-\sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n};$$

b) $|z|<3$, yani $\left|\frac{z}{3}\right|<1$ bo'lishi uchun quyidagi ko'rinishda yozamiz:

$$\frac{1}{z-3}=-\frac{1}{3-z}=-\frac{1}{3}\cdot\frac{1}{1-\frac{z}{3}}=-\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n.$$

Demak,

$$\frac{1}{(z-2)(z-3)}=-\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{2}{z}\right)^n-\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n.$$

Bulardan birinchisi qatorning bosh qismi bo'lib, ikkinchsi esa to'g'ri qismidir.

13-misol. $f(z)=\frac{1}{z-2}$ ni $z=0$ va $z=\infty$ nuqtalarda qatorga yoying.

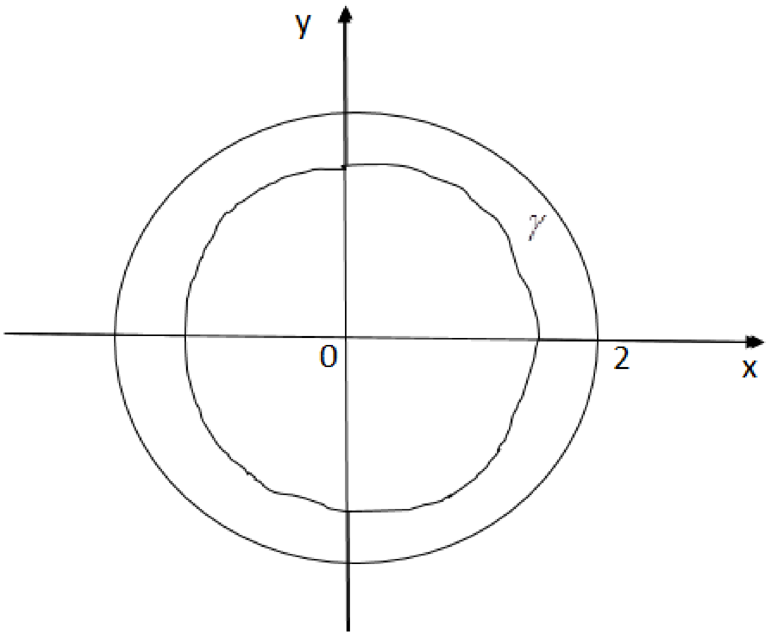
Yechilishi: Qatorga yoyishni ikki xil usul bilan bajaramiz :

1–usul. $f(z)=\frac{1}{z-2}$, $z=2$ nuqtadan boshqa hamma nuqtalarda analitik. Avval $|z|<2$ aylanada qatorga yoyamiz. Aylana markazi $a=0$, shuning uchun (6.46) formuladan foydalansak, quyidagiga ega bo'lamiz:

$$f(z)=\sum_{-\infty}^{\infty}C_n z^n, \qquad C_n=\frac{1}{2\pi i}\int_{\gamma}\frac{\frac{1}{\xi-2}d\xi}{\xi^{n+1}}$$

larni hisoblaylik. Manfiy n larda $(n<0)$ $C_n=0$, chunki bu holda integral ostidagi ifoda γ - kontur ichida analitik (6.5-rasm). Endi $n>0$ bo'lsin

$$C_n=\frac{1}{2\pi i}\int_{\gamma}\frac{\frac{1}{\xi-2}d\xi}{\xi^{n+1}}$$



6.5-rasm

Yuqori tartibli hosilalar uchun Koshi integrali formulasidan foydalanishimiz mumkin, chunki $f = \frac{1}{\xi - 2}$ γ - kontur ichida analitik:

$$C_n = \frac{1}{n!}\left(\frac{1}{\xi - 2}\right)^{(n)}_{|\xi=0} = \frac{(-1)^n n!}{n!(-2)^{n+1}} = -\frac{1}{2^{n+1}}$$

Shunday qilib $|z| < 2$ ichida $f(z) = \frac{1}{z-2}$ funksiya Loran qatoriga quyidagicha yoyiladi:

$$\frac{1}{z-2} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} .$$

Hosil bo'lgan Loran yoyilmasining bosh qismi yo'q, chunki $f(z) = \frac{1}{z-2}$, $|z| < 2$ sohaning hamma nuqtalarida analitik.

Bu holda Loran qatori Teylor qatori bilan mos tushadi.

2–usul. $f(z) = \frac{1}{z-2}$ ning Loran qatorini cheksiz kamayuvchi geometrik progressiya formulasidan foydalanib hosil qilamiz :

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q} , \quad |q| < 1 \tag{6.47}$$

$\frac{1}{z-2}$ ni quyidagicha yozib olaylik :

$$\frac{1}{z-2} = \frac{-1}{2\left(1 - \frac{z}{2}\right)} ; \qquad \left|\frac{z}{2}\right| < 1$$

chunki biz $|z|<2$ sohadagi yoyilmani qidirayapmiz. (6.47) formulani qo’llab, quyidagiga ega bo’lamiz :

$$-\frac{1}{2}\frac{1}{\left(1-\frac{z}{2}\right)}=-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^{n}=-\sum_{n=0}^{\infty}\frac{z^{n}}{2^{n+1}}$$

Shunday qilib, yagonalik teoremasiga mos ravishda ikkala usul bilan ham bir xil Loran qatorini hosil qildik.

Endi $z=\infty$ atrofida yoyilmani topamiz. Buning uchun $z=\frac{1}{\xi}$ almashtirish bajaramiz. U holda $z=\infty$ atrofidagi yoyilma $\zeta=0$ atrofidagi yoyilmaga to’g’ri keladi :

$$\frac{1}{z-2}=\frac{1}{\frac{1}{\xi}-2}=\frac{\xi}{1-2\xi}$$

Biz $\frac{\xi}{1-2\xi}$ funksiyani $\xi=0$ atrofida qatorga yoyish masalasiga kelamiz. Yana geometrik progressiya yig’indisi formulasidan foydalanamiz :

$$\frac{\xi}{1-2\xi}=\xi\sum_{n=0}^{\infty}(2\xi)^{n}=\sum_{n=0}^{\infty}2^{n}\xi^{n+1}$$

Nol nuqta atrofida $|2\xi|<1$. O’zgaruvchi z ga qaytib, quyidagiga ega bo’lamiz :

$$\frac{1}{z-2}=\sum_{n=o}^{\infty}2^{n}\xi^{n+1}=\sum_{n=0}^{\infty}2^{n}\frac{1}{z^{n+1}}$$

Bu ifoda $z=\infty$ atrofidagi yoyilmadir.

14-misol. Quyidagi funksiyani $z=0$, $z=a$ nuqtalar atrofida va $|a|<|z|<|b|$ halqada qatorga yoying :

$$f(z)=\frac{1}{(z-a)(z-b)},\quad 0<|a|<|b|$$

Yechilishi: Funksiyani o’zgartirib yozib olaylik :

$$\frac{1}{(z-a)(z-b)}=\frac{1}{(b-a)}\left(\frac{1}{(z-b)}-\frac{1}{(z-a)}\right) \qquad (6.48)$$

I. $z=0$ atrofida qatorga yoyaylik:
Yoyilma quyidagi ko’rinishda bo’ladi :

$$f(z)=\frac{1}{(z-a)(z-b)}=\sum_{n=0}^{\infty}C_{n}z^{n}$$

Loran yoyilmasining bosh qismi nolga teng, chunki $f(z)$ funksiya $z=0$ nuqtada analitik. Quyidagi o'zgartirishlarni bajaraylik :

$$\frac{1}{z-b}=\frac{1}{-b\left(1-\frac{z}{b}\right)}, \qquad \frac{1}{z-a}=-\frac{1}{a\left(1-\frac{z}{a}\right)}$$

$z=0$ atrofida $\left|\frac{z}{b}\right|<1$, $\left|\frac{z}{a}\right|<1$ ekanligidan va (6.47) formuladan foydalanib, berilgan funksiyaning qidirilayotgan yoyilmani topamiz :

$$\frac{1}{(z-a)(z-b)}=\frac{1}{(b-a)}\left\{-\frac{1}{b}\sum_{n=0}^{\infty}\left(\frac{z}{b}\right)^n+\frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{z}{a}\right)^n\right\}=-\frac{1}{b-a}\sum_{n=0}^{\infty}\left[\frac{1}{b^{n+1}}+\frac{1}{a^{n+1}}\right]z^n$$

2. (6.48) formuladan foydalanib $z=a$ atrofida Loran yoyilmasini topamiz. U quyidagi ko'rinishda bo'ladi:

$$f(z)=\sum_{n=-1}^{\infty}C_n(z-a)^n$$

Bu yerda bosh qismi bitta haddan iborat, chunki $f(z)$ funksiya $z=a$ nuqtada 1-tartibli qutbga ega. $\frac{1}{z-b}$ ni o'zgartirib yozib olaylik.

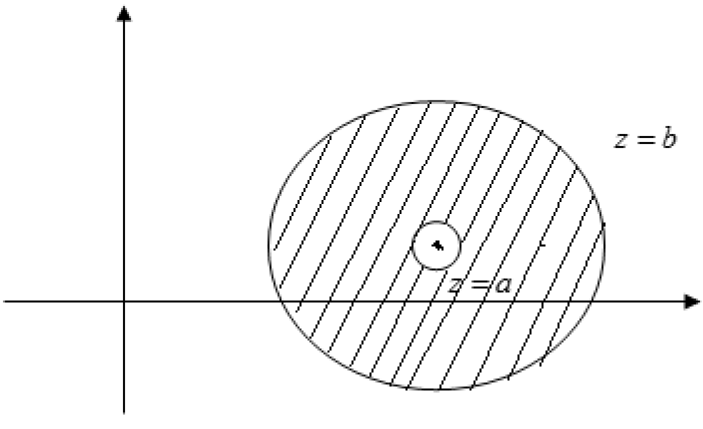
$$\frac{1}{z-b}=\frac{1}{z-a-(b-a)}=\frac{1}{(b-a)\left[1-\frac{z-a}{b-a}\right]}=-\frac{1}{b-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{b-a}\right) \qquad (6.49)$$

$z=a$ atrofida, $\left|\frac{z-a}{b-a}\right|<1$.

(6.49) ni (6.48) ga qo'yib, $z=a$ nuqta atrofida Loran yoyilmasini olamiz:

$$\frac{1}{(z-a)(z-b)}=\frac{1}{(b-a)}\left\{-\frac{1}{(b-a)}\sum_{n=0}^{\infty}\left(\frac{z-a}{b-a}\right)^n-\frac{1}{z-a}\right\}=-\sum_{n=-1}^{\infty}\frac{(z-a)^n}{(b-a)^{n+2}}$$

Hosil bo'lgan yoyilma $0<|z-a|<|b-a|$ halqada ma'noga ega, chunki $z=a$, $z=b$ nuqtalar funksiya uchun maxsus nuqtalar (6.6-rasm)



6.6-rasm

3. $|a|<|z|<|b|$ halqada yoyilmani topaylik. Qator quyidagi ko'rinishda bo'lishi kerak:

$$\frac{1}{(z-a)(z-b)}=\sum_{-\infty}^{\infty}C_n^n z^n$$ chunki halqaning markazi nol nuqtada.

$|a|<|z|<|b|$ halqada $\left|\frac{z}{b}\right|<1$, $\left|\frac{a}{z}\right|<1$ ekanligidan quyidagicha o'zgartirish qilamiz:

$$\frac{1}{z-b}=-\frac{1}{b\left(1-\frac{z}{b}\right)},\quad \frac{1}{z-a}=\frac{1}{z\left(1-\frac{a}{z}\right)}$$

U holda

$$\frac{1}{(z-a)(z-b)}=\frac{1}{b-a}\left\{-\frac{1}{b}\sum_{n=0}^{\infty}\frac{z^n}{b^n}-\frac{1}{z}\sum_{n=0}^{\infty}\frac{a^n}{z^n}\right\}==\frac{1}{a-b}\sum_{n=0}^{\infty}\frac{z^n}{b^{n+1}}+\frac{1}{a-b}\sum_{n=0}^{\infty}\frac{a^n}{z^{n+1}}$$

15-misol. $\frac{1}{z+z^2}$ funksiya $0<|z|<1$ halqada Loran qatoriga yoying.

Yechilishi: Ma'lumki,

$$\frac{1}{z+z^2}=\frac{1}{z(1+z)}=\frac{1}{z}-\frac{1}{1+z}=\frac{1}{z}-\sum_{n=0}^{\infty}(-1)^n z^n.$$

Bu qatorning bosh qismi birgina haddan iborat.

16-misol. $\frac{2}{z^2-1}$ funksiyani $1<|z+2|<2$ halqada Loran qatoriga yoying.

Yechilishi:

$$\frac{2}{z^2-1}=\frac{2}{(z-1)(z+1)}=\frac{A}{z-1}+\frac{B}{z+1},$$

$$2=A(z+1)+B(z-1);$$

$z=1$ bo'lganda $A=1$, $z=-1$ bo'lganda $B=-1$. U holda

a) $$\frac{1}{z-1}=\frac{1}{(z+2)-3}=-\frac{1}{3}\cdot\frac{1}{1-\frac{z+2}{3}}=-\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z+2}{3}\right)^n;$$

b) $$-\frac{1}{z+1}=-\frac{1}{(z+2)-1}=-\frac{\frac{1}{z+2}}{1-\frac{1}{z+2}}=$$

$$=-\frac{1}{z+2}\left[1+\frac{1}{(z+2)}+\frac{1}{(z+2)^2}+...\right]=-\sum_{n=1}^{\infty}\frac{1}{(z+2)^n}.$$

Demak,

$$\frac{2}{z^2-1}=-\sum_{n=1}^{\infty}\frac{1}{(z+2)^n}-\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z+2}{3}\right)^n.$$

Buning birinchisi qatorning bosh qismi bo'lib, ikkinchsi to'gri qismidir.

17-misol. $f(z)=\frac{1}{1+z^2}$ funksiyani $1<|z-i|<2$ halqada Loran qatoriga yoying.

Yechilishi: Ma'lumki,

$$\frac{1}{1+z^2}=\frac{A}{z+i}+\frac{B}{z-i},\ 1=A(z+i)+B(z-i);$$

$$z=i;\ A=\frac{1}{2i};\ z=-i\ B=-\frac{1}{2i};$$

$$\frac{1}{1+z^2}=\frac{1}{2i}\cdot\frac{1}{z-i}-\frac{1}{2i}\cdot\frac{1}{z+i}.$$

Bunda birinch had $z-i$ ga nisbatan qatorga yoyilgan bo'lgani uchun ikkinch hadni qatorga yoyamiz:

$$-\frac{1}{2i}\cdot\frac{1}{z+i}=-\frac{1}{2i}\cdot\frac{1}{(z-i)+2i}=\frac{1}{4}.\frac{1}{1+\frac{z-i}{2i}}=\frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z-i}{2i}\right)^n;$$

$$\left|\frac{z-i}{2i}\right|=\frac{|z-i|}{2}<1,\ |z-i|<2.$$

Demak,

$$\frac{1}{1+z^2}=-\frac{i}{2(z-i)}+\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{2i}\right)^n;\ 0<|z-i|<2.$$

18-misol. $f(z)=\frac{z+2}{z^2-4z+3}$ funksiyani $2<|z-1|<\infty$ halqada Loran qatoriga yoying.

Yechilishi: Maxrajni ko'paytuvchilarga ajratamiz:

$$z^2-4z+3=z^2-z-3z+3=z(z-1)-3(z-1)=(z-1)(z-3);$$
$$\frac{z+2}{(z-1)(z-3)}=\frac{A}{(z-1)}+\frac{B}{(z-3)};\ z+2=A(z-3)+B(z-1);$$
$$z=1 \text{ da } A=-\frac{3}{2},\ z=3 \text{ da } B=\frac{5}{2}.$$
Demak,
$$\frac{z+2}{z^2-4z+3}=-\frac{3}{2}\cdot\frac{1}{z-1}+\frac{5}{2}\cdot\frac{1}{z-3};$$
Bunda birinch had z-1 ga nisbatan qatorga yoyilgan bo'lib,
Ikkinch hadni z-1 ning darajalari orqali yozamiz.
$$\frac{5}{2}\cdot\frac{1}{z-3}=\frac{5}{2}\cdot\frac{1}{(z-1)-2}=\frac{5}{2}\cdot\frac{1}{z-1}\cdot\frac{1}{1-\frac{2}{z-1}}=$$
$$=\frac{5}{2}\cdot\frac{1}{z-1}\left[1+\frac{2}{z-1}+\left(\frac{2}{z-1}\right)^2+\left(\frac{2}{z-1}\right)^3+\ldots\right]$$
$$\left|\frac{2}{z-1}\right|<1,\ 2<|z-1|.$$
Shunga ko'ra
$$\frac{z+2}{z^2-4z+3}=\frac{1}{z-1}+5\sum_{n=2}^{\infty}\frac{2^{n-2}}{(z-1)^n}.$$
Bu qatorning yaqinlashish sohasi $2<|z-1|<\infty.$

Mashqlar

Quyidagi Loran qatorlarning yaqinlashish soxalarini aniqlang.

66. $\sum_{n=1}^{\infty}\frac{(3+4i)^n}{(z+2i)^n}+\sum_{n=0}^{\infty}\frac{(z+2i)^n}{6^n}$; 67. $\sum_{n=1}^{\infty}\frac{1}{n^n(z-2+i)^n}+$
$+\sum_{n=0}^{\infty}(1+in)(z-2+i)^n$; 68. $\sum_{n=1}^{\infty}\frac{2}{z^n}+\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$;
69. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4z^4}+\sum_{n=0}^{\infty}\frac{z^n}{n2^n}$; $(-1)^n$ 70. $-\frac{1}{z-1}+\sum_{n=0}^{\infty}(-1)^n(z-1)^n$;

71. $\sum_{n=1}^{\infty}\frac{1}{z^n}+\sum_{n=1}^{\infty}\left(\frac{z}{2}\right)^n$; 72. $\sum_{n=-\infty}^{-1}\frac{(-1)^{n-1}}{9}(z-1)^n+$

$+\sum_{n=0}^{\infty}\frac{3n+5}{9\cdot 2^{n+2}}(z-1)^n$; 73. $\sum_{n=-\infty}^{-1}(n+2)i^{n+1}(z-i)^n$.

Quyidagi funksiyalarni Loren qatoriga yoying:

74. $\frac{e^z}{z}$ funksiyani $z=0$ nuqta atrofida.

75. $\frac{e^z}{z^8}$ funksiyani $z=0$ nuqta atrofida.

76. $\frac{1-\cos x}{z^2}$ funksiyani $z=0$ nuqta atrofida.

77. $\frac{1}{z}\sin^2\frac{2}{z}$ funksiyani $z=0$ nuqta atrofida.

78. $\frac{1-e^{-z}}{z^3}$ funksiyani $z=0$ nuqta atrofida.

79. $\frac{2z+3}{z^2+3z+2}$ funksiyani $1<|z|<2$ halqada.

80. $\frac{z^2-z+3}{z^2-3z+2}$ funksiyani $1<|z|<2$ halqada.

81. $\frac{1}{z^2+2z-8}$ funksiyani $1<|z+2|<4$ halqada.

82. $\frac{1}{(z^2-4)^2}$ funksiyani $1<|z+2|<\infty$ halqada.

83. $\frac{2z-3}{z^2-3z+2}$ funksiyani $0<|z-2|<1$ halqada.

84. $\frac{2z-3}{z^2-3z+2}$ funksiyani $2<|z|<\infty$ halqada.

85. $\frac{1}{(z-1)^2(z+2)}$ funksiyani $1<|z|<2$ halqada.

86. $\frac{1}{(z^2+1)(z^2-4)}$ funksiyani $1<|z|<2$ halqada

87. $z^3 e^{\frac{1}{z}}$ funksiyani $0<|z|<\infty$ halqada.

88. $z^2 sin\dfrac{(z+1)\pi}{z}$ funksiyani $0<|z|<\infty$ halqada.

Berilgan funksiyalarni yo ko'rsatilgan halqada, yo ko'rsatilgan nuqta atrofida Loran qatoriga yoying.

89. $f(z)=\dfrac{1}{z(1-z)}$; $z=0$, $z=1$, $z=\infty$ nuqtalar atrofida.

90. $f(z)=\dfrac{\sin z}{z^2}$ $z=0$ nuqta atrofida.

91. $f(z)=\dfrac{e^z}{z^3}$; $z=0$ nuqta atrofida.

92. $f(z)=\dfrac{1}{z^2+z}$; $0<|z|<1$ halqada.

93. $f(z)=\dfrac{2}{z^2-1}$; $1<|z+2|<3$ halqada.

94. $f(z)=\dfrac{1}{z}\sin^2\dfrac{2}{z}$; $z=0$ nuqta atrofida

95. $f(z)=\dfrac{1-e^z}{z^3}$; $z=0$ nuqta atrofida

96. $f(z)=\dfrac{2z-3}{z^2+3z+2}$; $0<|z|<2$ halqada.

97. $f(z)=\dfrac{1}{(z^2+1)(z^2-4)}$; $0<|z|<2$ halqada.

98. $f(z)=z^2\sin\dfrac{(z+1)\pi}{z}$; $0<|z|<\infty$ halqada.

99. $f(z)=\dfrac{1}{1+z^2}$; $0<|z-i|<2$ halqada.

100. $f(z)=\dfrac{z+2}{z^2-4z+3}$; $2<|z-1|<+\infty$ halqada.

101. $f(z)=\dfrac{1}{(z-4)(z-9)}$; $4<|z|<9$ halqada.

102. $f(z)=z^4\cos\dfrac{1}{z}$; $z=0$ nuqta atrofida

103. $f(z)=\dfrac{1+\cos z}{z^4}$; $z=0$ nuqta atrofida (javobini topolmadim)

JAVOBLAR

1.Yaqinlashuvchi 2. Yaqinlashuvchi 3. Uzoqlashuvchi 4. Yaqinlashuvchi 5. Yaqinlashuvchi 6. Uzoqlashuvchi 7. Yaqinlashuvchi

8. Uzoqlashuvchi 9. Yaqinlashuvchi 10. Yaqinlashuvchi 11. Uzoqlashuvchi 12. Yaqinlashuvchi

13. $\alpha > 0$ 14. $\alpha > 0$ 15. α-xar qanday xaqiqiy son. 16. α<0. 17. R=1. 18.R=∞. 20. R=2. 21. $R=e;$ 22. $R=1;$ 23. $R=1;$ 24. $R=\frac{1}{e};$ 25. $R=1;$ 26. $R=1;$

27. $R=1;$ 28. $R=\infty;$ 29. $R=1;$ 30. $R=k^{-k};$ 31. $R=0;$ 32. $|z-2i|<3;$ 33. $R=\sqrt{2};$

34. $R=1;$ 35. $R=1;$ 36. $R=1;$ 37. $R=1;$ 38. $R=\infty;$ 39. $\sum_{n=0}^{\infty}(-1)^n\frac{(az)^n}{b^{n+1}};$ $\mathrm{R}=\left|\frac{b}{a}\right|;$ 40. $\frac{1}{2}+\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!};$ $R=\infty;$ 41. $2\sum_{n=0}^{\infty}\frac{z^{2n}+1}{2n+1},$ $R=\infty;$ 42. $\frac{1}{2}+\sum_{n=0}^{\infty}\frac{2^{2n}-\lg^{2n}}{(2n)!},$ $R=\infty;$ 43. $2\sum_{n=0}^{\infty}\frac{z^{2n}+1}{2n+1},$ $R=1;$

44. $\ln 2-\sum_{n=0}^{\infty}\left(1+\frac{1}{2^n}\right)\frac{z^n}{n;}$ $R=1;$ 45. $\sum_{n=1}^{\infty}\frac{n^2z^n}{a^n+1},$ $R=|a|;$ 46. $\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{(2n)!},$ $R=\infty;$

47. $-iz+z^3+iz^5-z^7-...,$ $R=1;$ 48. $\frac{1}{2}\sum_{n=1}^{\infty}\frac{z^{2n}}{(2n)!},$ $R=\infty;$

49. $\ln 2-\frac{1}{2}\left(\frac{z}{2}+\frac{z^2}{2\cdot 4}+\frac{z^3}{3\cdot 8}+...\right),$ $R=2;$

50. $-i\left[\left(z-1+\frac{\pi i}{2}\right)-\frac{1}{3!}\left(z-1+\frac{\pi i}{2}\right)^3+\frac{1}{5!}\left(z-1+\frac{\pi i}{2}\right)^5+...\right];$

51. $\frac{1}{\sqrt{2}}\left[1+\left(z+\frac{\pi}{4}\right)-\frac{1}{2!}\left(z+\frac{\pi}{4}\right)^2-\frac{1}{3!}\left(z+\frac{\pi}{4}\right)^3+...\right],$ $R=\infty;$

52. $-\frac{1}{3}+\frac{2}{3^2}(z-3)-\frac{2^2}{8^3}(z-3)^2+\frac{2^3}{8^3}(z-3)^3-...,$ $R=\frac{3}{2};$ 5 $-\sum_{n=0}^{\infty}3\ z-\ n\ .^{2n}+$

54. $\frac{1}{3}\sum_{n=0}^{\infty}\left[2+(-1)^{n+1}\right]\left(z-1\right)^n;$ 55. $|z|>\frac{1}{\sqrt{2}}$ 56. $|z+1|>\frac{1}{4}$ 57. $|z+1-i|>$ 58. $|z|>2$

59. $|z|>|b|;$ 6 $|z0-|$. $|a>|$ 6 $|z1>|a|.$ 62. $\sum_{n=1}^{\infty}\frac{3^{n-1}}{2^n3^{n^2}},$ $|z|>\frac{5}{2};$

63. $z^2-\frac{1}{2!}+\frac{1}{4!z^2}+\frac{1}{6!z^4}+\frac{1}{8!z^6}+...,$ 64. $\sum_{n=-\infty}^{-1}a^{-2(n+1)}z^{2n},$ $|z|>a$

65. $\sum_{n=-\infty}^{-1}(a-b)^{-n-1}(z-b)^n,$ $b\neq a,$ $|z-b|>|a-b|;$ 6 6 $<|z5+\ i|<$ 6 7 $<|z0-\ +i|<2$

68. $1<|z|<2;$ 69. $1<|z|<2;$ 70. $0<|z-i|<1;$ 71. $1<|z|<2;$ 72. $1<|z-1|<2;$ 73. $0<|z-1|<1;$

74. $\frac{1}{z}+\sum_{n=1}^{\infty}\frac{z^{n-1}}{n!};$ 75. $\frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{2!z}+\frac{1}{3!}+\frac{z}{4!}+\frac{z^2}{5!}+...;$ 76. $\frac{1}{2!}-\frac{z^2}{4!}+\frac{z^4}{6!}-\frac{z^6}{8!}+...;$

77. $\frac{4^2}{2!2z^3}-\frac{4^4}{4!2z^5}+\frac{4^6}{6!2z^7}-...;$ 78. $\frac{1}{z^2}-\frac{1}{2!z}+\frac{1}{3!}-\frac{z}{4!}+\frac{z^2}{5!}-...;$

79. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{z^n}+\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}z^n;$ 80. $\sum_{n=1}^{\infty}\frac{n}{z^{n+1}}+\sum_{n=0}^{\infty}\frac{(-1)^{n_2 n}}{2^{n+1}};$ 81. Qatorga yoyilmaydi;

82. $\sum_{n=1}^{\infty}\frac{n4^{n-1}}{(z+2)^{n+3}};$ 8 $3\frac{1}{z-2}.+\sum_{n=0}^{\infty} - \quad ^{n} \quad z-(\quad ^{n}$ 84. $\sum_{n=1}(-1)^{n+1}\frac{2^{n-1}}{z^n}+\sum_{n=1}^{\infty}\frac{1}{z^n};$

85. $\sum_{n=-\infty}^{-1}\frac{-3n-4}{9}z^n+\sum_{n=0}^{\infty}\frac{(-1)^{n_2 n}}{9\cdot 2^{n+1}};$ 86. $\sum_{n=-\infty}^{-1}\frac{(-1)^n}{5}z^{2n}+\sum_{n=0}^{\infty}\frac{(-1)^n}{5\cdot 4^{n+1}}z^{2n};$

87. $z^3+z^2+\frac{z}{2}+\frac{1}{6}+\sum_{n=1}^{\infty}\frac{z^{-n}}{(n+3)!};$ 88. $-\pi z+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\pi^{2n+1}}{(2n+1)!}z^{-2n+1}.$

VII BOB

MAXSUS NUQTALARNING KLASSIFIKATSIYASI

Kompleks argumentli funksiyalar nazariyasidan ma'lumki, $\omega = f(z)$ funksiya z_0 nuqtada hosilaga ega bo'lsa, bu nuqta *tog'ri nuqta* deyiladi. Agar shu nuqtada analitiklik shartlari bajarilmasa, bu nuqta funksiyaning *maxsus nuqtasi* deyiladi. Masalan, $\omega = \frac{z-1}{z+i}$ funksiya uchun $z=-i$ maxsus nuqtadir, chunki bu nuqtada hosila mavjud emas.

7.1-§. FUNKSIYANING NOLLARI

Faraz qilaylik, $\omega = f(z)$ funksiya biror α nuqtada hosilaga ega bo'lsin (bu yerda $z = x+iy$, $\alpha -$kompleks son, $\omega = u(x,y) + i\upsilon(x,y)$ kompleks o'zgaruvchining funksiyadir). Agar o'sha α son $f(z)$ funksiya nolga aylantirsa, ya'ni $f(\alpha)=0$ bo'lsa, α son $f(z)$ ning *noli* deyiladi. Agar

$$f(\alpha)=0, \quad f'(\alpha)=0, \quad f''(\alpha)=0, ..., f^{(n-1)}(\alpha), \quad f^{(n)}(\alpha)\neq 0 \tag{7.1}$$

bo'lsa, u holda α son $f(z)$ funksiyaning $n-$*tartibli* yoki n-*karrali noli* deyiladi. Agar $n=1$ bo'lsa, α son *oddiy nol* deyiladi.

Demak, ta'rifga muvofiq α nuqtada analitik bo'lgan $f(z)$ funksiya uchun α nuqta $n-$ *karrali nol* bo'lishi uchun shu nuqtaning biror atrofida ushbu

$$f(z) = (z-\alpha)^n \varphi(z) \tag{7.2}$$

tenglik bajarilib, $\varphi(z)$ funksiya α nuqtada analitik va $\varphi(\alpha)\neq 0$ bo'lishi zarur va yetarlidir. Chunki $z=\alpha$ da $f(z)$ analitik funksiya bo'lganligi uchun u shu nuqta atrofida $(z=\alpha)$ ga nisbatan darajali qatorga yoyiladi.

Quyidagi funksiyalarning nollarning va ularning karraliklarini aniqlang.

1-misol. $f(z) = z - i.$

Yechilishi: $f(i) = i - i = 0, \quad \alpha = i, \; f'(z) = 1, \; f(i) = 1 \neq 0.$

Demak, $z=\alpha=i$ son funksiyaning oddiy nolidir.

2-misol. $f(z) = z^2+1.$

Yechilishi: Ma'lumki,

$$f(\pm i) = (\pm i)^2 + 1 = -1+1 = 0; \quad z_1 = i, \; z_2 = -i.$$

$$f'(z) = 2z; \quad f'(\pm i) = 2(\pm i) = \pm 2i \neq 0.$$

Demak, ikkala ildiz ham oddiy noldir.

3-misol. $f(z) = \sin z - 1.$

Yechilishi: $f(z)$ ning o’ng tomonini nolga tenglab, so’ngra tenglamani yechamiz:

$$\sin z-1=0,\quad \sin z=1;\quad z=\frac{\pi}{2}+2\pi n=(4\pi+1)\frac{\pi}{2},\ n=0,\pm1,\pm2,...$$

Endi

$$f'(z)=\cos z;\ \ f'\left[(4\pi+1)\frac{\pi}{2}\right]=\cos\left[(4\pi+1)\frac{\pi}{2}\right]=0;$$

$$f''(z)=-\sin z;\ \ f''\left[(4\pi+1)\frac{\pi}{2}\right]=-\sin\left[(4\pi+1)\frac{\pi}{2}\right]=-1\neq 0;$$

Demak, $(4\pi+1)\frac{\pi}{2},\ \ n=1,2,...$ funksiyaning ikki karrali nol.

4-misol. $f(z)=\dfrac{z^3}{\frac{z^3}{2}+\cos z}$.

Yechilishi: Ma’lumki,

$$\frac{z^3}{2}+\cos z=\frac{z^2}{2}+\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+...\right)=1+\frac{z^4}{4!}-\frac{z^6}{6!}+\frac{z^8}{8!}-...,$$

$$f(z)=z^3\varphi(z),\quad \varphi(z)=\frac{1}{1+\frac{z^4}{4!}-\frac{z^6}{6!}+...}$$

$f(0)=0,\ \ \varphi(0)=1\neq 0$ Demak, $z=0$ son funksiyaning uch karrali nolidir.

5-misol. $f(z)=z^4+4z^2$

Yechilishi. Ma’lumki,

$$z^4+4z^2=z^2\left(z^2+4\right)=0;\quad z^2=0;\quad z^2+4=0,\quad z=\pm 2i$$

$$f(z)=4z^3+8z=4z\left(z^2+2\right),\quad f'(0)=0,\quad f'(\pm 2i)\neq 0$$

$$f'(z)=12z^2+8;\quad f''(0)=8\neq 0$$

Demak, $z=\pm 2i$ oddiy nollar, $z=0$ esa ikki karrali noldir.

6-misol. $f(z)=z^2\sin z$

Yechilishi. Ma’lumki,

$$z^2\sin z=0,\quad z^2=0,\ z=0,\quad \sin z=0,\ z=\pi n,\ n=0,\pm1,\pm2,...$$

$$f'(z)=2z\sin z+z^2\cos z;\quad f''(z)=2\sin z+4z\cos z-z^2\sin z;$$

$$f'(0)=0,\quad f''(0)=0;\quad f'(\pi n)=(\pi n)^2\cos\pi n=(-1)^n(\pi n)^2\neq 0$$

Tekshirish ko’rsatadiki, $f'''(0)\neq 0$. Demak, $z=0$ uch karrali nol bo’lib, $z=\pi n$ lar oddiy nollar ekan.

7-misol. $f(z)=1+chz$

Yechilishi: Qo’yidagi tenglamani yechamiz:

$$1+chz=0,\quad chz=\frac{1}{2}\left(e^z+e^{-z}\right)=-1;\quad e^{2z}+2e^z+1=0$$

$$\left(e^{z}+1\right)^{2}=0, e^{z}=-1; z=Ln(-1)=\ln 1+\pi i+2\pi n i=(2n+1)\pi i, n=0,\pm 1,\pm 2,...;$$

$$f'(z)=shz, \quad f''(z)=chz.$$

Tekshirish ko'rsatadiki,

$$f'\left|(2n+1)\pi i\right|=0, \quad f''\left[(2n+1)\pi i\right]\neq 0.$$

Demak, $(2n+1)\pi i$ ikki karrali nollardir.

8-misol. $f(z)=\left(z^{2}+\pi^{2}\right)\left(1+e^{-z}\right).$

Yechilishi:

$$z^{2}+\pi^{2}=0, z=\pm\pi i; 1+e^{-z}=0, e^{z}=-1; \mathrm{z}=\mathrm{Ln}(-1)=(2n+1)\pi i;$$

$$f'(z)=2z\left(1+e^{-z}\right)-\left(z^{2}+\pi^{2}\right)e^{-z}; \quad e^{\pm\pi i}=\cos\pi\pm i\sin\pi=-1;$$

$$f'(\pm\pi i)=0; \quad f'\left[(2n+1)\pi i\right]\neq 0, \quad f''(\pm\pi i)\neq 0.$$

Demak , $\pm\pi i$ ikki karrali nollar bo'lib, $(2n+1)\pi i$ oddiy nollardir.

9-misol. $f(z)=2(chz-1)-z^{2}$ funksiya uchun $z=0$ necha karrali nol bo'ladi?

Yechilishi: $f(z)$ dan hosilalar olaylik:

$$f'(z)=2shz-2z; \quad f''(z)=2chz-2; \quad f'''(z)=2shz; \quad f^{IV}(z)=2chz$$

bulardan

$$f(0)=0, \quad f'(z)=0, \quad f''(0)=0, \quad f'''(0)=0, \quad f^{IV}(0)=2\neq 0$$

Demak, $z=0$ to'rt karrali noldir.

10-misol. $f(z)=z^{2}\left(e^{z^{2}}-1\right)$ funksiya uchun $z=0$ necha karrali nol bo'ladi?

Yechilishi: $f(z)=z^{2}\left(e^{z^{2}}-1\right)$ dan ketma–ket hosilalar olamiz ;

$$f'(z)=2z\left(e^{z^{2}}-1\right)+z^{2}e^{z^{2}}\cdot 2z=e^{z^{2}}\left(2z^{3}+2z\right)-2z;$$

$$f''(z)=e^{z^{2}}\left(4z^{4}+10z^{2}+2\right)-2; \quad f'''(z)=e^{z^{2}}\left(8z^{5}+36z^{8}+24z\right);$$

$$f^{IV}(z)=2ze^{z^{2}}\left(8z^{5}+36z^{8}+24z\right)+e^{z^{2}}\left(40z^{4}+108z^{2}+24\right),$$

Bulardan $f(0)=0, f'(0)=0, f''(0)=0, f'''(0)=0, f^{IV}(0)=24\neq 0$ demak, $z=0$ nuqta berilgan funksiyaning to'rt karrali noli ekan.

11-misol. $f(z)=\dfrac{\sin z}{z}.$

Yechilishi: $\dfrac{\sin z}{z}=0, \sin z=0, z=\pi n, n=\pm 1,\pm 2,...$

$$f'(z)=\frac{z\cos z-\sin z}{z^{2}}, \quad f'(\pi n)=\frac{(-1)^{n}}{\pi n}\neq 0.$$

Demak $z=\pi n$ berilgan funksiyaning oddiy nollaridir.

Mashqlar

Quyidagi funksiyalarning nollari hamda ularning karraliklarini aniqlang.

1. $f(z)=1+\cos z$; 2. $f(z)=1-e^{z}$; 3. $f(z)=\dfrac{z^2}{z-\sin z}$; 4. $f(z)=\left(z^2+1\right)^3 shz$

5. $f(z)=(z+\pi i)shz$; 6. $f(z)=\dfrac{sh^2 z}{z}$; 7. $f(z)=\dfrac{(1-shz)^2}{z}$; 8. $f(z)=\cos z^3$.

9. $f(z)=\dfrac{z^3}{1+z-6^2}$; 10. $f(z)=\dfrac{(1-\cos 2z)}{z-shz}$; 11. $f(z)=\left(e^{z}-e^{z^2}\right)\ln(1-z)$.

7.2-§ AJRATILGAN MAXSUS NUQTA

Agar $f(z)$ funksiya a nuqtada analitik bo’lmay, uning biror $0<|z-a|<\delta$ atrofida analitik bo’lsa, u holda a nuqta $f(z)$ funksiyaning *ajralgan maxsus nuqtasi* deyiladi. Masalan, $f(z)=\dfrac{1}{z}$ uchun $z=0$ ajralgan maxsus nuqtadir, chunki bu nuqtadan boshqa barcha nuqtalarda $f(z)$ hosilaga ega; $f_1(z)=\dfrac{z+1}{z-1}$ ning ajralgan maxsus nuqtasi $z=1$ dir; $f_2(z)=\dfrac{1}{z^2+1}$ funksiyaning ajralgan maxsus nuqtasi $z=\pm i$ dir;

Berilgan $f(z)$ funksiyaning boshqa xil maxsus nuqtalari ham bo’lishi mumkin, lekin ajralgan maxsus nuqtalar funksiya haqida asosiy informatsiyani beradi.

Ajralgan maxsus nuqtalar uch xil bo’ladi.

1. Agar $f(z)$ funksiyaning maxsus nuqtasida

$$\lim_{x\to a} f(z)=A \tag{7.3}$$

mavjud bo’lib, A aniq chekli son bo’lsa, u holda a nuqta $f(z)$ funksiyaning *chetlashtiriladigan (yoki tuzatiladigan) maxsus nuqtasi* deyiladi.

2.Agar $f(z)$ funksiyaning maxsus nuqtasida

$$\lim_{x\to a} f(z)=\infty \tag{7.4}$$

bo’lsa, u holda a son $f(z)$ funksiyaning *qutb nuqtasi* deyiladi.

3. Agar a nuqtada $f(z)$ funksiyaning limiti mavjud bo’lmasa, (ya’ni $f(z)$ funksiyaning a nuqtadagi limiti z ning a ga qanday qilib

yaqinlashishiga bog'liq bo'lsa), u holda a *muhim maxsus nuqta* deyiladi.

7.3-§. CHETLANTIRILADIGAN MAXSUS NUQTALAR

Biz bu paragrifda chetlantiriladigan (tuzatiladigan) nuqtalarga doir misollar bilan tanishamiz. Yuqoridagi ta'riflarga binoan ajralgan maxsus nuqtaning qaysi turga kirishini aniqlash uchun $f(z)$ funksiyaning o'sha nuqtadagi limitini izlash talab qilinadi. Ba'zan bu ish maxsus nuqtaning harakterini aniqlash deyiladi.

1-misol. $f(z)=\dfrac{1-\cos z}{z^2}$.

Yechilishi: Bu funksiya uchun $z=0$ ajralgan maxsus nuqtadir. Endi $f(z)$ dan limit olib, o'sha nuqtaning harakterini aniqlaymiz. Ma'lumki,

$$\cos z = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+...;\quad \frac{1-\cos z}{z^2}=\frac{1}{2!}-\frac{z^2}{4!}+\frac{z^4}{6!}-\frac{z^6}{8!}+...;$$

$$\lim_{z\to 0} f(z)=\lim_{z\to 0}\left(\frac{1}{2!}-\frac{z^2}{4!}+\frac{z^4}{6!}-\frac{z^6}{8!}+...\right)=\frac{1}{2}.$$

Demak, $A=\dfrac{1}{2}$ chekli son bo'lgani uchun $z=0$ nuqta berilgan $f(z)$ funksiyaning chetlantiriladigan nuqtasi ekan. $z=0$ ning maxsus nuqta deyilishiga sabab, formal ravishda $f(z)$ funksiyaning argumentiga 0 qo'yilsa, ushbu

$$f(0)=\frac{1-\cos 0}{0}=\frac{0}{0}.$$

Aniqmas ifoda hosil bo'ladi. Agar

$$f(z)=\begin{cases}\dfrac{1-\cos z}{z^2},\ z\neq 0\\[2mm] \dfrac{1}{2},\ z=0\end{cases}$$

deb qabul qilsak, $f(z)$ funksiya endi $z=0$ nuqtada ham analitik bo'ladi, ya'ni 0 nuqtadagi maxsuslik chetlantiriladi.

2-misol. $f(z)=\dfrac{e^2-1}{z}$.

Yechilishi: Ko'rinib turibdiki, $z=0$ nuqtada funksiya aniqlanmagan, chunki z ning o'rniga formal ravishda 0 qo'yilsa, qo'yidagi $f(0)=\dfrac{6^0-1}{0}=\dfrac{0}{0}$ aniqmas ifoda hosil bo'ladi. Endi shu nuqtadagi limitni izlaymiz;

$$e^z=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+...,\ f(z)=\frac{e^z-1}{z}=1+\frac{z}{2!}+\frac{z^2}{3!}+...;\ \lim_{z\to0}f(z)=A=1$$

Demak, $z=0$ chetlantiriladigan nuqta bo'lib,

$$f(0)=\frac{6^0-1}{0}=1$$

deb qabul qilamiz.

3-misol. $f(z)=\dfrac{\ln\left(1+z^3\right)}{z^2}$**.**

Yechilishi: Ravshanki, $z=0$ maxsus nuqtadir. Ma'lumki,

$$\ln\left(1+z^3\right)=z^3-\frac{\left(z^3\right)^2}{2}+\frac{\left(z^3\right)^3}{3}-\frac{\left(z^3\right)^4}{4}+...;$$

$$f(z)=\frac{\ln\left(1+z^3\right)}{z^2}=z-\frac{z^4}{2}+\frac{z^7}{3}-\frac{z^{10}}{4}+...;\quad \lim_{z\to0}=0.$$

4-misol. $f(z)=\dfrac{\sin^2 z}{z}$**.**

Yechilishi: $z=0$ ajralgan maxsus nuqtadir. Shu sababli,

$$f(z)=\frac{1-\cos2z}{2z}=\frac{1}{2z}\left[1-\left(1-\frac{(2z)^2}{2!}+\frac{(2z)^4}{4!}-\frac{(2z)^6}{6!}+...\right)\right]=$$

$$=z-\frac{(2z)^3}{4!}+\frac{(2z)^5}{6!}-\frac{(2z)^7}{8!}+...;\quad \lim_{z\to0}f(z)=A$$

Demak, $z=0$ chetlantiriladigan maxsus nuqta bo'lib,

$$f(0)=\frac{0}{0}=0$$

deb qabul qilinadi.

5-misol. $f(z)=\dfrac{z^2-1}{z-1}$**.**

Yechilishi: Bundan ko'rinadiki, $z=1$ chetlantiriladigan maxsus nuqta bo'lib, $f(1)=\dfrac{0}{0}=2$ deb qabul qilamiz.

Mashqlar

Quyidagi funksiyalar chetlantiriladigan maxsus nuqtalarga ega ekanligini ko'rsating.

12. $\dfrac{\sin z}{z}$; 13. $\dfrac{1-\cos z}{z^2}$; 14. $\dfrac{z^2-1}{z^2+1}$; 15. $\dfrac{z}{tgz}$; 16. $\dfrac{1}{\cos^2 z}-\dfrac{1}{\left(z-\dfrac{\pi}{2}\right)^2}$;

17. $ctgz-\frac{1}{z}$; 18. $\frac{1}{e^z-1}-\frac{1}{\sin z}$; 19. $\frac{z^2-1}{z+1}$;

7.4-§. QUTBLAR

Agar ajralgan maxsus a nuqta qutb bo'lsa, (7.4) tenglikning bajarilishi lozim ekanligini ta'rifdan ko'rgan edik. Endi qutbdga ega bo'lgan ba'zi funktsiyalar bilan tanishib o'tamiz. O'z o'zidan ma'lumki, a nuqta $f(z)$ funktsiyaning qutb nuqtasi bo'lsa, u holda a nuqta $\varphi(z)=\frac{1}{f(z)}$ funktsiyaning noli bo'ladi. Agar a son φ ning n karrali noli bo'lsa, u $f(z)$ ning n karrali qutbi deyiladi.

Berilgan a nuqta $f(z)$ funktsiyaning n karrali noli bo'lishi uchun $f(z)$ ni quyidagicha yozish mumkin bo'lishi zarur va yetarli:

$$f(z)=\frac{\varphi(z)}{(z-a)^n} \tag{7.5}$$

Bu yerda $\varphi(z)$ funktsiya a nuqtada analitik va $\varphi(a)\neq 0$.

Quyidagi funktsiyalarning qutblarini aniqlang.

1-misol. $f(z)=\frac{1}{z-\sin z}$

Yechilishi: $z=0$ ning ifodasidan ko'rinadiki, $z=0$ nuqtada qutbdir. $z=0$ ning necha karrali ekanligini tekshiraylik:

$$\varphi(z)=z-\sin z=z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+...\right)=\frac{z^3}{3!}-\frac{z^5}{5!}+....$$

Bundan uch marta hosila olib, $z=0$ nuqtada tekshirsak, ko'ramizki,

$$\varphi(0)=0,\quad \varphi'(0)=0,\quad \varphi''(0)=0,\quad \varphi'''(0)\neq 0.$$

Demak, $z=0$ $\varphi(z)$ ninguch karrali noli, ya'ni $f(z)$ ning uch karrali qutbidir.

2-misol. $f(z)=\frac{1}{\cos z-1+\frac{z^2}{2}}$

Yechilishi: Bu misolda $z=0$ nuqta (7.4) ni qanoatlantiradi, shu sababli u qutbdir, Endi uning necha karrali ekanligini tekshiraylik:

$$\cos z-1+\frac{z^2}{2}=\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+...\right)-1+\frac{z^2}{2}=\frac{z^4}{4!}-\frac{z^6}{6!}+\frac{z^8}{8!}-\frac{z^{10}}{10!}+...$$

Bundan (7.5) ga binoan

$$f(z)=\frac{\varphi(z)}{z^4}=\frac{1}{\frac{z^4}{4!}-\frac{z^6}{6!}+\frac{z^8}{8!}-\frac{z^{10}}{10!}}=\frac{1}{z^4\left(\frac{1}{4!}-\frac{z^2}{6!}+...\right)};$$

$$\varphi(z)=\frac{1}{\frac{1}{4!}-\frac{z^2}{6!}+...},\quad \varphi(0)=4!\neq 0.$$

Demak, $z=0$ nuqta $n=4$ karrali qutb ekan.

3-misol. $f(z)=\frac{shz}{z-shz}$.

Yechilishi: $f(z)$ ning qutbini izlash uchun maxrajining nollarini izlaymiz:

$$\psi(z)=\frac{1}{f(z)}=\frac{z-shz}{shz}=0;\quad z-shz=0;\quad shz=\frac{e^z-e^{-z}}{2};$$

ma'lumki,

$$e^z-e^{-z}=\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+...\right)-\left(1-z+\frac{z^2}{2!}-\frac{z^3}{3!}+...\right)=2\left(z+\frac{z^3}{3!}+\frac{z^5}{5!}+...\right),$$

$$z-shz=z-\frac{1}{2}\left(e^z-e^{-z}\right)=-\left(\frac{z^3}{3!}+\frac{z^5}{5!}+\frac{z^7}{7!+...}\right)$$

Bundan ko'rinadiki, $z=0$ nuqta ψ ning uch karrali noli, $f(z)$ ning esa uch karrali qutbdir.

4-misol. $f(z)=\frac{1}{1-\sin z}$

Yechilishi: $\psi(z)=\frac{1}{f(z)}=1-\sin z=0;\quad \sin z=1$

$$z=2\pi n+\frac{\pi}{2}=(4n+1)\frac{\pi}{2},\quad n=0,\pm1,\pm2,...$$

Bu nuqta necha karrali nol ekanligini tekshirish uchun $\psi(z)$ dan ketma-ket hosilalar olamiz:

$$\psi'(z)=\cos z;\quad \psi''(z)=\sin z;\quad \psi'\left((4n+1)\frac{\pi}{2}\right)=0;$$

$$\psi''\left[(4n+1)\frac{\pi}{2}\right]=\sin\left[(4n+1)\frac{\pi}{2}\right]=1\neq 0$$

Demak, $z=2n\pi+\frac{\pi}{2}=(4n+1)\frac{\pi}{2}$ lar $\psi(z)$ uchun ikki karrali nol, $f(z)$ uchun esa ikki karrali qutblardir.

5-misol. $f(z)=\dfrac{1}{z^4+2z^3+z^2}$

Yechilishi: $f(z)=\dfrac{1}{z^4+2z^3+z^2}=\dfrac{1}{z^2(z+1)^2}$

Demak, $z=0$, $z=-1$ nuqtalar ikki karrali qutblardir.

6-misol. $f(z)=\dfrac{z^2-3z+2}{z^2-2z+1}$

Yechilishi: Surat va maxrajini ko'paytiruvchilarga ajratamiz:

$z^2-3z+2=(z-1)(z-2)$, $z^2-2z+1=(z-1)^2$; $f(z)=\dfrac{z-2}{z-1}$; $z=1, z=2$

Demak, $z=1$ oddiy qutb, $z=2$ esa oddiy nol.

7-misol. $f(z)=\dfrac{e^{z+e}}{z+e}$

$\psi(z)=\dfrac{z+e}{e^{z+e}}=0$; $z+e=0$, $z=-e$; $\psi(z)=e^{z+e}$; $\psi(-e)=e^0=1\neq 0$

Demak, $z=-e$ oddiy qutb ekan.

8-misol. $f(z)=\dfrac{z^2-1}{z^6+2z^5+z^4}$

Yechilishi: Surat va maxrajni ko'paytuvchilarga ajratib, qisqartirsak,

$$f(z)=\frac{z-1}{z^4(z+1)^2},\quad z=1, z=0, z=-1$$

Demak, $z=0$ to'rt karrali qutb, $z=-1$ oddiy qutb, $z=1$ esa oddiy noldir.

Mashqlar

Quyidagi funktsiyalarning qutblarini toping.

20. $\dfrac{1}{z}$; 21. $\dfrac{1}{(z+1)^2}$; 22. $\dfrac{z}{e^z+1}$; 23. $\dfrac{z}{(e^z-1)^2}$; 24. $ctg\dfrac{\pi}{z}$; 25. $tg\pi z$; 26. $\dfrac{\sin z}{z^2}$; 27. $\dfrac{1}{z-z^3}$; 28. $\dfrac{z^4}{1+z^4}$; 29. $\dfrac{e^z}{1+z^2}$; 30. $\dfrac{1}{e^z-1}-\dfrac{1}{z}$.

7.5-§. MUHIM MAXSUS NUQTALAR

Ta'rifdan ma'lumki, agar biror a nuqtada $f(z)$ funktsiya hech qanday limitga ega bo'lmasa, ya'ni chekli ham, cheksiz ham limiti mavjud bo'lmasa, u holda $z=a$ nuqta $f(z)$ ning *muhim maxsus nuqtasi* deyilar edi. O'sha nuqtani aniqlash uchun o'zgaruvchi $z=x+iy$ ni ikki turli yo'l bilan a ga intiltirib, $f(z)$ ning limit sonlarini aniqlash lozim. Agar bu limit sonlar har xil bo'lsa, u holda a muhim maxsus nuqta bo'ladi.

Biz ikkitagina misol bilan chegaralanamiz, muhim maxsus nuqtani boshqa osonroq yo'l bilan aniqlash mumkin.

1-misol. $f(z)=e^{\frac{1}{z^2}}$

Yechilishi: $f(z)$ ning maxsus nuqtasi $z=0$ ekanligi ayon. Endi biz z ni 0 ga ikki xil yo'l bilan yaqinlashtiramiz:

a) $z=x\to 0$ bo'lsin, u holda

$$f(z)=f(x)=e^{\frac{1}{x^2}}\to+\infty;$$

b) $z=iy\to 0$ bo'lsin, u holda

$$f(z)=f(iy)=e^{-\frac{1}{y^2}}\to 0$$

Shunday qilib, ikki xil limit$(+\infty\ va\ 0)$ hosil bo'ldi. Demak, $z=0$ muhim maxsus nuqtadir.

2-misol. $f(z)=e^{\frac{1}{z}}$

Yechilishi: Bu holda ham $z=0$ maxsus nuqta bo'lib, z ni ikki hil yo'l bilan 0 ga yaqinlashtiramiz:

a) $x>0,\quad z=x\to 0$ bo'lsin, u holda

$$f(z)=f(x)=e^{\frac{1}{x}}\to+\infty;$$

b) $x<0,\quad z=x\to 0$ bo'lsin, u holda

$$f(z)=f(x)=e^{\frac{1}{x}}\to 0$$

Demak, $z=0$ muhim maxsus nuqtadir.

7.6-§. MAXSUS NUQTALAR VA LORAN QATORI

Oldingi boblardan ushbu

$$\sum_{n=-\infty}^{+\infty}c_n(z-a)^n \tag{7.6}$$

umumlashgan Loran qatori ma'lum bo'lib, bu qator to'g'ri va bosh qismlardan iborat edi. Ajralgan maxsus nuqtaning qaysi tipga kirishini Loran qatori yordami bilan aniqlash osonroq. Kompleks argumentli funktsiyalar nazariyasidan quyidagi teoremalar ma'lum.

1-teorema. *$f(z)$ funksiyaning ajralgan maxsus a nuqtasi chetlasntiriladigan maxsus nuqta bo'lishi uchun Loran qatori bosh qismiga ega bo'lmasligi zarur va yetarlidir:*

$$f(z)=c_0+c_1(z-a)+c_2(z-a)^2+...+c_n(z-a)^n+...=\sum_{n=0}^{\infty}c_n(z-a)^n \quad (7.7).$$

2-teorema. *$f(z)$ funktsiyaning ajralgan maxsus a nuqtaga ega bo'lishi uchun Loran qatoridagi bosh qismning hadlari soni chekli bo'lishi zarur va yetarlidir:*

$$f(z)=\frac{c_{-k}}{(z-a)^k}+\frac{c_{-(k-1)}}{(z-a)^{k-1}}+...+\frac{c_{-2}}{(z-a)^2}+\frac{c_{-1}}{(z-a)}+\sum_{n=0}^{\infty}c_n(z-a)^n \quad (7.8),$$

bunda $c_{-k}\neq 0$ va k -chekli natural son.

3-teorema. *$f(z)$ funktsiyaning ajralgan maxsus a nuqtagasi muhim maxsus nuqta bo'lishi uchun Loran qatorining bosh qismi cheksiz ko'p hadlarga ega bo'lishi zarur va yetarldir:*

$$f(z)=\sum_{n=-\infty}^{\infty}c_n(z-a)^n \quad (7.9).$$

Mana shu teoremalarga asoslanib, bir necha misollar ko'ramiz.

1-misol. $f(z)=\frac{\sin z}{z}$ funktsiya uchun $z=0$ maxsus nuqta.

Yechilishi: Buni Loran qatoriga yoysak,

$$f(z)=\frac{1}{z}\sin z=\frac{1}{z}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+...\right)=1-\frac{z^2}{3!}+\frac{z^4}{5!}-\frac{z^6}{7!}+...$$

hosil bo'lib, uning bosh qismi yo'q. Demak, birinchi teoremaga muvofiq, $z=0$ chetlantiriladigan nuqta ekan va $f(0)=\frac{0}{0}=1$

2-misol. $f(z)=\frac{shz}{z}$ funktsiya uchun $z=0$ maxsus nuqta.

Yechilishi: Buni Loran qatoriga yoysak,

$$f(z)=\frac{shz}{z}=\frac{1}{z}\left(z+\frac{z^3}{3!}+\frac{z^5}{5!}+...\right)=1+\frac{z^2}{3!}+\frac{z^4}{5!}+\frac{z^6}{7!}+...$$

hosil bo'lib, bosh qismi yo'q. Demak, $z=0$ chetlantiriladigan nuqta bo'lib, $z(0)=1$ ni qabul qilsak, bu funktsiya $z=0$ da ham analitik bo'ladi.

3-misol. $f(z)=\cos\frac{1}{z}+\sin\left(\frac{2-\pi z}{2z}\right)$ funktsiya uchun $z=0$ maxsus nuqtadir. Buning o'ng tomonini ixchamlaymiz:

$$f(z)=\cos\frac{1}{z}+\sin\left(\frac{1}{z}-\frac{\pi}{2}\right)=\cos\frac{1}{z}+\left(\sin\frac{1}{z}\cos\frac{\pi}{2}-\cos\frac{1}{z}\sin\frac{\pi}{2}\right)=\cos\frac{1}{z}-\cos\frac{1}{z}=0$$

Aniq son kelib chiqdi, demak, $z=0$ chetlantirilgan maxsus nuqtadir.

4-misol. $f(z)=zsh\frac{1}{z}$ funktsiya uchun $z=0$ maxsus nuqta.

Yechilishi: Oldingi 2-misolga ko'ra

$$zsh\frac{1}{z}=z\left(\frac{1}{z}+\frac{1}{3!z^3}+\frac{1}{5!z^5}+...\right)=1+\frac{1}{3!z^2}+\frac{1}{5!z^4}+...$$

Bu esa Loran qatorining to'la bosh qismidan iborat bo'lgani uchun $z=0$ muhim maxsus nuqtadir.

5-misol. $f(z)=e^{\frac{1}{z+2}}$ funktsiya uchun $z=-2$ maxsus nuqta.

Yechilishi: $f(z)$ ni Loran qatoriga yoyamiz. Ma'lumki,

$$e^z=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+...+\frac{z^n}{n!}+...$$

shunga asosan

$$e^{\frac{1}{z+2}}=1+\frac{1}{z+2}+\frac{1}{2!(z+2)^2}+...+\frac{1}{n!(z+2)^n}+....$$

Bu esa bosh qism bo'lgani uchun $z=a=-2$ son berilgan funktsiyaning muhim maxsus nuqtasidir.

6-misol. $f(z)=e^{\frac{z}{1-z}}$ funktsiyaning maxsus nuqtasi $z=1$.

Yechilishi: Ma'lumki,

$$\frac{z}{1-z}=-\frac{z}{z-1}=-\frac{(z-1)+1}{z-1}=-1-\frac{1}{z-1};\quad e^{\frac{z}{1-z}}=e^{-1}e^{-\frac{1}{z-1}}.$$

Shunga muvofiq,

$$f(z)=\left[1-\frac{1}{z-1}+\frac{1}{2!(z-1)^2}-\frac{1}{3!(z-1)^3}+\frac{1}{4!(z-1)^4}-...\right]$$

Bu esa Loran qatorining to'la bosh qismidan iborat bo'lgani uchun $z=-1$ muhim maxsus nuqtadir.

7-misol. $f(z)=\sin\frac{1}{1-z}$ funktsiyaning maxsus nuqtasi $z=1$.

Yechilishi: $f(z)$ ni qatorga yoysak,

$$\sin\frac{1}{1-z}=\frac{1}{1-z}-\frac{1}{3!(1-z)^2}+\frac{1}{5!(1-z)^5}-\frac{1}{7!(1-z)^7}+...$$

bosh qismdan iborat bo'lib, $z=1$ muhim maxsus nuqta ekan.

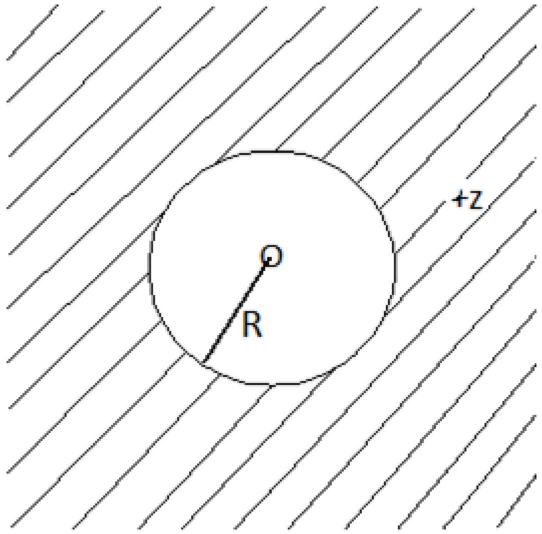
Mashqlar

Quyidagi funktsiyalarning muhim maxsus nuqtalarini toping:

31. $z^2\cos\frac{\pi}{z}$; 32. e^{tgz}; 33. $\sin\frac{\pi}{z}$; 34. $\cos\frac{z}{1+z}$; 35. $\sin\frac{\pi}{1+z^2}$; 36. $z^2\sin\frac{z}{z+1}$; 37. $\sin e^{\frac{1}{z}}$; 38. $\frac{1}{z^2-1}\cos\frac{\pi z}{1+z}$; 39. $e^{tg\frac{\pi}{z}}$.

7.7-§ . CHEKSIZ UZOQLASHGAN NUQTA.

Ta'rifdan ma'lumki, markaz nol nuqtaga joylashgan har qanday katta R radiusli doira tashqarisi cheksiz uzoqlashgan $z=\infty$ nuqtaning *atrofi* deyiladi (7.1-rasm, bu yerda aylana manfiy yo'nalishda olinadi).



7.1-rasm

Ba'zan berilgan $f(z)$ funksiya cheksiz uzoqlashgan nuqta atrofida bir qiymatki va analitik bo'lib, $z=\infty$ nuqtaning o'zi ajralgan maxsus nuqta bo'lib qoladi. Bu nuqta ham $z=a\neq\infty$ ajralgan maxsus nuqtaga o'xshash uch tipga ajratiladi: *cheklantiriladigan* (tuzatiladigan), *qutb, muhim maxsus* nuqtalar.

Bundan oldingi paragrafda biz $z=a\neq\infty$ ajralgan maxsus nuqtaning qaysi tipga kirishini Loran qatori vositasida aniqlashni ko'rgan edik. Endi cheksiz uzoqlashgan $z=\infty$ nuqtaning ham qaysi tipga kirishi o'sha Loran qatori yordamida topish mumkin ekanligini ko'raylik. Buning uchun biz dastlab (7.9) dan $z=a=0$ deb faraz etib, quyidagi Loran qatorini yozib olamiz:

$$f(z)=\sum_{n=-\infty}^{+\infty}c_nz^n=\sum_{n=0}^{\infty}c_nz^n+\sum_{n=-1}^{-\infty}c_nz^n=\left(c_0+c_1z+c_2z^2+...+c_nz^n+...\right)+$$

$$+\left(\frac{c_{-1}}{z}+\frac{c_{-2}}{z^2}+...+\frac{c_{-n}}{z^n}+...\right) \qquad (7.10)$$

Bu esa $f(z)$ funksiyaning $z=0$ atrofidagi Loran qatoridir. Endi $z=\frac{1}{\xi}$ deb belgilab olamiz, u holda $\xi=\frac{1}{z}$; $\lim\limits_{z\to0}\frac{1}{z}=\lim\limits_{\xi\to\infty}\xi=\infty$, ya'ni $z=0$ ga $\xi=\infty$ mos keladi. Agar

$$f(z)=f\left(\frac{1}{\xi}\right)=\varphi(\xi)$$

deb belgilab olsak, (7.10) dan $\xi=\infty$ nuqta atrofida Loran qatoriga ega bo'lamiz:

$$f(z)=\varphi(\xi)=f\left(\frac{1}{\xi}\right)=\left(c_0+\frac{c_1}{\xi}+\frac{c_2}{\xi^2}+\ldots+\frac{c_n}{\xi^n}+\ldots\right)+$$
$$+\left(c_{-1}\xi+c_{-2}\xi^2+\ldots+c_{-n}\xi^n+\ldots\right) \qquad (7.11)$$

Ma'lumki, $z=0$ ajralgan maxsus nuqtaning qaysi tipga kirishini biz (7.10) dan aniqlab olar edik. Endi ajralgan maxsus $z=\infty$ cheksiz uzoqlashgan nuqtaning qaysi tipiga kirishini esa (7.11) qatordan aniqlanadi. Lekin z bilan ξ bir-biriga teskari bo'lgani uchun $\xi=\infty$ nuqtaga tegishli (7.11) Loran qatorining bosh va to'gri qismlarining rollari almashadi, ya'ni

a) qatorning to'g'ri qismi:

$$c_0+\frac{c_1}{\xi}+\frac{c_2}{\xi^2}+\ldots+\frac{c_n}{\xi^n}+\ldots=\sum_{n=0}^{\infty}c_n\xi^{-n},$$

b) qatorning bosh qismi:

$$c_{-1}\xi+c_{-2}\xi^2+\ldots+c_{-n}\xi^n+\ldots=\sum_{n=1}^{\infty}c_{-n}\xi^n$$

Odatda misol yechishni yengillashtirish maqsadida ξ o'rniga z ishlatiladi. U holda $z=\infty$ nuqta atrofida $f(z)$ funksiyani Loran qatoriga yoysak, yana (7.10) qator yozilib, uning mazmuni o'zgaradi, ya'ni

$$f(z)=\left(c_0+c_1z+c_2z^2+\ldots+c_nz^n+\ldots\right)+\left(\frac{c_{-1}}{z}+\frac{c_{-2}}{z^2}+\ldots+\frac{c_{-n}}{z^n}+\ldots\right). \qquad (7.12)$$

Buning birinchisi $z=\infty$ nuqta atrofidagi Loran qatorining bosh qismi bo'lib, ikkinchisi to'g'ri qismidir. Shunday qilib, quyidagi qoidaga ega bo'ldik:

1. (7.12) Loran qatori bosh qismga ega bo'lmasa, u holda $z=\infty$ chetlantiriladigan (tuzatiladigan) nuqta bo'ladi.
2. Bosh qismdagi hadlar soni chekli bo'lsa, u holda $z=\infty$ qutb bo'ladi.
3. Bosh qism to'la bo'lsa, ya'ni cheksiz ko'phadlarga ega bo'lsa, u holda $z=\infty$ muhim maxsus nuqta bo'ladi.

1-misol. $f(z)=\dfrac{z}{5+z^2}$

Yechilishi: Bu funksiyani $z=\infty$ nuqta atrofida Loran qatoriga yoyish oson:

$$f(z)=\frac{1}{1+\frac{5}{z^2}}=1-\frac{5}{z^2}+\frac{5^2}{z^4}-\frac{5^3}{z^6}+\ldots+(-1)^n\left(\frac{5}{z^2}\right)^n+\ldots$$

Bu qator faqat to'g'ri qismdangina iborat bo'lganligi sababli $z=\infty$ chetlantiriladigan maxsus nuqtadir. Bu qatordan ko'rinadiki, $|t|\to\infty$ da $f(t)\to 1$, shu sababli $f(\infty)=1$ qabul qilinsa, $f(z)$ $z=\infty$ da ham analitik bo'ladi.

2-misol. $f(z)=e^{\frac{1}{z}}+z^2-4$

Yechilishi: Ma'lumki,

$$f(z)=z^2-4+1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+...+\frac{1}{n!z^n}...$$

Buning bosh qismi ikkitagina (z^2-4) haddan iborat bo'lib, 2-darajali bo'lgani sababli $z=\infty$ nuqta ikki karrali qutbdir.

3-misol. $f(z)=e^{-z}$

Yechilishi: Ma'lumki,

$$e^{-z}=1-z+\frac{z^2}{2!}-\frac{z^3}{3!}+\frac{z^4}{4!}-...+(-1)^n\frac{z^n}{n!}+...$$

bu qator faqat bosh qismdan iborat bo'lgani sababli $z=\infty$ nuqta muhim maxsus nuqtadir.

4-misol. $f(z)=e^{\frac{1}{1-z}}$

Yechilishi: $f(z)$ ni Loran qatoriga yoysak, u

$$f(z)=e^{\frac{1}{1-z}}=1+\frac{1}{1-z}+\frac{1}{2!(1-z)^2}+\frac{1}{3!(1-z)^3}+...+\frac{1}{n!(1-z)^n}+...$$

to'g'ri qismdan iborat bo'lgani sababli $z=\infty$ chetlantiriladigan maxsus nuqta bo'lib, $f(\infty)=1$ deb qabul qilinsa, $f(z)$ funksiya $z=\infty$ da ham analitik bo'ladi.

Javoblar

1. $z=(2n-1)\pi\,(n=0,\pm1,\pm2...)$ ikki karrali nollardir; 2. $z=2n\pi i,\ (n=0,\pm1,\pm2...)$ oddiy nollardir; 3. $z=0$ besh karrali noldir; 4. $z=\pm i$ lar uch karrali nollardir: $z=n\pi i(n=0,\pm1,\pm2...)$ lar esa oddiy nollardir; 5. $z=\pi i$ ikki karrali nol bo'lib, $z=n\pi i(n=0,\pm1,\pm2...)$ lar oddiy nollardir; 6. $z=0$ oddiy nol, $z=n\pi i(n=0,\pm1,\pm2...)$ lar esa ikki karrali nollardir; 7. $z=(4n+1)\frac{\pi}{2}i\ \ (n=0,\pm1,\pm2...)$ ikki karrali nollardir; 8. $z=\sqrt{(2n+1)\frac{\pi}{2}}$ $z=\sqrt{(2n+1)\frac{\pi}{2}}\frac{1\pm i\sqrt{3}}{2}$ $(n=0,\pm1,\pm2...)$ oddiy. 9. $z=0$ oddiy nol; 10. $z=0$ uch karrali nol; 11. $z=0$ ikki karrali nol; 12. $z=0$ uch

karrali nol; 13. $z=0$; 14. $z=\infty$; 15. $z=0$; 16. $z=\frac{\pi}{2}$; 17. $z=0$; 18. $z=0$; 19. $z=-1$; 20. $z=0$; 21. $z=\pm i$; 22. $z=\pi i$; 23. $z=0$; 24. $z=\infty$; 25. $z=\pm\frac{1}{2}$; $z=\pm\frac{3}{2}$; 26. $z=0$; 27. $z=0$ va $z=\pm 1$; 28. $z=\frac{1\pm i}{\sqrt{2}}$ va $z=\frac{1\pm i}{\sqrt{2}}$; 29. $z=\pm i$; 30. $z=2n\pi i$ $(n=0,\pm 1,\pm 2...)$; 31. $z=0$; 32. $z=\frac{\pi}{2}$; 33. $z=0$; 34. $z=-1$; 35. $z=-i$; 36. $z=-1$; 37. $z=0$; 38. $z=-1$; 39. $z=\pm 1;\pm\frac{1}{2};\pm\frac{1}{3};....$

VIII BOB

CHEGIRMALAR

8.1-§. Chegirmalarni hisoblash

Ta'rif: $f(z)$ funksiyaning a ajralgan maxsus nuqtadagi ***chegirmasi*** (qoldig'i) deb quyidagi ifodaga aytiladi:

$$res\ f(a) = \frac{1}{2\pi i}\int_{\gamma} f(\xi)\,d\xi \qquad (8.1)$$

Bu erda $\gamma - a$ nuqtani o'z ichiga olgan ixtiyoriy berk kontur.

Har qanday $a\ (a \neq \infty)$ ajralgan maxsus nuqta uchun quydagi tenglik o'rinli

$$res\ f(a) = C_{-1} \qquad (8.2)$$

$C_{-1} - a$ nuqta atrofidagi Loran qatorining koeffisienti.

Agar $a = \infty$ bo'lsa ,

$$res\ f(a) = -C_{-1} \qquad (8.3)$$

a - ajralgan maxsus nuqta bo'lsin $(a \neq \infty)$.

1. Agar a qutulib bo'ladigan maxsus nuqta bo'lsa

$$res\ f(a) = 0$$

2.Agar a $n-$ tartibli qutb bo'lsa, u holda chegirma quyidagicha hisoblanadi:

$$res\ f(a) = \lim_{z\to a}\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left\{(z-a)^n\ f(z)\right\} \qquad (8.4)$$

Xususan, 1- tartibli qutb uchun

$$res\ f(a) = \lim_{z\to a}\{(z-a)\ f(z)\} \qquad (8.5)$$

3. Agar a muhim maxsus nuqta bo'lsa , chegirma yoki (8.1) ta'rif, yoki (8.2) formula bo'yicha hisolanadi.

4. Cheksiz uzoqlashgan ajralgan maxsus nuqta uchun chegirma yoki Loran qatori (8.3) yoki ta'rif (8.1) yordamida topiladi.

1-misol. Barcha ajratilgan maxsus nuqtalarga nisbatan chegirmalarni toping.

$$f(z) = \frac{z^2}{1+z^2}.$$

Yechilishi. $\frac{z^2}{1+z^2}$ funksiya uchun, maxraj nolga aylanadigan nuqtalar $z = \pm i$ ajralgan maxsus nuqtalar hisoblanadi:

Bular 1-tartibli qutblardir. Chegirmani (8.5) formula bo'yicha topamiz:

$$res\ f(i)=\lim_{z\to i}\left\{(z-i)\frac{z^2}{(z-i)(z+i)}\right\}=\frac{i}{2}$$

$$res\ f(-i)=\lim_{z\to -i}\left\{(z+i)\frac{z^2}{(z+i)(z-i)}\right\}=-\frac{i}{2}$$

$z=\infty$ nuqta regulyar oddiy nuqta hisoblanadi, chunki

$$\lim_{z\to\infty} f(z)=1.$$

2-misol. Barcha ajralgan maxsus nuqtalarga nisbatan chegirmalarni toping.

$$f(z)=\frac{e^z}{z^2(z^2+9)}.$$

Yechilishi. Berilgan funksiya uchun quydagilar ajralgan maxsus nuqtalar hisoblanadi:

$z=0$ - 2-tartibli qutb,

$z=\pm 3i$ - 1-tartibli qutb,

$z=\infty$ - muhim maxsus nuqta.

Chegirmalarni topaylik: $z=0$ nuqtada

$$res\ f(0)=\lim_{z\to 0}\frac{d}{dz}\left\{z^2\cdot\frac{e^z}{z^2(z^2+9)}\right\}=\frac{1}{9}$$

$z=\pm 3i$ nuqtalarda chegirmalar quyidagilarga teng:

$$res\ f(3i)=\lim_{z\to 3i}\left\{(z-3i)\frac{e^z}{z^2(z^2+9)}\right\}=\frac{-e^{3i}}{2\cdot 3^3\cdot i},$$

$$res\ f(-3i)=\lim_{z\to -3i}\left\{(z+3i)\frac{e^z}{z^2(z^2+9)}\right\}=\frac{e^{-3i}}{2\cdot 3^3\cdot i},$$

$z=\infty$ nuqtadagi chegirmani Loran qatori yordamida topamiz. Buning uchun $z=\frac{1}{\xi}$ almashtirish bajaramiz.

$$f(z)=\frac{e^z}{z^2(z^2+9)}=\frac{\xi^4 e^{\frac{1}{\xi}}}{(1+9\xi^2)}=\varphi(\xi)$$

va $\varphi(\xi)$ ni $\xi=0$ atrofida yoyamiz.

$$\varphi(\xi)=\xi^4\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{\xi^n}\cdot\sum_{k=0}^{\infty}(-1)^k\left(9\xi^2\right)^k$$

Bu yerda biz eksponentaning yoyilmasidan foydalandik, $\frac{1}{1+9\xi^2}$ ni esa geometrik progressiya yig'indisi deb oldik.

$$\varphi(\xi)=\sum_{n,k=0}^{\infty}\frac{(-1)^k\cdot 3^{2k}}{n!}\cdot\xi^{2k-n+4}$$

z o'zgaruvchiga qaytib , $z=\infty$ atrofida $f(z)$ funksiyaning Loran yoyilmasini hosil qilamiz.

$$f(z)=\sum_{n,k=0}^{\infty}\frac{(-1)\cdot 3^{2k}}{n!}z^{n-2k-4}$$

Chegirma C_{-1} koeffitsientga teng, bu esa o'z navbatida z^{-1} ning oldidagi koeffitsientdir. Darajaning tartibi minus (-1) teng bo'lishi uchun, n va k lar quyidagi shartni qanoatlantirishi kerak

$$n-2k-4=-1.$$

Bu yerdan

$$C_{-1}=\sum_{n,k=0}^{\infty}\frac{(-1)^k\cdot 3^{2k}}{n!}.$$

Ixtiyoriy o'zgaruvchi k ni tanlab olib, n ni esa k orqali ifodalab quyidagini hosil qilamiz:

$$C_{-1}=\sum_{k=0}^{\infty}\frac{(-1)^k\cdot 3^{2k}}{(2k+3)!}\quad .$$

Bu yig'indini hisoblash uchun $k=m-1$ almashtirish bajaramiz.

$$C_{-1}=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}\cdot 3^{2m-2}}{(2m+1)!}=-\frac{1}{3^3}\sum_{m=1}^{\infty}\frac{(-1)^m 3^{2m+1}}{(2m+1)!}\quad .$$

Agar olingan summani $sin3$ bilan solishtirsak:

$$sin3=\sum_{0}^{\infty}\frac{(-1)^m\cdot 3^{2m+1}}{(2m+1)!}$$

C_{-1} osonlikcha topiladi:

$$C_{-1}=-\frac{1}{3^3}\left\{sin3-3\right\}.$$

Shunday qilib, $z=\infty$ bo'lganda:

$$resf(\infty)=-C_{-1}=\frac{1}{27}\{sin3-3\}.$$

Buni tekshirish uchun hamma chegirmalarning yig'indisini 0 ga teng ekanligidan foydalanamiz:

$$res\ f(0)+res\ f(3i)+res\ f(-3i)+res\ f(\infty)=\frac{1}{9}+\frac{1}{2\cdot 3^3\cdot i}(e^{-3i}-e^{3i})+\frac{1}{27}\sin 3-\frac{1}{9}=0$$

3-misol. $f(z)=\dfrac{1}{\sin z}$ funksiyaning ajralgan maxsus nuqtalardagi chegirmalarini toping.

Yechilishi. $z_k=k\pi,\quad k=0,\pm1,\pm2,...$ nuqtalar ajralgan maxsus nuqtalardir. Bu nuqtalar birinchi tartibli qutblardir. $z=\infty$ nuqta $z_k=k\pi$

maxsus nuqtalarning quyulish nuqtasidir, shuning uchun, ajralgan maxsus nuqta emas. $f(z)$ ning z_k dagi chegirmasi quyidagiga teng:

$$resf(z_k)=\lim_{z\to z_k}\left\{(z-z_k)\cdot\frac{1}{\sin z}\right\}=\lim_{z\to z\pi}\left\{(z-k\pi)\cdot\frac{(-1)^k}{\sin(z-k\pi)}\right\}.$$

Biz $\sin z=(-1)^k\sin(z-k\pi)$ ifodadan foydalandik. $z\to 0,\quad \frac{\sin z}{z}=1$ ajoyib limitni qo'llab quyidagini olamiz: $resf(k\pi)=(-1)^k$.

Mashqlar

Quyidagi funksiyalarning maxsus nuqtalardagi chegirmalarini toping.

1. $f(z)=\dfrac{1}{z^3+z^5}$; 2. $f(z)=\dfrac{z^2}{(z^2+1)^2}$; 3. $f(z)=ctg^2 z$;

4. $f(z)=\dfrac{z}{(z+1)^5\cdot(z-2)^2}$; 5. $f(z)=\dfrac{\cos z}{z^3-\frac{\pi}{2}\cdot z^2}$; 6. $f(z)=z^3\cdot e^{\frac{1}{2}}$;

7. $f(z)=\dfrac{z^{2n}}{(z-1)^4}\quad (n>0, butun)$; 8. $f(z)=thz$.

8.2-§. CHEGIRMALAR TEOREMASINING ANIQ INTEGRALLARNI HISOBLASHGA TATBIQI

Integrallarni chegirmalar nazariyasi yordamida hisoblash Koshining ushbu teoremasiga asoslangan: agar $f(z)$ funksiya G sohaning ichida chekli sondagi maxsus nuqtalardan boshqa hamma nuqtalarda analitik va C sohaning chegarasi C da uzluksiz bo'lsa, unda:

$$\int_C f(z)dz=2\pi i\sum_{k=}^{n}resf(z_k). \tag{8.6}$$

Bu erda z_k lar $f(z)$ ning maxsus nuqtalari $z_k\in G$. Bu paragrafning hamma misollarida integralni hisoblash talab etiladi.

1-misol: $J=\dfrac{1}{2\pi i}\displaystyle\int_{|z|=1}\sin\frac{1}{z}dz$.

Yechilishi: Integral ostidagi funksiya $z=0$ nuqtada muhim maxsus nuqtaga ega. Bu nuqta konturning ichida yotadi, shuning uchun:

$$J=resf(0).$$

Chegirmani $z=0$ atrofida Loran qatoriga yoyilmasidan topamiz:

$$\sin\frac{1}{z}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\cdot\frac{1}{z^{2k+1}} \tag{8.7}$$

(8.7) dan $C_{-1}=1$ ekanligi ko'rinadi. Shuning uchun:

$$resf(0)=C_{-1}=I=1.$$

2-misol. $J=\frac{1}{2\pi i}\int\limits_{|z|=1} z^n e^{\frac{2}{z}}dz$.

Yechilishi: Integral ostidagi funksiya 2 xil maxsus nuqtaga ega: $z=0$-muhim maxsus nuqta, $z=\infty$ esa $n-$tartibli qutbdir. Integrallash konturining ichida faqat $z=0$ maxsus nuqta yotadi, shuning uchun:

$$J=resf(0).$$

Chegirmani $z=0$ atrofida Loran qatoriga yoyilmasidan topamiz :

$$f(z)=z^n\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{2}{z}\right)^k=\sum_{k=0}^{\infty}\frac{2^k}{k!}z^{n-k} \qquad (8.8)$$

(8.8) dan: $C_{-1}=\frac{2^{n+1}}{(n+1)!}$ bundan :

$$J=resf(0)=C_{-1}=\frac{2^{n+1}}{(n+1)!}$$

3-misol. $J=\int\limits_0^{2\pi} e^{\cos\varphi}\cos(n\varphi-\sin\varphi)\,d\varphi$ bu yerda n -butun son.

Yechilishi: Integral ostidagi funksiyani Eyler formulasi yordamida boshqa ko'rinishda yozib olamiz:

$$\cos(n\varphi-\sin\varphi)\,e^{\cos\varphi}=\frac{1}{2}\left[e^{in\varphi-i\sin\varphi}+e^{-in\varphi+i\sin\varphi}\right]e^{\cos\varphi}$$

$z=e^{i\varphi}$ almashtirishdan keyin integral quyidagi ko'rinishga keladi:

$$J=\frac{1}{2}\int\limits_0^{2\pi}\left[e^{in\varphi+e^{-i\varphi}}+e^{-in\varphi+e^{i\varphi}}\right]d\varphi=\frac{1}{2i}\int\limits_{|z|=1}\left[z^n e^{\frac{1}{z}}+z^{-n}e^{z}\right]\frac{dz}{z}=$$

$$=\frac{1}{2i}\int\limits_{|z|=1} z^{n-1}e^{\frac{1}{z}}dz+\frac{1}{2i}\int\limits_{|z|=1} z^{-(n+1)}e^{z}dz \qquad (8.9)$$

$n\geq 0$ bo`lgan holni ko`raylik: (8.9) dagi birinchi integral quyidagiga teng:

$$\int\limits_{|z|=1} e^{n-1}e^{\frac{1}{z}}dz=2\pi i\,resf(0)=\frac{2\pi i}{n!}$$

$z=0$ muhim maxsus nuqta va chegirma quyidagiga teng (12-misolga qarang)

$$res\ f(0)=C_{-1}=\frac{1}{n!}$$

(8.9) dagi ikkinchi integral esa quyidagichadir :

$$\int\limits_{|z|=1} z^{-(n+1)}e^{z}dz=2\pi i resf(0)=\frac{2\pi i}{n!}$$

Bu yerda $z=0$ $(n+1)$ – tartiibli qutb. Chegirmani $z=0$ atrofida Loran yoyilmasi yordamida topish qulay :

$$z^{-(n+1)}e^{z}=z^{-(n+1)}\sum_{k=0}^{\infty}\frac{z^{k}}{k!}=\sum_{k=0}^{\infty}\frac{z^{k-n-1}}{k!}$$

Natijada:

$$J=\frac{1}{2i}2\pi i\left(\frac{1}{n!}+\frac{1}{n!}\right)=\frac{2\pi}{n!}$$

Endi $n<0$ bo'lgan holni ko'raylik. (8.9) dagi 1 - va 2 - integrallar nolga teng. 1-integral ostidagi funksiyaning $z=0$ dagi chegirmasi nolga teng. 2-integral ostidagi funksiya esa $z=0$ da analitik, shuning uchun integralning qiymati nolga teng.

4–misol. $J=\int\limits_{0}^{\infty}\frac{dx}{(1+x^{2})^{n}}$ n – natural son.

Yechilishi : Integral ostidagi funksiyaning juftligidan :

$$J=\frac{1}{2}\int\limits_{-\infty}^{\infty}\frac{dx}{(1+x^{2})^{n}}$$

Bu integralni hisoblash uchun kompleks tekisligida ushbu integralni ko'ramiz:

$$\int\limits_{C}\frac{dz}{(1+z^{2})^{n}}$$

Kontur 8.1–rasmda chizilgan. Kontur ichida n - tartibli $z=i$ qutb maxsus nuqta bor. Bu nuqtadagi chegirmani topamiz:

$$res f(i)=\lim_{z\to i}\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left\{(z-i)^{n}\frac{1}{(z-i)^{n}(z+i)^{n}}\right\}=$$

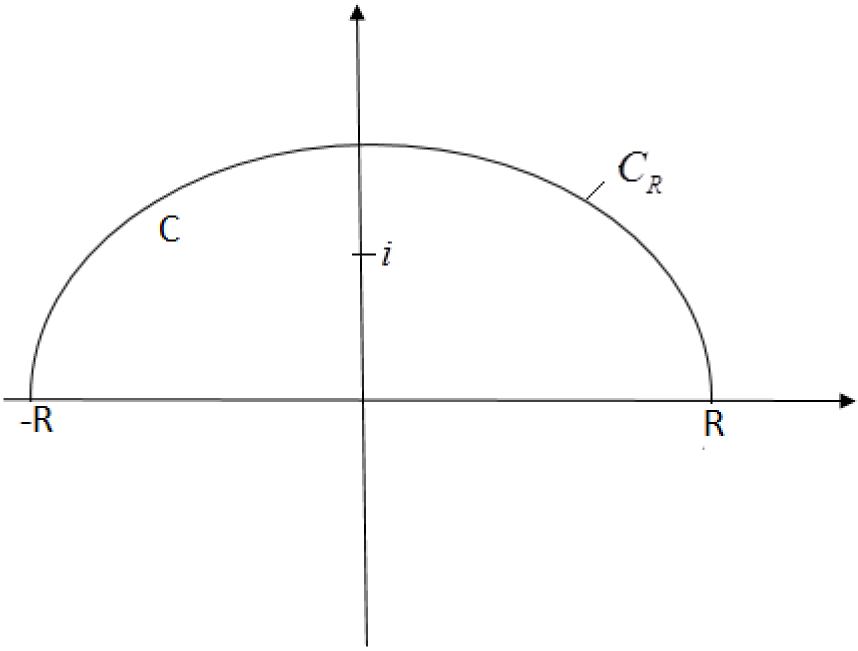
$$=\lim_{z\to i}\frac{1}{(n-1)!}\frac{(-1)^{n-1}n(n+1)...(2n-2)}{(z+i)^{2n-1}}=-i\frac{n(n+1)...(2n-2)}{(n-1)!\,2^{2n-1}}.$$

Endi integralni C konturning qismlari bo'yicha yoyib yozamiz:

$$\int\limits_{C}\frac{dz}{(1+z^{2})^{n}}=\int\limits_{-R}^{R}\frac{dx}{(1+x^{2})^{n}}+\int\limits_{C_R}\frac{dz}{(1+z^{2})^{n}}$$

Koshi teoremasini qo'llab buni olamiz:

$$\int\limits_{C}\frac{dz}{(1+z^{2})^{n}}=2\pi i\, res f(i)=2\pi\frac{n(n+1)...(2n-2)}{(n-1)!\,2^{2n-1}} \qquad (8.10)$$



8.1-rasm

$R\to\infty$ C_R yarim aylana bo'yicha integral nolga intiladi, ya'ni integral ostidagi funksiya $\frac{1}{|z|}$ ga nisbatan tezroq kamayadi. Shuning uchun $R\to\infty$ da

$$\int_C \frac{dz}{(1+z^2)^n} = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^n}$$

(8.10) ni qo'llab quyidagini olamiz:

$$J = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^n} = \frac{\pi}{2}\frac{(2n-2)!}{2^{2n-1}\left[(n-1)!\right]^2}$$

5 – misol. Integralning bosh qiymatini toping :

$$J = \int_{-\infty}^{\infty} \frac{\cos tx}{1+x^3}dx$$

Yechilishi: Integralning bosh qiymati quyidagicha aniqlanadi:

$$V.p.J = \lim_{\varepsilon\to 0}\left\{\int_{-\infty}^{-1-\varepsilon} \frac{\cos tx}{1+x^3}dx + \int_{-1+\varepsilon}^{\infty} \frac{\cos tx}{1+x^3}dx\right\}$$

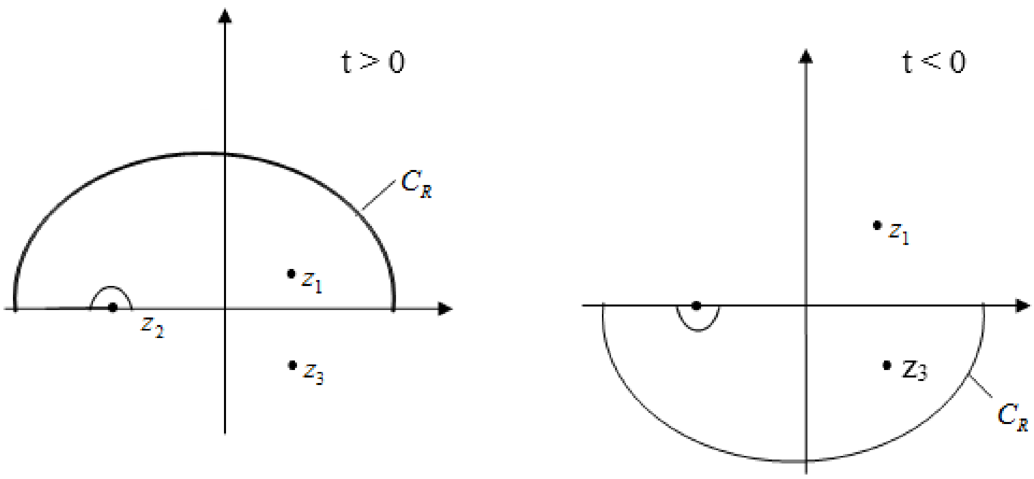
Integralning bosh qiymatini chegirmalar nazariyasi yordamida topamiz. Keyinchalik Jordano lemmasidan foydalanishga yordam beradigan qo'shimcha almashtirish qilamiz:

$$V.p.J = \operatorname{Re}\lim_{\varepsilon\to 0}\left\{\int_{-\infty}^{-1-\varepsilon} \frac{e^{itx}}{1+x^3}dx + \int_{-1+\varepsilon}^{\infty} \frac{e^{itx}}{1+x^3}dx\right\} \qquad (8.11)$$

z kompleks tekisligida bu integralni ko'raylik:

$$\int_C \frac{e^{itz}}{1+z^3}dz$$

C kontur 8.2-rasmda ko'rsatilgan :



8.2-rasm.

Avval $t>0$ holni ko'rib chiqamiz. Jordano lemmasidan foydalanish uchun kontur yuqori yarim tekislikda olinadi.

Integral ostidagi funksiyaning 3 ta maxsus nuqtasi bor

$$z_1 = e^{i\frac{\pi}{3}}, \quad z_2 = e^{i\frac{\pi+2\pi}{3}} = e^{i\pi}, \quad z_3 = e^{i\frac{5\pi}{3}}$$

Kontur ichida faqatgina z_1 yotadi, shuning uchun:

$$\int_C \frac{e^{itz}}{1+z^3}dz = 2\pi i res f(z_1) = \frac{2}{3}\pi i e^{-\frac{\sqrt{3}}{2}t} e^{i\left(\frac{t}{2}-\frac{2}{3}\pi\right)}$$

Endi integralni qismlarga ajratib yozib olaylik :

$$\int_C \frac{e^{itz}}{1+z^3}dz = \int_{-R}^{1-\varepsilon} \frac{e^{itz}}{1+x^3}dx + \int_{1+\varepsilon}^{R} \frac{e^{itz}}{1+x^3}dx + \int_{C_\varepsilon} \frac{e^{itz}}{1+z^3}dz + \int_{C_R} \frac{e^{itz}}{1+z^3}dz \qquad (8.12)$$

O'ng tomondagi birinchi ikki had bizga $R\to\infty$ $\varepsilon\to 0$ kerakli bosh qiymatni beradi. C_R bo'yicha integral Jardano lemmasi bo'yicha $R\to\infty$ nolga intiladi. Demak, faqatgina 3-integralni hisoblash qoladi:

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} \frac{e^{itz}}{1+z^3}dz = \lim_{\varepsilon\to 0}\int_{\pi}^{0} \frac{e^{it\left(-1+\varepsilon e^{i\varphi}\right)} i\varepsilon e^{i\varphi}d\varphi}{(-1+\varepsilon e^{i\varphi})^3+1} = \lim_{\varepsilon\to 0} i\int_{\pi}^{0} \frac{e^{it\left(-1+\varepsilon e^{i\varphi}\right)}\varepsilon e^{i\varphi}d\varphi}{3\varepsilon e^{i\varphi} - 3\varepsilon^2 e^{2i\varphi} + \varepsilon^3 e^{3i\varphi}} \Rightarrow$$

Integral ostidagi funksiyaning tekis yaqinlashuvchi ekanligidan limitni integral ostiga kiritishimiz mumkin. Shunday qilib quyidagini olamiz:

$$\Rightarrow i\int_{\pi}^{0} \frac{e^{-it}}{3}d\varphi = -\frac{i\pi}{3}e^{-it}$$

(8.12) formulaga qaytsak:

$$V.p.J = \mathrm{Re}\left\{\int_C \frac{e^{itz}}{1+z^3}dz - \int_{C_\varepsilon} \frac{e^{itz}}{1+z^3}dz\right\} =$$

$$= \mathrm{Re}\left\{\frac{2}{3}\pi i e^{-\frac{\sqrt{3}}{2}t} e^{i\left(\frac{t}{2}-\frac{2}{3}\pi\right)} + \frac{i\pi}{3}e^{-it}\right\} = \frac{2}{3}\pi e^{-\frac{\sqrt{3}}{2}t}\sin\left(-\frac{t}{2}+\frac{2}{3}\pi\right) + \frac{\pi}{3}\sin t$$

(8.13)

Endi $t<0$ bo'lgan holni ko'raylik. Jordano lemmasidan foydalanish uchun konturni pastki yarim tekislikda berkitamiz. Konturning ichida z_3 maxsus nuqta bor. Shuning uchun

$$\int_C \frac{e^{itz}}{1+z^3}dz = 2\pi i resf\left(z_3\right) = \frac{2}{3}\pi i e^{it\left(\cos\frac{5}{3}\pi + i\sin\frac{5}{3}\pi\right)} \cdot e^{-i\frac{10}{3}\pi} = \frac{2}{3}\pi i e^{\frac{\sqrt{3}}{2}t} e^{i\left[\frac{2}{3}\pi+\frac{t}{2}\right]}$$

Integralni qismlarga bo'lib yozamiz :

$$\int_C \frac{e^{itz}}{1+z^3}dz = -\left\{\int_{-R}^{-1-\varepsilon} \frac{e^{itz}}{1+z^3}dz + \int_{-1+\varepsilon}^{R} \frac{e^{itz}}{1+z^3}dz\right\} + \int_{C_\varepsilon} \frac{e^{itz}}{1+z^3}dz + \int_{C_R} \frac{e^{itz}}{1+z^3}dz \quad (8.14)$$

Bu holda integrallash yo'nalishi soat strelkasiga qarama-qarshi bo'lgani uchun o'ng tomondagi integrallar oldiga "-" ishorasini qo'ydik. Jordano lemmasi bo'yicha $R\to\infty$ da C_R bo'yicha integralning qiymati nolga intiladi. C_ε bo'yicha integral $t>0$ bo'lgan holdagiday hisoblanadi. Faqatgina chegaralar $0\le\varphi\le-\pi$ bo'ladi. Shularni e'tiborga olsak :

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} \frac{e^{itz}}{1+z^3}dz = -\frac{i\pi}{3}e^{-it},$$

topilgan qiymatlarni (8.14) ga olib borib qo'ysak :

$$V.p.J = \mathrm{Re}\left\{-\frac{2}{3}\pi i e^{\frac{\sqrt{3}}{2}t} e^{i\left[\frac{2}{3}\pi+\frac{1}{2}\right]} - \frac{i\pi}{3}e^{-it}\right\} = \frac{2}{3}\pi e^{\frac{\sqrt{3}}{2}t}\sin\left(\frac{2}{3}\pi+\frac{t}{2}\right) - \frac{\pi}{3}\sin t$$

(8.15)

Integrallarning $t>0$ va $t<0$ holdagi qiymatlarini bitta umumiy formula bilan yozish mumkin:

$$V.p.J = \frac{\pi}{3}\left\{2e^{-\frac{\sqrt{3}}{2}|t|}\sin\left(\frac{2}{3}\pi - \frac{|t|}{2}\right) + \sin|t|\right\}$$

6-misol. Integralning bosh qiymatini toping:

$$J = \int_0^\infty \frac{x^p dx}{1-x} \qquad -1<p<0$$

Yechilishi. Bosh qiymat quyidagicha aniqlanadi.

$$V.p.J = \lim_{\varepsilon\to 0}\left\{\int_0^{1-\varepsilon} \frac{x^p dx}{1-x} + \int_{1+c}^{\infty} \frac{x^p dx}{1-x}\right\} \quad (8.16)$$

(8.16) ni hisoblash uchun kompleks tekislikda ushbu integralni ko'rib chiqamiz:

$$\int_C \frac{z^p dz}{1-z}$$

bunda C kontur 8.3-rasmda ko'rsatilgan.

$f(z)=\frac{z^p}{1-z}$ ning $z=1$ da 1-tartibli qutbi, va $z=0$ da tarmoqlanish nuqtalari bor.

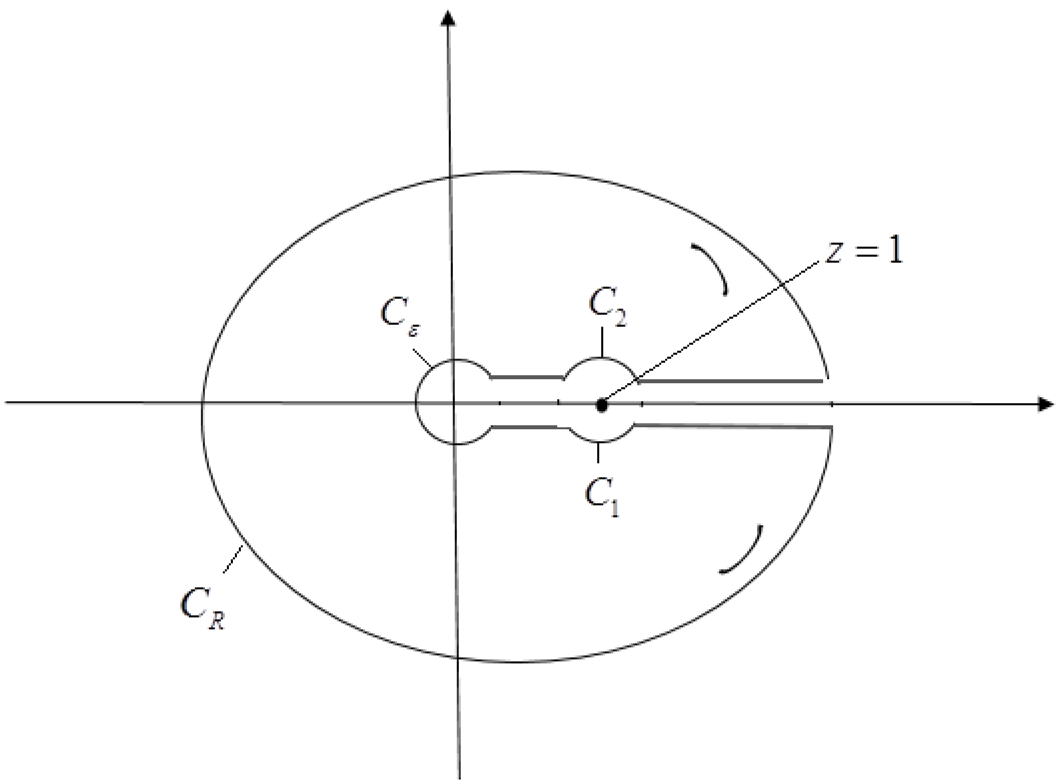
Ko'p qiymatli funksiya z^p ning tarmog'ini shunday tanlaylikki musbat haqiqiy yarim o'q $0<x<\infty$ da

$$z^p = x^p \qquad (8.17)$$

$$z^p = \rho^p e^{i(\varphi+2k\pi)p} \qquad (8.18)$$

(8.18) ni $\varphi=0$ dagi qiymatini (8.17) ga olib borib qo'yib, funksiyaning tarmog'ini tanlaymiz. $\rho^p e^{i(\varphi+2\pi k)p} = x^p$

Bu ifoda faqatgina $k=0$ bo'lgandagina o'rinlidir. Shuning uchun, bizga kerakli tarmoq qo'yidagi ko'rinishga ega: $z^p = \rho^p e^{i\varphi p}$



8.3-rasm.

Shunday qilib,bizga ushbu integralni hisoblash kerak:

$$\int_C \frac{z^p dz}{1-z}$$

Bu yerdagi $z^p = \rho^p e^{i\varphi p}$ deb tushunamiz. C kontur ichida maxsus nuqtalar yo'q. Shuning uchun

$$\int_C \frac{z^p dz}{1-z} = 0 \qquad (8.19)$$

Integralni C konturning bo'laklari bo'yicha yozib chiqamiz:

$$\int_c \frac{z^p dz}{1-z} = \int_R^{1+\varepsilon_1} \frac{x^p e^{2\pi ip} dx}{1-x} + \int_{1-\varepsilon_1}^{\varepsilon} \frac{x^p e^{2\pi ip} dx}{1-x} + \int_{\varepsilon}^{1+\varepsilon_1} \frac{x^p dx}{1-x} + \int_{1+\varepsilon_1}^{R} \frac{x^p dx}{1-x} + \int_{C_\varepsilon} \frac{z^p dz}{1-z} + \int_{C_1} \frac{z^p dz}{1-z} +$$

$$+ \int_{C_2} \frac{z^p dz}{1-z} + \int_{C_3} \frac{z^p dz}{1-z} \qquad (8.20)$$

Bu ifodaga izoh beraylik. Konturning $(R_1;\, 1+\varepsilon_1)$ va $(1-\varepsilon_1;\varepsilon)$ bo'lagida

$$z^p = |z|^p e^{2\pi ip} = x^p e^{2\pi ip}$$

Chunki $\varphi = 2\pi$

Shuning uchun, o'ng tomondagi 1-va 2- integrallarning ostida fazoviy ko'paytuvchi paydo bo'ldi. C_R bo'yicha integral esa, $R \to \infty$ da nolga intiladi, chunki integral ostidagi funksiya $|z| \to \infty$ da $\frac{1}{|z|}$ ga nisbatan tezroq nolga intiladi. Endi integralni hisoblaylik :

$$\lim_{\varepsilon_1 \to o} \int_{C_1} \frac{z^p dz}{1-z} = \lim_{\varepsilon_1 \to o} \int_{2\pi}^{o} \frac{\varepsilon^p e^{i\varphi p} \varepsilon i e^{i\varphi} d\varphi}{1-\varepsilon e^{i\varphi}} = 0$$

$$\lim_{\varepsilon_1 \to 0} \int_{C_2} \frac{z^p dz}{1-z} = \lim_{\varepsilon_1 \to 0} \int_{\pi}^{0} \frac{\left(1+\varepsilon_1 e^{i\varphi}\right)^p \varepsilon_1 i e^{i\varphi} d\varphi}{-\varepsilon_1 e^{i\varphi}} = \pi i$$

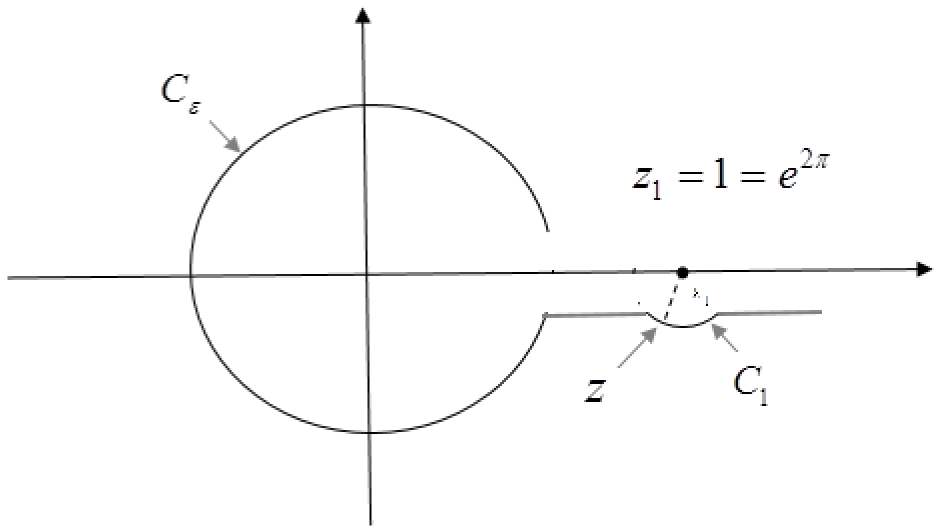
$$\lim_{\varepsilon_1 \to 0} \int_{C_3} \frac{z^p dz}{1-z} = \lim_{\varepsilon \to 0} \int_{0}^{-\pi} \frac{\left(e^{2\pi i} + \varepsilon_1 e^{i\varphi}\right)^p \varepsilon_1 i e^{i\varphi} d\varphi}{-\varepsilon_1 e^{i\varphi}} = \pi i e^{2i\pi p}$$

Oxirgi intgralning hisoblashga izoh beraylik.

C_1 konturdagi z kompleks sonni quyidagicha yozish mumkin.

$$z = z_1 + \varepsilon_1 e^{i\varphi}$$

z_1 kompleks son birga teng, modulga ega, uning argumenti 2π ga teng, chunki z_1 nuqta qirqimining pastki qirg'og'ida joylashadi. Shuning uchun, $z \in C_1$ ni quyidagi ko'rinishda yozish mumkin.



8.4-rasm

$$z = e^{2\pi i} + \varepsilon_1 e^{i\varphi}$$

Yuqoridagi intgralni hisoblashda shu tenglikdan foydalanilgan. Hisoblangan integrallarning qiymatlarini (8.20) ga qo'yib, quydagicha yozamiz:

$$\left(1-e^{2\pi i\rho}\right)\lim_{\varepsilon_1\to 0}\left\{\int_0^{1-\varepsilon_1}\frac{x^p dx}{1-x}+\int_{1+\varepsilon_1}^{\infty}\frac{x^p}{1-x}dx\right\} = -\pi i\left(1+e^{2\pi pi}\right)$$
$$V.p.J = \frac{\pi i\left(1+e^{2\pi\rho i}\right)}{e^{2\pi i\rho}-1} = \pi ctg\pi\rho \tag{8.21}$$

7-misol. Integralni hisoblang.

$$\mathrm{I} = \int_0^{\infty}\frac{\ln x dx}{\sqrt{x}\left(x^2+a^2\right)^2}$$

Yechilishi. Kompleks tekisligida integral ostidagi funksiya ko'p qiymatlidir. Shuning uchun biz, birinchidan, bu funksiyaning tarmog'ini tanlab olishimiz kerak. Ikkinchidan konturni shunday tanlashimiz kerakki, bu funksiyaning tarmoqlanish nuqtasi $z=0$ kontur ichiga tushib qolmasin.

Tarmoqni shunday tanlab olaylik.

$$\ln z = \ln|z| + i\left(\arg z + 2\pi\right), \quad \sqrt{z} = \sqrt{|z|}\exp\left\{\frac{i\arg z + 2\pi mi}{2}\right\}$$

$(0,\infty)$ yarim o'qda bu funksiyalar $\ln x$ va $\sqrt{x}$ bilan mos kelishi kerak. Shuning uchun

$$\ln z \Rightarrow \ln x + 2\pi ki = \ln x \quad \sqrt{z} \Rightarrow \sqrt{x}e^{i\pi m} = \sqrt{x}$$

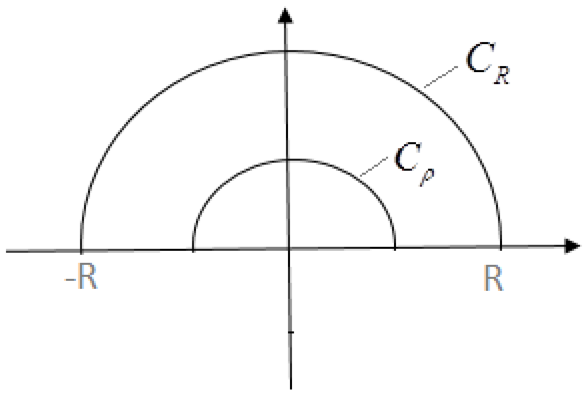
Oxirgi tengliklardan $k=0,\ \ m=0$ teng ekanligini aniqlaymiz, ya'ni tanlangan tarmoqlarni quyidagi ko'rinishda yozish mumkin:

$$\ln z=\ln|z|+i\arg z,\quad \sqrt{z}=\sqrt{|z|}e^{i\frac{\varphi}{2}}$$

Kompleks tekislikda quyidagi integralni ko'rib chiqaylik:

$$\int_C \frac{\ln z dz}{\sqrt{z}\left(z^2+a^2\right)^{2,}} \qquad 8.22)$$

C kontur 8.5-rasmda ko'rsatilgan



8.5-rasm.

C kontur ichida integral ostidagi ifoda $z=ai$ maxsus nuqtada, ikkinchi tartibli qutbga ega. Koshi teoremasiga asosan

$$\int_C \frac{\ln z}{\sqrt{z}(z^2+a^2)^2}dz=2\pi i\, res(ia) \qquad (8.23),$$

$z=ai$ nuqtada chegirma quyidagiga teng:

$$res(ia)=\lim_{z\to ia}\frac{d}{dz}\left\{(z-ia)^2\frac{\ln z}{\sqrt{z}(z-ia)^2(z+ia)^2}\right\}=\frac{2-3\ln ia}{\sqrt{2}\,2^3a^{\frac{z}{2}}}(1+i)=$$

$$=\frac{2-3\left[\ln a+\dfrac{i\pi}{2}\right]}{(2a)^{\frac{z}{2}}}(1+i) \qquad (8.24)$$

endi (8.22) integralni kontur bo'laklari bo'yicha qaytadan yozamiz:

$$\int_C \frac{\ln z}{\sqrt{z}(z^2+a^2)^2}dz=\int_{-R}^{-\rho}\frac{(\ln|x|+i\pi)}{\sqrt{|x|e^{\frac{i\pi}{2}}}(x^2+a^2)^2}dx+\int_{C_\rho}\frac{\ln z dx}{\sqrt{z}(z^2+a^2)^2}+\int_{\rho}^{R}\frac{\ln x dx}{\sqrt{x}(x^2+a^2)^2}+\int_{C_R}\frac{\ln z dz}{\sqrt{z}(z^2+a^2)}$$

(8.25)

C_R kontur bo'yicha integral $R\to\infty$ da lemmaga muvofiq nolga intiladi. C_ρ kontur bo'yicha ham $\rho\to\infty$ da integral nolga intilishini ko'rsataylik. Bu integralni quyidagi ko'rinishda yozish mumkun:

$$\lim_{\rho\to\infty}\int_{C_\rho}\frac{(\ln\rho+i\pi-i\rho e^{i\varphi})d\varphi}{\sqrt{\rho}e^{\frac{i\varphi}{2}}(\rho^2e^{2i\varphi}+a^2)^2}=0$$

Bu yerda $\rho^{1/2}\ln\rho$ aniqmaslikning $\rho\to 0$ dagi limiti Lapital qoidasi yordamida osonlikcha topiladi va

$$\lim_{\rho\longrightarrow 0}\sqrt{\rho}\ln\rho=0$$

Bu hisoblashlardan keyin (8.25) ni (8.23) ni hisobga olgan holda quyidagicha yozish mumkin.

$$I=\frac{2\pi i res f(ia)-\pi\int\limits_0^\infty\frac{dx}{\sqrt{x}(x^2+a^2)^2}}{1-i} \tag{8.26}$$

Endi quyidagi integralni hisoblash qoldi

$$I_2=\int\limits_0^R\frac{dx}{\sqrt{x}(x^2+a^2)^2}.$$

Bu integral quyidagi ko'rinishdagi

$$\int\limits_0^\infty x^\alpha f(x)dx$$

Biz yuqorida hisoblagan integrallar turiga kitadi (6-misolga qarang)

$$I_2=\frac{2\pi i}{1-e^{\pi i}}\left\{res\frac{1}{\sqrt{z}(z^2+a^2)_{|z|=ia}}+res\frac{1}{\sqrt{z(z^2+a^2)_{|z|=-ia}}}\right\}$$

$$res\frac{1}{\sqrt{z}(z^2+a^2)^2_{z=-ai}}=\lim_{z\to -ai}\frac{d}{dz}\left\{(z-ai)^2\frac{1}{\sqrt{z}(z^2+a^2)^2}\right\}=-\frac{3}{2^3}(-ia)^{-\frac{z}{2}}$$

$$res\frac{1}{\sqrt{z}(z^2+a^2)^2_{z=-ai}}=\lim_{z\to -ai}\frac{d}{dz}\left\{(z+ai)^2\frac{1}{\sqrt{z}(z^2+a^2)^2}\right\}=-\frac{3}{2}(-ai)^{-\frac{z}{2}}$$

$$I_2=\left(-\frac{3}{2^3}\right)\frac{\pi i}{a^{\frac{7}{2}}}\left\{i^{\frac{7}{2}}+(-i^{-\frac{7}{2}})\right\}=\frac{3\pi}{a^{\frac{7}{2}}4\sqrt{2}}.$$

Topilgan qiymatlarni (8.26) ga qo'yib, quyidagiga ega bo'lamiz:

$$I=\frac{2\pi i\left[2-3(\ln a-\frac{\pi i}{2})\right](1+i)-6\pi^2}{(1-i)a^{7/2}2^{7/2}}=\frac{\pi}{2\sqrt{2}a^{7/2}}\left\{\frac{3}{4}\pi-1+\frac{3}{2}\ln a\right\}$$

Mashqlar

Integrallarni hisoblang

9. $\int\limits_C\frac{dz}{z^4+1}$, bu yerda C kontur $x^2+y^2=2x$ aylana.

10. $\int\limits_C\frac{zdz}{(z-1)(z-2)^2}$ bu yerda C kontur $|z-2|=\frac{1}{2}$ aylana.

11. Aniq integralni hisoblang. $\int_0^{2\pi} \frac{d\phi}{a+\cos\phi}$, $a>1$.

12. $\int_0^{2\pi} \frac{\sin^2\varphi}{a+b\cos\varphi} d\varphi$, $a>b>0$.

13. $\int_{-\infty}^{\infty} \frac{xdx}{(x^2+4x+13)^2}$.

14. $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$, $(a>0,\ b>0)$.

15. $\int_0^{\infty} \frac{\cos xdx}{(x^2+1)(x^2+4)}$.

16. $\int_0^{\infty} \frac{\cos mx}{x^2+a^2} dx$, $(a>0, m>0)$.

17. $\int_0^{2\pi} \frac{\cos xdx}{1-2p\sin x+p^2}$, $(0<p<1)$.

18. $\int_0^{\infty} \frac{x\sin x}{1+x^2+x^4} dx$.

8.3-§. ZOMMERFELD-VATSON METODI.

Bu metodning batafsil bayonini A.N.Tixonov, A.G.Sveshnikovning "Kompleks o'zgaruvchining funksiyalari nazariyasi" kitobidan topish mumkin.

1-misol. Yig'indini toping.

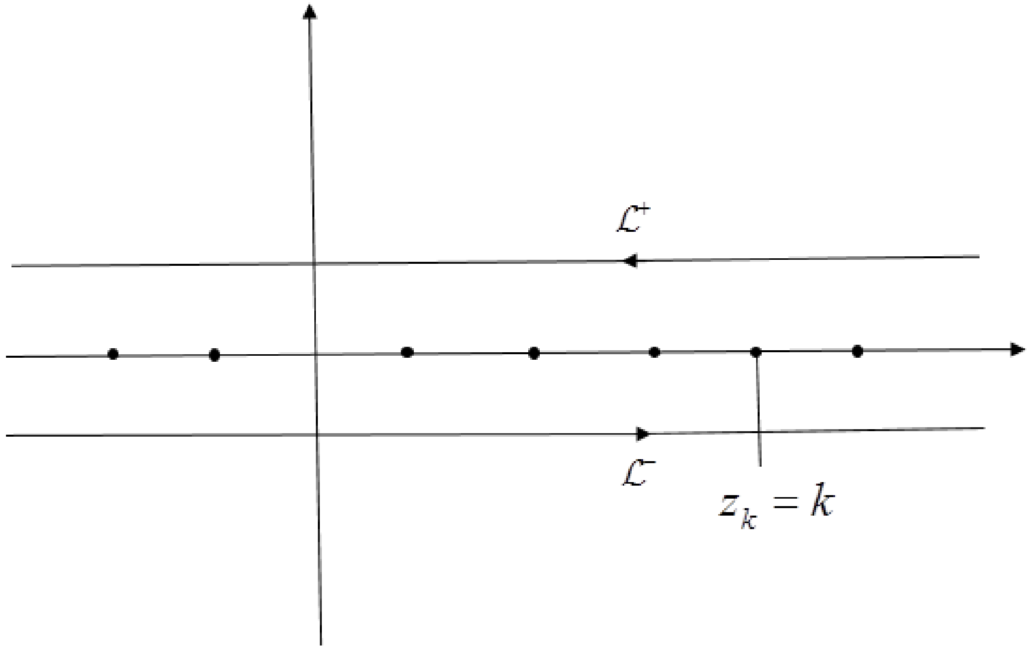
$$S=\sum_{-\infty}^{\infty} \frac{1}{(a+n)^2} .$$

Yechilishi:

I. Aytaylik a haqiqiy va har qanday n uchun $a\neq n$ bo'lsin.

$$f(z)=\frac{1}{(a+n)^2}\frac{e^{inz}}{2i\sin\pi z}$$

funksiyani va undan $\mathcal{L}^+ + \mathcal{L}^-$ kontur bo'yicha olingan integralni qaraymiz .



8.6-rasm.

$f(z)$ funksiya $z = 0, \pm 1, \pm 2 .... \pm k$ nuqtalarda birinchi tartibli qutbga va $z = -a$ nuqtada ikkinchi tartibli qutbga ega. Shuning uchun konturdan olingan integral

$$\int\limits_{\mathcal{L}^+ + \mathcal{L}^-} f(z)dz = 2\pi i \sum_{k=-\infty}^{\infty} res\{f(z_k), z_k = k\} + 2\pi i res f(-a) \qquad (8.27)$$

Chegirmalarini hisoblaymiz:

$$res[f(z_k), z_k = k] = \lim_{z \to k} \frac{e^{i\pi z}}{(a+z)^2 2\pi i \cos \pi z} = \frac{1}{2\pi i (a+k)^2}$$

$$res f(-a) = \lim_{z \to -a} \frac{d}{dz} \{ \frac{e^{i\pi z}}{2i \sin \pi z} \} = -\frac{\pi}{2i \sin^2 \pi a}$$

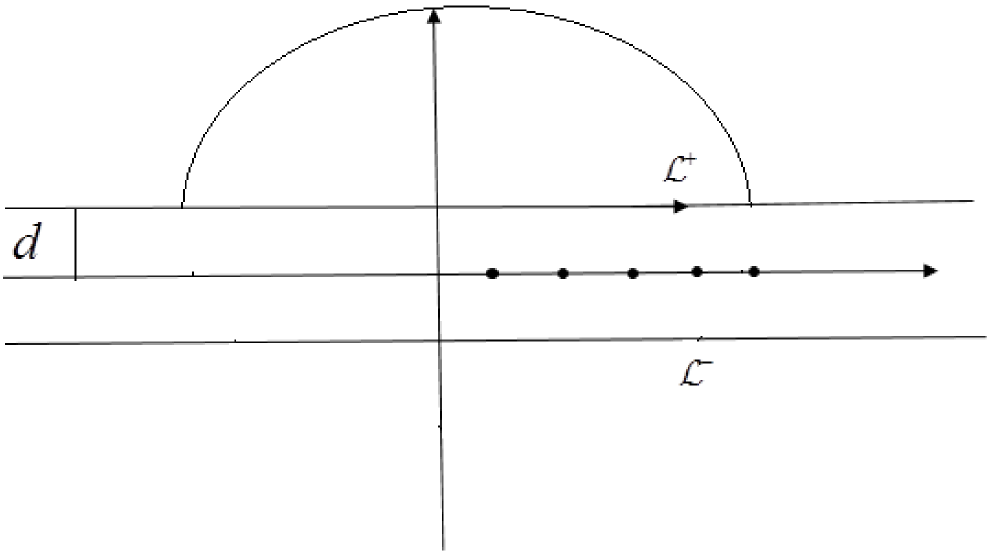
Shunday qilib, quyidagiga ega bo'lamiz:

$$\int\limits_{\mathcal{L}^+ + \mathcal{L}^-} f(z)dz = 2\pi i \sum_{-\infty}^{\infty} \frac{1}{2\pi i} \frac{1}{(a+k)^2} - 2\pi i \frac{\pi}{2i \sin^2 \pi a} = \sum_{-\infty}^{\infty} \frac{1}{(a+k)^2} - \frac{\pi^2}{\sin^2 \pi a}$$

Endi integralni hisoblaymiz. Buning uchun integralni qaraymiz:

$$\int\limits_{c} f(z)dx = -\int\limits_{\mathcal{L}^+} f(x)dx + \int\limits_{c_R} f(z) \qquad (8.28)$$

Biz $\mathcal{L}^+$ kontur bo'yicha yo'nalishini teskarisiga tanlab olganimiz uchun integral oldiga minus ishora qo'yiladi.



8.7-rasm.

C chiziq bo'yicha olingan integral nolga teng, chunki bu chiziq ichida maxsus nuqtalar yo'q. $R\to\infty$ bulganda C bo'yicha integiral ham nolga teng.

$$\left|\frac{1}{2!}\int_{C_R}\frac{e^{i\pi z}dx}{(a+z)^2\sin\pi z}\right| \le R\int_0^{\pi}\frac{e^{-y\pi}dx}{|a+z|^2|\sin\pi z|}$$

$$\frac{e^{-y\pi}}{|\sin\pi z|}=\frac{2e^{-y\pi}}{|e^{i\pi z}||1-e^{2\pi iz}|}=\frac{2}{|1-e^{-2\pi iz}|}\le\frac{2}{e^{2\pi d}-1}$$

Bu yerda d haqiqiy o'qdan $\mathcal{L}^+$ chiziqqacha bo'lgan masofa. Shuning uchun

$$\frac{e^{i\pi z}}{\sin\pi z}$$

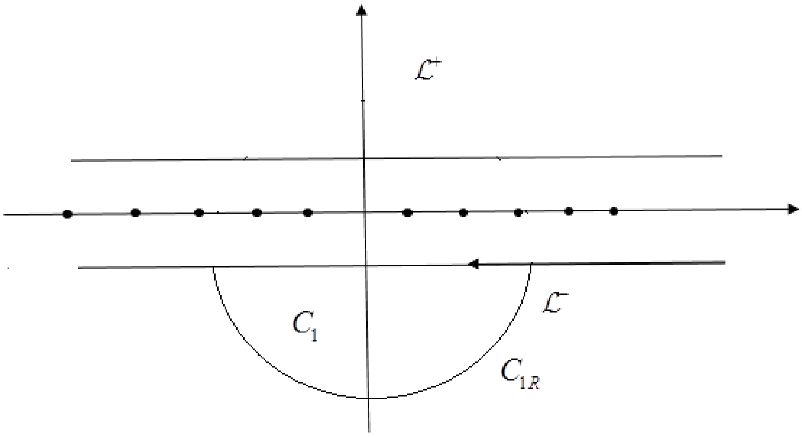
funksiya C_R chiziqda chegaralangan, $\frac{1}{(a+z)^2}$ funksiya esa $R\to\infty$ da kamayuvchi. Demak, (8.28) dan

$$-\int_{\mathcal{L}^+} f(x)dx=0.$$

Xuddi shuningdek $\mathcal{L}^-$ chiziqdan olingan integral ham nolga tengdir.

$$\int_{C_1} f(z)dz=-\int_{\mathcal{L}^-} f(z)dz+\int_{C_{1R}} f(z)dz=0$$

Chunki kontur ichida maxsus nuqtalar yo'q. $\mathcal{L}^-$ - chiziqdan olingan integral C_1 kontur bo'yicha olingan integralga teng. Ammo $R\to\infty$ bo'lganda C_{1R} bo'yicha integral nolga intiladi.



8.8-rasm.

$$\left|\frac{1}{2\imath}\int_{C_{1R}}\frac{e^{i\pi z}dz}{(a+z)^2\sin\pi z}\right|\le\frac{1}{2}\int_{\pi}^{2\pi}\left|\frac{e^{i\pi z}}{\sin\pi z}\right|\frac{Rdz}{|(a+z)|^2}$$

Quydagi funkisiyani baholaymiz:

$$\left|\frac{e^{i\pi z}}{\sin\pi z}\right|=\frac{2}{\left|1-e^{-2\pi iz}\right|}=\frac{2}{\left|1-e^{-2ix\pi-2y\pi}\right|}$$

$$|z_1-z_2|\ge\big||z_1|-|z_2|\big|$$

bundan,

$$\left|1-e^{-2ix\pi-2y\pi}\right|\ge\left|1-e^{-2y\pi}\right|=\left|e^{-2y\pi}-1\right|>e^{2\pi d}-1$$

$\left|\frac{e^{\pi iz}}{\sin z\pi}\right|$ chegaralangan, $\frac{1}{(a+z)^2}$ esa $z\to\infty$ da kamayuvchi funksiya shunung uchun $R\to\infty$ da C_{1R} bo'yicha olingan integral nolga teng.

Ammo bu bo'yicha

$$\int_{\mathcal{L}^+}f(z)dz=0 \text{ ni beradi.}$$

Shunday qilib quyidagiga ega bo'lamiz

$$\int_{\mathcal{L}^+ +\mathcal{L}^-}f(z)dz=0$$

(8.27) ga qaytib, quyidagiga ega bo'lamiz:

$$\sum_{n=-\infty}^{\infty}\frac{1}{(a+n)^2}=\frac{\pi^2}{\sin^2\pi a}$$

II. a -kompleks son va yuqori yarim tekislikda yotsin.

U holda

$$\int_{\mathcal{L}^+ +\mathcal{L}^-}f(z)dz=\sum_{n=-\infty}^{\infty}\frac{1}{(a+n)^2} \qquad (8.29)$$

chunki $z=0$ $\mathcal{L}^{+}+\mathcal{L}^{-}$ konturdan tashqarida yotadi, u holda $\mathcal{L}^{+}$ bo'yicha integralni hisoblayotganda C bo'yicha integralni noldan farqli deb hisoblaymiz, bu integral $(-C)$ nuqtadagi chegirmaga teng:

$$\int_{C} f(z)dz = -\int_{\mathcal{L}^{+}} f(z)dz + \int_{C_R} f(z)dz = 2\pi i\left(-\frac{\pi}{2i\sin^2 \pi a}\right) = -\frac{\pi^2}{\sin^2 \pi a}.$$

Ammo $R\to\infty$da $\int_{C_R} \to 0$. Shuning uchun

$$-\int_{\mathcal{L}^{+}} f(z)dz = -\frac{\pi^2}{\sin^2 \pi a}$$

$\mathcal{L}^{+}$ bo'yicha olingan integral esa birinchi holdagi kabi nolga teng, chunki $(-a)$ maxsus nuqta yuqori yarim tekislikda yotibdi. Hisoblangan integralni (8.28) ga qo'yib quyidagini olamiz .

$$S=\sum_{n=-\infty}^{\infty}\frac{1}{(a+n)^2}=\frac{\pi^2}{\sin^2 \pi a}$$

2-misol . Yig'indisini toping .

$$S=\sum_{-\infty}^{\infty}\frac{(-1)^n}{(a+n)^2}$$

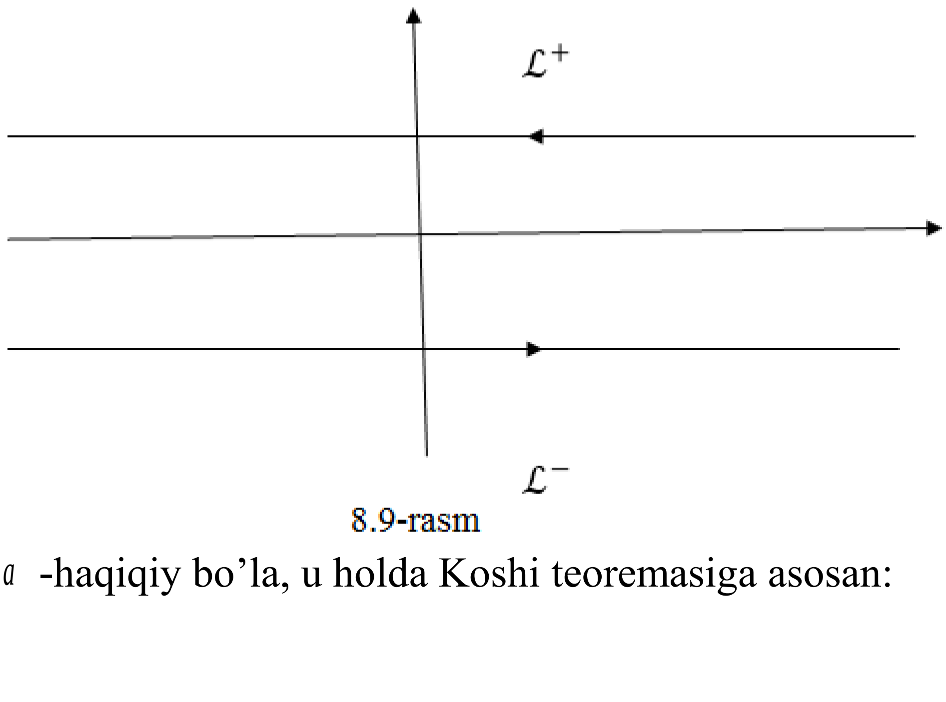
Yechilishi: Quyidagi funksiyani

$$f(z)=\frac{1}{2i(a+z)^2\sin \pi z}$$

va $f(z)$ dan olingan

$$\int_{C} f(z)dz, \quad C=\mathcal{L}^{+}+\mathcal{L}^{-} \qquad (8.30)$$

integralni qaraylik. Bu yerda C kontur [8.9-rasmda ko'rsatilgan].



8.9-rasm

Agar a -haqiqiy bo'la, u holda Koshi teoremasiga asosan:

$$\int_C f(z)dz = 2\pi i \sum_{-\infty}^{\infty} res\{f(z), z = n\} + 2\pi i \; res \; f(-a)$$

$$res \; f(n) = \frac{1}{2i} \frac{(-1)^n}{(a+n)^2 \pi}$$

$$res \; f(-a) = \lim_{z \to -a} \frac{d}{dz}\left[\frac{(z+a)^2}{2i(z+a)^2} \frac{1}{\sin \pi z}\right] = -\frac{\pi}{2i} \frac{ctg \pi a}{\sin \pi a}.$$

Shuning uchun

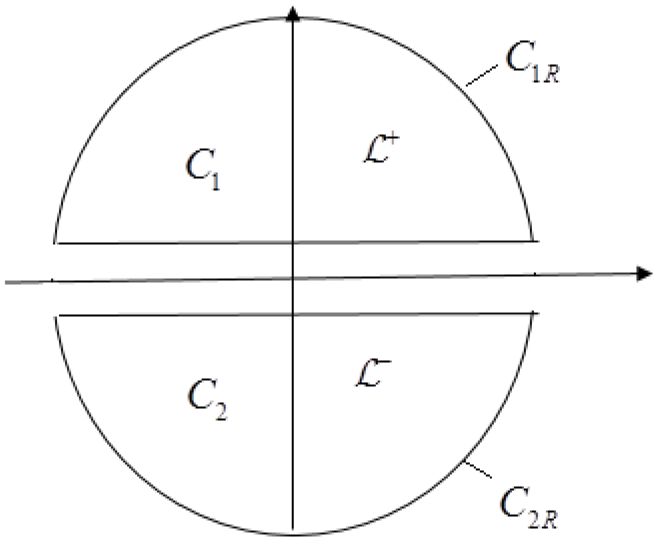
$$\int_C f(z)dz = \sum_{-\infty}^{\infty} \frac{(-1)^n}{(a+n)^2} - \frac{\pi^2 ctg \pi a}{\sin \pi a} \tag{8.31}$$

(8.30) ni konturning qismlari bo'yicha bo'lib yozamiz:

$$\int_C f(z)dz = \int_{\mathcal{L}^+} f(z)dz + \int_{\mathcal{L}^-} f(z)dz \tag{8.32}$$

$\mathcal{L}^+$ va $\mathcal{L}^-$ lar bo'yicha integrallarni hisoblash uchun C_1 va C_2 konturlar bo'yicha integrallarni qaraymiz.

$$\int_C f(z)dz = -\int_{\mathcal{L}^+} f(z)dz + \int_{C_{1R}} f(z)dz = 0$$



8.10-rasm

Chunki C_1 kontor ichida mahsus nuqtalar yo'q. $R \to \infty$ da $\int_{C_{1R}} f(z)dz = 0$ bo'lishini ko'rsatamiz.

$$\left| \int_{C_{1R}} f(z)dz \right| \le \int_0^{\pi} \frac{rd\varphi}{2|a+z|^2 |\sin \pi z|}$$

$$\left|\frac{1}{\sin \pi z}\right| = \left|\frac{2}{e^{i\pi z} - e^{-i\pi z}}\right| \le \frac{2}{|e^{-y\pi} - e^{y\pi}|} \le \frac{2}{|e^{y\pi} - 1|} \le \frac{2}{e^{\pi d} - 1}$$

Ya'ni biz $\frac{1}{|\sin \pi z|}=C_{1R}$ cheklanganligini, $\frac{1}{|a+z|^2}$ funksiya esa $R\to\infty$ da $\frac{1}{R^2}$ kabi bo'lishini aniqladik. Shuning uchun

$$\frac{1}{2}\int_0^{\pi}\frac{Rd\varphi}{|a+z|^2|\sin \pi z|}$$

integral ham yuqoridagiga o'xshash baholanadi.

Shunday qilib:

$$\int_{\mathcal{L}^+} f(z)dz=0\ ,\quad \int_{\mathcal{L}^-} f(z)dz=0 \tag{8.33}$$

(8.33) ni (8.32) va (8.31) larga qo'yib quyidagini olamiz.

$$\sum_{-\infty}^{\infty}\frac{(-1)^n}{(a+n)^2}=\frac{\pi^2 ctg\pi a}{\sin \pi a}$$

Javoblar

1. $res\left[f(z)\right]_{z=\pm 1}=-\frac{1}{2}$; $res\left[f(z)\right]_{z=0}=1$

2. $res\left[f(z)\right]_{z=i}=-\frac{i}{4}$; $res\left[f(z)\right]_{z=-i}=\frac{i}{4}$

3. $res\left[f(z)\right]_{z=k\pi}=0$

4. $res[f(z)]_{z=-1}=\frac{1}{27}$; $res[f(z)]_{z=2}=-\frac{1}{27}$

5. $res[f(z)]_{z=0}=-\frac{1}{\pi^2}$; $res[f(z)]_{z=\frac{\pi}{2}}=0$

6. $res[f(z)]_{z=0}=-\frac{1}{24}$

7. $res[f(z)]_{z=1}=\frac{2n(2n-1)...(2n-(n-2))}{(n-1)!}$

8. $res[f(z)]_{z=(n+\frac{1}{2}\pi i)}=1,\quad (n=0,\pm1,\pm2,\pm3,...)$

9. $-\frac{\pi i}{\sqrt{2}}$; 10. $-2\pi i$; 11. $\frac{2\pi}{\sqrt{a^2-1}}$; 12. $\frac{2\pi}{b^2}(a-\sqrt{a^2-b^2})$; 13. $-\frac{\pi}{27}$; 14. $\frac{\pi}{ab(a+b)}$;

15. $\frac{\pi}{12}e^{-2}(2e-1)$; 16. $\frac{\pi}{2a}e^{-ma}$; 17. 0; 18. $\frac{\pi}{\sqrt{13}}e^{-\frac{\sqrt{3}}{2}}\sin\frac{1}{2}$;

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FIZIKADA KOMPLEKS O'ZGARUVCHILI FUNKSIYALAR

O‘quv qo‘llanma

Muharrir M.A.Xakimov

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