III-BOB. Differensial tenglamalar sistemasi.

1-§. Differensial tenglamalarning normal sistemasi.Umumiy tushunchalar.

1-Ta'rif. Ushbu

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, ..., x_n) \quad (i = \overline{1,n}) \qquad (1.1)$$

ko'rinishdagi, $x_i = x_i(t) \quad (i = \overline{1,n})$ noma'lum funksiyalarning hosilalariga nisbatan yechilgan sistemaga **normal tipdagi sistema** deyiladi. (1.1) sistemaning biror $I \subset R$ intervaldagi yechimi deb, I intervalda uzluksiz differensiallanuvchi shunday $x_i = \varphi_i(t) \quad (i = \overline{1,n})$ funksiyalar majmuasiga aytiladiki, barcha $t \in I$ larda

$\frac{d\varphi_i}{dt} = f_i(t, \varphi_1(t), \varphi_2(t), ..., \varphi_n(t))$ tenglik o'rinli bo'ladi.

2-**Ta'rif.** (1.1) sistemaning sistemaning aniqlanish sohasida uzluksiz differensiallanuvchi bo'lgan $\Phi(t, x_1, x_2, ..., x_n)$ funksiya uchun,

$$\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i} \frac{\partial x_i}{\partial t} = \frac{\partial \Phi}{\partial t} + \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i} f_i(t, x_1, x_2, ..., x_n) = 0$$

shart bajarilsa, u holda $\Phi(t, x_1, x_2, ..., x_n)$ funksiya (1.1) sistemaning **birinchi integrali** deyiladi.

Agar (1.1) sistemaning n ta $\Phi_1(t, x_1, x_2, ..., x_n)$, $\Phi_2(t, x_1, x_2, ..., x_n)$, $\Phi_3(t, x_1, x_2, ..., x_n)$,…, $\Phi_n(t, x_1, x_2, ..., x_n)$ birinchi integrallari mavjud bo'lsa,

$$\Phi_i(t, x_1, x_2, ..., x_n) = C_i \quad (i = \overline{1,n}) \qquad (1.2)$$

tengliklar majmuasi (1.1) sistemaning umumiy integralini aniqlaydi, bu yerda C_i -ixtiyoriy o'zgarmas.

Normal tipdagi differensial tenglamalar sistemasini yechishning ikkita usulini keltiramiz.

Birinchi usulida (1.1) sistema bitta n-tartibli differensial tenglamaga keltiriladi va integrallanadi. Buning uchun (1.1) sistemadagi tenglamalarning birinchisini $n-1$ marta ketma-ket differensiallab va har gal differensiallashda $\frac{dx_i}{dt}$ ning o'rniga uning boshqa tenglamalardagi qiymati qo'yib boriladi, ya'ni

$$\begin{cases} \dfrac{dx_1}{dt} = F_1(t, x_1, x_2, ..., x_n) = f_1(t, x_1, x_2, ..., x_n) \\ \dfrac{d^2x_1}{dt^2} = F_2(t, x_1, x_2, ..., x_n) = \dfrac{\partial f_1}{\partial t} + \sum\limits_{i=1}^{n} \dfrac{\partial f_1}{\partial x_i}\dfrac{\partial x_i}{\partial t} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{d^{n-1}x_1}{dt^{n-1}} = F_{n-1}(t, x_1, x_2, ..., x_n) \\ \dfrac{d^n x_1}{dt^n} = F_n(t, x_1, x_2, ..., x_n) \end{cases} \qquad (1.3)$$

(1.3) sistemaning birinchi $n-1$ ta tenglamasidan $x_2, x_3, ..., x_n$ larni topib, oxirgi tenglamaga qo'ysak

$$\frac{d^n x_1}{dt^n} = F\left(t, x_1, \frac{dx_1}{dt}, ..., \frac{d^{n-1}x_1}{dt^{n-1}}\right)$$

ko'rinishga ega bo'lgan *n*- tartibli differensial tenglama hosil bo'ladi.

Ikkinchi usul integrallanuvchi kombinatsiyalar topish usuli deyiladi. Bu usulda berilgan sistema arifmetik operatsiyalar orqali, yangi $u = u(t, x_1, x_2, ..., x_n)$ noma'lum funksiyaga nisbatan oson integrallanadigan tenglamalarga keltiriladi.

Yuqorida keltirilgan usullarni

$$\frac{dx_1}{X_1(x_1, x_2, ..., x_n)} = \frac{dx_2}{X_2(x_1, x_2, ..., x_n)} = ... = \frac{dx_n}{X_n(x_1, x_2, ..., x_n)} \qquad (1.4)$$

ko'rinishga ega bo'lgan simmetrik formadagi differensial tenglamalar sistemasiga ham qo'llash mumkin, buning uchun kasrlarning

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_n}{b_n} \Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_n}{b_n} = \frac{k_1a_1 + k_2a_2 + ... + k_na_n}{k_1b_1 + k_2b_2 + ... + k_nb_n}. \qquad (1.5)$$

xossasini ishlatish qo'l keladi, bu yerda $k_1, k_2, ..., k_n$ - ixtiyoriy o'zgarmaslar.

1-***Misol***. $\begin{cases} \dfrac{dx}{dt} = y^2 + \sin t, \\ \dfrac{dy}{dt} = \dfrac{x}{2y}. \end{cases}$ sistemani yeching.

Yechish. Berilgan sistemaning birinchi tenglamasini ikkala tomonini t bo'yicha differensiallab, ikkinchi tenglamadagi $\dfrac{dy}{dt} = \dfrac{x}{2y}$ ni hosil bo'lgan tenglamaga qo'ysak $\dfrac{d^2x}{dt^2} = 2y\dfrac{dy}{dt} + \cos t = x + \cos t$ yoki $\dfrac{d^2x}{dt^2} - x = \cos t$ tenglamani olamiz.

Ma'lumki, oxirgi tenglama maxsus o'ng tomonli o'zgarmas koeffisiyentli chiziqli differensial tenglama bo'lib, uning umumiy yechimi $x = C_1 e^t + C_2 e^{-t} - \frac{1}{2}\cos t$ ko'rinishda bo'ladi. Topilgan yechimni sistemaning birinchi tenglamasiga qo'ysak,
$y^2 = \frac{dx}{dt} - \sin t = C_1 e^t - C_2 e^{-t} - \frac{1}{2}\sin t$ yechimni olamiz. Demak berilgan sistema yechimi

$$\begin{cases} x = C_1 e^t + C_2 e^{-t} - \frac{1}{2}\cos t \\ y = C_1 e^t - C_2 e^{-t} - \frac{1}{2}\sin t \end{cases}$$

bo'ladi.

2-*Misol*. $\begin{cases} \dot{x} = y, \\ \dot{y} = x, \\ \dot{z} = x + y + z. \end{cases}$ sistemasini yeching.

Yechish. Birinchi tenglamani ikkala tomonini t bo'yicha differensiallaymiz va
$\ddot{x} = \dot{y}$ ni hosil qilamiz. Berilgan sistemaning ikkinchi tenglamadagi $\dot{y}$ ni hosil bo'lgan tenglamaga qo'yib, $\ddot{x} - x = 0$ olamiz. Bu tenglama umumiy yechimi $x = C_1 e^t + C_2 e^{-t}$ bo'lib, uni sistemaning ikkinchi tenglamasiga qo'ysak, $y = \frac{dx}{dt} = C_1 e^t - C_2 e^{-t}$ ni topamiz. Topilgan $x(t),\ y(t)$ funksiyalarni sistemaning uchinchi tenglamasiga qo'yamiz va $\frac{dz}{dt} - z = 2C_1 e^t$ tenglamaga ega bo'lamiz. Bu tenglamani yechib $z = C_3 e^t + 2C_1 t e^t$ ni topamiz.

3-***Misol*.** $\begin{cases} \dot{x} = y - z, \\ \dot{y} = x^2 + y, \\ \dot{z} = x^2 + z. \end{cases}$ sistemasini yeching.

Yechish. Sistemadagi tenglamalarni barchasini qo'shib,

$$\dot{x} - \dot{y} + \dot{z} = y - z - x^2 - y + x^2 + z = 0$$

ya'ni $\frac{d}{dt}(x - y + z) = 0$ tenglamani hosil qilamiz, bundan $x - y + z = C_1$.
Sistemaning birinchi tenglamasidan va oxirgi munosabatdan

$\frac{dx}{dt}=x-C_1$ tenglamani uni integrallab esa $x=C_2e^t+C_1$ yechimni olamiz. Topligan $x$ning qiymatini ikkinchi tenglamaga qo'yib, $\frac{dy}{dt}-y=C_2^2e^{2t}+2C_1C_2e^t+C_1^2$ tenglamani hosil qilamiz. Bu tenglamani integrallab, $y=C_2^2e^{2t}+\left(2C_1C_2t+C_3\right)e^t-C_1^2$ yechimni olamiz. Topligan x va y ning qiymatlarini $x-y+z=C_1$ munosabatga qo'yib, $z=C_1-x+y=C_2^2e^{2t}+\left(2C_1C_2t+C_3-C_2\right)e^t-C_1^2$ yechimni topamiz.

4-***Misol***. $\frac{dx}{z+y}=\frac{dy}{x+z}=\frac{dz}{x+y}$ sistemani yeching.

Yechish. Teng kasrlar xossasiga asosan, quyidagi integrallanuvchi kombinasiyalarni tuzamiz:

$$\frac{dx-dy}{z+y-(x+z)}=\frac{dy-dz}{x+z-(x+y)} \quad \text{va} \quad \frac{dx+dy+dz}{z+y+x+z+x+y}=\frac{dx-dy}{z+y-(x+z)}.$$

Birinchi tenglamani soddalashtirib, $\frac{d(x-y)}{x-y}=\frac{d(y-z)}{y-z}$ munosabatni, bundan ikkala tomonini integrallab $x-y=C_1(y-z)$ birinchi integralni olamiz.

Ikkinchi tenglamani soddalashtirib, $\frac{d(x+y+z)}{2(y+x+z)}=\frac{-d(x-y)}{x-y}$, bundan $\frac{1}{2}\ln|x+y+z|+\ln|x-y|=\frac{1}{2}\ln C_2$, ya'ni $(x+y+z)(x-y)^2=C_2$ yana bir birinchi integalni olamiz. Demak ikkita birinchi integrallar chiqwli erkli bo'lgani uchun berilgan sistemaning barcha yechimlari $x-y=C_1(y-z)$ va $(x+y+z)(x-y)^2=C_2$ munosabatlardan aniqlanadi.

Mustaqil yechish uchun mashqlar.

Quyidagi sistemalarni yeching (488-502):

488. $\begin{cases} y'=\frac{x}{z} \\ z'=-\frac{x}{y} \end{cases}$

489. $\begin{cases} y'=\frac{z}{x} \\ z'=\frac{z(y+2z-1)}{x(y-1)} \end{cases}$

490. $\begin{cases} y'=y^2z \\ z'=\frac{z}{x}-yz^2 \end{cases}$

491. $\begin{cases} \frac{dx}{dt}=\frac{y}{(x-y)^2} \\ \frac{dy}{dt}=\frac{x}{(x-y)^2} \end{cases}$

492. $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dx}{dt} - \dfrac{dy}{dt} = x + y, \end{cases}$

$x(\pi) = -1, \quad y(\pi) = 0.$

493. $\begin{cases} \dfrac{dx}{dt} + \dfrac{dy}{dt} = e^{-t} - y \\ 2\dfrac{dx}{dt} + \dfrac{dy}{dt} = \sin t - 2y \end{cases}$

$x(0) = -2, \quad y(0) = 1.$

494. $\begin{cases} 2\dfrac{dx}{dt} = 6x - y - 6t^2 - t + 3 \\ \dfrac{dy}{dt} = 2y - 2t - 1, \end{cases}$

$x(0) = 2, \quad y(0) = 3.$

495. $\dfrac{dx}{2y - z} = \dfrac{dy}{y} = \dfrac{dz}{z}$

496. $\dfrac{dx}{y - x} = \dfrac{dy}{x + y + z} = \dfrac{dz}{x - y}$

497. $\dfrac{dx}{xz} = \dfrac{dy}{yz} = \dfrac{dz}{xy\sqrt{z^2 + 1}}$

498. $\dfrac{dx}{z^2 - y^2} = \dfrac{dy}{z} = -\dfrac{dz}{y}$

499. $\dfrac{dx}{y - u} = \dfrac{dy}{z - x} = \dfrac{dz}{u - y} = \dfrac{du}{x - z}$

500. $\dfrac{dx}{x(y + z)} = \dfrac{dy}{z(z - y)} = \dfrac{dz}{y(y - z)}$

501. $-\dfrac{dx}{x^2} = \dfrac{dy}{xy - 2z^2} = \dfrac{dz}{xz}$

502. $\dfrac{dx}{x(z - y)} = \dfrac{dy}{y(y - x)} = \dfrac{dz}{y^2 - xz}$

2-§. Chiziqli differensial tenglamalar sistemasi.

1-Ta'rif. Ushbu

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} x_j + f_i(t) \ \ (i = \overline{1,n}) \tag{2.1}$$

ko'rinishdagi sistemaga **o'zgarmas koeffisiyentli chiziqli differensial tenglamalar** sistemasi deyiladi, bu yerda a_{ij}-berilgan o'zgarmaslar, $f_i(t) \ (i = \overline{1,n})$-berilgan funksiyalar.

Agar (2.1) sistemada $f_i(t) \equiv 0$ bo'lsa, ya'ni

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} x_j, \ (i = \overline{1,n}) \tag{2.1$_1$}$$

sistemaga **bir jinsli** sistema deyiladi. (1.1) sistemaning biror $I \subset R$ intervaldagi **yechimi** deb, I intervalda uzluksiz differensiallanuvchi shunday

$$x_1 = \varphi_1(t), \quad x_2 = \varphi_2(t), \quad x_3 = \varphi_3(t), \ \ldots, x_n = \varphi_n(t) \tag{2.2}$$

funksiyalar majmuasiga aytiladiki, barcha $t \in I$ larda, (2.2) funksiyalar majmuasi (2.1) sistemaning tenglamalarini ayniyatga (to'ri tenglikga) aylantiradi.

2-Ta'rif. (2.1) sistemaning

$$x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \quad x_3(t_0) = x_3^0, \; \dots \;, x_n(t_0) = x_n^0 \qquad (2.3)$$

boshlang'ich shartlarni qanoatlantiruvchi

$$x_1 = x_1(t), \quad x_2 = x_2(t), \quad x_3 = x_3(t), \; \dots \;, x_n = x_n(t) \qquad (2.4)$$

yechimlarni topish masalasi **Koshi masalasi** deyiladi.

Quyida biz, (2.1) ko'rinishdagi o'zgarmas koeffisiyentli chiziqli differensial tenglamalar sistemasini yechish usullariga to'xtalamiz.

a) Sistemani bitta *n* – tartibli differensial tenglamaga keltirish usuli. Birinchi paragrafda ta'kidlaganimizdek, (2.1) sistemani ham bitta *n*- tartibli chiziqli differensial tenglamaga keltirib integallash mumkin:

Quyidagi sistemani qaraylik:

$$\begin{cases} \dfrac{dx}{dt} = ax + by + f(t) \\ \dfrac{dy}{dt} = a_1 x + b_1 y + g(t) \end{cases}$$

Bu yerda a, b, a_1, b_1-berilgan o'zgarmaslar, $f(t), g(t)$-berilgan funksiyalar, $x(t)$ va $y(t)$ esa noma'lum funksiyalar. Berilgan sistemaning birinchi tenglamasigan t bo'yicha hosila olamiz va $\dfrac{dy}{dt}$ ning o'rniga ikkinchi tenglamadagi qiymatini, y ning o'rniga esa birinchi tenglamadan aniqlanadigan qiymatni qo'yamiz va

$$\frac{d^2x}{dt^2} = a\frac{dx}{dt} + b\frac{dy}{dt} + f'(t) = a\frac{dx}{dt} + b\left(a_1 x + b_1 y + g(t)\right) + f'(t) =$$

$$= a\frac{dx}{dt} + ba_1 x + b_1\left(\frac{dx}{dt} - ax - f(t)\right) + bg(t) + f'(t),$$

yoki $\dfrac{d^2x}{dt^2} + A\dfrac{dx}{dt} + Bx = F(t)$ ko'rinishdagi ikkinchi tartibli o'zgarmas koeffisiyentli chiziqli differensial tenglamani olamiz. Bu tenglamadan $x(t)$ ni topib, berilgan sistemadagi brinchi tenglamaga qo'yib, $y(t)$ ni topamiz.

5-*Misol*. $\begin{cases} \dot{x} + y = t^2 + 6t + 1 \\ \dot{y} - x = -3t^2 + 3t + 1 \end{cases}$ sistemani yeching.

Yechish. Sistemadagi birinchi tenglamadan y ni topib, ikkinchi tenglamaga qo'ysak $\ddot{x}+\dot{x}=3t^2-t+5$ tenglamani hosil qilamiz, bu tenglama maxsus o'ng tomonli o'zgarmas koeffisiyentli chizqli differensial tenglama bo'lib, uning umumiy yechimi $x(t)=C_1\sin t+C_2\cos t+3t^2-t-1$ bo'ladi. y ni topish uchun topilgan $x(t)$ ning qiymatini berilgan sistemadagi birinchi tenglamaga qo'yamiz, demak

$$y(t)=t^2+6t+1-C_1\cos t+C_2\sin t-3t^2+t+1=-C_1\cos t+C_2\sin t-2t^2+7t+2$$

Shunday qilib, berilgan sistema yechimi

$$\left[\begin{array}{l} x(t)=C_1\sin t+C_2\cos t+3t^2-t-1 \\ y(t)=-C_1\cos t+C_2\sin t-2t^2+7t+2 \end{array}\right. \quad \text{bo'ladi.}$$

b) Sistemani integrallashning Eyler usuli.

Bu usulni uchta chiziqli differensial tenmglamalarning sistemasi uchun ko'ramiz, ya'ni

$$\begin{cases} \dfrac{dx}{dt}=a_0x+b_0y+c_0z \\ \dfrac{dy}{dt}=a_1x+b_1y+c_1z \\ \dfrac{dz}{dt}=a_2x+b_2y+c_2z \end{cases} \tag{2.5}$$

sistemani integrallashga to'xtalamiz, bu yerda a_j, b_j, c_j $(j=0,1,2.)$- berilgan o'zgarmaslar. (2.5) sistemaning xususiy yechimlarini

$$x=\alpha e^{\lambda t},\ y=\beta e^{\lambda t},\ z=\gamma e^{\lambda t} \tag{2.6}$$

ko'rinishda izlaymiz, bu yerda $\alpha, \beta, \gamma, \lambda$-o'zgarmas sonlar.

(2.6) ni (2.5) ga qo'yib, hamda $e^{\lambda t}$ ga qisqartirib, α, β, γ larni aniqlash mumkin bo'lgan

$$\begin{cases} (a_0-\lambda)\alpha+b_0\beta+c_0\gamma=0 \\ a_1\alpha+(b_1-\lambda)\beta+c_1\gamma=0 \\ a_2\alpha+b_2\beta+(c_2-\lambda)\gamma=0 \end{cases} \tag{2.7}$$

sistemani hosil qilamiz.Bu sistemadan noldan farqli α, β, γ larni topish uchun

$$\begin{vmatrix} a_0-\lambda & b_0 & c_0 \\ a_1 & b_1-\lambda & c_1 \\ a_2 & b_2 & c_2-\lambda \end{vmatrix}=0 \qquad (2.8)$$

shart bajarilishi kerak. (2.8) tenglamaga **xaraktiristik tenglama** deyiladi, yani (2.6) ga asosan bu tenglamaning uchta $\lambda_1, \lambda_2, \lambda_3$ ildizlariga mos berilgan differensial tenglamalar sistemasining uchta chiiziqli erkli yechimlar sistemasini topamiz. (2.8) tenglama λ ga ninsbatan uchunchi tartibli algebraik tenglama bo'lib, bu tenglamaning ildizlari turli hollarda bo'lishi mumkin, ya'ni 1) *barchasi haqiqiy va turli*, 2) *kompleks ildizlarga ega* va 3) *karrali ildizlarga ega*.

b₁). (2.8) tenglamaning ildizlari barchasi haqiqiy va turli bo'lsin, u holda $\lambda=\lambda_1, \lambda=\lambda_2$ va $\lambda=\lambda_3$ xos qiymatlarni ketma-ket (2.7) ga qo'yib, ularga mos $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$ va $\alpha_3, \beta_3, \gamma_3$ xos vektorlarni topamiz. Demak $\lambda_1, \lambda_2, \lambda_3$ xarakteristik sonlarga mos chiqli erkli $h_1(\alpha_1, \beta_1, \gamma_1), h_2(\alpha_2, \beta_2, \gamma_2)$ va $h_3(\alpha_3, \beta_3, \gamma_3)$ xos vektorlar aniqlanadi, ya'ni (2.6) ga asosan

$$\begin{cases} x_1=\alpha_1 e^{\lambda_1 t},\ y_1=\beta_1 e^{\lambda_1 t},\ z_1=\gamma_1 e^{\lambda_1 t} \\ x_2=\alpha_2 e^{\lambda_2 t},\ y_2=\beta_2 e^{\lambda_2 t},\ z_2=\gamma_2 e^{\lambda_2 t} \\ x_3=\alpha_3 e^{\lambda_3 t},\ y_3=\beta_3 e^{\lambda_3 t},\ z_3=\gamma_3 e^{\lambda_3 t} \end{cases} \qquad (2.9)$$

chiziqli erkli xususiy yechimlar sistemasini hosil qilamiz. Shunday qilib, (2.5) sistemaning umumiy yechimi

$$\begin{cases} x=C_1\alpha_1 e^{\lambda_1 t}+C_2\alpha_2 e^{\lambda_2 t}+C_3\alpha_3 e^{\lambda_3 t} \\ y=C_1\beta_1 e^{\lambda_1 t}+C_2\beta_2 e^{\lambda_2 t}+C_3\beta_3 e^{\lambda_3 t} \\ z=C_1\gamma_1 e^{\lambda_1 t}+C_2\gamma_2 e^{\lambda_2 t}+C_3\gamma_3 e^{\lambda_3 t} \end{cases} \qquad (2.10)$$

bo'ladi.

6-*Misol*. $\begin{cases} \dot{x}=2x-y-z \\ \dot{y}=x-z \\ \dot{z}=3x-y-2z \end{cases}$ sistemani yeching.

Yechish. Berilgan sistemaning xususiy yechimlarini (2.6) ko'rishda izlab,

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ 1 & -\lambda & -1 \\ 3 & -1 & -2-\lambda \end{vmatrix}=0$$

xarakterstik tenglamani olamiz. Bu tenglamani soddalashtirib, $\lambda\left(4-\lambda^2\right)+1+3-3\lambda+\lambda-2-2-\lambda=0$ yoki $-\lambda^3+\lambda=0$ tenglamani hosil qilamiz. Demak xos sonlar $\lambda_1=0,\ \lambda_{2/3}=\pm1$, hamda bu xos sonlarga mos xos vektorlar $h_1(1,1,1)$, $h_2(1,0,1)$ va $h_3(1,1,2)$ bo'ladi. Shunday qilib berilgan sistema yechimi $x(t)=C_1+C_2e^t+C_3e^{-t}$, $y(t)=C_1+C_3e^{-t}$, $z(t)=C_1+C_2e^t+2C_3e^{-t}$ bo'ladi.

b₂). (2.8) xaraktiristik tenglamaning ildizlari kompleks bo'lsin, ya'ni $\lambda=\lambda_{1/2}=a\pm ib$, $\lambda=\lambda_3\ (\lambda_3\in R)$. Bu holda $\lambda=\lambda_1=a+ib$ ni (2.7) ga qo'yib, unga mos $\alpha_1,\beta_1,\gamma_1$ ni topamiz (tabiiyki $\alpha_1,\beta_1,\gamma_1$lar haqiqiy bo'lmasligi mumkin). (2.6) ga asosan $x=\alpha_1e^{\lambda_1t}=\alpha_1e^{(a+ib)t}$, $y=\beta_1e^{(a+ib)t}$, $z=\gamma_1e^{(a+ib)t}$bo'ladi. Bundan Eyler formulasiga ko'ra
$x=\alpha_1e^{at}(\cos bt+i\sin bt)$, $y=\beta_1\alpha_1e^{at}(\cos bt+i\sin bt)$, $z=\gamma_1\alpha_1e^{at}(\cos bt+i\sin bt)$

xususiy yechimlarni topamiz. $\alpha_1,\beta_1,\gamma_1$ larning qiymatlari kompleks bo'lgan holda ularning qiymatlarini $(\cos bt+i\sin bt)$ ga ko'paytirib (qavslarni ochib), so'ngra ma'lum (II-BOB. 4-§. 4.2-teorema) teoremaga asosan,

$$x_1=\tilde\alpha_1e^{at}\cos bt,\qquad y_1=\tilde\beta_1e^{at}\cos bt,\qquad z_1=\tilde\gamma_1e^{at}\cos bt$$
$$x_2=\tilde\alpha_2e^{at}\sin bt,\qquad y_2=\tilde\beta_2e^{at}\sin bt,\qquad z_2=\tilde\gamma_2e^{at}\sin bt$$

chiziqli erkli xususiy yechimlarni topamiz. $\lambda=\lambda_2=a-ib$ qiymatlarga mos xususiy yechimlar topilgan xususiy yechimlar bilan chiziqli erli bo'lishi ma'lum, shuning uchun $\lambda=\lambda_3$qiymatga mos $x_3=\alpha_3e^{\lambda_3t}$, $y_3=\beta_3e^{\lambda_3t}$, $z_3=\gamma_3e^{\lambda_3t}$ xususiy yechimlarni olamiz.
Demak bu holda umumiy yechim:

$$\begin{cases} x=e^{at}\left(C_1\tilde\alpha_1\cos bt+C_2\tilde\alpha_2\sin bt\right)+C_3\alpha_3e^{\lambda_3t}\\ y=e^{at}\left(C_1\tilde\beta_1\cos bt+C_2\tilde\beta_2\sin bt\right)+C_3\beta_3e^{\lambda_3t}\\ z=e^{at}\left(C_1\tilde\gamma_1\cos bt+C_2\tilde\gamma_2\sin bt\right)+C_3\gamma_3e^{\lambda_3t}\end{cases}\qquad(2.11)$$

bo'ladi.

7-*Misol*. $\begin{cases}\dot x=2x-y+2z\\ \dot y=x+2z\\ \dot z=-2x+y-z\end{cases}$ sistemaning xos sonlari $\lambda_1=1$ va $\lambda_{2/3}=\pm i$

ekanligii ma'lum bo'lsa, sistemani yeching.

Yechish. Berilgan sistemaning xos sonlari uchun xos vektorlarni topamiz:

$\lambda_1 = 1$ bo'lsin, u holda $\begin{cases} \alpha_1 - \beta_1 + 2\gamma_1 = 0 \\ \alpha_1 - \beta_1 + 2\gamma_1 = 0 \\ -2\alpha_1 + \beta_1 - 2\gamma_1 = 0 \end{cases}$ sistemani yechib,

$\alpha_1 = 0,\ \beta_1 = 2$ va $\gamma_1 = 1$ ya'ni $h_1(0,2,1)$ xos vektorni topamiz. Demak $x_1 = 0,\ y_1 = 2C_1 e^t$ va $z_1 = C_1 e^t$.

$\lambda_2 = i$ bo'lsin, bu holda $\begin{cases} (2-i)\alpha_1 - \beta_1 + 2\gamma_1 = 0 \\ \alpha_1 - i\beta_1 + 2\gamma_1 = 0 \\ -2\alpha_1 + \beta_1 - (1+i)\gamma_1 = 0 \end{cases}$ sistemani yechamiz.

Sistemadagi birinchi tenglamadan ikkinchi tenglamani ayiramiz va $(1-i)\alpha_1 - (1-i)\beta_1 = 0$ yoki $\alpha_1 = \beta_1$ ni topamiz. Uchinchi tenglamadan esa $\gamma_1 = \dfrac{i-1}{2}\alpha_1$ ni olamiz, ya'ni $\alpha_1 = \beta_1 = 2$ bo'lsin deb olsak, $h_2(2,2,i-1)$ xos vektorni topamiz. (2.6) ga asosan $x = 2e^{it} = 2(\cos t + i\sin t)$, $y = 2e^{it} = 2(\cos t + i\sin t)$ va $z = (i-1)e^{it}$ ya'ni $z = (i-1)(\cos t + i\sin t) = -(\cos t + \sin t) + i(\cos t - \sin t)$.

Ma'lum teoremaga asosan sistemaning boshqa xususiy yechimlari $x_2 = 2C_2\cos t,\ y_2 = 2C_2\cos t$, $z_1 = -C_2(\cos t + \sin t)$ va $x_3 = 2C_3\sin t,\ y_3 = 2C_3\sin t$, $z_3 = C_3(\cos t - \sin t)$ bo'ladi. Shunday qilib, berilgan sistema umumiy yechimi $x(t) = 2C_2\cos t + 2C_3\sin t$, $y(t) = 2C_1e^t + 2C_2\cos t + 2C_3\sin t$ va $z(t) = C_1e^t - C_2(\cos t + \sin t) + C_3(\cos t - \sin t)$ bo'ladi.

Izox. *Berilgan sistemani yechishda $\lambda_3 = -i$ xos songa mos xos vektorni topishga zarurat yoq edi, chunki $\lambda_2 = i$ xos songa mos $h_2(2,2,i-1)$ xos vektor yordamida ikkitadan chiziqli erkli yechimlar aniqlandi.*

b₃). Ildizlari orasida karralilari ham mavjud bo'lsin. Agar $h_1, h_2, ..., h_k$ chiziqli erkli xos vektorlar soni, λ xarakteristik ildizning karralilik darajasiga teng bo'lsa, unga mos yechim $C_1h_1e^{\lambda t} + C_2h_2e^{\lambda t} + ... + C_kh_ke^{\lambda t}$ ko'rishda bo'ladi.

Agar λ xarakteristik ildiz karrali bo'lib, unga mos chiziqli erkli xos vektorlar soni $m\ (m < k)$ bo'lsa, u holda shu λ xarakteristik ildizga mos yechimni

$$\begin{cases} x_1 = \left(b_{10} + b_{11}t + b_{12}t^2 + ... + b_{1k-m}t^{k-m}\right)e^{\lambda t}, \\ .... \\ x_n = \left(b_{n0} + b_{n1}t + b_{n2}t^2 + ... + b_{nk-m}t^{k-m}\right)e^{\lambda t}. \end{cases} \qquad (2.12)$$

ko'rinishda izlanadi, bu yerda $b_{ij}\ (i=\overline{1.n},\ j=\overline{0.k-m})$-noma'lum koeffesiyentlar bo'lib, ularni topish uchun (2.12) ni (2.1_1) ga qo'yamiz va mos hadlar oldidagi koeffisiyentlarni tenglab b_{ij} larga nisbatan algebraik tenglamalar sistemasini hosil qilamiz. b_{ij} koeffisiyentlar k ta ixtiyoriy o'zgarmaslarga nisbatan aniqlanadi.

8-***Misol***. $\begin{cases} \dot{x} = x - 2y \\ \dot{y} = 2x - 3y \end{cases}$ sistemani yeching.

Yechish. Berilgan sistemaning xarakteristik (xos) sonlari $\begin{vmatrix} 1-\lambda & -2 \\ 2 & -3-\lambda \end{vmatrix}$=0 yoki $\lambda^2 + 2\lambda + 1 = 0$ tenglamaning $\lambda_1 = \lambda_2 = -1$ yechimlaridan iborat. Demak sistemanig birinchi xususiy yechimlari $x_1 = C_1 e^{-t},\ y_1 = C_1 e^{-t}$ bo'lib, shu yechimlarga tayanib, berilgan sistema umumiy yechimini

$$\begin{cases} x = (\alpha_1 + \beta_1 t)e^{-t} \\ y = (\alpha_2 + \beta_2 t)e^{-t} \end{cases} \qquad (*)$$

ko'rinishda izlaymiz. $\begin{cases} \dot{x} = (\beta_1 - \alpha_1 - \beta_1 t)e^{-t} \\ \dot{y} = (\beta_2 - \alpha_2 - \beta_2 t)e^{-t} \end{cases}$ va (*) ni berilgan sistemaga qo'yib,

$\begin{cases} \beta_1 - 2\alpha_1 - 2\beta_1 t + 2\alpha_2 + 2\beta_2 t = 0 \\ \beta_2 + 2\alpha_2 + 2\beta_2 t - 2\alpha_1 - 2\beta_1 t = 0 \end{cases}$ yoki $\begin{cases} \beta_2 = \beta_1 \\ 2\alpha_2 = 2\alpha_1 - \beta_1 \end{cases}$ ga ega bo'lamiz.

$\alpha_1,\ \beta_1$

lar ixtiyoriy o'zgarmaslar bo'lgani uchun $\alpha_1 = C_1,\ \beta_1 = C_2$ deb tanlaymiz, natijada (*) ga asosan to'g'ridan-to'g'ri berilgan sistemaning $\begin{cases} x = (C_1 + C_2 t)e^{-t} \\ y = \left(C_1 - \dfrac{C_2}{2} + C_2 t\right)e^{-t} \end{cases}$ umumiy yechimini topamiz.

c) **Qo'shib olingan vektorlar usuli.**

Malumki, jordan formaga ega bo'lgan ixtiyoriy matritsa uchun bazis vektor mavjud. Jordan formasining $k \geq 1$tartibli har bir kletkasi (katagi) uchun

$$
\begin{aligned}
&Ah_1 = \lambda h_1\\
&Ah_2 = \lambda h_2 + h_1\\
&Ah_3 = \lambda h_3 + h_2\\
&\dots\dots\dots\dots\dots\dots\\
&Ah_k = \lambda h_k + h_k - 1
\end{aligned}
\tag{2.13}
$$

tenglamalarni qanaotlantiruvchi $h_1, h_2,...,h_k$ bazis vektorlar mos keladi. Bu yerda h_1-***xos vektor***, $h_2,...,h_k$ lar esa h_1-***xos vektorga qo'shib olingan vektorlar*** deyiladi. Har bir $h_1, h_2,...,h_k$ vektorlar uchun $\dot{x} = Ax$ sistemaning k ta quyidagi chiziqli erkli yechimlar mos keladi:

$$
\begin{aligned}
&x^1 = h_1 e^{\lambda t},\\
&x^2 = e^{\lambda t}\left(\frac{t}{1!}h_1 + h_2\right),\\
&x^3 = e^{\lambda t}\left(\frac{t^2}{2!}h_1 + \frac{t}{1!}h_2 + h_3\right),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&x^k = e^{\lambda t}\left(\frac{t^{k-1}}{(k-1)!}h_1 + \frac{t^{k-2}}{(k-2)!}h_2 + ... + \frac{t}{1!}h_{k-1} + h_k\right)
\end{aligned}
\tag{2.14}
$$

bu yerda x larning daraja ko'rsatkichlari yechimlarning nomerini ko'rsatadi. Bunday yechimlarning barchasining soni barcha kletkalar (kataklar) tartiblari yig'indisiga ya'ni matrisa tartibiga teng bo'ladi.

9-***Misol***. $\begin{cases} \dot{x} = y + z \\ \dot{y} = x + z \\ \dot{z} = x + y \end{cases}$ sistemaning xos sonlari $\lambda_1 = 2$ va $\lambda_2 = \lambda_3 = -1$, ekanligii ma'lum bo'lsa, sistemani yeching.

Yechish. Berilgan sistemaning xos sonlariga mos xos vektorlarni topamiz:

$\lambda_1 = 2$ bo'lsin, u holda $\begin{cases} -2\alpha_1 + \beta_1 + \gamma_1 = 0 \\ \alpha_1 - 2\beta_1 + \gamma_1 = 0 \\ \alpha_1 + \beta_1 - 2\gamma_1 = 0 \end{cases}$ yoki $\begin{cases} -2\alpha_1 + \beta_1 + \gamma_1 = 0 \\ \alpha_1 - 2\beta_1 + \gamma_1 = 0 \end{cases}$

sistemani yechib, $\alpha_1 = \beta_1 = \gamma_1 = 1$ ya'ni $h_1(1,1,1)$ xos vektorni topamiz. Demak $x_1 = C_1 e^{2t}$, $y_1 = C_1 e^{2t}$ va $z_1 = C_1 e^{2t}$.

$\lambda_2 = \lambda_3 = -1$ bo'lsin, bu holda

$$\begin{cases} \alpha_1 + \beta_1 + \gamma_1 = 0 \\ \alpha_1 + \beta_1 + \gamma_1 = 0 \\ \alpha_1 + \beta_1 + \gamma_1 = 0 \end{cases} \qquad (**)$$

yoki $\alpha_1 + \beta_1 + \gamma_1 = 0$ tenglamaga ega bo'lamiz, (*) algebraic tenglamalar sistemasining matritsasi rangi bir ga teng, demak $\alpha_1 + \beta_1 + \gamma_1 = 0$ tenglama ikkita chiziqli erkli yechimlarga ega, ya'ni $h_2(1,0,-1)$ va $h_1(0,-1,1)$ xos vektorlar chiziqli erkli. Demak $x_2 = C_2 e^{-t}$, $y_2 = 0$, $z_2 = -C_2 e^{-t}$ va $x_3 = 0$, $y_3 = -C_3 e^{-t}$, $z_3 = C_3 e^{-t}$ yechimlarga ega bo'lamiz. Ishonch hosil qilish mumkinki, topilgan yechimlar chiqli erkli, ya'ni

$$\begin{vmatrix} e^{2t} & e^{2t} & e^{2t} \\ e^{-t} & 0 & -e^{-t} \\ 0 & -e^{-t} & e^{-t} \end{vmatrix}_{t=0} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{vmatrix} = -3 \neq 0 .$$

Shunday qilib, berilgan sistema umumiy yechimi $x = C_1 e^{2t} + C_2 e^{-t}$, $y = C_1 e^{2t} - C_3 e^{-t}$, $z = C_1 e^{2t} + (C_3 - C_2) e^{-t}$ bo'ladi.

10-***Misol***. $\begin{cases} \dot{x} = 4x - y \\ \dot{y} = 3x + y - z \\ \dot{z} = x + z \end{cases}$ sistemaning xos sonlari $\lambda_1 = \lambda_2 = \lambda_3 = 2$ ekanligii ma'lum bo'lsa, sistemani umumiy yechimini toping.

Yechish. $\lambda = 2$ xos songa $\begin{cases} 2\alpha_1 - \beta_1 = 0 \\ 3\alpha_1 - \beta_1 - \gamma_1 = 0 \\ \alpha_1 - \gamma_1 = 0 \end{cases}$ sistemaning yechimlaridan tuzilgan $h_1(1,2,1)$ xos vektor mos keladi, ya'ni $x_1 = C_1 e^{2t}$, $y_1 = 2C_1 e^{2t}$ va $z_1 = C_1 e^{2t}$. Endi (2.13) ga asosan $h_2(\alpha_2, \beta_2, \gamma_2)$ qo'shib olingan vektorni tuzamiz: $Ah_2 = \lambda h_2 + h_1$ ya'ni

$\begin{cases} 2\alpha_2 - \beta_2 = 1 \\ 3\alpha_2 - \beta_2 - \gamma_2 = 2 \\ \alpha_2 - \gamma_2 = 1 \end{cases}$, bundan $3\alpha_2 - \beta_2 - \gamma_2 = 2$ tenglamani hosil qilamiz, ya'ni oxirgi sistemaning rangi ikkiga teng bo'lagani uchun, $h_2^1(1,1,0)$ yoki $h_2^2(1,0,1)$ (daraja ko'rsatkichi uning tartibini bildiradi) vektorlarga egamiz. $Ah_3=\lambda h_3+h_2$ tenglikga ko'ra $\begin{cases}2\alpha_3-\beta_3=1\\ 3\alpha_3-\beta_3-\gamma_3=1\\ \alpha_3-\gamma_3=0\end{cases}$, bundan

$\begin{cases}2\alpha_3-\beta_3=1\\ \alpha_3=\gamma_3\end{cases}$ ya'ni $h_3^1(1,1,1)$. Xuddi shunday $Ah_3=\lambda h_3+h_2$ tenglikdan

$h_2^2(1,0,1)$ vektorga mos h_3 vektorni topamiz, $\begin{cases}2\alpha_3-\beta_3=1\\ 3\alpha_3-\beta_3-\gamma_3=0\\ \alpha_3-\gamma_3=1\end{cases}$, bundan

$\begin{cases}2\alpha_3-\beta_3=1\\ 2\alpha_3-\beta_3=-1\end{cases}$ ya'ni sistema yechimga ega emas, demak $h_2^2(1,0,1)$ vektorga mos h_3 vektor mavjud emas. Shunday qilib, (2.14) ga asosan

$$x_2=C_2e^{2t}\left(\frac{t}{1!}+1\right)=C_2(t+1)e^{2t},\ y_2=C_2e^{2t}\left(\frac{t}{1!}\cdot 2+1\right)=C_2(2t+1)e^{2t},\ z_2=C_2e^{2t}t$$

va $x_3=C_3e^{2t}\left(\frac{t^2}{2}+t+1\right),\ y_3=C_3e^{2t}\left(t^2+t+1\right),\ z_3=C_3e^{2t}\left(\frac{t^2}{2}+1\right)$ ya'ni berilgan sistema umumiy yechimi

$$x(t)=e^{2t}\left(\frac{C_3}{2}t^2+(C_2+C_3)t+C_1+C_2+C_3\right),$$

$$y(t)=e^{2t}\left(C_3t^2+(2C_2+C_3)t+2C_1+C_2+C_3\right),\ z(t)=e^{2t}\left(\frac{C_3}{2}t^2+C_2t+C_1+C_3\right)$$

bo'ladi. Agar $\begin{cases}C_1+C_2+C_3=C_1^*\\ C_2+C_3=C_2^*\\ \frac{C_3}{2}=C_3^*\end{cases}$ (keyin yana $C_1^*\to C_1,\ C_2^*\to C_2,\ C_3^*\to C_3$) almashtitrish kiritsak, berilgan sistema umumiy yechimi $x(t)=e^{2t}\left(C_3t^2+C_2t+C_1\right)$,

$$y(t)=e^{2t}\left(2C_3t^2+2(C_2-C_3)t+2C_1-C_2\right),$$

$z(t)=e^{2t}\left(C_3t^2+(C_2-2C_3)t+C_1-C_2+2C_3\right)$ ko'rinishga ega bo'ladi.

d). O'garmasni variatsiyalash usuli.

Agar sistemada qatnashayotgan tenglamalar maxsus o'ng tomonli chizqli differensial tenglamalar bo'lsa, bunday sistemalarni integrallashda bitta yuqori tartibli maxsus o'ng tomonli o'zgarmas koeffisiyentli chizqli differensial tenglamalar uchun qo'llanilgan usuldan foydalanish mumkin. Biroq biz bu yerda ixtiyoriy o'n'g tomonli tenglamalar sistemasi uchun umumiyroq bo'lgan usulni keltiramiz.

Ushbu

$$\begin{cases} \frac{dx}{dt}+a_0x+b_0y+c_0z=f_0(t) \\ \frac{dy}{dt}+a_1x+b_1y+c_1z=f_1(t) \\ \frac{dz}{dt}+a_2x+b_2y+c_2z=f_2(t) \end{cases} \tag{2.15}$$

ko'rinishga ega bo'lgan chiziqli birhinsli bo'lmagan tenglamalar sistemasi berilgan bo'lsin, bu yerda $f_j(t)$ $(j=0,1,2)$-berilgan funksiyalar. (2.15) sistemaga mos birjinsli sistemani integrallash usuli bizga ma'lum bo'lgani uchun, mos bir jinsli sitemaning umumiy yechimini

$$\begin{cases} x=C_1x_1+C_2x_2+C_3x_3 \\ y=C_1y_1+C_2y_2+C_3y_3 \\ z=C_1z_1+C_2z_2+C_3z_3 \end{cases} \tag{2.16}$$

bo'lsin deb faraz qilamiz. U holda (2.15) sistemaning yechimini

$$\begin{cases} x=C_1(t)x_1+C_2(t)x_2+C_3(t)x_3 \\ y=C_1(t)y_1+C_2(t)y_2+C_3(t)y_3 \\ z=C_1(t)z_1+C_2(t)z_2+C_3(t)z_3 \end{cases} \tag{2.17}$$

bu yerda $C_1(t), C_2(t), C_3(t)$- noma'lum funksiyalar. Bu noma'lum funksiyalarni topish uchun (2.16) ni (2.15) ga qo'yamiz, u holda (2.15) sistemaning birinchi tenglamasidan

$$C_1'(t)x_1+C_2'(t)x_2+C_3'(t)x_3+C_1(t)\left(x_1'+a_0x_1+b_0y_1+c_0z_1\right)+C_2(t)\left(x_2'+a_0x_2+b_0y_2+c_0z_2\right)+$$
$$+C_3(t)\left(x_3'+a_0x_3+b_0y_3+c_0z_3\right)=f_1(t)$$

ni olamiz. (2.17) ga asosan qavs ichidagilar nolga teng, ya'ni oxirgi tenglikdan

$$C_1'(t)x_1+C_2'(t)x_2+C_3'(t)x_3=f_0(t)$$

ga ega bo'lamiz. Xuddi shunday usul bilan (2.15) sistemaning ikkinchi va uchinchi tenglamalaridan $C_1'(t)x_1+C_2'(t)x_2+C_3'(t)x_3=f_1(t)$ va $C_1'(t)x_1+C_2'(t)x_2+C_3'(t)x_3=f_2(t)$ munosabatlarni olamiz. Demak $C_1(t), C_2(t), C_3(t)$- noma'lum funksiyalarni topish uchun

$$\begin{cases} C_1'(t)x_1+C_2'(t)x_2+C_3'(t)x_3=f_0(t) \\ C_1'(t)x_1+C_2'(t)x_2+C_3'(t)x_3=f_1(t) \\ C_1'(t)x_1+C_2'(t)x_2+C_3'(t)x_3=f_2(t) \end{cases} \tag{2.18}$$

sistemani $C_1'(t), C_2'(t), C_3'(t)$ noma'lumlarga nisbatan yechish yetarli. Ma'lumki (2.17) dagi xususiy yechimlar chiziqli erkli bo'lgani uchun

$$\Delta = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \neq 0$$

bo'ladi, ya'ni (2.18) sistema $C_1'(t), C_2'(t), C_3'(t)$ larga nisbatan yagona yechimga ega.

11-***Misol***. $\begin{cases} \dot{x} = y + tg^2 t - 1 \\ \dot{y} = -x + tgt \end{cases}$ sistemani integrallang.

Yechish. Berilgan sistemaga mos quyidagi $\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$ bir jinsli sistemaning umumiy yechimini topamiz, buning uchun a). usuldan foydalanamiz: $\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases} \Rightarrow \ddot{x} + x = 0, \quad x(t) = C_1 \cos t + C_2 \sin t$. Topilgan $x(t)$ ning qiymatini sistemaning birinchi tenglamasiga qo'yib, $y(t) = -C_1 \sin t + C_2 \cos t$ yechimni topamiz.

Berilgan bir jinslimas sistemaning yechimini $x(t) = C_1(t)\cos t + C_2(t)\sin t$, $y(t) = -C_1(t)\sin t + C_2(t)\cos t$ ko'rinishda izlaymiz. Noma'lum $C_1(t), C_2(t)$ funksiyalarni topish uchun $x(t)$ va $y(t)$ larning qiymatlarini berilgan sistemaga qo'yamiz va $C_1'(t), C_2'(t)$ funksiyalarga nisbatan $\begin{cases} C_1'(t)\cos t + C_2'(t)\sin t = tg^2 t - 1 \\ -C_1'(t)\sin t + C_2'(t)\cos t = tgt - 1 \end{cases}$ sistemani hosil qilamiz. Bu sistemani yechib, $C_1(t) = C_1 - \sin t,\ C_2(t) = C_2 + \frac{1}{\cos t} + \cos t$ larni topamiz. Shunday qilib, berilgan sistema umumiy yechimi

$$x(t) = \left(C_1 - \sin t\right)\cos t + \left(C_2 + \frac{1}{\cos t} + \cos t\right)\sin t = C_1 \cos t + C_2 \sin t + tgt,$$

$$y(t) = -\left(C_1 - \sin t\right)\sin t + \left(C_2 + \frac{1}{\cos t} + \cos t\right)\cos t = -C_1 \sin t + C_2 \cos t + 2.$$

e). Normal ko'rinishga keltirilmagan sistemalarni integrallash.

Ushbu

$$\begin{cases} a_{10}x^{(n)} + a_{11}x^{(n-1)} + \ldots + a_{1n}x + b_{10}y^{(n)} + b_{11}y^{(n-1)} + \ldots + b_{1n}y = 0 \\ a_{20}x^{(n)} + a_{21}x^{(n-1)} + \ldots + a_{2n}x + b_{20}y^{(n)} + b_{21}y^{(n-1)} + \ldots + b_{2n}y = 0 \end{cases} \qquad (2.19)$$

normal tipda bo'lmagan sistemaga mos xarakteristik tenglama

$$\begin{vmatrix} a_{10}\lambda^n + a_{11}\lambda^{n-1} + ... + a_{1n} & b_{10}\lambda^n + b_{11}\lambda^{n-1} + ... + b_{1n} \\ a_{20}\lambda^n + a_{21}\lambda^{n-1} + ... + a_{2n} & b_{20}\lambda^n + b_{21}\lambda^{n-1} + ... + b_{2n} \end{vmatrix} = 0 \qquad (2.20)$$

ko'rinshda yoziladi. Bu xarakteristik tenglama yechilib λ xarakteristik sonlar topilgandan keyin (2.19) sistema yechimni Eyler usuli (b). punktdagi usul) orqali topiladi.

12-***Misol***. $\begin{cases} \ddot{x} + 5\dot{x} + 2\dot{y} = -y \\ 3\ddot{x} + \dot{y} = -5x - 3y \end{cases}$ sistemani integrallang.

Yechish. Berilgan sistema (2.19) ko'rinishdagi sistema bo'lib, (2.20) ga ko'ra uning xarakteristik tenglamasi $\begin{vmatrix} \lambda^2 + 5\lambda & 2\lambda + 1 \\ 3\lambda^2 + 5 & \lambda + 3 \end{vmatrix} = 0$ yoki $-5\lambda^3 + 5\lambda^2 + 5\lambda - 5 = 0$ ko'rinishga ega bo'ladi. Demak berilgan sistemaning xarakteristik (xos) sonlari $\lambda_1 = \lambda_2 = 1, \lambda_3 = -1$. $\lambda_1 = 1$ xos songa mos xos vektor $\begin{cases} 6\alpha_1 + 3\beta_1 = 0 \\ 8\alpha_1 + 4\beta_1 = 0 \end{cases}$ sisatema yoki $2\alpha_1 + \beta_1 = 0$ tenglama yechimlaridan bir bo'ladi, ya'ni $h_1(1,-2)$ bo'lib unga mos xususiy yechim $x_1 = C_1 e^t$, $y_1 = -2C_1 e^t$ bo'ladi. $\lambda_1 = 1$ xos son ikki karrali bo'lgani uchun ikkinchi yechimni

$$x_2 = (a + bt)e^t, \qquad y_2 = (c + dt)e^t \qquad (*)$$

ko'rinishda izlaymiz. a, b, c, d noma'lum koeffisiyentlarni topish uchun (*) ni va $\dot{x}_2 = (a + b + bt)e^t$, $\dot{y}_2 = (c + d + dt)e^t$, $\ddot{x}_2 = (a + 2b + bt)e^t$ hosilalarni berilgan siste-maga qo'yamiz, hamda bir hadlar oldidagi koeffisiyentlarni tenglashtirib $\begin{cases} 6a + 7b + 3c + 2d = 0 \\ 2b + d = 0 \\ 8a + 6b + 4c + d = 0 \end{cases}$ sistemani hosil qilamiz. Bu sistemani yechib, $\begin{cases} d = -2b \\ c = -2a - b \end{cases}$ munosabatlarga ega bo'lamiz. a va b sonlarining ixtiyoriy o'zgarmas ekanligidan foydalanib, $a = C_1$, $b = C_2$ deb olsak , $\lambda_1 = \lambda_2 = 1$ xos sonlarga mos yechimni quyidagicha yozamiz: $x_1(t) + x_2(t) = (C_1 + C_2 t)e^t$, $y_1(t) + y_2(t) = \left(-2C_1 - C_2 - 2C_1\right)e^t$.

$\lambda_3 = -1$ xos songa mos xos vektor $\begin{cases} -4\alpha_1 - \beta_1 = 0 \\ 8\alpha_1 + 2\beta_1 = 0 \end{cases}$ sistemani yechimlaridan bir bo'ladi, ya'ni $h_3(1,-4)$ bo'lib unga mos xususiy yechim $x_3 = C_1 e^{-t}$, $y_3 = -4C_1 e^{-t}$ bo'ladi. Shunday qilib berilgan sistema umumiy yechimi $x(t) = (C_1 + C_2 t)e^t + C_1 e^{-t}$, $y(t) = \left(-2C_1 - C_2 - 2C_1\right)e^t - 4C_1 e^{-t}$ bo'ladi.

Mustaqil yechish uchun mashqlar.

Quyidagi bir jinsli sistemalarni yeching (503-525):

503. $\begin{cases} \dot{x} - 2x - y = 0 \\ \dot{y} - 3x - 4y = 0 \end{cases}$

504. $\begin{cases} \dot{x} = 2y \\ \dot{y} = 2x \end{cases}$

505. $\begin{cases} \dot{x} = 3x - y \\ \dot{y} = 4x - y \end{cases}$

506. $\begin{cases} \dot{x} = x - y \\ \dot{y} = y - 4x \end{cases}$

507. $\begin{cases} \dot{x} = 2x - 9y \\ \dot{y} = x + 8y \end{cases}$

508. $\begin{cases} \dot{x} + x - 8y = 0 \\ \dot{y} = x + y \end{cases}$

509. $\begin{cases} \dot{x} = x - 3y \\ \dot{y} = 3x + y \end{cases}$

510. $\begin{cases} \dot{x} = 2x + y \\ \dot{y} = 4y - x \end{cases}$

511. $\begin{cases} \dot{x} = 2y - 3x \\ \dot{y} = y - 2x \end{cases}$

512. $\begin{cases} \dot{x} = 5x + 3y \\ \dot{y} = -y - 3x \end{cases}$

513. $\begin{cases} \dot{x} = 2x - y + z \\ \dot{y} = x + 2y - z \\ \dot{z} = x - y + 2z \end{cases}$

$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$

514. $\begin{cases} \dot{x} = -3x + 4y - 2z \\ \dot{y} = x + z \\ \dot{z} = 6x - 6y + 5z \end{cases}$

$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1$

515. $\begin{cases} \dot{x} = 3x - y + z \\ \dot{y} = x + y + z \\ \dot{z} = 4x - y + 4z \end{cases}$

$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 5$

516. $\begin{cases} \dot{x} = x - 2y - z \\ \dot{y} = y - x + z \\ \dot{z} = x - z \end{cases}$

$\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = -1$

517. $\begin{cases} \dot{x} = 8y \\ \dot{y} = -2z \\ \dot{z} = 2x + 8y - 2z \end{cases}$

$\lambda_1 = -2, \lambda_{2/3} = \pm 4i$

518. $\begin{cases} \dot{x} = x - y - z \\ \dot{y} = y + x \\ \dot{z} = 3x + z \end{cases}$

$\lambda_1 = 1, \lambda_{2/3} = 1 \pm 2i$

519. $\begin{cases} \dot{x} = 2x + y \\ \dot{y} = x + 3y - z \\ \dot{z} = -x + 2y + 3z \end{cases}$

$\lambda_1 = 2, \lambda_{2/3} = 3 \pm i$

520. $\begin{cases} \dot{x} = -3x + 2y + 2z \\ \dot{y} = -3x - y - z \\ \dot{z} = -x + 2y \end{cases}$

$\lambda_1 = -2, \lambda_{2/3} = -1 \pm 2i$

521. $\begin{cases} \dot{x} = 2x + y + z \\ \dot{y} = -2x - z \\ \dot{z} = 2x + y + 2z \end{cases}$

$\lambda_1 = \lambda_2 = 1, \lambda_3 = 2$

522. $\begin{cases} \dot{x} = 2x - y - z \\ \dot{y} = 3x - 2y - 3z \\ \dot{z} = -x + y + 2z \end{cases}$

$\lambda_1 = 0, \lambda_{2/3} = 1$

523. $\begin{cases} \dot{x} = y - x - 2z \\ \dot{y} = 4x + y \\ \dot{z} = 2x + y - z \end{cases}$

$\lambda_1 = 1, \lambda_{2/3} = -1$

524. $\begin{cases} \dot{x} = 2x - y - z \\ \dot{y} = 2x - y - 2z \\ \dot{z} = 2z - x + y \end{cases}$

$\lambda_1 = \lambda_2 = \lambda_3 = 1$

525. $\begin{cases} \dot{x} = y - z \\ \dot{y} = x - z \\ \dot{z} = 2x + 2y - 3z \end{cases}$

$\lambda_1 = \lambda_2 = \lambda_3 = -1$

Quyidagi bir jinsli bo’lmagan sistemalarni yeching (526-545):

526. $\begin{cases} \dot{x} = -2y + 3 \\ \dot{y} = 2x - 2t \end{cases}$

527. $\begin{cases} \dot{x} = 2x + 4y - 8 \\ \dot{y} = 3x + 6y \end{cases}$

528. $\begin{cases} \dot{x} = 3x - \frac{1}{2}y - 3t^2 - \frac{1}{2}t + \frac{3}{2} \\ \dot{y} = 2y - 2t - 1 \end{cases}$

529. $\begin{cases} \dot{x} = x - y + 2\sin t \\ \dot{y} = 2x - y \end{cases}$

530. $\begin{cases} \dot{x} = 4x - 3y + \sin t \\ \dot{y} = 2x - y - 2\cos t \end{cases}$

531. $\begin{cases} \dot{x} = 2x - y \\ \dot{y} = 2y - x - 5e^t \sin t \end{cases}$

532. $\begin{cases} \dot{x} = x + 2y + 16te^t \\ \dot{y} = 2x - 2y \end{cases}$

533. $\begin{cases} \dot{x} = 2x + y + 2e^t \\ \dot{y} = x + 2y - 3e^{4t} \end{cases}$

534. $\begin{cases} \dot{x} = 2y - x \\ \dot{y} = 4y - 3x + \dfrac{e^{3t}}{e^{2t} + 1} \end{cases}$

535. $\begin{cases} \dot{x} = y \\ \dot{y} = x + \dfrac{1}{t^2} + \ln t \end{cases}$

536. $\begin{cases} \dot{x} = -4x - 2y + \dfrac{2}{e^t - 1} \\ \dot{y} = 3y + 6x - \dfrac{3}{e^t - 1} \end{cases}$

537. $\begin{cases} \dot{x} = x - y + \dfrac{1}{\cos t} \\ \dot{y} = 2x - y \end{cases}$

540. $\begin{cases} \dot{x} = -2x + 3y + 4z - 3t \\ \dot{y} = -6x + 7y + 6z + 1 - 7t \\ \dot{z} = x - y + z + t \end{cases}$

538. $\begin{cases} \dot{x} = 3x - 2y + \sin t \\ \dot{y} = 2x - y + 15e^t \sqrt{t} \end{cases}$

541. $\begin{cases} \dot{x} = -x + y + z + e^t \\ \dot{y} = x - y + z + e^{3t} \\ \dot{z} = x + y + z + 4 \end{cases}$

539. $\begin{cases} \dot{x} = 2x + y - 2z - t + 2 \\ \dot{y} = 1 - x \\ \dot{z} = x + y - z - t + 1 \end{cases}$

542. $\begin{cases} \dot{x} = x + y + t \\ \dot{y} = x - 2y + 2t \end{cases}$; $x(0) = -\dfrac{7}{9},\ y(0) = -\dfrac{5}{9}$

543. $\begin{cases} \dot{x} + \dot{y} = e^{-t} - y \\ 2\dot{x} + \dot{y} = \sin t - 2y \end{cases}$; $x(0) = -2,\ y(0) = 1.$

544. $\begin{cases} \dot{x} = e^t - y - 5x \\ \dot{y} = e^{2t} + x - 3y \end{cases}$; $x(0) = \dfrac{119}{900},\ y(0) = \dfrac{211}{900}.$

545. $\begin{cases} 2\dot{x} = 6x - y - 6t^2 - t + 3 \\ \dot{y} = 2y - 2t - 1 \end{cases}$; $x(0) = 2,\ \ y(0) = 3.$

Normal tipda bo’lmagan sistemalarni integrallang (546-558):

546. $\begin{cases} \ddot{x} = y \\ \ddot{y} = x \end{cases}$;

553. $\begin{cases} \ddot{x} = 2x - 3y \\ \ddot{y} = x - 2y \end{cases}$;

547. $\begin{cases} 2\dot{x} - 5\dot{y} = 4y - x \\ 3\dot{x} - 4\dot{y} = 2x - y \end{cases}$;

550. $\begin{cases} \ddot{x} + \dot{x} + \dot{y} = 2y \\ \dot{x} - \dot{y} + x = 0 \end{cases}$;

548. $\begin{cases} \dot{x} = -4x - 4y \\ \dot{x} + 4\dot{y} = -4y \end{cases}$;

$x(0) = 1,\ y(0) = 0$

551. $\begin{cases} \ddot{x} - 2\dot{y} + 2x = 0 \\ 3\dot{x} + \ddot{y} - 8y = 0 \end{cases}$;

552. $\begin{cases} \ddot{x} + 3\ddot{y} = x \\ \dot{x} + 3\dot{y} = 2y \end{cases}$;

549. $\begin{cases} \dot{x} + 2\dot{y} = 17x + 8y \\ 13\dot{x} = 53x + 2y \end{cases}$;

$x(0) = 2,\ y(0) = -1$

554. $\begin{cases} \ddot{x} + 2\ddot{y} = x + 2y \\ \dot{x} + \dot{y} = x - y \end{cases}$;

555. $\begin{cases} 2\ddot{x}+2\dot{x}+3\ddot{y}+\dot{y}=-x-y \\ \ddot{x}+4\dot{x}+3\ddot{y}+2\dot{y}=x+y \end{cases}$;

556. $\begin{cases} \ddot{x}+4\dot{x}-2x-2\dot{y}-y=0 \\ \ddot{x}-4\dot{x}-\ddot{y}+2\dot{y}+2y=0 \end{cases}$;

557. $\begin{cases} \ddot{x}=3x-y-z \\ \ddot{y}=-x+3y-z \\ \ddot{z}=-x-y+3z \end{cases}$

558. $\begin{cases} \ddot{x}=-x+y+z \\ \ddot{y}=x-y+z \\ \ddot{z}=x+y-z \end{cases}$

IV-BOB. Turg'unlik nazaryasi.

1-§. Lyapunov ma'nosidagi Turg'unlik.

1.1. Asosiy ta'rif va tushunchalar.

Quyidagi

$$\frac{dx_i}{dt}=f_i(t,x_1,...,x_n), \quad i=1,2,...,n \tag{1.1}$$

yoki

$$\frac{dx}{dt}=f(t,x), \;\; x=(x_1,...,x_n), \;\; f=(f_1,...,f_n) \tag{1.2}$$

vektor ko'rinishdagi sistema berilgan bo'lib, $f_i(t,x_1,...,x_n)$ va $\frac{df_i}{dx_k}$ $(k=\overline{1,n})$ funksiyalar $t_0\le t<\infty$ oraliqda uzluksiz bo'lsin.

1-Ta'rif. Agar ixtiyoriy $\varepsilon>0$ son uchun, shunday $\delta(\varepsilon)>0$ son mavjud bo'lsaki, boshlang'ich qiymati

$$\left|x(t_0)-\varphi(t_0)\right|<\delta \tag{1.3}$$

shartni qanoatlantiradigan (1.2) sistemaning ixtiyoriy $x(t)$ yechimi uchun barcha $t\ge t_0$ larda

$$\left|x(t)-\varphi(t)\right|<\varepsilon \tag{1.4}$$

shart bajarilsa, bu sistemaning $x=\varphi(t)$ yechimi **Lyapunov ma'nosida turg'un** deyiladi.

Agar $\delta>0$ yetarlicha kichik qiymatlarida ham, (1.4) shart bajarilmaydigan hech bo'lmaganda bitta $x(t)$ yechim topilsa, ya'ni $\varepsilon>0$ ning qandaydir qiymati uchun, $\delta(\varepsilon)>0$ son mavjud bo'lmasa, u holda $x=\varphi(t)$ yechim **Lyapunov ma'nosida noturg'un** deyiladi.

2-Ta'rif. Agar sistemaning, boshlang'ich shartlarlari yetarlicha yaqin bo'lgan barcha yechimlari $t\to+\infty$da Lyapunov ma'nosida turg'un bo'lgan $\varphi(t)$ yechimga cheksiz yaqinlashsa, ya'ni

$$\lim_{t\to+\infty}|x(t)-\varphi(t)|=0 \qquad (1.5)$$

shart bajarilsa $x=\varphi(t)$ yechim **asimptotik turg'un** deyiladi.

Yechimning turg'unligi yoki turg'un emasligi t_0 ni tanlashga bog'liq emas.

(1.2) sistemaning $x=\varphi(t)$ yechiminig turg'unligini tekshirish, (1.2) sistemadan $x-\varphi(t)=y$ almashtirish orqali hosil qilingan

$$\frac{dy}{dt}=F(t,y),\ \ y=(y_1,...,y_n)\ \ (\text{bu yerda } F(t,0)\equiv 0) \qquad (1.2')$$

sistemani $y(t)\equiv 0$ yechimning turg'unligini teksirishga keltiriladi. $y_i\equiv 0\ (i=1,...,n)$ nuqta (1.2') sistemaning **muvozanat nuqtasi** deyiladi

1-Misol. $\frac{dx}{dt}=4x-t^2x$ tenglamanig $x(0)=0$ shartni qanoatlantiruvchi yechimi Lyapunov ma'nosida turg'un ekanligini tekshiring.

Yechish. Tenglamaning berilishidan ma'lumki, $x=\varphi(t)\equiv 0$ funksiya boshlang'ich shartni va tenglamani qanoatlantiradi. Endi berilgan tenglamaning umumiy yechimini topamiz:

$\frac{dx}{dt}=x\left(4-t^2\right)\Rightarrow \frac{dx}{x}=\left(4-t^2\right)dt$, demak umumiy yechim $x(t)=C\cdot e^{4t-\frac{t^3}{3}}$.

1-Ta'rifga asosan (1.3) dan $|x(t_0)-\varphi(t_0)|=|x(0)-\varphi(0)|=|C-0|=|C|$,

hamda (1.4) dan $|x(t)-\varphi(t)|=|x(t)|=|C|\cdot e^{4t-\frac{t^3}{3}}$ ga ega bo'lamiz, ya'ni

(1.4) ga asosan $\varepsilon>0$ ma'lum. Faraz qilaylik, $M=\max_{t\ge 0}\left|e^{4t-\frac{t^3}{3}}\right|=e^{\frac{16}{3}}$

bo'lsin, u holda agar $\delta(\varepsilon)=\frac{\varepsilon}{M}$ tanlasak, $\delta(\varepsilon)=\frac{\varepsilon}{M}=\varepsilon e^{-16/3}$ topiladi.

Demak $x=\varphi(t)\equiv 0$ yechim Lyapunov ma'nosida turg'un. Endi 2-Ta'rifga ko'ra, (1.5) shartni tekshiramiz:

$\lim_{t\to+\infty}|x(t)-\varphi(t)|=\lim_{t\to+\infty}|x(t)|=\lim_{t\to+\infty}\left|C\cdot e^{4t-\frac{t^3}{3}}\right|=0$, demak $\varphi(t)\equiv 0$ yechim asimptotik turg'un.

2-*Misol.* $\begin{cases}\dot{x}=-y\\ \dot{y}=2x^3\end{cases}$ sistemaning $x(0)=y(0)=0$ shartni qanoatlantiruvchi yechimi Lyapunov ma'nosida turg'unlikga tekshiring.

Yechish. Berilgan sistemadagi birinchi tenglamani ikkinchi tenglamaga bo'lib, $\frac{\dot{x}}{\dot{y}} = \frac{dx}{dy} = \frac{-y}{2x^3}$ tenglamani hosil qilamiz. Bu tenglama umumiy yechimi $y^2 + x^4 = C$ ko'rinishga ega bo'ladi, bu yeda C-ixtiyoriy o'zgarmas son. $y^2 + x^4 = C$ chiziqlar oilasini, xoy tekisligida biror moddiy nuqtaning harakatlanadigan trayektoriyalar oilasi deb hisoblasak, bu moddiy nuqtani turg'unlikka tekshirish uchun, koordinatalar boshini moddiy nuqtaning tinch holati deb qabul qilib, moddiy nuqtaning bu nuqtadan biror (x_0, y_0) nuqtaga ixtiyoriy kichik qo'zg'alishni qaraymiz. U holda moddiy nuqta $(0,0)$ nuqtadan (x_0, y_0) nuqtaga $y^2 + x^4 = y_0^2 + x_0^4$ trayektoriya bo'yicha harakatlanadi. $|x_0| \le 1$ da $y_0^2 + x_0^4 = y_0^2 + x_0^2 \cdot x_0^2 \le y_0^2 + x_0^2$ tengsizlik o'rinli bo'lgani uchun, bu trayetoriya yopiq hamda, yetarlicha kichik x_0 va y_0 larda, radiusi r_0 $\left(r_0 \le \sqrt{x_0^2 + y_0^2}\right)$ bo'lgan doiradan tashqariga chiqmaydi, ya'ni $(0,0)$ nuqta turg'un bo'ladi.

3-*Misol.* Umumiy yechimi $x_1(t) = (C_1 - C_2 t)e^{-t}$, $x_2(t) = \frac{C_1\sqrt[3]{t}}{\ln\left(t^2+1\right)} + C_2$ bo'lgan sistemaning nol yechimini turg'unlikga tekshiring.

Yechish. $|x(t_0)| = \sqrt{x_1^2(t_0) + x_2^2(t_0)} = \sqrt{(C_1 - C_2 t_0)^2 e^{-2t_0} + \left(\frac{C_1\sqrt[3]{t_0}}{\ln\left(t_0^2+1\right)} + C_2\right)^2}$

tenglikdan $C_1^2 + C_2^2 \to 0$ (ya'ni $C_1 \to 0,\ C_2 \to 0$) da $|x(t_0)| \to 0$ ni olamiz.

Ikkinchi tomondan $\lim\limits_{t_0 \to +\infty} \frac{C_1\sqrt[3]{t_0}}{\ln\left(t_0^2+1\right)} = \lim\limits_{t_0 \to +\infty} \frac{\frac{1}{3}C_1\left(t_0^2+1\right)}{2t_0^{5/3}} = +\infty$ bo'gani uchun, $|C_1| \ne 0$ va $|C_2|$ larni har qancha kichik tanlamaylik, shunday $t_1 > 0$ topiladiki, avvaldan tanlangan $\varepsilon > 0$ uchun

$$\sqrt{(C_1 - C_2 t_t)^2 e^{-2t_0} + \left(\frac{C_1\sqrt[3]{t_0}}{\ln(t_1^2+1)} + C_2\right)^2} > \varepsilon$$

bo'ladi. Demak nol yechim noturg'un.

1.2. Birinchi yaqinlashish bo'yicha turg'unlikka tekshirish.

(1.2) sistema berilgan bo'lib, $x_i \equiv 0\ (i = 1,...,n)$ berilgan sistemaning muvozanat nuqtasi bo'lsin, ya'ni $f_i(t,0,0,...,0) = 0\ \ (i = 1,...,n)$. Faraz qilaylik $f_i(t, x_1,...,x_n)$ funksiyalar koordinata boshida yetarlicha differensiallanuvchu bo'lsin.

$f_i(t, x_1, ..., x_n)$ funksiyani x lar bo'yicha koortdinata boshi atrofida Teylor (Makloren) qatoriga yoyamiz:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} x_j + R_i(t, x_1, ..., x_n), \quad i = 1, 2, ..., n \qquad (1.6)$$

bu yerda a_{ij} – o'zgarmaslar, R_i esa - $x_1, ..., x_n$ o'zgaruvchilar bo'yicha ikkinchi tartibli kichik hadlar, ya'ni $\lim\limits_{|x| \to 0} \frac{R_i(t, x)}{|x|} = 0$ bo'lib, $|x| = \sqrt{|x_1|^2 + |x_2|^2 + ... + |x_n|^2}$.

Ushbu $\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} x_j$ sistemaga, (1.2) sistema uchun **1-yaqinlashishdagi tenglamalar sistemasi** deyiladi.

1-Teorema. (Lyapunov teoremasi). Agar (1.6) sistemadagi A=$\left(a_{ij}\right)$ matritsaning barcha xos sonlarining haqiqiy qismi manfiy bo'lsa, u holda (1.6) sistemaning nol yechimi asimptotik turg'un, agar hech bo'lmagan bitta xos soni haqiqiy qismi musbat bo'lsa, nol yechim noturg'un bo'ladi.

4-*Misol.* $\begin{cases} \dot{x} = tg(y - x) \\ \dot{y} = 2^y - 2\cos\left(\frac{\pi}{3} - x\right) \end{cases}$ sistemaning nol yechimlarini birinchi yaqinlashish yordamida turg'unlikka tekshiring.

Yechish. $tg(y - x)$ va $2^y - 2\cos\left(\frac{\pi}{3} - x\right)$ funksiyalarni Makloren qatoriga yoyib, ularning chiqli qismini ajratamiz:

$$tg(y - x) = y - x + o\left(y^2 + x^2\right)$$

$$2^y - 2\cos\left(\frac{\pi}{3} - x\right) = 1 + y\ln 2 + \frac{y^2 \ln^2 2}{2} + o\left(y^2\right) - 2\cos\frac{\pi}{3} \cdot \left(1 - \frac{y^2}{2} + o\left(y^2\right)\right) -$$

$$-2\sin\frac{\pi}{3} \cdot \left(x + o\left(x^2\right)\right) = -\sqrt{3}x + y\ln 2 + \frac{1}{2}\left(x^2 + y^2 \ln^2 2\right) + o\left(x^2 + y^2\right).$$

Demak $\lim\limits_{\sqrt{x^2+y^2} \to 0} \frac{\frac{1}{2}\left(x^2 + y^2 \ln^2 2\right)}{\sqrt{x^2 + y^2}} = 0$ bo'lganin uchun, berilgan sistemaning chiziqli qismi $\begin{cases} \dot{x} = y - x \\ \dot{y} = -\sqrt{3}x + y\ln 2 \end{cases}$ ko'rinishga ega bo'lib, bu sistemaga mos $\begin{vmatrix} -1 - \lambda & 1 \\ -\sqrt{3} & \ln 2 - \lambda \end{vmatrix} = 0$ yoki $\lambda^2 + (1 - \ln 2)\lambda + \sqrt{3} - \ln 2 = 0$

xarakteristik tenglamani yechib, $\lambda_{1/2} = \dfrac{\ln 2 - 1 \pm i\sqrt{4\sqrt{3} - (\ln 2 + 1)^2}}{2}$ xos sonlarni topamiz. Ma'lumki $\mathrm{Re}(\lambda_{1/2}) = \frac{1}{2}(\ln 2 - 1) < 0$, demak 1-teoremaga asosan berilgan sistemaning nol yechimlari asimptotik turg'un bo'ladi.

5-Misol. $\begin{cases} \dot{x} = e^x - e^{-3z} \\ \dot{y} = 4z - 3\sin(x+y) \\ \dot{z} = \ln(1+z-3x) \end{cases}$ sistemaning nol yechimlarini birinchi yaqinlashish yordamida turg'unlikka tekshiring.

Yechish. Berilgan sistemadagi tenglamalarning o'ng tomonidagi funksiyalarni Makloren qatoriga yoyamiz:

$$e^x - e^{-3z} = x + 3z + \frac{1}{2}x^2 - \frac{9}{2}z^2 + o\left(x^2 + z^2\right)$$

$$4z - 3\sin(x+y) = 4z - 3(x+y) + o\left(x^2 + y^2 + z^2\right)$$

$$\ln(1+z-3x) = z - 3x - \frac{1}{2}(z-3x)^2 + o\left(x^2 + z^2\right)$$

Demak berilgan funksiyalarning birinchi yaqinlashishdagi chiziqli qismlaridan iborat $\begin{cases} \dot{x} = x + 3z \\ \dot{y} = -3x - 3y + 4z \\ \dot{z} = -3x + z \end{cases}$ sistemaning nol yechimini turg'unlikka tekshiramiz, buning uchun $\begin{vmatrix} 1-\lambda & 0 & 3 \\ -3 & -3-\lambda & 4 \\ -3 & 0 & 1-\lambda \end{vmatrix} = 0$ tenglamani yechib, $\lambda_1 = -3,\ \lambda_{2/3} = 1 \pm 3i$ xos sonlarni topamiz. $\mathrm{Re}(\lambda_{2/3}) = 1 > 0$ bo'lgani uchun 1-teoremaga asosan berilgan sistemaning nol yechimlari noturg'un bo'ladi.

6-*Misol.* $\begin{cases} \dot{x} = \ln(e + ax) - e^y \\ \dot{y} = bx + tgy \end{cases}$ sistemaning nol yechimlari a va b ning qanday qiymatlarida asimptotik turg'un bo'ladi.

Yechish. Berilgan sistemadagi tenglamalarning o'ng tomonidagi funksiyalarni Makloren qatoriga yoyamiz:

$$\ln(e+ax) - e^y = 1 - 1 + \frac{a}{e}x - y + o\left(x^2 + y^2\right) = \frac{a}{e}x - y + o\left(x^2 + y^2\right)$$

$$bx + tgy = bx + y + o\left(x^2 + y^2\right)$$

va birinchi yaqinlashishdagi chiziqli qismlaridan iborat $\begin{cases}\dot{x}=\frac{a}{e}x-y\\ \dot{y}=bx+y\end{cases}$

sistemaning nol yechimini turg'unlikka tekshiramiz. Oxirgi sistemaga mos xarakteristik tenglama $\left(\frac{a}{e}-\lambda\right)(1-\lambda)+b=0$ yoki $\lambda^2-\left(\frac{a}{e}+1\right)\lambda+\frac{a}{e}+b=0$ bo'lib, uning yechimlari $\lambda_{1/2}=\frac{1}{2}\left(\frac{a}{e}+1\right)\pm\frac{1}{2}\sqrt{\left(\frac{a}{e}-1\right)^2-4b}$ bo'ladi. Bundan nol yechim asimptotik turg'un bo'lishi uchun $\mathrm{Re}(\lambda_1)<0$ va $\mathrm{Re}(\lambda_2)<0$, ya'ni

$$\begin{cases}\frac{1}{2}\left(\frac{a}{e}+1\right)<0\\ \frac{1}{4}\left(\frac{a}{e}-1\right)^2\le b\end{cases} \quad \text{va} \quad \begin{cases}\frac{1}{2}\left(\frac{a}{e}+1\right)+\frac{1}{2}\sqrt{\left(\frac{a}{e}-1\right)^2-4b<0}\\ \frac{1}{4}\left(\frac{a}{e}-1\right)^2\ge b\end{cases}$$

tengsizlillar sistemasi bajarilashi kerak. Birinchi sistemaning yechimi $\begin{cases}a<-e\\ b>0\end{cases}$ bo'ladi, ikkinchi sistemadan esa, faqat $a<-e$ da $\begin{cases}\left(\frac{a}{e}-1\right)^2-4b<\left(\frac{a}{e}+1\right)^2,\\ \frac{1}{4}\left(\frac{a}{e}-1\right)^2\ge b\end{cases}$

sistemani yechish mumkinligi aniqlash qiyin emas. Ikkila sistema uchun ham umumiy yechim $\begin{cases}a<-e\\ b>0\end{cases}$ bo'ladi, ya'ni $\begin{cases}a<-e\\ b>0\end{cases}$ da berilgan sistemaning nol yechimlari asimptotik turg'un bo'ladi.

7-*Misol.* $\begin{cases}\dot{x}=-\sin y\\ \dot{y}=2x+\sqrt{1-3x-\sin y}\end{cases}$ sistemaning barcha muvozanat holatlarini toping va ularni turg'unlikka tekshiring.

Yechish. Berilgan sistemadagi tenglamalarning o'ng tomonidagi funksiyalarni nolga tenglashtirib, $\begin{cases}-\sin y=0\\ 2x+\sqrt{1-3x-\sin y}=0\end{cases}$, bundan $\begin{cases}y=\pi k,\ k\in Z\\ x=-1\end{cases}$ muvozanat holatlarini topamiz. Demak $(-1,\pi k)$, $k\in Z$ nuqtalar to'plamiga ega bo'ldik. Endi berilgan sistemada

$x=-1+x_1,\ y=\pi k+y_1$ almashtirish bajarib, muvozanat holatlari $(0,0)$ bo'lgan

$$\begin{cases}\dot{x}_1=(-1)^{k+1}\sin y_1\\ \dot{y}_1=2x_1-2+\sqrt{4-3x_1-(-1)^k\sin y_1}\end{cases}$$

sistemani hosil qilamiz. Hosil bo'lgan sistemadagi tenglamalarning o'ng tomonidagi funksiyalarni Makloren qatoriga yoyib, birichi yaqinlashishdan sistemaning chiziqli qismini ajratamiz:

$$(-1)^{k+1}\sin y_1=(-1)^{k+1}y_1+o\left(x_1^2+y_1^2\right),$$

$$2x_1-2+\sqrt{4-3x_1-(-1)^k\sin y_1}=\frac{5}{4}x_1+\frac{(-1)^{k+1}}{4}y_1+o\left(x_1^2+y_1^2\right).$$

Demak $\begin{cases}\dot{x}_1=\left(-1\right)^{k+1}y_1\\ \dot{y}_1=\dfrac{5}{4}x_1+\dfrac{\left(-1\right)^{k+1}}{4}y_1\end{cases}$ sistemaga mos $\lambda\left(\lambda-(-1)^{k+1}\dfrac{1}{4}\right)-(-1)^{k+1}\dfrac{5}{4}=0$

xarakteristik tenglamani yechib, $\lambda_{1/2}=\left(-1\right)^{k+1}\cdot\dfrac{1}{8}\pm\dfrac{1}{8}\sqrt{1-(-1)^{k+1}\cdot 20}$ xos sonlarni topamiz. Agar $k=2n,\ n\in Z$ bo'lsa, $\mathrm{Re}\left(\lambda_{1/2}\right)<0$, ya'ni $\left(-1,\,2n\pi\right)$, $n\in Z$ nuqta turg'un, agar $k=2n+1,\ n\in Z$ bo'lsa, $\mathrm{Re}(\lambda_1)>0$ bo'ladi, ya'ni $\left(-1,\,(2n+1)\pi\right)$, $n\in Z$ nuqta noturg'un.

1.3. Lyapunov funksiyasi yordamida turg'unlikka tekshirish.

$S_h=\left\{x\in R^n:|x|<h,\ h>0\right\}$ sharda aniqlangan va uzluksiz differensiallanuvchi $v(x)=v(x_1,...,x_n)$-skalyar funksiya berilgan bo'lib, $v(0)=0$ bo'lsin.

3-Ta'rif. Agar barcha $x\in S_h$ larda ($x=0$ dan tashqari) $v(x)>0$ bo'lsa, $v(x)$ funksiya S_h sharda musbat aniqlangan, $v(x)<0$ bo'lsa $v(x)$ funksiya S_h sharda manfiy aniqlangan deyiladi. Ikkila holda ham $v(x)$ funksiyaga ishorasi aniqlangan deyiladi.

4-Ta'rif. Agar barcha $x\in S_h$ larda $v(x)\geq 0$ yoki $v(x)\leq 0$ bo'lsa, $v(x)$ funksiya S_h sharda ishorasi o'zgarmas deyiladi. Birinchi holda $v(x)$ funksiya S_h sharda ishorasi musbat o'zgarmas, ikkinchi holda ishorasi manfiy o'zgarmas deyiladi.

4-Ta'rif. Agar $v(x)$ funksiya S_h sharda musbat va manfiy qiymatlar qabul qilsa, u holda $v(x)$ funksiyaga S_h sharda ishorasi o'zgaruvchan deyiladi.

(1.2) sistemada $f(t,x)$ funksiya S_h sharda aniqlangan va uzluksiz bo'lib, $f(t,0)=0$ ya'ni $x=0$ funksiya (1.2) sistamaning yechimi bo'lsin. (1.2) sistemaning biror $x(t)$ yechimi bo'yicha aniqlangan $v=v(x(t))$ funksiya t o'zgaruvchi bo'yicha uzluksiz differensiallanuvchi bo'lib, uning t bo'yicha hosilasi ((1.2) ga asosan)

$$\frac{dv}{dt}=\frac{dv(x(t))}{dt}=\sum_{i=1}^{n}\frac{\partial v}{\partial x_i}\frac{dx_i}{dt}=\sum_{i=1}^{n}\frac{\partial v}{\partial x_i}f_i(x) \qquad (1.7)$$

bo'lsin.

1-Teorema. **(***Lyapunovning turg'unlik haqidagi teoremasi***)**. Agar (1.2) sistema uchun S_h sharda ishorasi aniqlangan $v(x)$ funksiya mavjud va (1.7) formula bilan aniqlangan $\frac{dv}{dt}$ hosila ishorasi o'zgarmas funksiya bo'lib, $v(x)$ funksiya ishorasiga qarama-qarshi yoki aynan nolga teng bo'lsa, (1.2) sistemaning nol yechimi Lyapunov ma'nosida turg'un bo'ladi.

2-Teorema. **(***Lyapunovning asimptotik turg'unlik haqidagi teoremasi***)**. Agar (1.2) sistema uchun S_h sharda ishorasi aniqlangan $v(x)$ funksiya mavjud va (1.7) formula bilan aniqlangan $\frac{dv}{dt}$ hosila ham ishorasi aniqlangan funksiya bo'lib, $v(x)$ funksiya ishorasiga qarama-qarshi bo'lsa, (1.2) sistemaning nol yechimi Lyapunov ma'nosida asimptotik turg'un bo'ladi.

3-Teorema. **(***Lyapunovning noturg'unlik haqidagi teoremasi***)**. Agar (1.2) sistema uchun S_h sharda $v(x)$ funksiya mavjud va (1.7) formula bilan aniqlangan $\frac{dv}{dt}$ hosila, ishorasi aniqlangan funksiya, $v(x)$ funksiyaning o'zi esa $x=0$ nuqtaning ixtiyoriy atrofida ishorasi o'zgarmas bo'lib, $\frac{dv}{dt}$ ishorasiga qarama-qarshi bo'lsa, (1.2) sistemaning nol yechimi noturg'un bo'ladi.

4-Teorema. **(***Chetayevning noturg'unlik haqidagi teoremasi***)**. Agar (1.2) sistema uchun $x_1,...,x_n$ o'zgaruvchilar fazosining biror V sohasida uzluksiz differensiallanuvchi $v(x)=v(x_1,...,x_n)$ funksiya mavjud bo'lib,

1) $x=0$ nuqta V soha chegarasiga tegishli bo'lsin,
2) $|x|<\varepsilon$ da, V soha chegarasiga $v(x)=0$ bo'lsin,

3) V soha ichida $v(x)>0,\ \frac{dv(x)}{dt}>0$ bo'lsa, u holda (1.2) sistemaning nol yechimi noturg'un bo'ladi.

8-*Misol.* $\begin{cases}\dot{x}=x-y-xy^2\\ \dot{y}=2x-y-y^3\end{cases}$ sistemaning nol yechimlarini Lyapunov funksiyasini qurib, turg'unlikka tekshiring.

Yechish. Quyidagi funksiyani tekshiramiz:

$v(x,y)=x^2-xy+\frac{1}{2}y^2=\left(x-\frac{1}{2}y\right)^2+\frac{1}{4}y^2$. Ma'lumki bu funksiya $x^2+y^2\neq 0$ da musbat, ya'ni $v(x,y)>0$. Demak $v(x,y)$ musbat aniqlangan funksiya. Endi (1.7) formula asosida $\frac{dv(x,y)}{dt}$ hosilani hisoblaymiz.

$$\frac{dv(x,y)}{dt}=(2x-y)\frac{dx}{dt}+(y-x)\frac{dy}{dt}=(2x-y)\left(x-y-xy^2\right)+(y-x)\left(2x-y-y^3\right)=$$
$$=-2x^2y^2+2xy^3-y^4=-y^2\left(x^2-2xy+y^2+x^2\right)=-y^2\left((x-y)^2+x^2\right)\leq 0.$$

Demak $\frac{dv(x,y)}{dt}$ hosila ishorasi $v(x,y)$ musbat aniqlangan funksiya ishorasiga qarama-qarshi, ya'ni 1-teoremaga asosan berilgan sistemaning nol yechimlari turg'un.

9-*Misol.* $\begin{cases}\dot{x}=x^3-y\\ \dot{y}=x+y^3\end{cases}$ sistemaning nol yechimlarini Lyapunov funksiyasini qurib, turg'unlikka tekshiring.

Yechish. Quyidagi $v(x,y)$ funksiyani tekshiramiz: $v(x,y)=x^2+y^2>0$ $x^2+y^2\neq 0$ da. $\frac{dv(x,y)}{dt}$ hosilaning ishorasini aniqlaymiz.

$$\frac{dv(x,y)}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt}=2x\left(x^3-y\right)+2y\left(x+y^3\right)=2x^4+2y^4.$$

Demak $x^2+y^2\neq 0$ da $\frac{dv(x,y)}{dt}>0$, ya'ni $v(x,y)$ funksiya va uning $\frac{dv(x,y)}{dt}$ hosilasi bir xil ishorali bo'lgani uchun 4-teoremaga asosan nol yechim noturg'un.

Mustaqil yechish uchun mashqlar.

Quyidagi tenglama va tenglamalar sistemalarning yechimlarini ta'rif bo'ticha Lyapunov ma'nosida turg'unlikga tekshiring (559-569):

559. $\dot{x} = x + t, \quad x(0) = 1.$

560. $\dot{x} = 2t(x+1), \quad x(0) = 0.$

561. $\dot{x} = -x + t^2, \quad x(1) = 1.$

562. $2t\dot{x} = x - x^3, \quad x(1) = 0.$

563. $\begin{cases} \dot{x} = -x \\ \dot{y} = -2y \end{cases}$

$x(0) = y(0) = 0.$

564. $\begin{cases} \dot{x} = -x \\ \dot{y} = y \end{cases}$

$x(0) = y(0) = 0.$

565. $\begin{cases} \dot{x} = y \\ \dot{y} = -\sin x \end{cases}$

$x(0) = y(0) = 0.$

566. $\begin{cases} \dot{x} = x - 13y \\ \dot{y} = \frac{1}{4}x - 2y \end{cases}$

$x(0) = y(0) = 0.$

567. $\begin{cases} \dot{x} = -x - 3y \\ \dot{y} = x - y \end{cases}$

$x(0) = y(0) = 0.$

568. $\begin{cases} \dot{x} = -y\cos x \\ \dot{y} = -\sin x \end{cases}$

$x(0) = y(0) = 0.$

569. $\begin{cases} \dot{x} = y \\ \dot{y} = x^3\left(1 + y^2\right) \end{cases}$

$x(0) = y(0) = 0.$

Quyidagi sistemalarning nol yechimlarini birinchi yaqinlashish yordamida turg'unlikga tekshiring (569-580):

569. $\begin{cases} \dot{x} = 2xy - x + y, \\ \dot{y} = 5x^4 + y^3 + 2x - 3y. \end{cases}$

570. $\begin{cases} \dot{x} = x^2 + y^2 - 2x, \\ \dot{y} = 3x^2 - x + 3y. \end{cases}$

571. $\begin{cases} \dot{x} = e^{x+2y} - \cos 3x, \\ \dot{y} = \sqrt{4+8x} - 2e^y. \end{cases}$

572. $\begin{cases} \dot{x} = \ln(4y + e^{-3x}), \\ \dot{y} = 2y - 1 + \sqrt[3]{1-6x.} \end{cases}$

573. $\begin{cases} \dot{x} = \ln(3e^y - 2\cos x), \\ \dot{y} = 2e^x - \sqrt[3]{8+12y.} \end{cases}$

574. $\begin{cases} \dot{x} = x + 2y - \sin y^2, \\ \dot{y} = -x - 3y + x\left(e^{x^2/2} - 1\right). \end{cases}$

575. $\begin{cases} \dot{x} = -x + 3y + x^2 \sin y, \\ \dot{y} = -x - 4y + 1 - \cos y^2. \end{cases}$

576. $\begin{cases} \dot{x} = -2x + 8\sin^2 y, \\ \dot{y} = x - 3y + 4x^3. \end{cases}$

577. $\begin{cases} \dot{x} = 7x + 2\sin y - y^4, \\ \dot{y} = e^x - 3y - 1 + \frac{5}{2}x^2. \end{cases}$

578. $\begin{cases} \dot{x} = \frac{5}{2}xe^x - 3y + \sin x^2, \\ \dot{y} = 2x + ye^{-y^2/2} - y^4\cos x. \end{cases}$

579. $\begin{cases} \dot{x} = -\sin(x - z), \\ \dot{y} = \sin^2 x - y - \sin z, \\ \dot{z} = tg(y - z). \end{cases}$

580. $\begin{cases} \dot{x} = tg(z-y) - 2x, \\ \dot{y} = \sqrt{9+12x} - 3e^{y}, \\ \dot{z} = -3y \end{cases}$

Quyidagi sistemalarning nol yechimlari a va b ning qanday qiymatlarida asimptotik turg'un bo'ladi. (581-585):

581. $\begin{cases} \dot{x} = ax - 2y + x^2, \\ \dot{y} = x + y + xy. \end{cases}$

582. $\begin{cases} \dot{x} = ax + y + x^2, \\ \dot{y} = x + ay + y^2. \end{cases}$

583. $\begin{cases} \dot{x} = x + ay + y^2, \\ \dot{y} = bx - 3y - x^2. \end{cases}$

584. $\begin{cases} \dot{x} = y + \sin x, \\ \dot{y} = ax + by. \end{cases}$

585. $\begin{cases} \dot{x} = 2e^{-x} - \sqrt{4+ay}, \\ \dot{y} = \ln(1+9x+ay). \end{cases}$

586. $\begin{cases} \dot{x} = y^2 - 2ty - 2y - x, \\ \dot{y} = 2x + 2t^2 + e^{2t-2y}. \end{cases}$ sistemaning $x = -t^2, \quad y = t$ yechimlari turg'unmi?

587. $\begin{cases} \dot{x} = \ln\left(x + 2\sin^2\dfrac{t}{2}\right) - \dfrac{y}{2}, \\ \dot{y} = (4 - x^2)\cos t - 2x\sin^2 t - \cos^3 t. \end{cases}$ sistemaning $x = \cos t, \quad y = 2\sin t$ yechimlari turg'unmi?

Quyidagi sistemalarning barcha muvozanat holatlarini (maxsus nuqtalarini) topib, ularni turg'unlikga tekshiring (588-59).

588. $\begin{cases} \dot{x} = y - x^2 - x, \\ \dot{y} = 3x - x^2 - y. \end{cases}$

589. $\begin{cases} \dot{x} = (x-1)(y-1), \\ \dot{y} = xy - 2. \end{cases}$

590. $\begin{cases} \dot{x} = y, \\ \dot{y} = \sin(x+y). \end{cases}$

591. $\begin{cases} \dot{x} = \ln(y^2 - x), \\ \dot{y} = x - y - 1. \end{cases}$

592. $\begin{cases} \dot{x} = 3 - \sqrt{4 + x^2 + y}, \\ \dot{y} = \ln(x^2 - 3). \end{cases}$

593. $\begin{cases} \dot{x} = e^{y} - e^{x}, \\ \dot{y} = \sqrt{3x + y^2} - 2. \end{cases}$

594. $\begin{cases} \dot{x} = \ln(1 + y + \sin x), \\ \dot{y} = 2 + \sqrt[3]{3\sin x - 8}. \end{cases}$

2-§. Maxsus nuqtalar.

1-Ta'rif. Ushbu

$$\begin{cases} \dfrac{dx}{dt} = P(x,y) \\ \dfrac{dy}{dt} = Q(x,y) \end{cases} \qquad (2.1)$$

sistema yoki

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)} \qquad (2.2)$$

tenglamaning **maxsus (muvozanat) nuqtasi** deb, $\begin{cases} P(x,y)=0 \\ Q(x,y)=0 \end{cases}$ sistemaning yechimiga aytiladi. Bu yerda $P(x,y)$ va $Q(x,y)$ berilgan uzluksiz differensiallanuvchi funksiyalar.

Agar (2.1) sistemada $P(x,y) = a_{11}x + a_{12}y$ va $Q(x,y) = a_{21}x + a_{22}y$ bo'lsa, u holda bu sistema uchun $x=0,\ y=0$ nuqta maxsus nuqta bo'ladi, bu yerda $a_{ij}\ (i,j=1,2)$ -berilgan o'zgarmas sonlar bo'lib,

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0 .$$

Demak

$$\begin{cases} \dfrac{dx}{dt} = a_{11}x + a_{12}y \\ \dfrac{dy}{dt} = a_{21}x + a_{22}y \end{cases} \qquad (2.3)$$

sistema yoki

$$\frac{dy}{dx} = \frac{a_{11}x + a_{12}y}{a_{21}x + a_{22}y} \quad \left(\frac{dx}{dy} = \frac{a_{21}x + a_{22}y}{a_{11}x + a_{12}y} \right) \qquad (2.4)$$

differensial tenglamaning maxsus nuqtalarini o'rganish uchun, (2.3) sistemaning xarakteristik sonlarini, ya'ni

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0 \qquad (2.5)$$

tenglamaning $\lambda_1,\ \lambda_2$ ildizlarini topamiz, va shu ildizlarning qabul qiladigan qiymatlariga qarab maxsus nuqta turlarga ajraladi.

Quyidagi hollar bo'lishi mumkin.

I. (2.5) xarakteristik tenglamaning $\lambda_1,\ \lambda_2$ ildizlari haqiqiy va har xil:

a) $\lambda_1 < 0,\ \ \lambda_2 < 0$ bo'lsa, maxsus nuqta asimptotik turg'un (**turg'un tugun nuqta,** 1-rasm).

b) $\lambda_1 > 0,\ \ \lambda_2 > 0$ bo'lsa, maxsus nuqta noturg'un (**noturg'un tugun nuqta, 2**-rasm).

c) $\lambda_1 > 0, \quad \lambda_2 < 0$ bo’lsa, maxsus nuqta noturg’un (**noturg’un egar nuqta,** 3-rasm).

II. (2.5) xarakteristik tenglamaning ildizlari $\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta$ kompleks:

a) $\alpha < 0, \quad \beta \neq 0$ bo’lsa, maxsus nuqta asimptotik turg’un (**turg’un fokus nuqta,** 4-rasm**).**

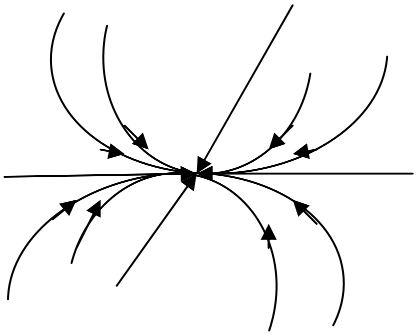
b) $\alpha > 0, \quad \beta \neq 0$ bo’lsa, maxsus nuqta noturg’un (**noturg’un fokus nuqta, 5**-rasm).

c) $\alpha = 0, \quad \beta \neq 0$ bo’lsa, maxsus nuqta turg’un (**turg’un markaz nuqta,** 6-rasm**).**

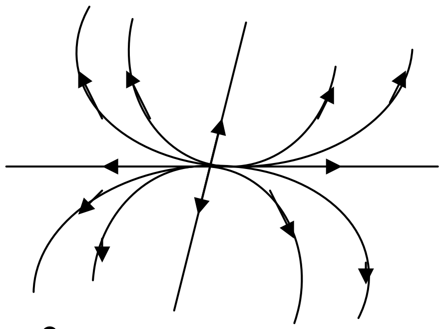
III. (2.5) xarakteristik tenglamaning ildizlari $\lambda_1 = \lambda_2 \neq 0$ karrali:

a) $\lambda_1 = \lambda_2 < 0$ bo’lsa, maxsus nuqta asimptotik turg’un (**turg’un tugun nuqta, 7**-, 8-rasm**).**

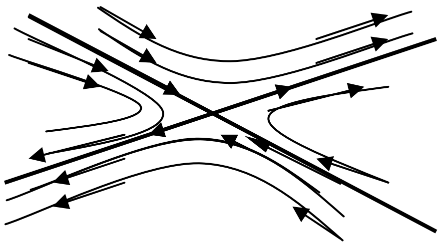
b) $\lambda_1 = \lambda_2 > 0$ bo’lsa, maxsus nuqta noturg’un (**noturg’un tugun nuqta,** 9-, 10-rasm **).**



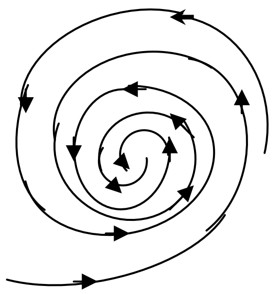
1-*rasm.*



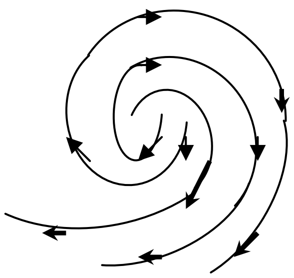
2-*rasm.*



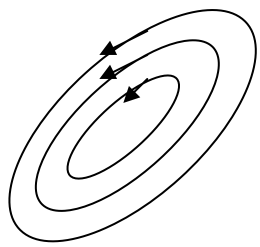
3-*rasm.*



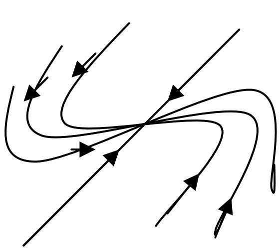
4-*rasm.*



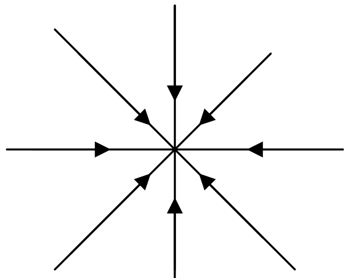
5-*rasm.*



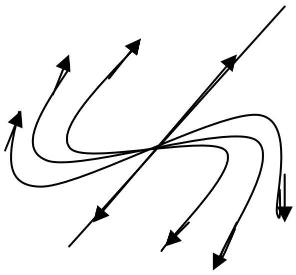
6-*rasm.*



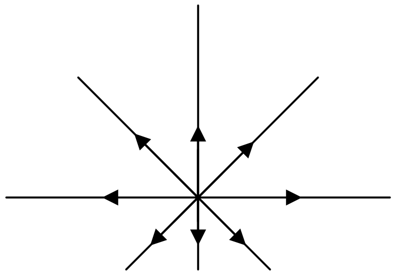
7-*rasm.*



8-*rasm.*



9-*rasm.*



10-rasm.

1-Misol.
Yechish.

2-Misol.
Yechish.

3-Misol.
Yechish.

4-Misol.
Yechish.

5-Misol.
Yechish.

6-Misol.
Yechish.

Mustaqil yechish uchun mashqlar.

Quyidagi differensial tenglama va sistemalarning maxsus nuqtasini va uning xarakterini aniqlang (595-612):

595. $y' = \frac{2x+y}{3x+4y}$.

596. $y' = \frac{-2x+y}{3x+y}$.

597. $y' = \frac{y-2x}{y}$.

598. $y' = \frac{-x+y}{3y-x}$.

599. $y' = \frac{x-2y}{3x-4y}$.

600. $y' = \frac{2x+5y}{3x+7y}$.

601. $y' = \frac{y}{x}$.

602. $y' = \frac{2x+y}{5x-y}$.

603. $\begin{cases} \dot{x} = 2y - x \\ \dot{y} = x + y \end{cases}$

604. $\begin{cases} \dot{x} = x + 3y \\ \dot{y} = -6x - 5y \end{cases}$

605. $\begin{cases} \dot{x} = -2x - y \\ \dot{y} = 3x - y \end{cases}$

606. $\begin{cases} \dot{x} = -2x - 5y \\ \dot{y} = 2x + 2y \end{cases}$

607. $\begin{cases} \dot{x} = \frac{5}{7}y - 2x \\ \dot{y} = 7x - 3y \end{cases}$

608. $\begin{cases} \dot{x} = 3x - y \\ \dot{y} = x + y \end{cases}$

609. $\begin{cases} \dot{x} = -x - y \\ \dot{y} = x - 3y \end{cases}$

610. $\begin{cases} \dot{x} = 3x - 2y \\ \dot{y} = -6x + 4y \end{cases}$

611. $\begin{cases} \dot{x} = 3x \\ \dot{y} = 2x + y \end{cases}$

612. $\begin{cases} \dot{x} = 3x - 2y \\ \dot{y} = 4x - y \end{cases}$

Quyidagi tenglamalar sistemasining maxsus nuqtasini va uning xarakterini birinchi yaqinlashishdan foydalanib, aniqlang (613-625):

613. $\begin{cases} \dot{x} = 3x + 6 \\ \dot{y} = 2y - x \end{cases}$

614. $\begin{cases} \dot{x} = x - 2y - 5 \\ \dot{y} = 2x + y \end{cases}$

615. $\begin{cases} \dot{x} = 2xy - 4y - 8 \\ \dot{y} = -x^2 + 4y^2 \end{cases}$

616. $\begin{cases} \dot{x} = x - y \\ \dot{y} = x^2 + y^2 - 2 \end{cases}$

617. $\begin{cases} \dot{x} = -2y(x - y) \\ \dot{y} = 2 + x - y^2 \end{cases}$

618. $\begin{cases} \dot{x} = (x-2)(2x-y) \\ \dot{y} = -2 + xy \end{cases}$

619. $\begin{cases} \dot{x} = (y-1)(3x+y-5) \\ \dot{y} = -5 + x^2 + y^2 \end{cases}$

620. $\begin{cases} \dot{x} = x + y + 1 \\ \dot{y} = \sqrt{1+2x^2} + y \end{cases}$

621. $\begin{cases} \dot{x} = \ln\left(2 - y^2\right) \\ \dot{y} = e^x - e^y \end{cases}$

622. $\begin{cases} \dot{x} = \sqrt{2 + x^2 - y} - 2 \\ \dot{y} = arctg\left(x^2 + xy\right) \end{cases}$

623. $\begin{cases} \dot{x} = \ln\frac{y^2 - y + 4}{3} \\ \dot{y} = x^2 - y^2 \end{cases}$

624. $\begin{cases} \dot{x} = \ln\left(y^2 - y + 1\right) \\ \dot{y} = 3 - \sqrt{x^2 + 8y} \end{cases}$ 625. $\begin{cases} \dot{x} = x + y + 1 \\ \dot{y} = \sqrt[4]{1 + 20x^2} + y \end{cases}$

3-§. *n*-tartibli o'zgarmas koeffisiyentli differensial tenglamalar uchun turg'unlik nazariyasi.

Ushbu

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0 \ \left(a_i = const \ \ (i = 0,1,...,n),\ a_n > 0\right) \quad (3.1)$$

koeffisiyentlari haqiqiy sonlardan iborat bo'lgan chiziqli differensial tenglama berilgan bo'lsin.

(3.1) tenglamaning $y \equiv 0$ nol yechimi asimtotik turg'un bo'ladi, agar

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + ... + a_1 \lambda + a_0 = 0 \quad (3.2)$$

xarakteristik tenglamaning barcha ildizlari manfiy haqiqiy qismga ega bo'lsa.

Raus-Gurvits kriteriysi. (3.2) xarakteristik tenglamaning barcha ildizlari manfiy haqiqiy qismga ega bo'lishi uchun, ushbu

$$\begin{pmatrix} a_{n-1} & a_n & 0 & 0 & 0 & 0 \ ... \ 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n & 0 & 0 \ ... \ 0 \\ a_{n-5} & a_{n-4} & a_{n-3} & a_{n-2} & a_{n-1} & a_n \ ... \ 0 \\ \multicolumn{6}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ 0 & 0 & 0 & 0 & 0 & 0 \ ... \ a_0 \end{pmatrix}$$

Gurvits matritsasining bosh diagonal minorlari musbat bo'lishi zarur va yetarli.

Gurvits matritsasini tuzish uchun uning bosh diagonaliga a_{n-1} dan a_0 ga koeffesiyentlar qo'yib chiqiladi, so'ngra har bir qatordagi element indeksi o'zidan oldingi element indeksidan 1 ga kam qilib yozib chiqiladi, bu yerda agar a_i elementning i indeksi n dan katta yoki $i < 0$ bo'lsa, bu element o'rniga nol soni yozib boriladi.

Gurvits matritsasining bosh diagonal minorlari quyidagi ko'rinishga ega bo'ladi.

$$\Delta_1 = a_{n-1}, \quad \Delta_2 = \begin{vmatrix} a_{n-1} & a_n \\ a_{n-3} & a_{n-2} \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_{n-1} & a_n & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} \\ a_{n-5} & a_{n-4} & a_{n-3} \end{vmatrix}, \ \ldots,$$

$$\Delta_n = \begin{vmatrix} a_{n-1} & a_n & 0 & 0 & 0 & 0 \ldots 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n & 0 & 0 \ldots 0 \\ a_{n-5} & a_{n-4} & a_{n-3} & a_{n-2} & a_{n-1} & a_n \ldots 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 \ldots a_0 \end{vmatrix}.$$

Demak (3.1) tenglamaning $y \equiv 0$ nol yechimi asimptotik turg'un bo'lishi uchun, $\Delta_1 > 0, \Delta_2 > 0, \ldots, \Delta_n > 0$ bo'lishi zarur va yetarli. $\Delta_n = a_0 \Delta_{n-1}$ tenglikni e'tiborga olsak, $\Delta_{n-1} > 0$ bo'lganda $a_0 > 0$ dan $\Delta_n > 0$ kelib chiqadi.

1-Misol. a ning qanday qiymatlarida $y''' + 2y'' + ay' + 3y = 0$ tenglamaning nol yechimi turg'un bo'ladi.

Yechish. Raus-Gurvits kriteriysiga ko'ra berilgan tenglamaga mos Gurvits matritsasining bosh dioganal minorlarini musbat deb ishlaymiz.

$\Delta_1 = 2 > 0, \quad \Delta_2 = \begin{vmatrix} 2 & 1 \\ 3 & a \end{vmatrix} > 0 \;\Rightarrow\; 2a - 3 > 0$, ya'ni $a > \frac{3}{2}$ da berilgan tenglama nol yechimi turg'un bo'ladi, chunki $\Delta_3 = 3 \cdot \Delta_2 > 0$.

2-*Misol.* $y^{V} + 2y^{IY} + 4y''' + 6y'' + 5y' + 4y = 0$ tenglamaning nol yechimini turg'unlikga tekshiring.

Yechish. Gurvits matritsasining bosh dioganal minorlarini musbat deb ishlaymiz.

$$\Delta_1 = 2 > 0,\ \Delta_2 = \begin{vmatrix} 2 & 1 \\ 6 & 4 \end{vmatrix} = 2 > 0,\ \Delta_3 = \begin{vmatrix} 2 & 1 & 0 \\ 6 & 4 & 2 \\ 4 & 5 & 6 \end{vmatrix} = 48 + 8 - 20 - 36 = 0,\ \Delta_4 = 3 \cdot \Delta_3 = 0$$

Demak berigan tenglama nol yechimi noturg'un.

Mustaqil yechish uchun mashqlar.

Quyidagi differensial tenglamalarning nol yechimlarini yurg'unlikka tekshiring (626-640):

626. $y''' - 3y' + 2y = 0$

627. $y''' + 5y'' + 9y' + 5y = 0$

628. $y''' - 3y'' + 12y' - 10y = 0$

629. $y''' + y'' + y' + 2y = 0$

630. $y''' + 2y'' + 2y' + 3y = 0$

631. $y^{IY} + 4y''' + 7y'' + 6y' + 2y = 0$

632. $y^{IY} - 2y''' + y'' + 2y' - 2y = 0$

633. $y^{IY} + 7y''' + 17y'' + 17y' + 6y = 0$

634. $y^{IY} + 2y''' + 4y'' + 3y' + 2y = 0$

635. $y^{IY} + 2y''' + 3y'' + 7y' + 2y = 0$

636. $y^{IY} + 11y''' + 41y'' + 61y' + 30y = 0$

637. $y^{V} + 2y^{IY} + 5y''' + 6y'' + 5y' + 2y = 0$

638. $y^{V} + 3y^{IY} - 5y''' - 15y'' + 4y' + 12y = 0$

639. $y^{V} + 3y^{IY} + 10y''' + 22y'' + 23y' + 12y = 0$

640. $y^{V} + 7y^{IY} + 33y''' + 88y'' + 122y' + 60y = 0$

Quyidagi differensial tenglamalarning nol yechimlari a va b ning qanday qiymatlarida turg'un bo'ladi (641-652):

641. $y''' + ay'' + by' + y = 0$

642. $y''' + 3y'' + ay' + by = 0$

643. $y''' + ay'' + by' + 2y = 0$

644. $y^{IV} + ay''' + 2y'' + y' + 3y = 0$

645. $y^{IV} + 2y''' + ay'' + y' + y = 0$

646. $y^{IV} + 2y''' + 3y'' + 2y' + ay = 0$

647. $y^{IV} + 3y''' + ay'' + 2y' + by = 0$

648. $ay^{IV} + y''' + y'' + y' + by = 0$

649. $y^{IV} + y''' + ay'' + y' + by = 0$

650. $y^{IV} + ay''' + 4y'' + 2y' + by = 0$

651. $y^{IV} + 2y''' + ay'' + by' + y = 0$

652. $y^{IV}+2y'''+4y''+ay'+by=0$

V. BOB. Birinchi tartibli xususiy hosilali differensial tenglamalar.

1-§. Birinchi tartibli chiziqli bir jinsli xususiy hosilali differensial tenglama.

Xususiy hosilali differensial tenglamalar tushunchasini ushbu qo'llanmaning kirish qismida keltirib o'tgan edik. Ushbu bobda biz, birinchi tartibli chiziqli xususiy hosilali differensial tenglamalar va ularning yechimlari tushunchasini o'rganamiz.

1-Ta'rif. Ushbu

$$a_1(x_1,...,x_n,u)\frac{\partial u}{\partial x_1}+a_2(x_1,...,x_n,u)\frac{\partial u}{\partial x_2}+...+a_n(x_1,...,x_n,u)\frac{\partial u}{\partial x_n}=f(x_1,...,x_n,u)\quad(1.1)$$

ko'rinishdagi tenglamaga **birinchi tartibli kvazichiziqli xususiy hosilali differensial tenglama** deyiladi, bu yerda $u(x_1,...,x_n)$- noma'lum funksiya, $a_i(x_1,...,x_n,u)$ (i=1,2,..n), $f(x_1,...,x_n,u)$-biror $D^{n+1}\subset R^{n+1}$ sohada aniqlangan, ma'lum funsiyalar.

Agar $a_i(x_1,...,x_n,u)$ (i=1,2,..n) koeffisiyentlar $u(x_1,...,x_n)$- noma'lum funksiyaga bog'liq emas, hamda $f(x_1,...,x_n,u)\equiv 0$ bo'lsa, (1.1) tenglamaga **birinchi tartibli chiziqli bir jinsli xususiy hosilali differensial tenglama** deyiladi, ya'ni

$$a_1(x_1,...,x_n)\frac{\partial u}{\partial x_1}+a_2(x_1,...,x_n)\frac{\partial u}{\partial x_2}+...+a_n(x_1,...,x_n)\frac{\partial u}{\partial x_n}=0\qquad(1.2)$$

ko'rinishdagi tenglamaga birinchi tartibli chiziqli bir jinsli xususiy hosilali differensial tenglama deyiladi.

2-Ta'rif. Ushbu

$$\frac{dx_1}{a_1(x_1,...,x_n)}=\frac{dx_2}{a_2(x_1,...,x_n)}=...=\frac{dx_n}{a_n(x_1,...,x_n)}\qquad(1.3)$$

simmetrik korinishdagi oddiy differensial tenglamalar sistemasiga (1.2) birinchi tartibli chiziqli bir jinsli xususiy hosilali differensial tenglama uchun **xarakteristik tenglamalar sistemasi** deyiladi.

Teorema. $u(x_1,...,x_n)$ funksiya (1.2) birinchi tartibli chiziqli bir jinsli xususiy hosilali differensial tenglamaning yechimi bo'lishi uchun, bu funksiyaning (1.3) xarakteristik tenglamalar sistemasining birinchi integrali bo'lishi zarur va yetarli.

(1.3) sistema $n-1$ ta $\varphi_1(x_1,...,x_n), \varphi_2(x_1,...,x_n),...,\varphi_n(x_1,...,x_n)$ chiziqli erkli integrallarga ega, ularning har biri (1.2) tenglamaning yechimi bo'ladi. Bubdan tashqari, bu funksiyalarning ixtiyoriy $\Phi = F(\varphi_1, \varphi_2,...,\varphi_n)$ uzluksiz differensial-lanuvchi funksiyasi (1.3) sistemaning integrali shu bilan birga (1.2) tenglamaning yechimi ham bo'ladi. Shunday qilib, (1.2) tenglamaning umumiy yechimi, ixtiyoriy funksiya F ga bog'liq bo'lgan

$$u = F(\varphi_1, \varphi_2,...,\varphi_n) \tag{1.4}$$

yechimlar oilasidan iborat.

Koshi masalasi. (1.2) tenglamaning

$$u(x)\big|_{x\in\gamma} = \psi(x), \quad x = (x_1,...,x_n) \tag{1.5}$$

shartni qanoatlantiruvchi yechimini toping, bu yerda γ- $D^n \subset R^n$ sohadagi biror silliq gipersirt, $\psi(x)$- shu gipersirtda aniqlangan ma'lum funksiya. γ- gipersirtga boshlang'ich gipersirt deyiladi. γ- boshlang'ich gipersirtdagi x nuqta noxarakteristik nuqta deyiladi, agar bu nuqta orqali o'tuvchi xarakteristika boshlang'ich gipersirtga urunmasa.

1-*Misol.* $u = \frac{x^2}{y^2} + \frac{y^2}{z^2}$ funksiyaning $x>0,\ y>0,\ z>0$ sohada $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$ tenglamaning yechimi ekanligini ko'rsating.

Yechish. $u(x,y,z)$ funksiyaning birinchi tartibli xususiy hosilalarini hisoblaymiz.

$$\frac{\partial u}{\partial x} = \left(\frac{x^2}{y^2} + \frac{y^2}{z^2}\right)'_x = \frac{2x}{y^2}, \quad \frac{\partial u}{\partial y} = -\frac{2x^2}{y^3} + \frac{2y}{z^2}, \quad \frac{\partial u}{\partial z} = -\frac{2y^2}{z^3}.$$

Olingan hosilalar $x>0,\ y>0,\ z>0$ sohada uzluksiz, demak ularni berilgan tenglamaga qo'yib, ayniyat hosil bo'lishini tekshiramiz.

$$x\frac{2x}{y^2} + y\left(-\frac{2x^2}{y^3} + \frac{2y}{z^2}\right) + z\left(-\frac{2y^2}{z^3}\right) = \frac{2x^2}{y^2} - \frac{2x^2}{y^2} + \frac{2y^2}{z^2} - \frac{2y^2}{z^2} \equiv 0.$$

Ayniyat hosil bo'ldi, demak berilgan funksiya tegishli tenglamaning yechimi.

2-*Misol.* $(x-z)\frac{\partial u}{\partial x} + (y-z)\frac{\partial u}{\partial y} + 2z\frac{\partial u}{\partial z} = 0$ tenglamani yeching**.**

Yechish. (1.3) ga ko'ra, berilgan tenglama uchun simmetrik formadagi xarakteristik tenglamalar sistemasi $\frac{dx}{x-z}=\frac{dy}{y-z}=\frac{dz}{2z}$ ko'rinishga ega. Bu sistemani integrallash uchun, teng kasrlar xossasidan foydalanamiz:

$\frac{d(x-y)}{x-y}=\frac{dz}{2z} \Rightarrow \frac{(x-y)^2}{z}=C_1$ **-** birinchi yechim.

$\frac{d(x+z)}{x+z}=\frac{dz}{2z} \Rightarrow \frac{(x+z)^2}{z}=C_2$-ikkinchi yechim**.** Shunday qilib, (1.4) ga asosan berilgan tenglama yechimi $u=F\left(\frac{(x-y)^2}{z},\frac{(x+z)^2}{z}\right)$ bo'ladi, bu yerda F -ixtiyoriy uzluksiz differensiallanuvchi funksiya.

3-*Misol.* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+xy\frac{\partial u}{\partial z}=0$ tenglamaning $u(x,y,0)=x^2+y^2$ shartni qanoatlantiruvchi yechimini toping.

Yechish. Berilgan tenglamaga mos xarakteristik tenglamalar sistemasi $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{xy}$ ko'rinishga ega, bo'lib bundan $\frac{x}{y}=C_1,\ xy-2z=C_2$ yechimlarni olamiz. Demak berilgan tenglama umumiy yechimi $u=F\left(\frac{x}{y},xy-2z\right)$ bo'ladi. Endi boshlang'ich shartni qanoatlantiramiz: $u(x,y,0)=x^2+y^2 \quad\Rightarrow\quad x^2+y^2=F\left(\frac{x}{y},xy\right)$. Quyidagicha almashtirish bajaramiz: $\frac{x}{y}=s,\ xy=t$ natijada $x^2=st,\ y^2=\frac{t}{s}$ bo'ladi, bundan $F(s,t)=st+\frac{t}{s}$, ya'ni

$$u=F\left(\frac{x}{y},xy-2z\right)=F\left(\frac{x}{y},xy\left(1-\frac{2z}{xy}\right)\right)=F\left(s,t\left(1-\frac{2z}{t}\right)\right)=$$
$$=st\left(1-\frac{2z}{t}\right)+\frac{1}{s}t\left(1-\frac{2z}{t}\right)=\frac{t}{s}\left(1-\frac{2z}{t}\right)\left(s^2+1\right)=\left(x^2+y^2\right)\left(1-\frac{2z}{xy}\right).$$

Shunday qilib, qo'yilgan Koshi masalasi yechimi $u=\left(x^2+y^2\right)\left(1-\frac{2z}{xy}\right)$ bo'ladi.

4-*Misol.* $x=y=z$ to'g'ri chiziqdan o'tuvchi tekisliklar oilasining ortogonal traektoriyasini toping.

Yechish. $x=y=z$ to'g'ri chiziq tenglamasini $\begin{cases} x-y=0 \\ x-z=0 \end{cases}$ sistema orqali yozish mumkin. Demak bu to'g'ri chiqdan o'tuvchi tekisliklar oilasining tenglamasi $x-y+(x-z)C=0$ yoki $F(x,y,z)\equiv\frac{x-y}{x-z}=C$ ko'rinishga ega bo'ladi, bu yerda C-ixtiyoriy o'zgarmas son.
Ma'lumki, biror egri chiziq $F(x,y,z)=C$ sirtlar oialsiga ortogonal trayektoriya bo'lishi uchun uning har bir urinmasi ko'rsatilgan (tayinlangan yoki mos) sirtning normali bo'lishi shart.

Hosil bo'lgan tenglamani differensiallab, $\frac{\partial F}{\partial x}=\frac{y-z}{(x-z)^2}$, $\frac{\partial F}{\partial y}=-\frac{1}{x-z}$,
$\frac{\partial F}{\partial z}=\frac{x-y}{(x-z)^2}$ ni topamiz. Demak

$$\frac{y-z}{(x-z)^2}u_x-\frac{1}{x-z}u_y+\frac{x-y}{(x-z)^2}u_z=0 \qquad (*)$$

xususiy hosilali differensial tenglamaning xarakteristikalari, $x=y=z$ to'g'ri chiziqdan o'tuvchi tekisliklar oilasining ortogonal traektoriyasi bo'ladi. (*) tenglamaga mos xarakteristik tenglamalar sistemasini yozamiz:

$$\frac{dx}{y-z}=\frac{dy}{z-x}=\frac{dz}{x-y}.$$

Bu sistemani integrallab, ikkita chiziqli erkli $x+y+z=C_1$ va $x^2+y^2+z^2=C_2$ birinchi integrallarni topamiz. Shunday qilib, topilgan $\begin{cases} x+y+z=C_1 \\ x^2+y^2+z^2=C_2 \end{cases}$ chiziqlar markazlari $x=y=z$ to'g'ri chiziqlarda bo'lgan aylanalardan iborat bo'lib, ular $x=y=z$ to'g'ri chiziqqa perpendikulyar bo'ladi, ya'ni $x=y=z$ to'g'ri chiziqdan o'tuvchi tekisliklar oilasining ortogonal traektoriyasi bo'ladi.

Mustaqil yechish uchun mashqlar.

Quyidagi bir jinsli tenglamalarni yeching (653-662):

653. $y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}=0$

654. $\frac{\partial u}{\partial x}-(y+2z)\frac{\partial u}{\partial y}+(3y+4z)\frac{\partial u}{\partial z}=0$

655. $(x+2y)\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=0$

656. $(z-y)^2\frac{\partial u}{\partial x}+z\frac{\partial u}{\partial y}+y\frac{\partial u}{\partial z}=0;$

$u(0,y,z)=2y(y-z)$

657. $\left(x^2+1\right)\frac{\partial u}{\partial x}+xy\frac{\partial u}{\partial y}=0,\ u(0,y)=y^2$

658. $x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=0,\ u(x,1)=2x$

659. $2\sqrt{x}\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=0,\ u(1,y)=y^2$

660. $\left(x-2e^y\right)\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=0,\ u(x,0)=x$

661. $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+2\frac{\partial u}{\partial z}=0,\ u(1,y,z)=yz$

662. $y\frac{\partial u}{\partial x}+z\frac{\partial u}{\partial z}=0,\ u(1,y,z)=\ln z-\frac{1}{y}$

663. $x(y^2-z^2)\frac{\partial u}{\partial x}+y(x^2+z^2)\frac{\partial u}{\partial y}+z(x^2+y^2)\frac{\partial u}{\partial z}=0;$

2-§. Birinchi tartibli bir jinli bo'lmagan chiziqli xususiy hosilali differensial tenglama.

1-§ da ta'kidlaganimizdek (1.1) ko'rinishdagi tenglamaga bir jinli bo'lmagan kvazichiziqli xususiy hosilali differensial tenglama deyiladi. Agar (1.1) tenglamada $a_i(x_1,...,x_n,u)\,(i=1,2,..n)$ koeffisiyentlar, hamda $f(x_1,...,x_n,u)$ funksiya $u(x_1,...,x_n)$- noma'lum funksiyaga bog'liq bo'lmasa, bu tenglamaga birinchi tartibli bir jinli bo'lmagan chiziqli xususiy hosilali differensial tenglama deyiladi.

Faraz qilaylik $a_n(x_1^0,...,x_n^0,u^0)\neq 0\left((x_1^0,...,x_n^0,u^0)\in D^{n+1}\right)$ bo'lsin, u holda (1.1) tenglamani yechimini topish masalasi, unga mos

$$\frac{dx_1}{a_1(x_1,...,x_n,u)}=\frac{dx_2}{a_2(x_1,...,x_n,u)}=...=\frac{dx_n}{a_n(x_1,...,x_n,u)}=\frac{du}{f(x_1,...,x_n,u)}$$

(2.1)

simmetrik formadagi oddiy differensial tengalamalar sistemasini integrallash masalasi bilan ekvivalent bo'ladi. Ya'ni, agar $\varphi_1(x_1,...,x_n,u),\ \varphi_2(x_1,...,x_n,u),\ ...,$ $\varphi_n(x_1,...,x_n,u)$ funksiyalar (2.1) sistemaning integrallari bo'lsa,

$$F\left(\varphi_1(x_1,...,x_n,u),\ \varphi_2(x_1,...,x_n,u),...,\varphi_n(x_1,...,x_n,u)\right)=0 \qquad (2.2)$$

tenglikga (1.1) tenglamaning oshkormas ko'rinishdagi umumiy yechimi deyiladi, bu yerda F ixtiyoriy uzluksiz differensiallanuvchi funksiya. (2.2) munosabatni u ga nisbatan yechib, (1.1) tenglamaning oshkor ko'rinishdagi umumiy yechimini topamiz.

(2.2) simmetrik formadagi oddiy differensial tengalamalar sistemasiga (1.1) bir jinli bo'lmagan kvazichiziqli xususiy hosilali differensial tenglama uchun **xarakteristik tenglamalar sistemasi** deyiladi.

Eslatma. Bir jinsli bo'lmagan chiziqli tenglama uchun Koshi masalasi, bir jinsli tenglama uchun qo'yilgan Koshi masalasi bilan bir xil qo'yiladi.

Teorema. Ushbu

$$a_1(x_1,...,x_n)\frac{\partial u}{\partial x_1}+a_2(x_1,...,x_n)\frac{\partial u}{\partial x_2}+...+a_n(x_1,...,x_n)\frac{\partial u}{\partial x_n}=f(x_1,...,x_n)$$

tenglama uchun γ boshlang'ich gipersirtning biror x_0 noxarakteristik nuqtasida qo'yilgan Koshi masalasi yechimi mavjud va yagona, hamda bu yechim

$$u\left(g(x,t)\right)=\varphi(x)+\int_0^t f\left(g(x,s)\right)ds$$

Formula orqali topiladi, bu yerda $g(x,t)$-xarakteristik tenglamalarning boshlang'ich gipersirtdagi $g(x,0)=x$ boshlang'ich shartni qanoatlantiruvchi yechimlarining t momentidagi qiymati.

1-*Misol.* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u-x^2-y^2$ tenglamani qanoatlantiruvchi $u=u(x,y)$ sirtlarni toping.

Yechish. (2.1) ga asosan, berilgan tenglamaga mos xarakteristik tenglamalar sistemasi $\frac{dx}{x}=\frac{dy}{y}=\frac{du}{u-x^2-y^2}$ bo'ladi. Bu sistemani integrallab,

$\frac{dx}{x}=\frac{dy}{y}\Rightarrow\frac{x}{y}=C_1$,

hamda $\frac{d\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}=\frac{du}{u-(x^2+y^2)}=\frac{dy}{y}\Rightarrow\frac{d\left(u+x^2+y^2\right)}{u+x^2+y^2}=\frac{dy}{y}$

ya'ni $\frac{u+x^2+y^2}{y}=C_2$ ni olamiz. Demak berilgan tenglama umumiy yechimi $F\left(\frac{x}{y},\frac{x^2+y^2+u}{y}\right)=0$, ya'ni berilgan tenglamani qanoatlantiruvchi sirt tenlamalari $u=yf\left(\frac{x}{y}\right)-x^2-y^2$ bo'ladi, bu yerda f-ixtiyoriy uzluksiz differensiallanuvchi funksiya.

2-*Misol.* $xu\frac{\partial u}{\partial x}+yu\frac{\partial u}{\partial y}+xy=0$ tenglamaning $u\big|_{xy=1}=1$ shartni qanoatlantiruvchi $u=u(x,y)$ yechimini toping.

Yechish. Berilgan tenglamaga mos $\frac{dx}{xu}=\frac{dy}{yu}=\frac{du}{-xy}$ xarakteristik tenglamalar sistemasini integrallab, $\frac{y}{x}=C_1$, hamda $\frac{dy}{u}=\frac{du}{-x} \Rightarrow \frac{dy}{u}=\frac{C_1 du}{-y} \Rightarrow \Rightarrow C_1u^2+y^2=C_2$, bundan esa $C_1=\frac{y}{x}$ ni o'rniga qo'yib, $\frac{yu^2}{x}+y^2=C_2$ ni topamiz. $\begin{cases}\frac{y}{x}=C_1\\ y^2+\frac{yu^2}{x}=C_2\end{cases}$ sistemaga $u\big|_{xy=1}=1$ shartni qanoatlantirib, $\begin{cases}y^2=C_1\\ 2y^2=C_2\end{cases}$ ya'ni $C_2=2C_1$ munosabatni olamiz. Demak $\frac{yu^2}{x}+y^2=C_2=2C_1=\frac{2y}{x}$, bundan $u^2=2-xy$ yechimni olamiz. Bu yechimdan $2>xy$ sohada $u\big|_{xy=1}=1$ shartni qanoatlantiruvchi $u=\sqrt{2-xy}$ yechimni olamiz.

Mustaqil yechish uchun mashqlar.

Quyidagi tenglamalarni yeching (664-675):

664. $y\frac{\partial u}{\partial x}=u$.

665. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u$

666. $e^x\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}=ye^x$

667. $y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}=x^2+y^2$

668. $\frac{1}{x}\frac{\partial u}{\partial x}+\frac{1}{y}\frac{\partial u}{\partial y}=\frac{u}{y^2}$

669. $xy\frac{\partial u}{\partial x}+(x-2u)\frac{\partial u}{\partial y}=yu$

670. $2x\frac{\partial u}{\partial x}+(y-x)\frac{\partial u}{\partial y}-x^2=0$

671. $2y^4\frac{\partial u}{\partial x}-yx\frac{\partial u}{\partial y}=x\sqrt{z^2+1}$

672. $(z-y)^2\frac{\partial u}{\partial x}+xz\frac{\partial u}{\partial y}=xy$

673. $\sin^2 x\frac{\partial u}{\partial x}+tgu\frac{\partial u}{\partial y}=\cos^2 u$

674. $(xu+y)\frac{\partial u}{\partial x}+(x+uy)\frac{\partial u}{\partial y}=1-u^2$

675. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+(z+u)\frac{\partial u}{\partial z}=xy$

Quyidagi Koshi masalalarini yeching (676-684):

676. $xy\frac{\partial u}{\partial x}+x^2\frac{\partial u}{\partial y}=y, \quad u(x,o)=x^2.$

677. $x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=x-y, \quad u(1,y)=y+e^y.$

678. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u-xy, \quad u(x,2)=1+x^2.$

679. $y\frac{\partial u}{\partial x}-x\frac{\partial u}{\partial y}=y^2-x^2, \quad u\left(\frac{1}{y},y\right)=\frac{y^2}{1+y^4}.$

680. $tgx\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u, \quad u(x,x)=x^3.$

681. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2xy, \quad u(x,x)=x^2.$

682. $u\frac{\partial u}{\partial x}+(u^2-x^2)\frac{\partial u}{\partial y}+x=0, \quad u(x,x^2)=2x.$

683. $y^2\frac{\partial u}{\partial x}+yu\frac{\partial u}{\partial y}+u^2=0, \quad u(x,x)=\frac{x-1}{x}.$

684. $x\frac{\partial u}{\partial x}+2y\frac{\partial u}{\partial y}+3z\frac{\partial u}{\partial z}=4u, \quad u(x,x,z)=z.$