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THE MINKOWSKI FUNCTIONAL AS A METRIC ON THE SPACE OF PROBABILITY MEASURES

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Minkovskiy funksionali ehtimollik o‘lchovlari fazosidagi metrika sifatida

Ehtimollik o‘lchovlari va idempotent ehtimollik o‘lchovlari fazolari orasidagi bog‘lanish kategoriyalar nazariyasining muhim masalalaridan biri hisoblanadi. Ushbu maqolada idempotent ehtimollik o‘lchovlari fazosi ehtimollik o‘lchovlari fazosining qutbi orqali tavsiflandi. Bundan tashqari Minkovskiy funksionali ehtimollik o‘lchovlari fazosida metrika sifatida talqin etildi.

Kalit so‘zlar: Minkovskiy funksionali; ehtimollik o‘lchovi; qutb.

Функционал Минковского как метрика в пространстве вероятностных мер

Соответствие между пространствами вероятностных мер и идемпотентных вероятностных мер – один из актуальных вопросов теории категорий. В работе дано описание пространства идемпотентных вероятностных мер полюсом пространства вероятностных мер. Затем функционал Минковского интерпретируется как метрика на пространстве вероятностных мер.

Ключевые слова: Функционал Минковского; вероятностная мера; полюс.

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A subset C of $\mathbb{R}^n$ is said to be convex if $(1-\lambda)x+\lambda y \in C$ whenever $x,\, y \in C$ and $0 \le \lambda \le 1$. A subset K of $\mathbb{R}^n$ is called a cone if it is closed under positive scalar multiplication, i. e. $\lambda x \in K$ when $x \in K$ and $\lambda > 0$. Such a set is a union of half-lines emanating from the origin. The origin itself may or may not be included. A convex cone is a cone which is a convex set.

A correspondence between the spaces of probability measures and idempotent probability measures is an actual question of the category theory [1], [2], [3], [5], [7]. We give a description of the space of idempotent probability measures by the polar of the space of probability measures. Then the Minkowski functional interpreted as a metric on the space of probability measures.

Let f be a function whose values are real or $+\infty$, $-\infty$ and whose domain is a subset S of $\mathbb{R}^n$. The set $\{(x,\, \mu) : x \in S,\, \mu \in \mathbb{R},\, \mu \ge f(x)\}$ is called the epigraph of f and is denoted by epi f. We define f to be a convex function on S if epi f is convex as a subset of $\mathbb{R}^{n+1}$. A concave function on S is a function whose negative is convex. An affine function on S is a function which is finite, convex and concave.

The effective domain of a convex function f on S, which we denote by dom f, is the projection on $\mathbb{R}^n$ of the epigraph of f:

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$$\operatorname{dom} f = \{x | \exists \mu, (x, \mu) \in \operatorname{epi} f\} = \{x | f(x) < +\infty\}$$

A convex function f is said to be proper if its epigraph is non-empty and contains no vertical lines, i. e. if $f(x) < +\infty$ for at least one x and $f(x) > -\infty$ for every x. Thus f is proper if and only if the convex set $C = \operatorname{dom} f$ is non-empty and the restriction of f to C is finite. Put another way, a proper convex function on $\mathbb{R}^n$ is a function obtained by taking a finite convex function f on a non-empty convex set C and then extending it to all of $\mathbb{R}^n$ by setting $f(x) = +\infty$ for $x \notin C$.

A function f on $\mathbb{R}^n$ is said to be positively homogeneous (of degree p) if for every x one has

$$f(\lambda x) = \lambda^p f(x), \qquad 0 < \lambda < +\infty.$$

Obiously, positive homogeneity is equivalent to the epigraph being a cone in $\mathbb{R}^{n+1}$. An example of a positively homogeneous convex function which is not simply a linear function is $f(x) \equiv |x|$.

It is well known (Theorem 4.7 in [4]) that a positively homogeneous function f from $\mathbb{R}^n$ to $(-\infty, +\infty]$ is convex if and only if $f(x+y) \leq f(x) + f(y)$ for every x, $y \in \mathbb{R}^{n+1}$.

Let f be a closed convex function on $\mathbb{R}^n$. A function f^* defined as

$$f^*(x^*) = \sup_x \{\langle x, x^* \rangle - f(x)\}$$

is said [4] to be the conjugate function of f.

The support function $\delta^*(\cdot | C)$ convex set C in $\mathbb{R}^n$ is defined by

$$\delta^*(x | C) = \sup\{\langle x, y \rangle \ | y \in C\}$$

The indicator function [4] of a convex set C in $\mathbb{R}^n$ is $\delta(x|C) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases}$

For a non-empty convex set C the set

$$C^0 = \{x^* | \delta^*(x^*|C) - 1 \leq 0\} = \{x^* | \forall x \in C, \langle x, x^* \rangle \leq 1\}$$

is called [4] the polar of C.

Let $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ equip with two new operations $\oplus$ and $\odot$, which are defined as $u \oplus v = \max\{u, v\}$, $u \odot v = u + v$, u, $v \in \mathbb{R}_{\max}$. Then $\mathbf{0} := -\infty$ is the zero of $\mathbb{R}_{\max}$ according to $\oplus$, and $\mathbf{1} := 0$ is the unit of $\mathbb{R}_{\max}$ according to $\odot$. It is known [6] that $(\mathbb{R}_{\max}, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is idempotent ($u \oplus u = u$ for all $u \in \mathbb{R}_{\max}$) semifield.

Let X be a compact Hausdorff space, $C(X)$ the algebra of all continuous maps on X with respect to usual algebraic operation. Define on $C(X)$ operations $\oplus$ and $\odot$ by $\varphi \oplus \psi = \max\{\varphi, \psi\}$ and $\varphi \odot \psi = \varphi + \psi$, φ, $\psi \in C(X)$. The set of all probability measures on X (i. e. normed ($\mu(1_X) = 1$), additive ($\mu(\varphi+\psi) = \mu(\varphi)+\mu(\psi)$, φ, $\psi \in C(X)$) and homogeneous ($\mu(\lambda\varphi) = \lambda\mu(\varphi)$, $\lambda \in \mathbb{R}$, $\varphi \in C(X)$) functionals on $C(X)$) denotes by $P(X)$ [8], and the set of all idempotent probability measures on X (i. e. normed ($\mu(1_X) = 1$), max-plus-additive ($\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$, φ, $\psi \in C(X)$) and max-plus-homogeneous ($\mu(\lambda \odot \varphi) = \lambda \odot \mu(\varphi)$, $\lambda \in \mathbb{R}$, $\varphi \in C(X)$) functionals on $C(X)$) by $I(X)$ [6]. If X consists of n points then $P(X)$ is a closed convex subset of $\mathbb{R}^n$, and $I(X)$ is a closed max-plus-convex subset of $\mathbb{R}^n_{\max}$.

A set A is called max-plus-convex if the inclusion a, $b \in A$ imply $\alpha \odot a \oplus \beta \odot b \in A$ for every pair of α, $\beta \in \mathbb{R}_{\max}$ such that $\alpha \oplus \beta = \mathbf{1}$.

The following result establishes relation between the spaces $P(X)$ and $I(X)$.

Theorem 1. *Let $|X| = n$. Then*

$$I(X) = \{(\xi_1^*, \ldots, \xi_n^*) \in P(X)^0 - 1 : \bigoplus_{i=1}^{n} \xi_i^* = \mathbf{1}\},$$

and

$$P(X) = \{(\xi_1^*, \ldots, \xi_n^*) \in I(X)^0 - 1 : \sum_{i=1}^{n} \xi_i^* = 1\}.$$

Note that a function k on $\mathbb{R}^n$ is called a gauge or Minkowski functional of a set C if k is a non-negative positively homogeneous convex function such that $k(0) = 0$, i. e. if $\operatorname{epi} k$ is a convex cone in $\mathbb{R}^{n+1}$ containing the origin but not containing any vectors (x, μ) such that $\mu < 0$. Gauges are thus the functions k such that

$$k(x) = \gamma(x|C) = \inf\{\mu \geq 0 : x \in \mu C\}$$

for some non-emtpy convex set C. Of course, C is not uniquely determined by k in general, although one always has $\gamma(\cdot|C) = k$ for

$$C = \{x : k(x) \leq 1\}.$$

If k is closed, the latter C is the unique closed convex set containing the origin such that $\gamma(\cdot|C) = k$.

The polar of a gauge k is the function k^0 defined by

$$k^0(x^*) = \inf\{\mu^* \geq 0 : \langle x, x^* \rangle \leq \mu^* k(x), \forall x \in \mathbb{R}^n\}.$$

If k is finite everywhere and positive except at the origin, this formula can be written as

$$k^0(x^*) = \sup_{x \neq 0} \frac{\langle x, x^* \rangle}{k(x)}.$$

Note that, if k is the indicator function of a convex cone K, k^0 is the same as the conjugate of k, the indicator function of the polar convex cone K^0.

A set C is called balanced if $x \in C \Rightarrow \alpha x \in C$ for $|\alpha| \leq 1$. If $C \subset \mathbb{R}^n$ is a nonempty, balanced, closed convex set, then it is easily seen that

(1) $k(x) \in \Gamma(\mathbb{R}^n)$, where $\Gamma(\mathbb{R}^n)$ is set of proper, lower semicontinuous convex functions on $\mathbb{R}^n$;

(2) $\mathbb{R}^n_C = \{x \in \mathbb{R}^n : k(x) < \infty\} = \bigcup\{\alpha C : \alpha \geq 0\}$, the domain of $k(x)$, is an algebraic subspace of $\mathbb{R}^n$, and $k(x)$ is a seminorm on this subspace;

(3) If C is in addition compact, then $k(x)$ is a norm on $\mathbb{R}^n_C$

In [4] it was noted that if C is a closed convex set containing the origin, the gauge functions of C and the support function of C are gauges polar to each other (Corollary 15.1.2).

The concept of a norm is natural to the study of certain metric structures and corresponding approximation problems. By definition, a metric on $\mathbb{R}^n$ is a real-valued function ρ on $\mathbb{R}^n \times \mathbb{R}^n$ such that (M1) $\rho(x, y) > 0$ if $x \neq y$, and $\rho(x, y) = 0$ if $x = y$, $\forall x$, $\forall y$ (nonnegativity); (M2) $\rho(x, y) = \rho(y, x)$, $\forall x$, $\forall y$ (symmetry); (M3) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$, $\forall x$, $\forall y$, $\forall z$ (triangle axiom). The quantity $\rho(x, y)$ is interpreted as the distance between x and y with respect to ρ.

Generally speaking, a metric on $\mathbb{R}^n$ need not have any relation with the algebraic structure of $\mathbb{R}^n$. Two properties which may naturally be demanded of a metric ρ, in order that it be compatible with vector addition and scalar multiplication, are

(M4) $\rho(x + z, y + z) = \rho(x, y)$, $\forall x$, $\forall y$, $\forall z$,

(M5) $\rho(x, (1 - \lambda)x + \lambda y) = \lambda\rho(x, y)$, $\forall x$, $\forall y$ and $\forall \lambda \in [0, 1]$.

Property (M4) says that distances remain invariant under translation, and (M5) says that distances behave linearly along line segments. A metric which has these two extra properties is called a Minkowski metric on $\mathbb{R}^n$. There is a one-to-one correspondence between Minkowski metrics and norms. If k is a norm, then

$$\rho(x, y) = k(x - y)$$

defines a Minkowski metric; moreover, each Minkowski metric is defined in this way by a uniquely determined norm [4].

Theorem 2. *Let $X = \{x_1, \ldots, x_{n+1}\}$ be a discrete space. The Minkowski metric on the space $P(X) = \{(u_1, \ldots, u_{n+1}) : u_i \geq 0, i = 1, 2, \ldots, n + 1, u_1 + \ldots + u_{n+1} = 1\}$ of probability measures on X defined by a gauge $k(u) = |u_1| + \ldots + |u_{n+1}|$, $u \in P(X)$, generates the pointwise convergence topology on $P(X)$. If in addition we require $\rho_{P(X)}(u, v) = \frac{1}{2}k(u - v)$ then* $\operatorname{diam} P(X) = 1$.

Proof. At first note that points of the view $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0)$,..., $e_{n+1} = (0, 0, \ldots, 1) \in \mathbb{R}^{n+1}$ are extremal points of $P(X)$. Then $\frac{1}{2}(e_i - e_j) \in U(X)$, where $U(X) = \{(u_1, \ldots, u_{n+1}) : |u_1| + \ldots + |u_{n+1}| \leq 1\}$ is the unit ball $U(X) = \{(u_1, \ldots, u_{n+1}) : |u_1| + \ldots + |u_{n+1}| \leq 1\}$. That is why $k(e_i - e_j) = 2$, $i, j \in \{1, 2, \ldots, n + 1\}$, $i \neq j$.

Now the metric

$$\rho_{U(X)}(u, v) = \tfrac{1}{2}k(u - v)$$

on $U(X)$ induces a metric $\rho_{P(X)}(u, v)$ on $P(X)$, such that $\operatorname{diam} P(X) = 1$.

Theorem 2 is proved. □

Let X be a compact metric space, ρ_X the metric on it. Fix a countable subset N of X. Let $\mathcal{N}$ be the set of all finite subsets of N. Enumerate the elements of $\mathcal{N}$: $N_1, N_2, \ldots, N_k, \ldots$. Then $N = \bigcup_{k=1}^{\infty} N_k$ and $\bigcup_{k=1}^{\infty} U(N_k)$ is everywhere dense in $U(X)$.

We attribute a sequence $\{\mu_n\} \subset \bigcup_{k=1}^{\infty} U(N_k)$ to each measure $\mu \in U(X)$ such that $\mu_k \in U(N_k)$ and $\lim_{k\to\infty} \mu_k = \mu$.

For every pair of measures $\mu, \nu \in U(X)$ we put

$$\rho_{U(X)}(\mu, \nu) = \sum_{i=1}^{\infty} \frac{1}{2^i} \rho_{U(N_i)}(\mu_i, \nu_i). \tag{1}$$

Theorem 3. *Let (X, ρ_X) be a compact metric space. The function $\rho_{U(X)}\colon U(X) \times U(X) \to \mathbb{R}$ defined by (1) is a metric on $U(X)$ which generated the pointwise convergence topology on $U(X)$.*

Proof. Direct checking shows that $\rho_{U(X)}$ is a metric on $U(X)$. On the other hand $\rho_{U(X)}$ generates the product topology on a subspace $U(X)$ of $R^{C(X)}$. But the product topology and the pointwise convergence topology on $U(X)$ coincide. Theorem 3 is proved. □

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