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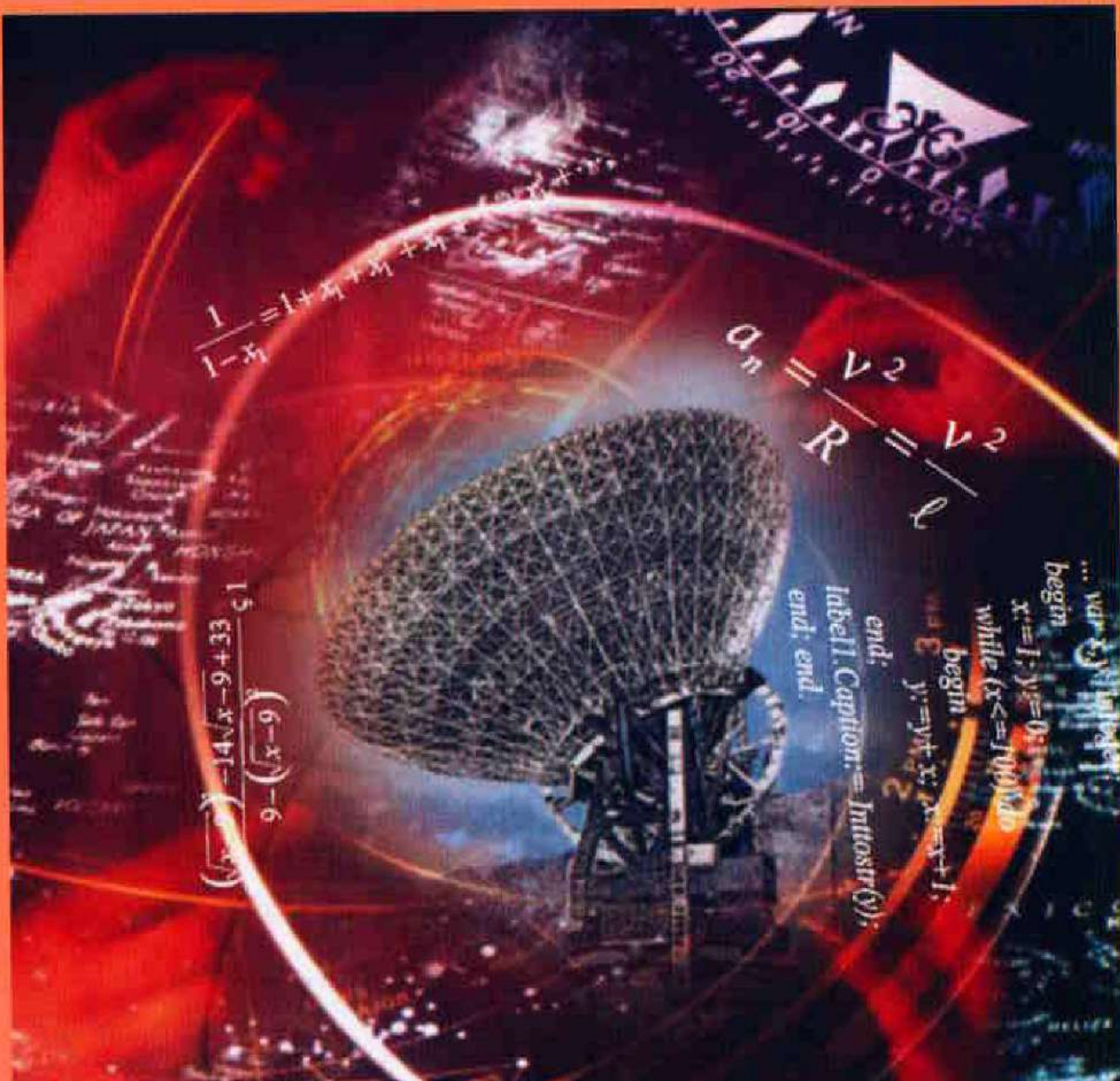

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TRIGONOMETRIK FUNSIYALAR UCHUN YOYISH FORMULALARI

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Ushbu maqolada kompleks analizni elementar matematikaning asosiy bo'limlaridan biri bo'lgan trigonometriyaga tatbiqini keltiramiz. Muavr formulasidan foydalanib, trigonometrik funksiyalar uchun yoyish formulalarini keltirib chiqaramiz.

Kalit so'zlar: *Kompleks son, Muavr formulasi, Nyuton binomi, trigonometriya, trigonometrik funksiyalar, yoyish formulalari.*

In this article, we will apply complex analysis to trigonometry, one of the main branches of elementary mathematics. Using Moivre's formula, we will derive expansion formulas for trigonometric functions.

Keywords: *Complex number, Moivre's formula, Newton's binomial, trigonometry, trigonometric functions, expansion formulas.*

В этой статье мы рассмотрим применение комплексного анализа в тригонометрии — одном из основных разделов элементарной математики. Используя формулу Муавра, выводим формулы разложения тригонометрических функций.

Ключевые слова: *Комплексное число, формула Муавра, бином Ньютона, тригонометрия, тригонометрические функции, формулы разложения.*

Ma'lumki, trigonometriya qadimiy tarixga ega. Hindistonliklar ilk marta trigonometrik funksiyalar qiymatlari jadvalini kashf qilganlar. Shumer astronomlari aylanalarni 360 gradusga bo'lish orqali, burchak o'lchovini o'rganishgan. Ular va keyinchalik bobilliklar o'xshash uchburchaklar tomonlari nisbatlarini o'rgandilar va bu nisbatlarning ba’zi xususiyatlarini kashf etdilar, lekin buni uchburchaklarning tomonlari va burchaklarini topishning tizimli usuliga aylantirmadilar. Miloddan avvalgi III asrda Yevklid va Arximed kabi yunon matematiklari aylanalarga chizilgan burchaklarning xossalarini o‘rganib, zamonaviy trigonometrik formulalarga ekvivalent bo‘lgan teoremalarni isbotladilar. Ular ushbu formulalarni algebraik jihatdan emas, balki geometrik jihatdan isbotlaganlar. Miloddan avvalgi 140-yilda Gipparx zamonaviy sinus qiymatlari jadvallariga o‘xshash akkordlarning birinchi jadvallarini bergan va ulardan trigonometriya va sferik trigonometriya masalalarini yechishda foydalangan [1].

Trigonometriya matematikaning asosiy bo‘limlaridan biri hisoblanib, uchburchak tomonlari va burchaklari orasidagi bog‘lanishlar, trigonometrik funksiyalarning xossalari va ular o‘rtasidagi bog‘lanishlarni o‘rganadi [2]. Trigonometrik funksiyalar elementar matematikaning asosiy hamda maktab o‘quvchilari tushunishlari uchun murakkab bo‘lgan bo‘limlaridan biri hisoblanadi. Trigonometrik funksiyalarni kompleks analiz jihatidan o‘rganish, trigonometriyani o‘ragnishni bir muncha osonlashtiradi [3]. Biz ushbu maqolada trigonometrik funksiyalar uchun yoyish formulalarini keltirib chiqaramiz.

Moduli 1 ga teng bo‘lgan kompleks sonni qaraylik. Ma’lumki bunday sonlar

$$z = \cos\varphi + i\sin\varphi$$

ko‘rinishida bo‘ladi. U holda kompleks sonlarni darajaga ko‘tarish formulasini qo‘llasak

$$\left(\cos\theta + i\sin\theta\right)^n = \cos n\theta + i\sin n\theta \qquad (1)$$

Shuningdek, Nyuton binomiga ko‘ra:

$$\left(\cos\theta + i\sin\theta\right)^n = C_n^0\cos^n\theta + C_n^1\cos^{n-1}\theta\cdot i\sin\theta + \ldots + C_n^n\left(i\sin\theta\right)^n \qquad (2)$$

Ma’lumki, (1) va (2) tengliklarning har ikki tomonida kompleks sonlar joylashgan va ular aynan bitta sonni ifodalaydi. Ikkita kompleks sonning haqiqiy va mavhum qismlari mos ravishda teng bo‘lsa ular teng deyiladi. Natijada quyidagi tenglikka ega bo‘lamiz

$$\cos n\theta + i\sin n\theta = C_n^0\cos^n\theta + C_n^1\cos^{n-1}\theta\cdot i\sin\theta + \cdots + C_n^n i^n\sin^n\theta. \qquad (3)$$

Haqiqiy va mavhum qismlarni mos ravishda tenglaymiz:

$$\cos n\theta = C_n^0\cos^n\theta - C_n^2\cos^{n-2}\theta\sin^2\theta + C_n^4\cos^{n-4}\theta\sin^4\theta - \ldots \qquad (4)$$

va

$$\sin n\theta = C_n^1\cos^{n-1}\theta\sin\theta - C_n^3\cos^{n-3}\theta\sin^3\theta + C_n^5\cos^{n-5}\theta\sin^5\theta + \ldots \qquad (5)$$

Yuqoridagi qatorlarning hadlarining oldidagi ishora, n ning toq yoki juft bo‘lishiga qarab, musbat va manfiy bo‘lishi mumkin. Shuningdek, yuqoridagi qatorlarning oxirgi hadlari ikki holatda ham turlicha bo‘ladi. (2) ifodaning oxirgi ikki hadlari quyidagilar: $C_n^{n-1}i^{n-1}\cos\theta\sin^{n-1}\theta$ va $C_n^n i^n\sin^n\theta$

Ushbu hadlar, mos ravishda, n ning toq yoki juft bo‘lishiga qarab, haqiqiy va mavhum yoki mavhum va haqiqiy bo‘ladi. Quyida biz bu hollarni alohida qarab chiqamiz.

1-hol. n toq bo‘lganda, quyidagilarga ega bo‘lamiz

$$C_n^{n-1}i^{n-1}\cos\theta\sin^{n-1}\theta = n(-1)^{\frac{n-1}{2}}\cos\theta\sin^{n-1}\theta,$$

$$C_n^n i^n\sin^n\theta = i(-1)^{\frac{n-1}{2}}\sin^n\theta.$$

U holda

$$\cos n\theta = C_n^0\cos^n\theta - C_n^2\cos^{n-2}\theta\sin^2\theta + C_n^4\cos^{n-4}\theta\sin^4\theta - \ldots + n(-1)^{\frac{n-1}{2}}\cos\theta\sin^{n-1}\theta.$$

$$\sin n\theta = C_n^1\cos^{n-1}\theta\sin\theta - C_n^3\cos^{n-3}\theta\sin^3\theta + C_n^5\cos^{n-5}\theta\sin^5\theta + \ldots + (-1)^{\frac{n-1}{2}}\sin^n\theta.$$

Misol 1. $\cos 7\theta$ ni $\cos\theta$ ning darajalari orqali ifodalang.

Yechish: Bizga ma’lumki n toq bo‘lganda

$$\cos n\theta = C_n^0\cos^n\theta - C_n^2\cos^{n-2}\theta\sin^2\theta + C_n^4\cos^{n-4}\theta\sin^4\theta - .$$

$-\ldots + n(-1)^{\frac{n-1}{2}}\cos\theta\sin^{n-1}\theta.$ $n-7$ bo‘lgani uchun quyidagi ifodani yozamiz

$$\cos 7\theta = \cos^7\theta - C_7^2\cos^5\theta\sin^2\theta + C_7^4\cos^3\theta\sin^4\theta - C_7^6\cos\theta\sin^6\theta =$$

$$= \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$$

$$= \cos^7\theta - 21\cos^5\theta\left(1-\cos^2\theta\right) + 35\cos^3\theta\left(1-\cos^2\theta\right)^2 - 7\cos\theta\left(1-\cos^2\theta\right)^3 =$$

$$= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

2-hol. n juft bo‘lganda, quyidagilar ega bo‘lamiz:

$$C_n^{n-1} i^{n-1}\cos\theta\sin^{n-1}\theta = n\,i\left(-1\right)^{\frac{n-2}{2}}\cos\theta\sin^{n-1}\theta,$$

$$C_n^n i^n \sin^n\theta = \left(-1\right)^{\frac{n}{2}}\sin^n\theta.$$

U holda

$$\cos n\theta = C_n^0\cos^n\theta - C_n^2\cos^{n-2}\theta\sin^2\theta + C_n^4\cos^{n-4}\theta\sin^4\theta - \ldots + \left(-1\right)^{\frac{n}{2}}\sin^n\theta.$$

$$\sin n\theta = C_n^1\cos^{n-1}\theta\sin\theta - C_n^3\cos^{n-3}\theta\sin^3\theta + \ldots + n\left(-1\right)^{\frac{n-2}{2}}\cos\theta\sin^{n-1}\theta.$$

Yuqoridagi ifodalardan foydalanib $tgn\theta$ uchun ham yoyilma yozish mumkin

$$tgn\theta = \frac{\sin n\theta}{\cos n\theta} = \frac{C_n^1\cos^{n-1}\theta\sin\theta - C_n^3\cos^{n-3}\theta\sin^3\theta + \ldots}{\cos^n\theta - C_n^2\cos^{n-2}\theta\sin^2\theta + C_n^4\cos^{n-4}\theta\sin^4\theta - \ldots}$$

Surat va maxrajni $\cos n\theta$ ga bo‘lib, bizga quyidagini hosil qilamiz:

$$tgn\theta = \frac{C_n^1 tg\theta - C_n^3 tg^3\theta + C_n^5 tg^5\theta - \ldots}{1 - C_n^2 tg^2\theta + C_n^4 tg^4\theta - \ldots}$$

Oxirgi hadlar (a) n toq bo‘lganda, suratning oxirgi hadi $\left(-1\right)^{\frac{n-1}{2}} tg^n\theta$ ga teng, maxrajning oxirgi hadi esa $n\left(-1\right)^{\frac{n-2}{2}} tg^{n-1}\theta$ ga teng. U holda

$$tgn\theta=\frac{C_n^1tg\theta-C_n^3tg^3\theta+C_n^5tg^5\theta-...+(-1)^{\frac{n-1}{2}}tg^n\theta}{1-C_n^2tg^2\theta+C_n^4tg^4\theta-...+n(-1)^{\frac{n-2}{2}}tg^{n-1}\theta}.$$

(b) n juft bo'lganda, suratning oxirgi hadi $n(-1)^{\frac{n-2}{2}}tg^{n-1}\theta$ ga teng, maxrajning oxirgi hadi esa $(-1)^{\frac{n-1}{2}}tg^n\theta$ ga teng. U holda

$$tgn\theta=\frac{C_n^1tg\theta-C_n^3tg^3\theta+C_n^5tg^5\theta-...+n(-1)^{\frac{n-2}{2}}tg^{n-1}\theta}{1-C_n^2tg^2\theta+C_n^4tg^4\theta-...+(-1)^{\frac{n-1}{2}}tg^n\theta}.$$

$tg(\theta_1+\theta_2+\theta_3+...+\theta_n)$ uchun kengaytma.

Kompleks analizdan ma'lumki:

$$\cos(\theta_1+\theta_2+\theta_3+...+\theta_n)+i\sin(\theta_1+\theta_2+\theta_3+...+\theta_n)=$$

$$=(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)(\cos\theta_3+i\sin\theta_3)....(\cos\theta_n+i\sin\theta_n) \quad (6)$$

Agar $\cos\theta_j+i\sin\theta_j=\cos\theta_j(1+i\,tg\theta_j),\ j=\overline{1...n}$

ekanligini inobatga olsak, (6) dan yozishimiz mumkin

$$\cos(\theta_1+\theta_2+\theta_3+...+\theta_n)+i\sin(\theta_1+\theta_2+\theta_3+...+\theta_n)=$$

$$=\cos\theta_1\cos\theta_2\cos\theta_3....\cos\theta_n(1+i\,tg\theta_1)(1+i\,tg\theta_2)(1+i\,tg\theta_3)...(1+i\,tg\theta_n)$$

$$=\cos\theta_1\cos\theta_2\cos\theta_3....\cos\theta_n\left[1+is_1+i^2s_2+i^3s_3+i^4s_4+...\right]=$$

$$=\cos\theta_1\cos\theta_2\cos\theta_3....\cos\theta_n\left[(1-s_2+s_4-....)+i(s_1-s_3+s_5-....)\right]$$

bunda s_r $\theta_1,\theta_2,\theta_3,...,\theta_n$ burchaklarning tangenslarining r tadan olingan ko'paytmalarining yig'indisini bildiradi, ya'ni

$$s_1=\sum_{j=1}^{n}tg\theta_j=tg\theta_1+tg\theta_2+tg\theta_3+...+tg\theta_n$$

$$s_2 = \sum_{i=1, j=1, i\neq j}^{n} tg\theta_i tg\theta_j = tg\theta_1 tg\theta_2 + tg\theta_1 tg\theta_3 + tg\theta_1 tg\theta_4 + ... + tg\theta_{n-1} tg\theta_n.$$

Haqiqiy va mavhum qismlarni tenglashtirib, biz quyidagilarni olamiz:

$$\cos\left(\theta_1+\theta_2+\theta_3+...+\theta_n\right)=\cos\theta_1\cos\theta_2\cos\theta_3....\cos\theta_n\left[1-s_2+s_4-....\right] \quad (7)$$

$$\sin\left(\theta_1+\theta_2+\theta_3+...+\theta_n\right)=\cos\theta_1\cos\theta_2\cos\theta_3....\cos\theta_n\left[s_1-s_3+s_5-....\right] \quad (8)$$

(8) ni (7) ga bo‘lib, biz quyidagilarni olamiz:

$$tg\left(\theta_1+\theta_2+\theta_3+...+\theta_n\right)=\frac{s_1-s_3+s_5-....}{1-s_2+s_4-....}$$

Eslatma 1. Surat va maxrajning oxirgi hadlari mos ravishda quyidagilardan iborat:

(a) n toq bo‘lsa, $\left(-1\right)^{\frac{n-1}{2}} s_n$ va $\left(-1\right)^{\frac{n-1}{2}} s_{n-1}$. U holda

$$tg\left(\theta_1+\theta_2+\theta_3+...+\theta_n\right)=\frac{s_1-s_3+s_5-...+\left(-1\right)^{\frac{n-1}{2}} s_n}{1-s_2+s_4-...+\left(-1\right)^{\frac{n-1}{2}} s_{n-1}}.$$

(b) n juft bo‘lsa, $\left(-1\right)^{\frac{n-2}{2}} s_{n-1}$ va $\left(-1\right)^{\frac{n}{2}} s_n$. U holda

Eslatma 2. Agar $\theta_1=\theta_2=\theta_3=...=\theta_n=\theta$ bo‘lsa, u holda $tgn\theta$ ning kengaytmasidan foydalanamiz.

Misol 2. Agar α, β, γ $x^3+px^2+qx+p=0$ tenglamaning ildizlari bo‘lsa, u holda $tg^{-1}\alpha+tg^{-1}\beta+tg^{-1}\gamma=n\pi$ radian ekanligini isbotlang, faqat bitta holatdan tashqari va uni ko‘rsating.

Yechim: Faraz qilaylik $\alpha=tg\theta_1,\ \beta=tg\theta_2,\ \gamma=tg\theta_3$. U holda

$$\theta_1=tg^{-1}\alpha,\ \ \theta_2=tg^{-1}\beta,\ \ \theta_3=tg^{-1}\gamma$$

bo‘ladi. Berilgan $x^3+px^2+qx+p=0$ tenglamaning ildizlari α, β, γ ya’ni $tg\theta_1, tg\theta_2, tg\theta_3$. Viyet teoremasidan foydalansak:

$$s_1=\sum tg\theta_i = \alpha+\beta+\gamma=-p.$$

$$s_2=\sum tg\theta_i tg\theta_j=\alpha\beta+\beta\gamma+\alpha\gamma=q.$$

$$s_3=\alpha\beta\gamma=tg\theta_1 tg\theta_2 tg\theta_3=-p.$$

Endi, $tg\left(\theta_1+\theta_2+\theta_3\right)=\frac{s_1-s_2}{1-s_2}=\frac{-p+p}{1-q}=0$.

Agar q=1 bo‘lsa, u holda kasr aniq bo‘lmagan 0/0 ko‘rinishni oladi.

Istisno holatini hisobga olmaganda, $tg\left(\theta_1+\theta_2+\theta_3\right)=tgn\pi$ bo‘ladi.

$\theta_1+\theta_2+\theta_3=n\pi$ radian, bundan $tg^{-1}\alpha+tg^{-1}\beta+tg^{-1}\gamma=n\pi$ ekanligi kelib chiqadi.

Xulosa qilib aytganda, kompleks analiz jihatidan trigonometriyani o‘rganish, trigonometriyani o‘quvchilarga oson tushunishga yordam beradi. Yana shuningdek, trigonometrik funksiyalarni biz sezmagan jihatlarini matematik ifodalar orqali bog‘lanishini ko‘rsatishga yordam beradi.

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