MIRZO ULUG'BEK NOMIDAGI
O'ZBEKISTON MILLIY UNIVERSITETI

100 YIL

N.M.Jabborov

OLIY MATEMATIKA VA UNING TATBIQLARIGA DOIR MASALALAR TO'PLAMI

(II-qism, IV-jild)

O‘ZBEKISTON RESPUBLIKASI
OLIY VA O‘RTA MAXSUS TA’LIM VAZIRLIGI

MIRZO ULUG‘BEK NOMIDAGI
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OLIY MATEMATIKA VA UNING TATBIQLARIGA DOIR MASALALAR TO‘PLAMI

(Bakalavr ta’lim yo‘nalishlari talabalari uchun o‘quv qo‘llanma)

II qism

IV-jild

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Mazkur o‘quv qo‘llanma bakalavr ta’lim yo‘nalish talabalari uchun mo‘ljallangan bo‘lib, unda fazoda analitik geometriya elementlari, ikki o‘zgaruvchili funksiyalar differensial va integral hisobining asoslari, birinchi va ikkinchi tartibli oddiy differensial tenglamalar, maydonlar nazariyasi, hamda ehtimollar nazariyasi va matematik statistika asoslariga oid mavzular bo‘yicha misol va masalalar keltirilgan.

O‘quv qo‘llanma O‘zbekiston Respublikasi Oliy va o‘rta-maxsus ta’lim vazirligi 2017-yil 28-iyundagi 434-sonli buyrug‘iga asosan nashrga tavsiya etilgan.

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SO‘ZBOSHI

Ko‘pchilik oliy o‘quv yurtlarida o‘rganiladigan dastlabki fanlardan biri oliy matematika hisoblanadi.

Oliy matematikaning asosiy vazifasi talabalarni shu fanning tushunchalari, tasdiqlari va boshqa matematik ma’lumotlar majmuasi bilan tanishtirishdangina iborat bo‘lmay, balki ularni mantiqiy fikrlash, matematik usullarni amaliy masalalarni yechishda qo‘llashni o‘rgatishdan iborat.

O‘zbekistonda kadrlar tayyorlash tizimini tubdan isloh qilish jarayonida talabalarni darslik hamda o‘quv qo‘llanmalar bilan ta’minlash muhim o‘rin egallaydi.

O‘liy matematika kursi bo‘yicha turli darajada yozilgan, maqsad hamda yo‘nalishlari xilma-xil bo‘lgan qator darslik va o‘quv qo‘llanmalar mavjud. Ammo davlat ta’lim standartlari o‘quv dasturini zamon talablariga moslashtirish va qayta ko‘rib chiqishni taqozo etmoqda.

Mazkur o‘quv qo‘llanma davlat ta’lim standartlari asosida yozilgan bo‘lib, u ma’lum tartibda bob va paragraflarga ajratilib bayon etilgan.

Ma’lumki, oliy matematika mutaxassisliklarga qarab, ularga mos hajmda o‘qitiladi. Binobarin, ularga mo‘ljallangan misol va masalalar ham mazmuni, ham hajmi, ham sodda va murakkabliga qarab turlicha bo‘lishi lozim.

O‘quv qo‘llanmaning barcha mutaxassisliklar uchun mos keladigan universial qo‘llanma bo‘lishiga harakat qilindi. Qo‘llanmani yozishda uzoq yillar davomida turli mutaxassisliklar bo‘yicha ta’lim oladigan talabalar bilan olib borilgan darslaridan, shuningdek, oliy matematika bo‘yicha xorijda chop etilgan adabiyotlardan foydalanildi. Jumladan, I.I.Bavrin, V.L.Matrosovning “Общий курс высшей математики” (1995), V.G.Skatetskiy va boshqalarning “Математические методы в химии” (2006), L.I.Lurening “Основы высшей математики” (2003), K.N.Lungu “Сборник задач по высшей математике” 1, 2 q. –M. (2007) kabi darslik va o‘quv qo‘llanmalari tahlil qilinib, ulardan foydalanildi.

O‘quv qo‘llanma ikki qsimdan iborat bo‘lib, uning mazkur ikkinchi qismida fazoda koordinatalar sistemasi, fazoda tekislik va to‘g‘ri chiziq, ikkinchi tartibli sirtlar, fazoda vektorlar va ularning ba’zi tatbiqlari, vektorlar analizining elementlari, ko‘p o‘zgaruvchili funksiya limiti, uzluksizligi, hosila va differensiallari, karrali integrallar, birinchi tartibli va ikkinchi tartibli oddiy differensial tenglamalar, egri chiziqli integrallar, sirt integrallari, maydonlar nazariyasining elementlari, matematik fizikaning ba’zi bir tenglamalari, ehtimollar nazariyasining asoslariga oid misol va masalalar keltirilgan.

Har bir bob (paragraf) zaruriy nazariy ma’lumotlarni keltirish bilan boshlangan. Muhim ta’riflar, teoremalar, formulalar keltirilgan. So‘ng bir nechta misol va masalalarning yechilish yo‘llari va batafsil yechimlari bayon etilgan. Bu keyingi misol va masalalarni mustaqil yechishga yordam beradi.

Har bir bob (paragraf)da mustaqil yechish uchun misol va masalalar keltirilgan. Ular tuzilishiga qarab (avval, sodda yechiladigan masalalar, keyin o'rtacha murakkablikka ega, pirovardida, mupakkabroq masalalar) joylashtirilgan.

Masalalar sonining ko'pligi va ularni yuqorida aytilgan tartibda joylashtirilishi oliy matematikani turli hajmdagi dastur bo'yicha o'qitilishida ularga mos keladiganlarini tanlash imkonini beradi.

Har bir bob so'ngida shu bobda keltirilgan mavzularni mustahkamlash maqsadida savollar keltirilgan bo'lib, ular talabalarning o'zini o'zi tekshirish imkoniyatini beradi.

Oliy matematikaning tatbiq doirasi nihoyatda keng. Fan va texnikaning turli sohalaridagi, jumladan, mexanika, fizika, texnika, iqtisoduyotdagi ko'pgina masalalar matematik usullar yordamida hal etiladi.

Qo'llanmada matematikaning tatbiqlariga doir masalalar keltirilib, ulardan ayrimlarining yechimi batafsil bayon etilgan.

Jamiyatda fan va texnikaning jadal rivojlanishi matematik masalalarni ham texnik vositalar yordamida yechishni talab etadi. Shuni e'tiborga olgan holda qo'llanmaning ba'zi boblarida oliy matematikaning ba'zi mavzulari bo'yicha Maple paketidan foydalanib, masalalar yechib ko'rsatilgan. Bu esa shu dastur asosida boshqa matematik masalalarni ham yechish imkonini beradi.

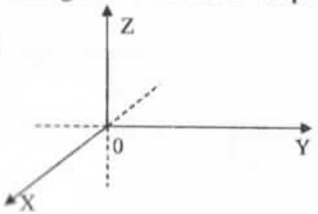
Muallif

11-bob

Fazoda tekislik, to'g'ri chiziq va ikkinchi tartibli sodda sirtlar

1-§. Fazoda dekart koordinatalari sistemasi

1^0. Asosiy tushunchalar. Aytaylik, fazodagi biror O nuqtadano'zaro bir-biri bilan perpendikulyar va yo'nalishga ega bo'lgan uchta to'g'ri chiziq o'tkazilgan bo'lsin. Bu O nuqta **koordinatalar boshi** deyiladi.



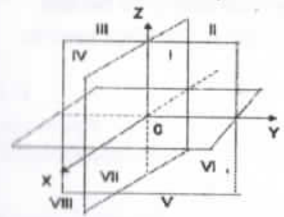
1-chizma

1-chizmada tasvirlangandek, to'g'ri chiziqlardan biri OX o'qi (abssissalar o'qi), ikkinchisi OY o'qi (ordinatalar o'qi), uchin-chisi esa OZ o'qi (applikatalar o'qi) deyiladi. Ularning musbat yo'nalishlari 1-chizmada strelkalar bilan ko'rsatilgan.

Barcha o'qlarda bir xil o'lchov birligi (masshtab) belgilanishi bilan sistema yuzaga keladi. Bu sistema (tekislikda Dekart koordinatalar sistemasi singari) fazoda **Dekart koordinatalar sistemasi** deyiladi.

OX va OY o'qlari orqali o'tgan tekislik XOY koordinata tekisligi, OY va OZ o'qlari orqali o'tgan tekislik YOZ koordinata tekisligi, OZ va OX o'qlari orqali o'tgan tekislik ZOX koordinata tekisligi deyiladi.

XOY, YOZ, ZOX koordinata tekisliklari fazoni 8 qismga ajratadi. Bu qismlar **oktantlar** deyiladi. Ular 2-chizmada ko'rsatilgandek raqamlanadi.



2-chizma

Bu sistemada fazodagi har bir M nuqta x, y, z sonlardan tuzilgan (x, y, z) uchlik bilan aniqlanadi. Odatda, (x, y, z) uchlik M nuqtaning koordinatalari (dekart koordinatalari) deyilib, $x - M$ nuqtaning birinchi koordinatasi, ya'ni abssissasi, $y - M$ nuqtaning ikkinchi koordinatasi, ya'ni ordinatasi, $z - M$ nuqtaning uchinchi koordinatasi, ya'ni applikatasi deyiladi va

$$M = M(x, y, z)$$

kabi belgilanadi.

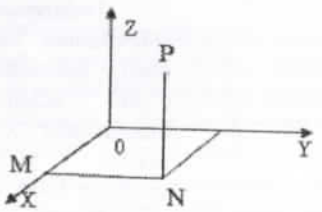
Ravshanki, koordinatalar boshi O nuqtaning koordinatalari $(0,0,0)$ bo'ladi.

Fazodagi nuqta koordinatalarining ishoralari uning oktantada joylashishiga qarab turlicha bo'ladi.

Nuqta koordinatalari	Oktantlar							
	I	II	III	IV	V	VI	VII	VIII
x	+	–	–	+	+	–	–	+
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

1-misol. Ushbu $P(3;2;6)$ nuqta fazoda tasvirlansin (yasalsin).

◀Fazoda Dekart koordinatalar sistemasini olamiz (3-chizma).



3-chizma

OX o‘qining musbat yo‘nalishi bo‘yicha uzunligi 3 birlikka teng bo‘lgan OM kesmani qo‘yib, M nuqtani topamiz. M nuqta orqali OY o‘qiga parallel to‘g‘ri chiziqo‘tkazib, uning musbat yo‘nalishi bo‘yicha uzunligi 2 birlikka teng bo‘lgan MN kesmani qo‘yib N nuqtani topamiz. N nuqta orqali OZ o‘qiga parallel to‘g‘ri chiziqo‘tkazib, XOY tekisligining yuqorisidagi qismi bo‘yicha uzunligi 6 birlikka teng bo‘lgan NP kesmani qo‘yib P nuqtani topamiz. Bu izlanayotgan nuqtaning tasviri bo‘ladi (3-chizma).▶

2^0. Fazoda ikki nuqta orasidagi masofa. Kesmani nisbatda bo‘lish

Fazodagi ikki $A(x_1,y_1,z_1)$ va $B(x_2,y_2,z_2)$ nuqtalar orasidagi masofa

$$|AB|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} \qquad (1)$$

bo‘ladi.

Fazodagi berilgan

$$A=A(x_1,y_1,z_1), \qquad B=B(x_2,y_2,z_2)$$

nuqtalarni to‘g‘ri chiziq bilan birlashtirib, hosil bo‘lgan kesmani AB deylik.

AB kesmada shunday C nuqta topingki, AC kesma uzunligini CB kesma uzunligiga nisbati berilgan λ songa teng, ya’ni:

$$\frac{AC}{CB}=\lambda \qquad (2)$$

bo‘lsin.

Izlanayotgan C nuqtaning koordinatalarini x,y,z deylik: $C=C(x,y,z)$.

Bu nuqtaning koordinatalari

$$x=\frac{x_1+\lambda x_2}{1+\lambda}, \qquad y=\frac{y_1+\lambda y_2}{1+\lambda}, \qquad z=\frac{z_1+\lambda z_2}{1+\lambda} \qquad (3)$$

bo‘ladi.

2-misol. Uchinchi oktantada koordinata o‘qlaridan

$$d_x=5, \quad d_y=3\sqrt{5}, \quad d_z=2\sqrt{13}$$

masofada joylashgan $M(x,y,z)$ nuqtani toping.

◀Aytaylik,

$$M_1(x,0,0), \quad M_2(0,y,0), \quad M_3(0,0,z)$$

bo‘lsin. Unda shartga ko‘ra

$$MM_1=d_x=5, \quad MM_2=d_y=3\sqrt{5}, \quad MM_3=d_z=2\sqrt{13}$$

bo‘ladi. Ikki nuqta orasidagi masofa formulasidan foydalanib topamiz:

$$MM_1=\sqrt{(x-x)^2+(0-y)^2+(0-z)^2}=\sqrt{y^2+z^2},$$

$$MM_2=\sqrt{(0-x)^2+(y-y)^2+(0-z)^2}=\sqrt{x^2+z^2},$$

$$MM_3=\sqrt{(0-x)^2+(0-y)^2+(z-z)^2}=\sqrt{x^2+y^2}.$$

Demak,

$$\sqrt{y^2+z^2}=5 \quad \text{ya'ni} \quad y^2+z^2=25,$$

$$\sqrt{x^2+z^2}=3\sqrt{5} \quad \text{ya'ni} \quad x^2+z^2=45,$$

$$\sqrt{x^2+y^2}=2\sqrt{13} \quad \text{ya'ni} \quad x^2+y^2=52.$$

Natijada, $M(x,y,z)$ nuqtaning koordinatalarini topish uchun

$$\begin{cases} y^2+z^2=25, \\ x^2+z^2=45, \\ x^2+y^2=52 \end{cases} \qquad (4)$$

sistemaga kelamiz. Bu sistemaning ikkinchi tenglamasidan birinchi tenglamasini ayirsak, unda:

$$x^2-y^2=20$$

tenglama hosil bo‘ladi. Demak,

$$\begin{cases} x^2-y^2=20, \\ x^2+y^2=52. \end{cases}$$

Keyingi sistemani yechamiz:

$$2x^2=72, \quad x^2=36, \quad x=\pm 6,$$

$$2y^2=32, \quad y^2=16, \quad y=\pm 4.$$

(4) sistemaning birinchi tenglamasidagi y ning o‘rniga ± 4 ni yoki (4) sistemaning ikkinchi tenglamasidagi x ning o‘rniga ± 6 ni qo‘yib

$$z=\pm 3$$

bo'lishini topamiz. Demak,

$$x=\pm 6, \quad y=\pm 4, \quad z=\pm 3.$$

Izlanayotgan $M(x,y,z)$ nuqta III oktantada bo'lishi kerakligini e'tiborga olib topamiz: $M(-6,-4,3)$. ▶

3-misol. Ushbu

$$A=A(2,-1,7), \quad B=B(4,5,-2)$$

nuqtalarni birlashtirishdan hosil bo'lgan AB kesmani XOY koordinata tekisligi qanday nisbatda bo'ladi?

◀Ravshanki, A nuqta IV oktantada, B nuqta V oktantada joylashadi. Unda AB kesma XOY koordinatalar tekisligini $C=C(x,y,0)$ nuqtada kesadi (4-chizma).

Endi XOY koordinata tekisligini AB kesma qanday nisbatda bo'lishini topamiz.

Agar

$$A=A(2,-1,7), \quad B=B(4,5,-2), C=C(x,y,0)$$

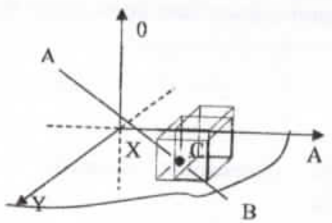
bo'lishini e'tiborga olsak va (2) formuladan foydalansak, unda ushbu

$$0=z=\frac{7+\lambda(-2)}{1+\lambda}$$

tenglikka kelamiz. Keyingi tenglikda

$$7+\lambda\cdot(-2)=0, \text{ ya'ni } \lambda=\frac{7}{2}$$

bo'lishi kelib chiqadi. ▶



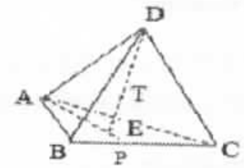
4-chizma

4-misol. Uchlari

$$A(x_1,y_1,z_1),\ B(x_2,y_2,z_2),\ C(x_3,y_3,z_3),\ D(x_4,y_4,z_4)$$

nuqtalarda bo'lgan piramidaning og'irlik markazini (og'irlik markazining ifodalovchi nuqtaning koordinatalari) toping.

◀Ma'lumki, piramidaning og'irlik markazi, uning ixtiyoriy uchini bu uchi qarshisidagi tomonining og'irlik markazini birlashtiruvchi to'g'ri chiziqda yotadi. Demak, izlanayotgan $T(x,y,z)$nuqta, masalan, DE to'g'ri chiziqda yotadi, bunda $E(x',y',z')-ABC$ tomonining og'irlik markazi (5-chizma).



5-chizma

Ma'lumki, $E(x',y',z')$ nuqta AP medianani $2:1$ nisbatda bo'ladi, ya'ni $AE:EP=2:1$ bo'lib,

$$P=P\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2},\frac{z_2+z_3}{2}\right)$$

bo'ladi.

Kesmani berilgan nisbatda bo'lish formulalari (3)dan foydalanib topamiz:

$$x'=\frac{x_1+2\frac{x_2+x_3}{2}}{1+2}=\frac{x_1+x_2+x_3}{3},$$

$$y'=\frac{y_1+2\frac{y_2+y_3}{2}}{1+2}=\frac{y_1+y_2+y_3}{3},$$

$$z'=\frac{z_1+2\frac{z_2+z_3}{2}}{1+2}=\frac{z_1+z_2+z_3}{3}.$$

Izlanayotgan $T(x,y,z)$nuqta DE kesmani $\lambda=3:1$ nisbatda bo'ladi: $DT:TE=3:1$.

Yana kesmani berilgan nisbatda bo'lish formulalari (3)dan foydalanib topamiz:

$$x=\frac{x_4+3\frac{x_1+x_2+x_3}{3}}{1+3}=\frac{x_1+x_2+x_3+x_4}{4},$$

$$y=\frac{y_4+3\frac{y_1+y_2+y_3}{3}}{1+3}=\frac{y_1+y_2+y_3+y_4}{4},$$

$$z=\frac{z_4+3\frac{z_1+z_2+z_3}{3}}{1+3}=\frac{z_1+z_2+z_3+z_4}{4}.$$

Demak, piramidaning og'irlik markazi

$$T=T\left(\frac{x_1+x_2+x_3+x_4}{4},\frac{y_1+y_2+y_3+y_4}{4},\frac{z_1+z_2+z_3+z_4}{4}\right)$$

bo'ladi. ▶

Quyidagi masalalarni yeching

1346. Agar

$$A=A(2,0,3),\ B=B(-5,2,1),\ C=C(3,2,1)$$

nuqtalar uchburchakning uchlari bo'lsa, A uchidan o'tkazilgan medananing uzunligini toping.

1347. Agar $M(x,y,z)$ nuqtaning

a) uchchala koordinatalari musbat,

b) uchchala koordinatalari manfiy

bo'lsa, huqta nechanchi oktantda joylashgan bo'ladi?

1348. Agar $M(x,y,z)$ nuqta

a) III oktantda joylashgan,

b) VI oktantda joylashgan,

c) VIII oktantda joylashgan,

bo'lsa, nuqtaning koordinatalari qanday ishorali bo'ladi?

1349. Agar $M(x,y,z)$ nuqta

a) OX o'qida joylashgan,

b) OY o'qida joylashgan,

c) OZ o'qida joylashgan

bo'lsa, nuqtaning koordinatalari qanday bo'ladi?

1350. Agar $M(x,y,z)$ nuqta

a) XOY tekisligida joylashgan,

b) YOZ tekisligida joylashgan,

c) XOZ tekisligida joylashgan,

bo'lsa, nuqtaning koordinatalari qanday bo'ladi?

1351. Quyidagi nuqtalar:

$$A(2,0,0),\quad B(0,-5,0),\quad C(0,0,-1), D(0,2,2),\quad E(5,-5,0)$$

koordinatalar sistemasida qanday joylashgan?

1352. Quyidagi nuqtalar orasidagi masofani toping:

a) $A(2,-3,-1)$ bilan $B(5,2,-5)$,

b) $A(1,-5,3)$ bilan $B(5,-1,7)$.

1353. AB kesma C, D, E va F nuqtalar yordamida 5 ta teng bo'lakka bo'lingan. Agar $C(3,-5,7)$, $F(-2,4,-8)$ bo'lsa, boshqa bo'luvchi nuqtalarning koordinatalarini toping.

1354. Uchinchi oktantda koordinat o'qlaridan

$$d_x=5,\quad d_y=3\sqrt{5},\quad d_z=\frac{2}{\sqrt{13}}$$

masofada joylashgan $M(x,y,z)$ nuqtani toping.

1355. Koordinatalar boshidan 8 ga teng masofada joylashgan $(ON=8)$ va ON kesma OX o'qi bilan $\frac{\pi}{4}$, OZ o'qi bilan $\frac{\pi}{3}$ burchak tashkil etuvchi $M(x,y,z)$ nuqtaning koordinatalarini toping.

1356. Uchlari $A=A(2,5,0)$, $B=B(11,3,8)$, $C=C(5,1,12)$ nuqtalarda bo'lgan uchburchakning og'irlik markazini toping.

1357. Bir jinsli sterjenning og'irlik markazi $M(1,-1,5)$ nuqtada bo'lib, uning bir uchi $A(-2,-1,7)$ nuqtada. Sterjen ikkinchi uchining koordinatalarini toping.

1358. Ushbu $E(0,y,0)$ nuqta fazoning qayerida joylashgan?

a) Koordinatalar boshida;

b) OY koordinata o'qida;

c) OX koordinata o'qida;

d) OZ koordinata o'qida.

1359. $A(-3,4,5)$ nuqtadan OZ o'qigacha bo'lgan masofani toping.

A) 4, B) 5,C) 6,D) $\sqrt{34}$.

2-§. Fazoda tekislik

1°. Tekislikning umumiy tenglamasi

Fazoda tekislikni ikki nuqtadan bir xil masofada joylashgan fazo nuqtalari to'plami (fazo nuqtalarining geometrik o'rni) sifatida qarash mumkin. Bunday nuqtalarning koordinatalari x,y,z lar ushbu:

$$Ax+By+Cz+D=0 \tag{1}$$

munosabatda bo'ladi, bunda A,B,C,D – o'zgarmas sonlar. (1) munosabat **tekislikning umumiy tenglamasi** deyiladi.

2°. Tekislikning kesmalar bo'yicha tenglamasi. Fazoda ushbu

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \tag{2}$$

tenglama tekislikning **kesmalar bo'yicha tenglamasi** deyiladi. Bunda a,b,c sonlar sodda geometrik ma'noga ega, ular tekislikning koordinatalar o'qlaridan ajratgan kesmalarning miqdorini bildiradi.

1-misol. Ushbu

$$3x+5y-7z+6=0$$

tekislikning umumiy tenglamasini uning kesmalar bo'yicha tenglamasiga keltiring.

◀Berilgan tekislik tenglamasini quyidagicha yozib,

$$3x+5y-7z=-6$$

so'ng bu tenglikning ikki tomonini -6 ga bo'lamiz:

Natijada,

$$\frac{3x}{-6}+\frac{5y}{-6}-\frac{7z}{-6}=1.$$

$$\frac{x}{-2}+\frac{y}{\frac{-6}{5}}+\frac{z}{\frac{6}{7}}=1$$

bo‘ladi. Bu tekislikning kesmalar bo‘yicha tenglamasidir.▶

3⁰. Uch nuqtadano‘tuvchi tekislik tenglamasi. Aytaylik, fazoda uchta $M_1(x_1,y_1,z_1)$, $M_2(x_2,y_2,z_2)$ va $M_3(x_3,y_3,z_3)$ nuqtalar berilgan bo‘lsin. Bu nuqtalar orqalio‘tuvchi tekislik tenglamasi quyidagicha bo‘ladi:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0 \qquad (3)$$

bo‘ladi.

2-misol. Ushbu

$$A_1(1,2,3),\ A_2(4,-1,-2),\ A_3(4,0,3)$$

nuqtalardan o‘tuvchi tekislik tenglamasini toping.

◀(3) formuladan foydalanib,

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 4-1 & -1-2 & -2-3 \\ 4-1 & 0-2 & 3-3 \end{vmatrix}=0,$$

ya’ni

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 3 & -3 & -5 \\ 3 & -2 & 0 \end{vmatrix}=0$$

bo‘lishini topamiz. Endi bu uchinchi tartibli determinantni hisoblaymiz:

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 3 & -3 & -5 \\ 3 & -2 & 0 \end{vmatrix}=(x-1)\cdot(-3)\cdot 0+$$

$$+(y-2)\cdot(-5)\cdot 3+3\cdot(-2)\cdot(z-3)-(z-3)\cdot(-3)\cdot 3-$$

$$-(x-1)\cdot(-5)\cdot(-2)-(y-2)\cdot 3\cdot 0=$$

$$=-10(x-1)-15(y-2)+3(z-3).$$

Demak, izlanayotgan tekislik tenglamasi

$$-10(x-1)-15(y-2)+3(z-3)=0,$$

ya’ni

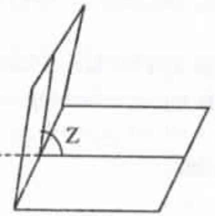
$$10x+15y-3z-31=0$$

bo‘ladi.▶

4⁰. Ikki tekislik orasidagi burchak. Ikki tekislik berilgan bo‘lib, ularning tenglamalari

$$A_1x+B_1y+C_1z+D_1=0,$$
$$A_2x+B_2y+C_2z+D_2=0$$

bo‘lsin. Bu tekisliklar hosil qilgan qo‘shni ikkiyoqli burchaklarda biri (bu qo‘shni burchaklar yig‘indisi π ga teng bo‘ladi) ikki tekislik orasidagi burchak deyiladi (6-chizma).



6-chizma

Bu burchak quyidagi formula bilan topiladi:

$$\cos\varphi=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\cdot\sqrt{A_2^2+B_2^2+C_2^2}} \qquad (4)$$

3-misol. Ushbu

$$x-z=0,$$
$$y-z=0$$

tekisliklar orasidagi burchakni toping.

◀Bu tekisliklar uchun

$$A_1=1,\quad B_1=0,\quad C_1=-1,$$
$$A_2=0,\quad B_2=1,\quad C_2=-1$$

bo‘ladi. (4) formuladan foydalanib topamiz:

$$\cos\varphi=\frac{1\cdot 0+0\cdot 1+(-1)\cdot(-1)}{\sqrt{1^2+0^2+(-1)^2}\cdot\sqrt{0^2+1^2+(-1)^2}}=\frac{1}{2}.$$

Demak, $\varphi=60^0$.▶

5⁰. Ikki tekislikning parallellik hamda perpendikulyarlik shartlari. Agar ikkita:

$$A_1x+B_1y+C_1z+D_1=0$$
$$A_2x+B_2y+C_2z+D_2=0$$

tekisliklar uchun

$$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2} \qquad (5)$$

shart bajarilsa, u holda tekisliklar o‘zaro parallel bo‘ladi.

Agar bu tekisliklar uchun

$$A_1\cdot A_2+B_1\cdot B_2+C_1\cdot C_2=0 \qquad (6)$$

shart bajarilsa, u holda tekisliklar o‘zaro perpendikulyar bo‘ladi.

4-misol. $M_1(3,-1,2)$ va $M_2(-1,2,5)$ nuqtalardan o'tuvchi hamda OZ o'qiga parallel bo'lgan tekislik tenglamasini toping.

◀Ma'lumki, tekislikning umumiy tenglamasi:

$$Ax+By+Cz+D=0$$

da $C=0$ bo'lsa, tekislik OZ o'qiga parallel bo'lib, u

$$Ax+By+D=0 \qquad (7)$$

ko'rinishga ega bo'ladi. Modomiki, bu tekislik M_1 va M_2 nuqtalar orqali o'tar ekan, unda bu nuqtalarning koordinatalari tekislik tenglamasini qanoatlantiradi. $x_1=3,\ y_1=-1$ va $x_2=-1,\ y_2=2$ larni (7) tenglamadagi x va y larning o'rniga qo'yib, ushbu:

$$\begin{cases}3A-B+D=0,\\-A+2B+D=0\end{cases}$$

sistemani hosil qilamiz. Bu sistemadan

$$A=-\frac{3}{5}D,\quad B=-\frac{4}{5}D$$

bo'lishini topamiz. Natijada,

$$Ax+By+D=0$$

tenglama ushbu

$$-\frac{3}{5}Dx+\left(-\frac{4}{5}D\right)y+D=0$$

ya'ni

$$3x+4y-5=0$$

ko'rinishga keladi. Bu izlanayotgan tekislik tenglamasidir.▶

6^0. Nuqtadan tekislikkacha bo'lgan masofa. Fazoda biror

$$Ax+By+Cz+D=0$$

tekislik va bu tekislikda yotmagan $M_0=M_0(x_0,y_0,z_0)$ nuqta berilgan bo'lsin. Bu nuqtadan tekislikka o'tkazilgan perpendikulyarning uzunligi berilgan nuqtadan berilgan tekislikkacha masofa deyiladi. U quyidagi formula:

$$d=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}} \qquad (8)$$

yordamida topiladi.

5-misol. $A(2,3,-4)$ nuqtadan ushbu $2x+6y-3z+16=0$ tekislikkacha bo'lgan masofani toping.

◀Bu masofani (8) formuladan foydalanib topamiz:

$$(x_0=2, y_0=3, z_0=-4,\ A=2,\ B=6,\ C=-3,\ D=16)$$

$$d=\frac{|2\cdot2+6\cdot3-3\cdot(-4)+16|}{\sqrt{4+36+9}}=\frac{50}{7}=7\frac{1}{7}.$$ ▶

Quyidagi masalalarni yeching

1360. Quyidagi uchta nuqtadan o'tuvchi tekislik tenglamasini toping.

a) $A_1(1,1,1),\ B_1(1,-1,0),\ C_1(2,1,3)$

b) $A_2(3,-1,2),\ B_2(4,-1,-1),\ C_2(2,0,2)$

1361. Quyidagi $A(-1,6,3),\ B(3,-2,-5),\ C(2,0,5), D(2,7,0),\ E(0,1,0)$ nuqtalardan qaysi biri ushbu:

$$4x-y+3z+1=0$$

tekislikda yotadi?

1362. Ushbu:

a) $2x+3z+1=0$;

b) $9z-3=0$;

c) $x-2y-5=0$;

d) $x+y+z=0$;

e) $8x-5y=0$;

f) $5y+3z-5=0$

tekisliklarning koordinata o'qlariga nisbatan joylashish holatini aniqlang.

1363. $M\{1,2,3\}$ nuqtadan o'tuvchi hamda XOY tekisligiga parallel bo'lgan tekislik tenglamasini toping.

1364. $M\{1,1,1\}$ nuqtadan o'tuvchi hamda

$$2x+4y+z-5=0$$

tekislikka

a) perpindikulyar bo'lgan,

b) parallel bo'lgan tekislik tenglamasini toping.

1365. OY o'qi orqali hamda $M\{-1,5,3\}$ nuqtadan o'tuvchi tekislik tenglamasini toping.

1366. Ikki $M_1\{2,-1,3\},\ M_2\{3,1,2\}$ nuqtalardan o'tuvchi hamda $3x-y-4z=0$ tekislikka perpindikulyar bo'lgan tekislik tenglamasini toping.

1367. $M\{1,2,3\}$ nuqtadan o'tuvchi hamda OZ o'qiga perpindikulyar bo'lgan tekislik tenglamasini toping.

1368. Ushbu

a) $M\{4,3,-2\}$ nuqtadan $3x-y+5z+1=0$ tekislikkacha,

b) $M\{3,1,-1\}$ nuqtadan $22x+4y-20z-41=0$ tekislikkacha,

c) $M\{1,2,1\}$ nuqtadan $2x-3y+6z-7=0$ tekislikkacha,

d) $M\{0,0,0\}$ nuqtadan $x-y+\sqrt{2}z-8=0$ tekislikkacha

masofalarni toping.

1369. Ushbu tekislik tenglamalarini

a) $3x+5y-7z+6=0$,

b) $2x+y-5z-6=0$,

c) $3x+4y-3z-12=0$.

kesmalar ko'rinishidagi tenglamalarga keltiring.

1370. Quyidagi tekisliklar orasidagi burchakni toping.

a) $4x-5y+3z+1=0$, $x-4y-z+9=0$;

b) $3x-y+2z+15=0$, $5x+9y-3z-1=0$;

c) $6x+2y-4z+5=0$, $9x+3y-6z-2=0$;

d) $2x-3y+6z-7=0$, $4x-y+8z-14=0$.

1371. Ushbu parallel tekisliklar orasidagi masofani toping.

a) $x-2y-2z-1=0$, $x-2y-2z-6=0$;

b) $2x-3y+6z-1=0$, $4x-6y+12z+1=0$.

1372. Ushbu

$$11x-8y-7z-15=0 \text{ va } 4x-10y+z-2=0$$

tekisliklar fazoda qanday munosabatda joylashgan.

a) parallel;

b) perpendikulyar;

c) 45^0 burchak tashkil etadi;

d) 60^0 burchak tashkil etadi.

1373. OY o'qida ushbu:

$$x+2y-2z+6=0 \text{ va } 2x+y+2z-9=0$$

tekisliklardan baravar uzoqlikda joylashgan nuqtani toping.

A) (0,1,0),B) (0,-15,0), C) (0,1,0),D) (0,-7,0).

1374. Ushbu $M(1,2,3)$ nuqtadan koordinatalar tekisligigacha bo'lgan masofalar yig'indisini toping.

A) (0,1,0),B) (0,-15,0), C) (0,1,0),D) (0,-7,0).

3-§. Fazoda to'g'ri chiziq

1^0. To'g'ri chiziqning umumiy tenglamasi. Fazoda to'g'ri chiziq parallel bo'lmagan ikki

$$A_1x+B_1y+C_1z+D_1=0, \quad A_2x+B_2y+C_2z+D_2=0$$

tekislikning kesish nuqtalari to'plami (nuqtalarning geometrik o'rni) sifatida qaraladi. Shuning uchun ushbu:

$$\begin{cases} A_1x+B_1y+C_1z+D_1=0, \\ A_2x+B_2y+C_2z+D_2=0 \end{cases}$$

sistema fazoda **to'g'ri chiziqning umumiy tenglamasi** deyiladi.

1-misol. Ushbu

$$\begin{cases} 2x-3y-3z+4=0, \\ x+2y+\ z-5=0 \end{cases} \qquad (1)$$

to'g'ri chiziq yasalsin (fazoda tasvirlang).

◄Bu to'g'ri chiziqni XOY koordinata tekisligi bilan kesishish nuqtasini topamiz. Ma'lumki, XOY tekisligida (x,y,z) nuqtaning uchinchi koordinatasi $z=0$ bo'ladi. (1) sistemada $z=0$ deb ushbu:

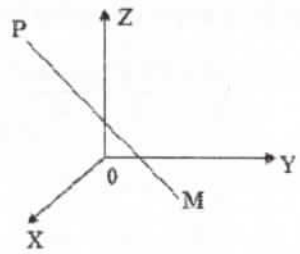
$$\begin{cases} 2x-3y+4=0, \\ x+2y-5=0 \end{cases}$$

sistemaga kelamiz va uni yechib, $x=1, y=2$ bo'lishini topamiz. Demak, to'g'ri chiziqning XOY tekisligi bilan kesishish nuqtasi $M(1,2,0)$ bo'ladi.

Endi, to'g'ri chiziqning XOZ tekisligi bilan kesishish nuqtasini topamiz. (1) sistemada $y=0$ deyilsa, unda:

$$\begin{cases} 2x-3z+4=0, \\ x+\ z-5=0 \end{cases}$$

bo'lib, bu sistemadan $x=2{,}2;\ z=2{,}8$ bo'lishini topamiz. Demak, to'g'ri chiziqning XOZ tekisligi bilan kesishish nuqtasi $P(2{,}2;0;2{,}8)$ bo'ladi. M va P nuqtalar orqali to'g'ri chiziqo'tkazamiz. Bu qaralayotgan to'g'ri chiziqning tasviri bo'ladi (7-chizma).



7-chizma

2^0. To'g'ri chiziqning kanonik (sodda) tenglamasi. Aytaylik, to'g'ri chiziqning umumiy tenglamasi:

$$\begin{cases} A_1x+B_1y+C_1z+D_1=0, \\ A_2x+B_2y+C_2z+D_2=0 \end{cases}$$

berilgan bo'lib, u (x_0,y_0,z_0) nuqta orqalio'tgan bo'lsin (ya'ni (x_0,y_0,z_0) tekisliklarning kesishish nuqtalaridan biri bo'lsin). U holda to'g'ri chiziq tenglamasini quyidagicha:

$$\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n} \qquad (2)$$

ko'rinishda yozish mumkin, bunda

$$l=\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix},\quad m=\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix},\quad n=\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \tag{3}$$

(2) tenglama to'g'ri chiziqning kanonik (sodda) tenglamasi deyiladi.

2-misol. To'g'ri chiziqning umumiy tenglamasi

$$\begin{cases} x+2y-3z+2=0, \\ 2x-2y+z-5=0 \end{cases} \tag{4}$$

ni kanonik ko'rinishidagi tenglamaga keltiring.

◀Avvalo, bu to'g'ri chiziqda yotuvchi $M(x_0, y_0, z_0)$ nuqtani topamiz. Buning uchun $z_0=0$ deb olamiz. Unda (4) sistema quyidagi:

$$\begin{cases} x+2y+2=0, \\ 2x-2y-5=0 \end{cases}$$

ko'rinishga keladi. Bu sistemaning yechimi $x_0=1,\ y_0=-\frac{3}{2}$ bo'ladi. Demak, $M\left(1;-\frac{3}{2};0\right)$ to'g'ri chiziq nuqtasi bo'ladi. $\left(x_0=1,\ y_0=-\frac{3}{2},\ z_0=0\right)$.

So'ng (3) va (4) munosabatlardan foydalanib l, m, n larni hisoblaymiz:

$$l=\begin{vmatrix} 2 & -3 \\ -2 & 1 \end{vmatrix}=-4,\quad m=\begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix}=-7,\quad n=\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix}=-6$$

Unda (2) formulaga ko'ra to'g'ri chiziqning kanonik tenglamasi:

$$\frac{x-1}{-4}=\frac{y+\frac{3}{2}}{-7}=\frac{z-0}{-6},$$

ya'ni:

$$\frac{x-1}{4}=\frac{y+\frac{3}{2}}{7}=\frac{z}{6}$$

bo'ladi.

3⁰. To'g'ri chiziqning parametrik tenglamasi. Fazodagi biror to'g'ri chiziqning kanonik tenglamasi

$$\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}$$

bo'lsin. Bu tenglikdagi har bir nisbat t deyilsa,

$$\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}=t$$

unda

$$\begin{cases} x=x_0+lt, \\ y=y_0+mt, \\ z=z_0+nt \end{cases} \tag{5}$$

sistema hosil bo'ladi.

(5) sistema to'g'ri chiziqning parametrik tenglamasi deyiladi (t – parametr).

3-misol. Ushbu:

$$\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{2}$$

to'g'ri chiziq bilan

$$2x+y+z-6=0$$

tekislikning kesishish nuqtasini toping.

◀Ravshanki, ko'rilayotgan to'g'ri chiziqning parametrik tenglamasi

$$x=2+t,$$
$$y=3+t,$$
$$z=4+t\cdot 2$$

bo'ladi. Bu – x, y, z lar qiymatlarini tekislik tenglamasidagi x, y, z lar o'rniga qo'yib,

$$2(2+t)+(3+t)+(4+2t)-6=0$$

bo'lishini, unda esa $t=-1$ ekanini topamiz.

Demak, $x=2-1=1,\ y=3-1=2,\ z=4-2=2$ bo'lib, kesishish nuqtasi $(1,2,2)$ bo'ladi.▶

4⁰. Ikki nuqtadano'tuvchi to'g'ri chiziq tenglamasi. Aytaylik, fazoda ikkita:

$$M_1=M_1(x_1,y_1,z_1)\ \ M_2=M_2(x_2,y_2,z_2)$$

nuqtalar berilgan bo'lsin. Bu nuqtalardan o'tuvchi to'g'ri chiziq tenglamasi quyidagicha:

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1} \tag{6}$$

bo'ladi.

4-misol. Ushbu:

$$2x+3y+5z-3=0,$$
$$x+y+2z-1=0$$

tekisliklarning kesishishidan hosil bo'lgan to'g'ri chiziqning sodda (kanonik) tenglamasini toping.

◀Aytaylik, $z=0$ bo'lsin. Unda masaladagi tenglamalar quyidagi

$$2x+3y=3,$$
$$x+y=1$$

ko'rinishga keladi. Bu sistemani yechib,

$$x=0, \quad y=1$$

bo'lishini topamiz.

Demak, $M_1(0,1,0)$ nuqta izlanayotgan to'g'ri chiziqda yotadi.

Endi $z=1$ deylik. Unda x va y larni aniqlaydigan

$$\begin{cases} 2x+3y=-2 \\ x+y=-1 \end{cases}$$

sistemaga kelamiz. Bu sistemaning yechimi $x=-1, \quad y=0$ bo'ladi. Demak, $M_2(-1,0,1)$ nuqta ham izlanayotgan to'g'ri chiziqda yotadi.

Ikki nuqtadan o'tuvchi to'g'ri chiziqning tenglamasini ifodalaydigan (6) formuladan foydalanib topamiz:

$$\frac{x}{-1}=\frac{y-1}{-1}=\frac{z}{1}.$$

Bu izlanayotgan to'g'ri chiziq tenglamasi bo'ladi.▶

5^0. Ikki to'g'ri chiziqning parallellik hamda perpendikulyarlik shartlari Agar fazodagi ikkita

$$\frac{x-x_1}{\ell_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}, \quad \frac{x-x_2}{\ell_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

to'g'ri chiziq uchun

$$\frac{\ell_1}{\ell_2}=\frac{m_1}{m_2}=\frac{n_1}{n_2} \qquad (7)$$

shart bajarilsa, u holda bu to'g'ri chiziqlar o'zaro parallel bo'ladi.

Agar yuqoridagi to'g'ri chiziqlar uchun:

$$\ell_1\cdot\ell_2+m_1\cdot m_2+n_1\cdot n_2=0 \qquad (8)$$

shart bajarilsa, u holda to'g'ri chiziqlar o'zaro perpendikulyar bo'ladi.

6^0. Ikki to'g'ri chiziq orasidagi burchak. Ikkita:

$$\frac{x-x_1}{\ell_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}, \quad \frac{x-x_2}{\ell_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

to'g'ri chiziqlar berilgan bo'lsin.

Fazoda biror nuqta olib, undan berilgan to'g'ri chiziqlarga parallel bo'lgan to'g'ri chiziqlarni o'tkazamiz. Ular orasidagi φ burchak berilgan to'g'ri chiziqlar orasidagi burchak deyiladi. U ushbu

$$\cos\varphi=\frac{\ell_1\cdot\ell_2+m_1\cdot m_2+n_1\cdot n_2}{\sqrt{\ell_1^2+m_1^2+n_1^2}\cdot\sqrt{\ell_2^2+m_2^2+n_2^2}} \qquad (9)$$

formula bilan topiladi.

5-misol. Ushbu:

$$\frac{x-1}{3}=\frac{y+2}{6}=\frac{z-5}{2} \text{ va } \frac{x}{2}=\frac{y-3}{9}=\frac{z+1}{6}$$

to'g'ri chiziqlar orasidagi o'tkir burchakni toping.

◀Bu to'g'ri chiziqlar uchun

$$l_1=3,\ m_1=6,\ n_1=2 \text{ va } l_2=2,\ m_2=9,\ n_2=6$$

bo'ladi. (9) formuladan foydalanib topamiz:

$$\cos\varphi=\frac{3\cdot2+6\cdot9+2\cdot6}{\sqrt{3^2+6^2+2^2}\cdot\sqrt{2^2+9^2+6^2}}=\pm\frac{72}{\sqrt{49}+\sqrt{121}}=\pm\frac{72}{77}$$

Demak, berilgan to'g'ri chiziqlar orasidagi o'tkir burchak

$$\varphi=arc\cos\left(\frac{72}{77}\right)$$

bo'ladi.▶

Quyidagi masalalarni yeching

1375. Ushubu to'g'ri chiziqlarning kanonik tenglamasini tuzing.

a) $\begin{cases} x-2y+3z+1=0, \\ 2x-y-4z-8=0 \end{cases}$ b) $\begin{cases} x-2y+3z-4=0, \\ 3x+2y-5z-4=0. \end{cases}$

c) $\begin{cases} 5x+y+z=0, \\ 2x+3y-2z+5=0. \end{cases}$ d) $\begin{cases} 2x-3y-3z=0, \\ x-2y+z=0. \end{cases}$

1376. Quyidagi to'g'ri chiziqlar orasidagi o'tkir burchakni toping.

a) $\frac{x-1}{3}=\frac{y+2}{6}=\frac{z-5}{2}, \quad \frac{x}{2}=\frac{y-3}{9}=\frac{z+1}{6};$

b) $\frac{x-2}{1}=\frac{y+2}{-1}=\frac{z-5}{\sqrt{2}}, \quad \frac{x+2}{1}=\frac{y-3}{1}=\frac{z+5}{\sqrt{2}}.$

1377. Ushbu

$$\begin{cases} x-2y+3z-4=0, \\ 3x-2y+z=0 \end{cases}$$

to'g'ri chiziqning yo'naltiruvchi kosinuslarini toping.

1378. Ushbu $M(2,5,4)$ nuqtadan o'tuvchi hamda quyidagi

$$\begin{cases} 11x-3y-3z+20=0, \\ x-3y-6z+1=0 \end{cases}$$

to'g'ri chiziqqa parallel bo'lgan to'g'ri chiziq tenglamasini toping.

1379. Ushbu

$$\frac{x-2}{1}=\frac{y-3}{-1}=\frac{z}{\sqrt{2}}, \quad \frac{x+1}{1}=\frac{y-2}{1}=\frac{z+5}{\sqrt{2}}$$

to'g'ri chiziqlar orasidagi φ burchakni toping.

1380. Quyidagi

$$\frac{x+2}{2}=\frac{y}{-3}=\frac{z-1}{4}, \quad \frac{x-3}{m}=\frac{y-1}{4}=\frac{z-7}{2}$$

to'g'ri chiziqlar m ning qanday qiymatida kesishadi?

1381. $M(-2,-3,5)$ nuqtadan o'tuvchi hamda OY o'qiga parallel bo'lgan to'g'ri chiziq tenglamasini toping.

1382. $M(1,-5,3)$ nuqtadan o‘tuvchi hamda koordinata o‘qlari bilan mos ravishda $\alpha=\frac{\pi}{3}$, $\beta=\frac{\pi}{4}$, $\gamma=\frac{2\pi}{3}$ burchak tashkil etuvchi to‘g‘ri chiziq tenglamasini toping.

1383. $M(2,3,4)$ nuqtadan ushbu

$$\frac{x}{1}=\frac{y-2}{3}=\frac{z-7}{0}$$

to‘g‘ri chiziqqacha bo‘lgan masofani toping.

1384. Ushbu

$$\frac{x}{11}=\frac{y+1}{8}=\frac{z-1}{7} \text{ va } \frac{x-4}{7}=\frac{y}{-2}=\frac{z+1}{8}$$

To‘g‘ri chiziqlar fazoda qanday munosabatda joylashgan?

A) perpendikulyar;
B) 30^0 burchak ostida kesishadi;
C) parallel;
D) 45^0 burchak ostida kesishadi.

1385. Ushbu $M(2,3,1)$ nuqtadan

$$\frac{x+5}{1}=\frac{y-4}{-3}=\frac{z-3}{-2}$$

to‘g‘ri chiziqqacha bo‘lgan masofani toping.

A) $2\sqrt{10}$, B) $3\sqrt{10}$, C) $0,5\sqrt{10}$, D) $\sqrt{10}$.

4-§. Fazoda tekislik va to‘g‘ri chiziq

Fazoda

$$\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n} \qquad (1)$$

to‘g‘ri chiziq hamda

$$Ax+By+Cz+D=0 \qquad (2)$$

tekislik berilgan bo‘lsin.

1^0. To‘g‘ri chiziq va tekislik orasidagi burchak. (1) to‘g‘ri chiziq bilan (2) tekislik orasidagi burchak quyidagi

$$\sin\varphi=\frac{Al+Bm+Cn}{\sqrt{A^2+B^2+C^2}\cdot\sqrt{l^2+m^2+n^2}} \qquad (3)$$

formula yordamida topiladi.

1-misol. $A(-1,0,-5)$, $B(1,2,0)$ nuqtalardan o‘tuvchi to‘g‘ri chiziqning

$$x-3y+z+5=0$$

tekislik bilan tashkil etgan burchakni toping.

◀3-§dagi (6) formuladan foydalanib, A va B nuqtalardan o‘tuvchi to‘g‘ri chiziqni topamiz:

$$\frac{x-(-1)}{1-(-1)}=\frac{y-0}{2-0}=\frac{z-(-5)}{0-(-5)}$$

ya’ni

$$\frac{x+1}{2}=\frac{y}{2}=\frac{z+5}{5} \qquad (l=2, m=2, n=5)$$

Bu to‘g‘ri chiziq bilan berilgan

$$x-3y+z+5=0 \qquad (A=1, B=-3, C=1)$$

tekislik orasidagi burchak (3) formulaga ko‘ra,

$$\sin\varphi=\frac{|1\cdot2+(-3)\cdot2+1\cdot5|}{\sqrt{1^2+(-3)^2+1^2}\cdot\sqrt{2^2+2^2+5^2}}=\frac{1}{\sqrt{11}+\sqrt{33}}=\frac{\sqrt{3}}{33}$$

bo‘ladi. Demak, $\varphi=arc\sin\left(\frac{\sqrt{3}}{33}\right)$.▶

2^0. To‘g‘ri chiziq va tekislikning parallellik sharti. Agar (1) to‘g‘ri chiziq hamda (2) tekislik uchun

$$Al+Bm+Cn=0 \qquad (4)$$

bo‘lsa, u holda to‘g‘ri chiziq bilan tekislik parallel bo‘ladi.

3^0. To‘g‘ri chiziq va tekislikning perpendikulyarlik sharti. Agar (1) to‘g‘ri chiziq hamda (2) tekislik uchun

$$\frac{A}{l}=\frac{B}{m}=\frac{C}{n} \qquad (5)$$

bo‘lsa, u holda to‘g‘ri chiziq bilan tekislik perpendikulyar bo‘ladi.

4^0. To‘g‘ri chiziq va tekislikning kesishishi. Agar $\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}$ to‘g‘ri chiziq hamda $Ax+By+Cz+D=0$ tekislik uchun

$$Al+Bm+Cn\neq0 \qquad (6)$$

bo‘lsa, u holda to‘g‘ri chiziq tekislik bilan bitta nuqtada kesishadi. Bu kesishish nuqtasi $P(x,y,z)$ning koordinatalari

$$x=x_0+lt,\ y=y_0+mt,\ z=z_0+nt$$

bo‘lib,

$$t=-\frac{Ax_0+By_0+Cz_0}{Al+Bm+Cn}$$

bo‘ladi.

<u>**Eslatma:**</u> a) agar $\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}$ to‘g‘ri chiziq hamda $Ax+By+Cz+D=0$ tekislik uchun

$$\begin{matrix} Al+Bm+Cn=0 \\ Ax_0+By_0+Cz_0=0 \end{matrix} \qquad (7)$$

bo‘lsa, u holda to‘g‘ri chiziq tekislikda butunlay yotadi;

b) agar $\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}$ to'g'ri chiziq hamda $Ax+By+Cz+D=0$ tekislik uchun

$$\begin{aligned} &Al+Bm+Cn=0 \\ &Ax_0+By_0+Cz_0\neq 0 \end{aligned} \qquad (8)$$

bo'lsa, u holda to'g'ri chiziq tekislikka parallel bo'ladi.

5^0. Ikki to'g'ri chiziqning bir tekislikda yotish sharti. Agar

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \text{ va } \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

hamda

$$Ax+By+Cz+D=0$$

tekislik uchun

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0 \qquad (9)$$

bo'lsa, u holda ikki to'g'ri chiziq bitta tekislikda yotadi.

2- misol. $M(2,3,1)$ nuqtadan ushbu

$$\frac{x+1}{2}=\frac{y}{-1}=\frac{z-2}{3} \qquad (*)$$

to'g'ri chiziqqa tushirilgan perpendikulyarning kanonik tenglamasini toping.

◀Ravshanki, $M(2,3,1)$ nuqtadano'tuvchi har qanday to'g'ri chiziqning kanonik tenglamasi

$$\frac{x-2}{l}=\frac{y-3}{m}=\frac{z-1}{n} \qquad (**)$$

bo'ladi.

(*) va (**) to'g'ri chiziqlar perpendikulyar bo'lishi kerakligi shartga ko'ra

$$2l-m+3n=0$$

ya'ni:

$$2\frac{l}{n}-\frac{m}{n}=-3$$

bo'ladi.

Modomiki, (*) va (**) to'g'ri chiziqlar kesishar ekan, unda ular bir tekislikda yotadi.

To'g'ri chiziqlarning bir tekislikda yotish shartidan foydalanib topamiz:

$$\begin{vmatrix} 2-(-1) & 3-0 & 1-2 \\ 2 & -1 & 3 \\ l & m & n \end{vmatrix}=0$$

ya'ni

$$\begin{vmatrix} 3 & 3 & -1 \\ 2 & -1 & 3 \\ l & m & n \end{vmatrix}=8l-11m-9n=0$$

Keyingi tenglamadan

$$8\frac{l}{n}-11\frac{m}{n}=9$$

bo'lishi kelib chiqadi.

Natijada,

$$\begin{cases} 2\frac{l}{n}-\frac{m}{n}=-3 \\ 8\frac{l}{n}-11\frac{m}{n}=9 \end{cases}$$

sistema hosil bo'ladi. Uni yechib,

$$\frac{l}{n}=-3,\ \frac{m}{n}=-3$$

bo'lishini topamiz. Demak, $l=3$, $m=3$, $n=-1$.

Bu qiymatlar (**) tenglikka qo'yilsa, unda:

$$\frac{x-2}{3}=\frac{y-3}{3}=\frac{z-1}{-1}$$

bo'ladi. Bu izlanayotgan to'g'ri chiziq tenglamasidir.▶

Quyidagi masalalarni yeching

1386. Ushbu

$$\frac{x-1}{4}=\frac{y}{12}=\frac{z-1}{-3}$$

to'g'ri chiziq hamda $6x-3y+2z=0$ tekislik orasidagi burchakni toping.

1387. $M(3,-2,4)$ nuqtadan o'tuvchi hamda ushbu

$$5x+3y-2z+1=0$$

tekislikka perpindikulyar bo'lgan to'g'ri chiziq tenglamasini toping.

1388. $M(3,-2,4)$ nuqtadan hamda ushbu

$$\frac{x-4}{5}=\frac{y+3}{2}=\frac{z}{1}$$

to'g'ri chiziq orqali o'tuvchi tekislik tenglamasini toping.

1389. $O(0,0,0)$ nuqtadan ushbu

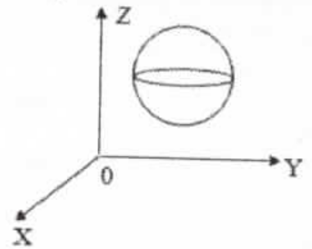
$$\frac{x-5}{4}=\frac{y-2}{3}=\frac{z+1}{-2}$$

to'g'ri chiziqqa tushirilgan perpindikulyarning tenglamasini toping.

5-§. Fazoda sodda ikkinchi tartibli sirtlar

Fazodagi sirtlar dekart koordinatalaridao'zgaruvchi nuqta $M(x,y,z)$ning koordinatalari x, y va z larga nisbatan ikkinchi darajali tenglamalar bilan ifodalanadi. Sfera, ellipsoid, giperbaloidlar, paraboloidlar, konus hamda silindrlar sodda ikkinchi tartibli sirtlar hisoblanadi.

1) **Sfera**:



8-chizma

Ushbu

$$(x-a)^2+(y-b)^2+(z-c)^2=r \qquad (1)$$

tenglama bilan aniqlanadigan sirt sfera deyiladi, bunda a, b, c lar sfera markazi, r-radius. (1) sfera tenglamasi bo'ladi.

Agar sfera markazi koordinatalar boshi bilan ustma-ust tushsa, unda $a=0,\ b=0,\ c=0$ bo'lib, sfera tenglamasi ushbu:

$$x^2+y^2+z^2=r^2 \qquad (2)$$

ko'rinishga keladi.

1-misol. Ushbu

$$x^2+y^2+z^2-2y-3z=0$$

sfera markazining koordinatalari hamda sfera radiusini toping.

◀Bu tenglikning chap tomonini quyidagicha yozib olamiz:

$$x^2+y^2+z^2-2y-3z=x^2+y^2-2y+1-1+z^2-2\frac{3}{2}z+\frac{9}{4}-\frac{9}{4}=$$

$$=x^2+(y-1)^2+\left(z-\frac{3}{2}\right)^2-\frac{13}{4}.$$

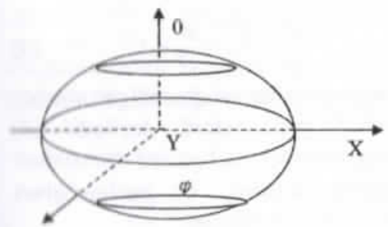
Natijada,

$$x^2+(y-1)^2+\left(z-\frac{3}{2}\right)^2=\frac{13}{4}$$

bo'lishi kelib chiqadi.

Demak, sferaning markazi $\left(0,1,\frac{3}{2}\right)$, radiusi esa $r=\frac{\sqrt{13}}{2}$ bo'ladi.▶

2) **Ellipsoid:**



9-chizma

Ushbu

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad (3)$$

tenglama bilan aniqlanadigan sirt **ellipsoid** deyiladi, bunda a, b, c lar ellipsoidning yarim o'qlari. (9-chizma)

2-misol. Ushbu

$$\frac{x^2}{36}+\frac{y^2}{16}+\frac{z^2}{9}=1$$

ellipsoid bilan $x=4$ tekislikning kesishishidan hosil bo'lgan chiziqni toping.

◀Ellipisoid tenglamasidan x ning o'rniga 4 ni qo'yamiz

$$\frac{16}{36}+\frac{y^2}{16}+\frac{z^2}{9}=1.$$

Natijada,

$$\frac{y^2}{16}+\frac{z^2}{9}=1-\frac{16}{36}=\frac{5}{9}$$

bo'lib, undan

$$\frac{y^2}{16\cdot\frac{5}{9}}+\frac{z^2}{9\cdot\frac{5}{9}}=1,\ \text{ya'ni}\ \frac{y^2}{\frac{80}{9}}+\frac{z^2}{5}=1$$

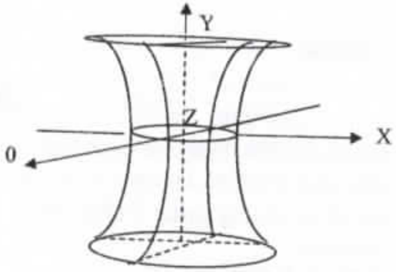
bo'lishi kelib chiqadi.

Demak, berilgan ellipsoid $x=4$ tekislik bilan

$$\frac{y^2}{\frac{80}{9}}+\frac{z^2}{5}=1$$

ellips bo'yicha kesishadi.▶

3) Giperboloidlar:



10-chizma

a) Ushbu

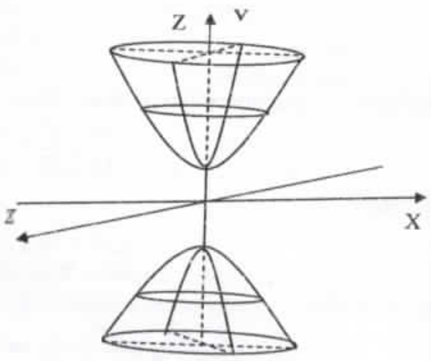
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad (4)$$

tenglama bilan aniqlanadigan sirt **bir pallali giperboloid** deyiladi. (4) tenglamadagi a,b,c lar bir pallali **giperboloidning yarimo'qlari** deyiladi (10-chizma).

b) Ushbu

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1 \qquad (5)$$

tenglama bilan aniqlanadigan sirt **ikki pallali giperboloid** deyiladi (11-chizma).



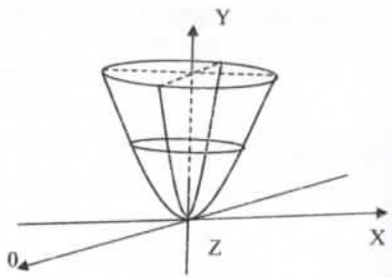
11-chizma

4) Paraboloid:

a) Ushbu

$$2z=\frac{x^2}{\rho}+\frac{y^2}{q} \quad (\rho>0,\ q>0) \qquad (6)$$

tenglama bilan aniqlanadigan sirt **elliptik paraboloid** deyiladi (12-chizma).

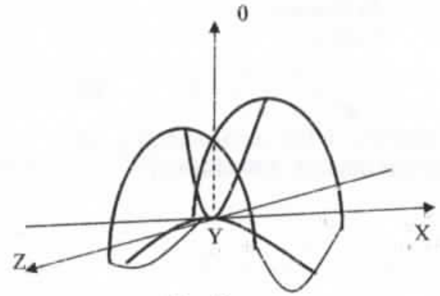


12-chizma

b) Ushbu:

$$2z=\frac{x^2}{\rho}-\frac{y^2}{q} \quad (\rho>0,\ q>0) \ (7)$$

tenglama bilan aniqlanadigan sirt **giperbolik paraboloid** deyiladi. (13-chizma)



13-chizma

3-misol. Ushbu:

$$\frac{x^2}{25}+\frac{y^2}{9}=z$$

elliptik paraboloid bilan

$$\frac{x}{10}=\frac{y+3}{3}=\frac{z-1}{3}$$

to'g'ri chiziqning kesishish nuqtalarini toping.

◀Avvalo, to'g'ri chiziqni, uning parametrik ko'rinishidagi tenglamasiga keltiramiz. U quyidagicha bo'ladi:

$$x=10t,$$
$$y=3t-3,$$
$$z=3t+1.$$

Bu $x,\ y,\ z$ larning qiymatlarini elliptik paraboloid tenglamasidagi $x,\ y,\ z$ lar o'rniga qo'yamiz. Natijada,

$$\frac{(10t)^2}{25}+\frac{(3t-3)^2}{9}=3t+1,$$

ya'ni

$$5t^2-5t=0$$

bo'lib, $t_1=0,\ t_2=1$ bo'ladi. Bu qiymatlarni to'g'ri chiziqning parametrik ko'rinishidagi tenglamasiga qo'yib topamiz:

$$x_1=0,\quad y_1=-3,\quad z_1=1,$$
$$x_2=10,\quad y_2=0,\quad z_2=4.$$

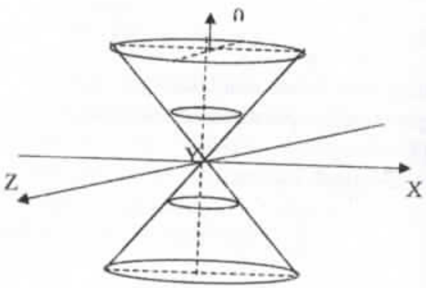
Demak, berilgan elliptik paraboloid va to'g'ri chiziq ikkita $M_1(0,-3,1)$ va $M_2(10,0,4)$ nuqtalarida kesishadi.▶

5) Konus:

Ushbu:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0 \qquad (8)$$

tenglama bilan aniqlanadigan sirt **konus** deyiladi. (14-chizma)



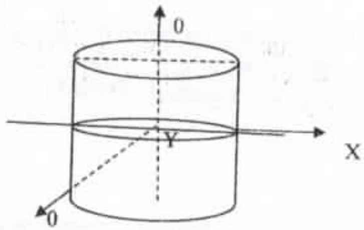
14-chizma

6) Silindr:

a) Ushbu:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (9)$$

tenglama bilan aniqlanadigan sirt **elliptik silindr** deyiladi (15-chizma).

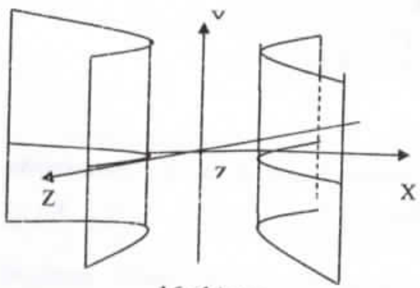


15-chizma

b). Ushbu

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad (10)$$

tenglama bilan aniqlanadigan sirt **giperbolik silindr** deyiladi (16-chizma).



16-chizna

Quyidagi masalalarni yeching

1390. Markazi $C(2,-2,1)$ nuqtada bo'lgan va koordinata boshidan o'tuvchi sferaning tenglamasini toping.

1391. Quyidagi sirtlarning umumiy tenglamalarini sodda (kanonik) ko'rinishiga keltirib, ularning nomlarini aniqlang.

a) $x^2+y^2+z^2+2x+4y-4=0$;

b) $x^2+2y^2+z^2+2x+4y-1=0$;

c) $x^2+2y^2-z^2+2x+4y-1=0$;

d) $x^2+2y^2+2x+4y-2z+3=0$;

e) $x^2-4y^2-z^2+8y-2z-9=0$;

f) $x^2+2y^2-2x-4y-1=0$.

1392. Fazoda quyidagi tenglamalar bilan ifodalangan sirtlarni aniqlang:

1) $x^2+z^2=9$, 2) $\frac{y^2}{25}-\frac{z^2}{16}=1$, 3) $y^2=6z$,

4) $x^2-z^2=0$, 5) $y^2+z^2=0$, 6) $x^2+4y^2+4=0$,

7) $x^2=0$, 8) $x^2+y^2=0$, 9) $x^2+y^2+z^2=0$

1393. Ushbu: $(1,0,0)$, $(0,4,0)$, $(1,1,1)$ nuqtalardan o'tuvchi giperbolidning kanonik tenglamasini toping.

1394. Ushbu:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

bir pallali giperbolidning $(0,0,c)$ nuqtasiga urinuvchi urinma tekislikning tenglamasini toping.

1395. Ko'rsatilgan sirtlar bilan to'g'ri chiziqlarning kesishish nuqtalari toping.

a) $\frac{x^2}{81}+\frac{y^2}{36}+\frac{z^2}{9}=1$, $\frac{x-3}{3}=\frac{y-4}{-6}=\frac{z+2}{4}$;

b) $\frac{x^2}{5}+\frac{y^2}{3}=z$, $\frac{x+1}{2}=\frac{y-2}{-1}=\frac{z+3}{-2}$.

1396. Ushbu:

$$\frac{x^2}{16}+\frac{y^2}{12}+\frac{z^2}{4}=1$$

ellipsid bilan $x-2=0$ tekislikning kesishishidan hosil bo'lgan ellipsning yarim o'qlarini toping.

Nazorat savollari

1. Fazoda ikki nuqtalar orasidagi masofa qanday hisoblanadi?
2. Kesmani nisbatda bo'lish qanday aniqlanadi?
3. Tekislikning umumiy tenglamasini izohlab bering.
4. Tekislikning kesmalar bo'yicha tenglamasini izohlab bering.
5. Uch nuqtadano'tuvchi tekislik tenglamasi qanday topiladi?
6. Ikki tekislik orasidagi burchak qanday aniqlanadi?
7. Ikki tekislikning parallellik hamda perpendikulyarlik shartlari qanday aniqlanadi?
8. Fazoda nuqtadan tekislikkacha bo'lgan masofa qanday topiladi?.
9. Fazoda to'g'ri chiziqning umumiy tenglamasi qanday aniqlanadi?
10. Fazoda to'g'ri chiziqning kanonik (sodda) tenglamasi qanday aniqlanadi?
11. Fazoda to'g'ri chiziqning parametrik tenglamasi qanday aniqlanadi?
12. Fazoda ikki nuqtadan o'tuvchi to'g'ri chiziq tenglamasi qanday aniqlanadi?.
13. Fazoda ikki to'g'ri chiziqning parallellik hamda perpendikulyarlik shartlari qanday aniqlanadi?
14. Fazoda ikki to'g'ri chiziq orasidagi burchak qanday aniqlanadi?
15. Fazoda to'g'ri chiziq va tekislik orasidagi burchak qanday aniqlanadi?
16. Fazoda to'g'ri chiziq va tekislikning parallellik sharti qanday aniqlanadi?
17. Fazoda to'g'ri chiziq va tekislikning perpendikulyarlik sharti qanday aniqlanadi?
18. Fazoda to'g'ri chiziq va tekislikning kesishishi qanday aniqlanadi?
19. Fazoda ikki to'g'ri chiziqning bir tekislikda yotish sharti qanday aniqlanadi?

12-bob
Vektorlar analizining elementlari
1-§. Vektor hisobining asosiy formulasi. Vektorlar ustida amallar

Tekislikda vektorlar va ular haqidagi asosiy ma'lumotlar hamda masalalar 1-bob 4-§da bayon etilgan. Vektorlarning fazoviy masalalarga tatbiqlari muhimligini e'tiborga olib, ushbu bobda fazoda vektorlar analizi va unga doir masalalarni keltiramiz. Bunda 1-bob 4-§da keltirilgan ma'lumotlardan foydalanamiz.

1°. Vektor hisobining asosiy formulasi. Fazoda $M = M(x_1, y_1, z_1)$ va $N = N(x_2, y_2, z_2)$ nuqtalar yordamida hosil qilingan

$$\vec{a} = \overline{MN}$$

vektorni qaraylik. Bu vektorning

$$OX, OY, OZ$$

koordinata o'qlaridagi proeksiyalari

$$a_x = x_2 - x_1, \quad a_y = y_2 - y_1, \quad a_z = z_2 - z_1 \quad (1)$$

bo'ladi. Unda $\vec{a}$ vektor quyidagicha:

$$\vec{a} = \{x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1\}$$

yozilib,

$$\vec{a} = a_x \cdot \vec{i} + a_y \cdot \vec{j} + a_z \cdot \vec{k} = (x_2 - x_1)\cdot \vec{i} + (y_2 - y_1)\cdot \vec{j} + (z_2 - z_1)\cdot \vec{k} \quad (2)$$

bo'ladi, bunda $\vec{i}, \vec{j}, \vec{k}$ – koordinata o'qlaridagi birlik vektorlar. (2) formula **vektor hisobining asosiy formulasi** deyiladi.

1-misol. Ikki $M_1(3,-4,1)$ va $M_2(4,6,-3)$ nuqtalar berilgan bo'lsin. $\vec{a} = \overline{M_1M_2}$ vektorning koordinatalarini toping.

$\vec{a}$ vektorning koordinatalari a_x, a_y, a_z lar (1) formulalar yordamida topiladi. Bu holda

$$x_1 = 3,\ y_1 = -4,\ z_1 = 1 \text{ va } x_2 = 4,\ y_2 = 6,\ z_2 = -3$$

bo'lib,

$$a_x = 4 - 3 = 1,\ a_y = 6 - (-4) = 10,\ a_z = -3 - 1 = -4$$

bo'ladi. Demak, $\vec{a} = \overline{M_1M_2} = \{1, 10, -4\}$

2-misol. Fazoda ikki – $M_1(x_1, y_1, z_1)$ va $M_2(x_2, y_2, z_2)$ nuqtalar berilgan bo'lsin. M_1M_2 kesmada shunday $M = M(x, y, z)$ nuqta topingki,

$$\frac{M_1M}{MM_2} = \lambda \qquad \lambda \text{ – berilgan son}$$

bo'lsin.

Ravshanki, $\overline{M_1M}$ va $\overline{MM_2}$ vektorlar uchun

$$\overline{M_1M} = \lambda \overline{MM_2}$$

bo'ladi. Vektor hisobining asosiy formulasi (2) dan foydalanib topamiz:

$$(x - x_1)\cdot \vec{i} + (y - y_1)\cdot \vec{j} + (z - z_1)\cdot \vec{k} = \lambda(x_2 - x)\cdot \vec{i} + \lambda(y_2 - y)\cdot \vec{j} + \lambda(z_2 - z)\cdot \vec{k}$$

Keyingi tenglikdan

$$x-x_1=\lambda(x_2-x),$$
$$y-y_1=\lambda(y_2-y),$$
$$z-z_1=\lambda(z_2-z)$$

bo'lib, undan

$$x=\frac{x_1+\lambda x_2}{1+\lambda},\ y=\frac{y_1+\lambda y_2}{1+\lambda},\ z=\frac{z_1+\lambda z_2}{1+\lambda}$$

bo'lishi kelib chiqadi.

2^0. Vektorning uzunligi va yo'nalishi. Vektorlar ustida chiziqli amallar. Ikki vektorlar orasidagi burchak. Aytaylik, $\vec{a}$ vektor koordinatalari orqali berilgan bo'lsin:

$$\vec{a}=\{a_x,a_y,a_z\}$$

Ushbu:

$$|\vec{a}|=\sqrt{a_x^2+a_y^2+a_z^2} \tag{3}$$

miqdor $\vec{a}$ vektorning uzunligi,

$$\cos\alpha=\frac{a_x}{|\vec{a}|},\ \cos\beta=\frac{a_y}{|\vec{a}|},\ \cos\gamma=\frac{a_z}{|\vec{a}|} \tag{4}$$

miqdorlar esa $\vec{a}$ vektorning yo'naltiruvchi kosinuslari deyiladi, bunda α,β,γ lar $\vec{a}$ vektorning mos ravishda OX, OY, OZ koordinata o'qlarining musbat yo'nalishlari orasidagi burchak.

Ikki

$$\vec{a}=a_x\cdot\vec{i}+a_y\cdot\vec{j}+a_z\cdot\vec{k},$$
$$\vec{b}=b_x\cdot\vec{i}+b_y\cdot\vec{j}+b_z\cdot\vec{k}$$

vektorlar berilgan bo'lsin. Bu vektorlarning yig'indisi, ayirmasi hamda vektorning songa ko'paytirish quyidagicha aniqlanadi:

$$\vec{a}+\vec{b}=(a_x+b_x)\cdot\vec{i}+(a_y+b_y)\cdot\vec{j}+(a_z+b_z)\cdot\vec{k} \tag{5}$$

$$\vec{a}-\vec{b}=(a_x-b_x)\cdot\vec{i}+(a_y-b_y)\cdot\vec{j}+(a_z-b_z)\cdot\vec{k} \tag{6}$$

$$\vec{a}\cdot\lambda=\lambda\cdot\vec{a}=(\lambda\cdot a_x)\cdot\vec{i}+(\lambda\cdot a_y)\cdot\vec{j}+(\lambda\cdot a_z)\cdot\vec{k} \tag{7}$$

Berilgan $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak ushbu:

$$\cos\varphi=\frac{a_xb_x+a_yb_y+a_zb_z}{\sqrt{a_x^2+a_y^2+a_z^2}\cdot\sqrt{b_x^2+b_y^2+b_z^2}} \tag{8}$$

formula yordamida topiladi.

Eslatma. Agar

$$a_xb_x+a_yb_y+a_zb_z=0 \tag{9}$$

bo'lsa, $\vec{a}$ va $\vec{b}$ vektorlar perpendikulyar,

$$\frac{a_x}{b_x}=\frac{a_y}{b_y}=\frac{a_z}{b_z} \tag{10}$$

bo'lsa, $\vec{a}$ va $\vec{b}$ vektorlar parallel bo'ladi.

3-misol. Agar $\vec{F}$ kuch

$$\vec{F}=\{4,4,-4\sqrt{2}\}$$

bo'lsa, bu kuchning miqdori va yo'nalishini toping.

◀Kuchning miqdorini, ya'ni $\vec{F}$ vektorning uzunligini (3) formuladan foydalanib topamiz:

$$|\vec{F}|=\sqrt{4^2+4^2+(-4\sqrt{2})^2}=\sqrt{16+16+32}=8.$$

$\vec{F}$ vektorning yo'naltiruvchi kosinuslari (4) formulaga ko'ra,

$$\cos\alpha=\frac{4}{|\vec{F}|}=\frac{4}{8}=\frac{1}{2},\ \cos\beta=\frac{1}{2},\ \cos\gamma=\frac{-4\sqrt{2}}{8}=-\frac{\sqrt{2}}{2}$$

bo'ladi. Keyingi tenglikdan $\alpha=60^0$, $\beta=60^0$ va $\gamma=135^0$ bo'lishini topamiz. Demak, $F=8$ bo'lib, bu vektor koordinata o'qlari bilan mos ravishda $\alpha=60^0$, $\beta=60^0$ va $\gamma=135^0$ burchaklar tashkil etadi. ▶

4-misol. Moddiy nuqta ushbu:

$$\vec{F_1}=2\vec{a}\ \text{ va }\ \vec{F_2}=-3\vec{b}$$

kuchlar ta'sirida bo'lib,

$$\vec{a}=7\vec{i}+2\vec{j}+3\vec{k},\ \vec{b}=3\vec{i}-2\vec{j}-3\vec{k}$$

bo'lsa, bu kuchlarga teng ta'sir etuvchi kuchni toping.

◀Bu kuchlarga teng ta'sir etuvchi bo'lgan $\vec{F}$ kuch (2) formulaga ko'ra,

$$\vec{F}=\vec{F_1}+\vec{F_2}=2\vec{a}-3\vec{b}$$

bo'ladi.

Ravshanki,

$$2\vec{a}=(2\cdot7)\cdot\vec{i}+(2\cdot2)\cdot\vec{j}+(2\cdot3)\cdot\vec{k}=14\vec{i}+4\vec{j}+6\vec{k},$$
$$-3\vec{b}=(-3)\cdot3\vec{i}+(-3)(-2)\vec{j}+(-3)(-3)\vec{k}=-9\vec{i}+6\vec{j}+9\vec{k}$$

Demak,

$$\vec{F}=2\vec{a}-3\vec{b}=(14-9)\cdot\vec{i}+(4+6)\cdot\vec{j}+(6+9)\cdot\vec{k}=5\vec{i}+10\vec{j}+15\vec{k}$$

bo'ladi. ▶

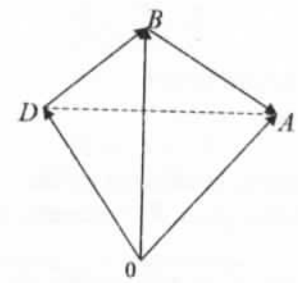
5-misol. Agar $ABCD$ to'rtburchakning tomonlari quyidagi

$$\overrightarrow{AB}=-\vec{i}+7\vec{j}-\vec{k},$$
$$\overrightarrow{BC}=-5\vec{i}-3\vec{j}+\vec{k},$$
$$\overrightarrow{AD}=-7\vec{i}-2\vec{j}$$

vektorlardan iborat bo'lsa, AC va BD diagonallarni o'zaro perpendikulyar bo'lishini isbotlang.

◀Ravshanki,

$$\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC},$$
$$\overrightarrow{BD}=\overrightarrow{AD}+\overrightarrow{AB}\qquad \text{(1-chizma)}$$



1-chizma

Ayni paytda,

$$\overrightarrow{AB}+\overrightarrow{BC}=(-1-5)\cdot\vec{i}+(7-3)\cdot\vec{j}+(-1+1)\vec{k}=-6\vec{i}+4\vec{j}$$
$$\overrightarrow{AD}+\overrightarrow{AB}=(-7-(-1))\cdot\vec{i}+(-2-7)\cdot\vec{j}+(0-(-1))\vec{k}=-6\vec{i}-9\vec{j}+\vec{k}$$

bo'ladi.

Demak,

$$\overrightarrow{AC}=-6\vec{i}+4\vec{j}$$
$$\overrightarrow{BD}=-6\vec{i}-9\vec{j}+\vec{k}$$

ya'ni koordinatalar shaklida

$$\overrightarrow{AC}=\{-6,4,0\}$$
$$\overrightarrow{BD}=\{-6,-9,1\}$$

bo'ladi.

Ravshanki, $-6\cdot(-6)+4\cdot(-9)+0\cdot1=0$ unda (6) shartga ko'ra, $\overline{AC}$ va $\overline{BD}$ vektorlar, ya'ni AC va BD diagonallar o'zaro perpendikulyar bo'ladi. ▶

Quyidagi masalalarni yeching

1397. Agar

a) $A(-1,5,2)$, $B(2,5,-2)$,

b) $A(1,3,0)$, $B(-2,3,0)$

bo'lsa, $\overrightarrow{AB}$ vektorning uzunligini toping.

1398. Agar $\vec{a}=3\vec{i}-2\vec{j}+6\vec{k}$, $\vec{b}=-2\vec{i}+\vec{j}$ bo'lsa,

$$\vec{a}+\vec{b},\ \vec{a}-\vec{b},\ 3\vec{a}+2\vec{b}$$

vektorlarni toping.

1399. Ushbu $\vec{a}=\{1,-1,\sqrt{2}\}$ vektorning uzunligi hamda yo'nalishini toping.

1400. Agar $\vec{a}$ vektorning uzunligi $|\vec{a}|=3$ bo'lib, uning koordinata o'qlari bilan tashkil etgan burchaklar uchun $\alpha=\beta=\gamma$ bo'lsa, $\vec{a}$ vektorning koordinatalarini toping.

1401. Ushbu:

a) $\vec{a}=\{2,-2,1\}$ va $\vec{b}=\{-4,1,1\}$

b) $\vec{a}=\{1,1,0\}$ va $\vec{b}=\{0,1,1\}$

vektorlar orasidagi burchakni toping.

1402. Agar

$$\vec{a}=3\vec{i}-4\vec{j}+6\vec{k}$$

bo'lsa, bu vektorning birlik vektori $\overline{a^0}$ uchun vektor hisobining asosiy formulasini yozing.

1403. Ushbu: $\vec{a}=\{1,1,1\}$ vektorning yo'naltiruvchi kosinuslarini toping.

1404. Agar O nuqta ABC uchburchakning og'irlik markazi bo'lsa, u holda

$$\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=0$$

bo'lishini isbotlang.

1405. Ushbu:

$$\overrightarrow{F_1}=\{10,20,0\},\overrightarrow{F_2}=\{0,-10,20\},\overrightarrow{F_3}=\{-10,0,-20\}$$

kuchlarning teng ta'sir etuvchi $\vec{F}$ ning miqdori va yo'nalishini toping.

1406. Agar $\vec{a}$ va $\vec{b}$ vektorlar 60° li burchak tashkil etib, $|\vec{a}|=5$, $|\vec{b}|=8$ bo'lsa, $|\vec{a}+\vec{b}|$ va $|\vec{a}-\vec{b}|$ larni toping.

1407. $N(1,3,5)$ nuqta $\vec{a}=12\vec{i}+16\vec{j}+21\vec{k}$ vektorga parallel bo'lgan $\overrightarrow{NK}$ vektorning boshi bo'lib, uning uzunligi $|\overrightarrow{NK}|=87$ ga teng. K nuqtani toping.

1408. Uchta $\vec{F_1}, \vec{F_2}, \vec{F_3}$ kuchlar bir nuqtaga ta'sir etuvchi kuchlar bo'lib, ular o'zaro perpendikulyar yo'nalishlarga ega. Agar $|\vec{F_1}|=2$, $|\vec{F_2}|=10$, $|\vec{F_3}|=11$ bo'lsa, bu kuchlarning teng ta'sir etuvchisining miqdorini toping.

1409. Ushbu:

$$\vec{a}=\{m,3,4\},\vec{b}=\{4,m,-7\}$$

vektorlar berilgan bo'lsin. m ning qanday qiymatida bu vektorlar perpendikulyar bo'ladi?

A) 4, B) 3, C) 5, D) 2.

1410. Moddiy nuqta quyidagi

$$\overrightarrow{F_1}=\vec{i}+2\vec{j}+3\vec{k},$$
$$\overrightarrow{F_2}=-2\vec{i}-4\vec{j}-4\vec{k}$$
$$\overrightarrow{F_3}=6\vec{i}+2\vec{j}+5\vec{k}$$

uchta kuch ta'sirida. Bu kuchlarning teng ta'sir etuvchisini toping.

A) $\vec{F}=5\vec{i}+4\vec{k}$, B) $\vec{F}=5\vec{i}+2\vec{j}+4\vec{k}$, C) $\vec{F}=4\vec{i}+\vec{j}+5\vec{k}$

D) $\vec{F}=3\vec{i}+4\vec{j}+5\vec{k}$.

2-§. Vektorlarning skalyar va vektor ko'paytmalari

1⁰. Vektorlarning skalyar ko'paytmasi va uning xossalari. Fazoda ikki – $\vec a$ va $\vec b$ vektorlar berilgan bo'lib, ular orasidagi burchak φ bo'lsin.

Ushbu:

$$|\vec a|\cdot|\vec b|\cdot\cos\varphi$$

miqdor $\vec a$ va $\vec b$ vektorlarning skalyar ko'paytmasi deyiladi va $(\vec a,\vec b)$ kabi yoziladi:

$$(\vec a,\vec b)=|\vec a|\cdot|\vec b|\cdot\cos\varphi \qquad (1)$$

Skalyar ko'paytma quyidagi xossalarga ega:

1) $(\vec a,\vec b)=(\vec b,\vec a)$,

2) $(\vec a,\vec b+\vec c)=(\vec a\,\vec b)+(\vec a,\vec c)$,

3) $(\lambda\vec a,\vec b)=\lambda(\vec a,\vec b)$

4) $\vec a^{\,2}=|\vec a|^2$

5) $(\vec a,\vec b)=0 \Leftrightarrow \vec a\perp\vec b$

Agar $\vec a$ va $\vec b$ vektorlar koordinatalari orqali berilgan bo'lsa,

$$\vec a=\{a_x,a_y,a_z\},\vec b=\{b_x,b_y,b_z\}$$

u holda bu vektorlarning skalyar ko'paytmasi

$$(\vec a,\vec b)=a_xb_x+a_yb_y+a_zb_z \qquad (2)$$

bo'ladi.

1-misol. Agar $\vec a$ va $\vec b$ vektorlar orasidagi burchak $\varphi=\frac{5\pi}{6}$ bo'lib, bu vektorlarning uzunliklari $|\vec a|=5$, $|\vec b|=8$ bo'lsa, $(\vec a,\vec b)$, $\vec a^{\,2}$, $\vec b^{\,2}$ larni toping.

◄(1) formulaga ko'ra,

$$(\vec a,\vec b)=|\vec a|\cdot|\vec b|\cdot\cos\varphi=5\cdot8\cdot\cos\frac{5\pi}{6}$$

bo'ladi. Ma'lumki,

$$\cos\frac{5\pi}{6}=-\frac{\sqrt3}{2}$$

Demak, $(\vec a,\vec b)=-20\sqrt3$

Ma'lumki,

$$\vec a^{\,2}=(\vec a,\vec a).$$

Shunga ko'ra,

$$\vec a^{\,2}=(\vec a,\vec a)=|\vec a|^2\cdot\cos0=5^2\cdot1=25,$$

$$\vec b^{\,2}=(\vec b,\vec b)=|\vec b|^2\cdot\cos0=8^2\cdot1=64$$

bo'ladi.►

2-misol. Agar $\vec a$ va $\vec b$ vektorlar orasidagi burchak $\varphi=\frac{2\pi}{3}$ ga teng bo'lib, bu vektorlarning uzunliklari $|\vec a|=10$, $|\vec b|=2$ bo'lsa, ushbu:

$$(\vec a+2\vec b,\ 3\vec a-\vec b)$$

skalyar ko'paytmani toping.

◄Skalyar ko'paytma xossalaridan foydalanib topamiz:

$$(\vec a+2\vec b,\ 3\vec a-\vec b)=(\vec a,3\vec a-\vec b)+(2\vec b,3\vec a-\vec b)=(\vec a,3\vec a)-(\vec a,\vec b)+(2\vec b,3\vec a)-(2\vec b,\vec b)=$$

$$=3(\vec a,\vec a)-(\vec a,\vec b)+6(\vec b,\vec a)-2(\vec b,\vec b)=3\vec a^{\,2}+5(\vec a,\vec b)-2\vec b^{\,2}=3\cdot|\vec a|^2+5|\vec a|\cdot|\vec b|\cdot\cos\varphi-2|\vec b|^2$$

Agar $|\vec a|=10$, $|\vec b|=2$, $\varphi=\frac{2\pi}{3}$ bo'lishini e'tiborga olsak, unda

$$(\vec a+2\vec b,\ 3\vec a-\vec b)=3\cdot100+5\cdot10\cdot2\cdot\cos\frac{2\pi}{3}-2\cdot4=300-50-8=242$$

bo'lishi kelib chiqadi. ►

2⁰. Vektorlarning vektor ko'paytmasi va uning xossalari. Ikki $\vec a$ va $\vec b$ vektorlar berilgan bo'lsin. Bu vektorlarga ko'ra $\vec c$ vektor quyidagicha aniqlansin:

1) $\vec a$ va $\vec b$ vektorlarning boshlari bitta N nuqtaga keltiriladi. $\vec c$ vektorning boshi shu N nuqtada bo'lib, u $\vec a$ va $\vec b$ vektorlar joylashgan tekislikka perpendikulyar bo'ladi;

2) $\vec c$ vektorning yo'nalishi shundayki, uning oxiridan qaralganda $\vec a$ dan $\vec b$ ga aylanish soat strelkasi aylanishiga qarama-qarshi bo'ladi;

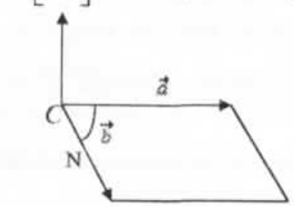
3) $\vec c$ vektorning uzunligi $\vec a$ va $\vec b$ vektorlarga yasalgan parallelogrammning yuziga teng bo'ladi.

$$|\vec c|=|\vec a|\cdot|\vec b|\cdot\sin\varphi \qquad (3)$$

(φ – $\vec a$ va $\vec b$ vektorlar orasidagi burchak).

Bunday aniqlangan $\vec c$ vektor $\vec a$ va $\vec b$ vektorlarning vektor ko'paytmasi deyiladi. Vektor ko'paytma quyidagicha belgilanadi:

$$\vec c=[\vec a,\vec b]. \qquad \text{(2-chizma)}$$



2-chizma

Vektor ko‘paytma quyidagi xossalarga ega:

1) $\left[\vec{b},\vec{a}\right]=-\left[\vec{a},\vec{b}\right]$,

2) $\lambda\cdot\left[\vec{a},\vec{b}\right]=\left[\lambda\vec{a},\vec{b}\right]=\left[\vec{a},\lambda\vec{b}\right]$,

3) $\left[\vec{a},\vec{b}+\vec{c}\right]=\left[\vec{a},\vec{b}\right]+\left[\vec{a},\vec{c}\right]$,

4) agar $\vec{a}=\{a_x, a_y, a_z\}, \vec{b}=\{b_x, b_y, b_z\}$ bo‘lsa,

$$\left[\vec{a},\vec{b}\right]=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix} \text{ yoki}$$

$$\left[\vec{a},\vec{b}\right]=\left(\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}, -\begin{vmatrix}a_x & a_z\\ b_x & b_z\end{vmatrix}, \begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}\right) \qquad (4)$$

bo‘ladi.

3-misol. Ushbu:

$$\vec{a}=2\vec{i}+\vec{j}-\vec{k}$$
$$\vec{b}=\vec{i}-3\vec{j}+\vec{k}$$

vektorlar bo‘yicha qurilgan parallelogrammning yuzini toping.

◀Izlanayotgan parallelogrammning yuzi S, $\vec{a}$ va $\vec{b}$ vektorlarning vektor ko‘paytmasi moduliga teng bo‘ladi:

$$S=\left|\left[\vec{a},\vec{b}\right]\right|.$$

$\vec{a}$ va $\vec{b}$ vektorlarning vektor ko‘paytmasini (4) formulaga ko‘ra hisoblaymiz:

$$[\vec{a},\vec{b}]=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 1 & -1\\ 1 & -3 & 1\end{vmatrix}=(1-3)\vec{i}-(2+1)\vec{j}+(-6-1)\vec{k}=-2\vec{i}-3\vec{j}-7\vec{k}.$$

Bu vektorning modulini topamiz:

$$\left|\left[\vec{a},\vec{b}\right]\right|=\sqrt{(-2)^2+(-3)^2+(-7)^2}=\sqrt{62}.$$

Demak, parallelogrammning yuzi $S=\sqrt{62}$ bo‘ladi. ▶

4-misol. Agar $\vec{a}$ va $\vec{b}$ vektorlar uchun $|\vec{a}|=2$, $|\vec{b}|=6$, bu vektorlar orasidagi burchak $\varphi=\frac{5\pi}{6}$ bo‘lsa, $[(2\vec{a}+3\vec{b}),\ (\vec{a}-4\vec{b})]$ vektorning modulini toping.

◀Avvalo, $[\vec{a},\vec{b}]$ vektor ko‘paytmaning modulini topamiz: (3) formulaga ko‘ra,

$$\left|\left[\vec{a},\vec{b}\right]\right|=|\vec{a}|\cdot|\vec{b}|\cdot\sin\frac{5\pi}{6}=2\cdot6\cdot\frac{1}{2}=6$$

bo‘ladi.

Endi, $[(2\vec{a}+3\vec{b}),\ (\vec{a}-4\vec{b})]$ vektorni vektor ko‘paytma xossalaridan foydalanib hisoblaymiz:

$[(2\vec{a}+3\vec{b}),\ (\vec{a}-4\vec{b})]=2[\vec{a},\vec{a}]-8[\vec{a},\vec{b}]+3[\vec{b},\vec{a}]-12[\vec{b},\vec{b}]=-8[\vec{a},\vec{b}]-3[\vec{a},\vec{b}]=-11[\vec{a},\vec{b}]$

Bu vektorning moduli

$$\left|[(2\vec{a}+3\vec{b}),\ (\vec{a}-4\vec{b})]\right|=\left|-11\cdot[\vec{a},\vec{b}]\right|=11\cdot\left|[\vec{a},\vec{b}]\right|=11\cdot6=66$$

bo‘ladi.◤

Quyidagi masalalarni yeching

1411. Ushbu:

$$\vec{a}=\{3, 1, -2\} \text{ va } \vec{b}=\{1, -4, -5\}$$

vektorning skalyar ko‘paytmasini toping.

1412. Aytaylik,

$$\overrightarrow{BA}=3\vec{i}-4\vec{j}+2\vec{k},\ \overrightarrow{CA}=2\vec{i}-2\vec{j}+10\vec{k}$$

vektorlar berilgan bo‘lsin. Quyidagi

$$2\overrightarrow{BA}-\overrightarrow{CA} \text{ va } 2\overrightarrow{AC}-\overrightarrow{AB}$$

vektorlarning skalyar ko‘paytmasi va $\overrightarrow{BA}^2$, $\overrightarrow{CA}^2$ larni toping.

1413. Agar $\vec{a}$ va $\vec{b}$ vektorlar uchun $|\vec{a}|=2$, $|\vec{b}|=1$, bo‘lib ular orasidagi burchak $\varphi=\frac{\pi}{3}$ bo‘lsa, $\vec{c}=2\vec{a}-3\vec{b}$ vektorning modulini toping.

1414. Ushbu:

$$\vec{a}=\{2, -2, 1\} \text{ va } \vec{b}=\{6, 0, -8\}$$

vektorlar orasidagi burchakni toping.

1415. Agar $A=A(2, 3, -1)$, $B=B(4, 1, -2)$, $C=C(1, 0, 2)$ lar uchburchakning uchlari bo‘lsa, uning C uchidagi burchakni toping.

1416. Agar $\vec{a}$ va $\vec{b}$ vektorlar uchun $|\vec{a}|=3$, $|\vec{b}|=2$ va ular orasidagi burchak $\varphi=\frac{\pi}{3}$ bo‘lsa, quyidagi

$$\vec{a}+\vec{b} \text{ va } \vec{a}-\vec{b}$$

vektorlar orasidagi burchakni toping.

1417. λ ning qanday qiymatida ushbu:

$$\vec{a}=\lambda\vec{a}-5\vec{j}+3\vec{k} \text{ va } \vec{b}=\vec{a}+2\vec{j}-\lambda\vec{k}$$

vektorlar o‘zaro perpendikulyar bo‘ladi?

1418. Ushbu $\vec{a}$ va $\vec{b}$ vektorlar uchun $|\vec{a}|=12$, $|\vec{b}|=\sqrt{27}$ bo‘lib, ular orasidagi burchak $\varphi=\frac{2\pi}{3}$ bo‘lsa, quyidagi $[\vec{a},\vec{b}]$ vektorning modulini toping.

1419. Agar $\vec{a}$ va $\vec{b}$ vektorlar uchun $|\vec{a}|=3$, $|\vec{b}|=26$ $|[\vec{a},\vec{b}]|=72$ bo‘lsa, $\vec{a}$ va $\vec{b}$ vektorlarning skalyar ko‘paytmasini toping.

1420. Ushbu:

$$\vec{a}=2\vec{i}-3\vec{j}+\vec{k} \text{ va } \vec{b}=\vec{i}+5\vec{j}-3\vec{k}$$

vektorlarning vektor ko‘paytmasini toping.

1421. Agar $\vec{a}$ va $\vec{b}$ vektorlar uchun $|\vec{a}|=1$, $|\vec{b}|=2$ bo‘lib, ular orasidagi burchak $\varphi=\frac{2\pi}{3}$ bo‘lsa, u holda $|[\vec{a},\vec{b}]|, |[\vec{a}+2\vec{b}, -\vec{a}+3\vec{b}]|$ larni toping.

1422. Uchlari $A(1,1,0)$, $B(1,0,1)$ va $C(0,1,1)$ nuqtalarda bo‘lgan uchburchakning yuzini toping.

1423. Ushbu:

$$\vec{a}=\{1,-2,3\} \text{ va } \vec{b}=\{3,2,1\}$$

vektorlarga qurilgan parallelogrammning yuzini toping.

1424. $O(0,2,1)$ nuqtaga $\vec{F}=\{2,-4,5\}$ kuch ta’sir ettirilgan. $A(-1,2,3)$ nuqtaga nisbatan kuch momentini aniqlang.

1425. Nuqtaga ta’sir ettirilgan $\vec{F}=\{2,-1,-4\}$ kuch $A(1,-2,3)$ nuqtadan $B(5,-6,1)$ nuqtaga to‘g‘ri chiziq bo‘ylab harakatlanganda qanday ish bajaradi?

1426. Nuqtaga ta’sir ettirilgan teng ta’sir etuvchi

$$\vec{F_1}=\vec{i}-\vec{j}+\vec{k} \text{ va } \vec{F_2}=2\vec{i}+\vec{j}+3\vec{k}$$

kuchlar koordinata boshidan $M(2,-1,-1)$ nuqtaga harakatlanganda qanday ish bajaradi?

1427. $A(2,-1,3)$ nuqta va unga ta’sir ettirilgan $\vec{F}=\{3,4,-2\}$ kuch berilgan bo‘lsin. $O(0,0,0)$ nuqtaga nisbatan kuch momentini va kuch momentining yo‘nalishini aniqlang.

1428. $A(3,-4,8)$ nuqtaga uchta

$$\vec{F_1}=\{2,4,6\}, \vec{F_2}=\{1,-2,3\}, \vec{F_3}=\{1,1,-7\}$$

kuch ta’sir ettirilgan bo‘lsin. $B(4,-2,6)$ nuqtaga nisbatan teng ta’sir etuvchi kuchning qiymati va yo‘naltiruvchi kosinusini toping.

1429. Ushbu:

$$\vec{a}=\vec{i}-3\vec{j}+\vec{k} \text{ va } \vec{b}=\vec{i}+\vec{j}-4\vec{k}$$

vektorlarning skalyar ko‘paytmasini toping.

A) –5, B) –6, C) –3, D) –4

1430. Ushbu:

$$\vec{a}=2\vec{i}+3\vec{j}+4\vec{k} \text{ va } \vec{b}=5\vec{i}+\vec{j}-\vec{k}$$

vektorlarning vektor ko‘paytmasini toping.

A) $7\vec{i}+22\vec{j}-13\vec{k}$, B) $7\vec{i}-22\vec{j}-13\vec{k}$,

C) $-7\vec{i}+22\vec{j}-13\vec{k}$, D) $7\vec{i}+22\vec{j}+13\vec{k}$

3-§. Vektorlarning ba’zi bir tatbiqlari

Ushbu paragrafda fazodagi eng muhim tushunchalardan bo‘lgan “tekislik” va “to‘g‘ri chiziq” tushunchalarini vektorlar yordamida ifodalanishi va ularga doir masalalar qaraladi.

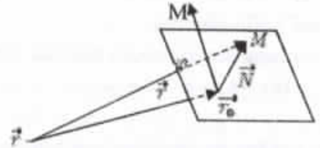
1⁰. Tekislikning vektor ko‘rinishidagi tenglamasi. Fazoda tekislikning vaziyati undagi biror $M_0(x_0,y_0,z_0)$ nuqta hamda bu tekislikka perpendikulyar bo‘lgan $\vec{N}$ vektor bilan aniqlanadi.

Aytaylik, $\vec{r_0}=\{x_0,y_0,z_0\}-M_0$ nuqtaning radius-vektori $\vec{r}=\{x,y,z\}$ esa tekislikdagi ixtiyoriy $M(x,y,z)$ nuqtaning radius vektori bo‘lsin.

Ushbu:

$$(\vec{N}, \vec{r}-\vec{r_0})=0$$

tenglama **tekislikning vektor ko‘rinishidagi tenglamasi** deyiladi (3-chizma)



3-chizma

$\vec{N}$ vektor tekislikning normali yoki tekislikning yo‘naltiruvchi vektori deyiladi.

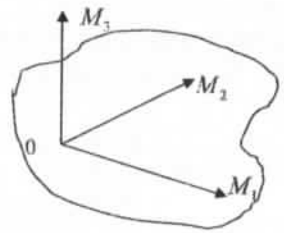
Agar tekislikning tenglamasi $Ax+By+Cz+D$ bo‘lsa, unda A,B,C lar $\vec{N}$ vektorning koordinatalari bo‘ladi: $\vec{N}=\{A,B,C\}$.

1-misol. Ushbu:

$$M_1(1,0,-1),\quad M_2(2,2,3),\quad M_3(0,-3,1)$$

nuqtalardan o‘tuvchi tekislik tenglamasini toping.

◀ M_1, M_2 va M_3 nuqtalar bitta tekislikda yotgani uchun $\overline{M_1M_2}$ va $\overline{M_1M_3}$ vektorlar ham shu tekislikda yotadi (4-chizma)



4-chizma

Demak, izlanayotgan tekislikning normali sifatida $\overline{M_1M_2}$, $\overline{M_1M_3}$ vektorlarning vektor ko‘paytmasi olinishi mumkin: $\vec{N}=[\overline{M_1M_2}, \overline{M_1M_3}]$.

Endi $\overrightarrow{M_1M_2}$, $\overrightarrow{M_1M_3}$ va $\vec{N}$ vektorlarning koordinatalarini topamiz:

$$\overrightarrow{M_1M_2}=\{2-1,\ 2-0,\ 3-(-1)\}=\{1,2,4\}$$

$$\overrightarrow{M_1M_3}=\{0-1,\ -3-0,\ 1-(-1)\}=\{-1,-3,2\}$$

$$\vec{N}=\left[\overrightarrow{M_1M_2},\ \overrightarrow{M_1M_3}\right]=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 2 & 4\\ -1 & -3 & 2\end{vmatrix}=\vec{i}(4-(-3)\cdot 4)-\vec{j}(1\cdot 2-(-1)\cdot 4)+\vec{k}(1\cdot(-3)-2\cdot(-1))=16\vec{i}-6\vec{j}-\vec{k}$$

Demak, tekislik normali $\vec{N}$ ning koordinatalari 16, –6, –1 bo‘lib, undan $A=16$, $B=-6$, $C=-1$ bo‘lishi kelib chiqadi. Unda izlanayotgan tekislik

$$16x-6y-z+D=0$$

ko‘rinishda bo‘ladi. Shartga ko‘ra, $M_1(1, 0, -1)$ nuqta tekislikda yotadi. Uning koordinatalarini keyingi tenglikka qo‘yamiz:

$$16\cdot 1-6\cdot 0-(-1)+D=0$$

Bu tenglikdan $D=-17$ bo‘lishi kelib chiqadi.

Demak, M_1, M_2, M_3 nuqtalardan o‘tuvchi tekislik tenglamasi

$$16x-6y-z-17=0$$

bo‘ladi.▶

2^0. To‘g‘ri chiziqning vektor ko‘rinishidagi tenglamasi. Fazoda to‘g‘ri chiziqning vaziyati undagi biror $M_0(x_0, y_0, z_0)$ nuqta hamda yo‘naltiruvchi vektor $\vec{S}=\{l, m, n\}$ bilan to‘liq aniqlanadi.

Aytaylik, $\vec{r_0}=\{x_0, y_0, z_0\}$ vektor $M_0(x_0, y_0, z_0)$ nuqtaning radius-vektori, $\vec{r}=\{x, y, z\}$ esa to‘g‘ri chiziqdagi ixtiyoriy $M(x, y, z)$ nuqtaning radius-vektori bo‘lsin.

Ushbu:

$$\vec{r}=\vec{r_0}+t\cdot\vec{S}\qquad(-\infty<t<+\infty)\tag{1}$$

tenglama to‘g‘ri chiziqning vektor ko‘rinishidagi tenglamasi deyiladi.

Agar fazoda to‘g‘ri chiziq ikki

$$A_1x+B_1y+C_1z+D_1=0$$

$$A_2x+B_2y+C_2z+D_2=0$$

tenglamalarning kesishishi sifatida qaraladigan bo‘lsa, bu to‘g‘ri chiziqning yo‘naltiruvchi vektori $\vec{S}$ ushbu $\vec{N_1}=\{A_1, B_1, C_1\}$ va $\vec{N_2}=\{A_2, B_2, C_2\}$ vektorlarning **vektor ko‘paytmasi** bo‘ladi:

$$\vec{S}=\left[\vec{N_1},\ \vec{N_2}\right]=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ A_1 & B_1 & C_1\\ A_2 & B_2 & C_2\end{vmatrix}\tag{2}$$

bo‘ladi.

2-misol. Ushbu:

$$\frac{x-1}{1}=\frac{y}{-4}=\frac{z+3}{1}\ \text{ va }\ \frac{x}{2}=\frac{y}{2}=\frac{z+2}{-2}=\frac{z}{-1}$$

to‘g‘ri chiziqlar orasidagi buvchakni toping.

◀Ravshanki, bu to‘g‘ri chiziqlar orasidagi burchak, ularning yo‘naltiruvchi vektorlari

$$\vec{S_1}=\{1, -4, 1\},\ \vec{S_2}=\{2, -2, -1\}$$

orasidagi burchak bo‘ladi.

Vektorlar orasidagi burchakning kosinusi formulasi

$$\cos\varphi=\frac{l_1\cdot l_2+m_1\cdot m_2+n_1\cdot n_2}{\sqrt{l_1^2+m_1^2+n_1^2}\cdot\sqrt{l_2^2+m_2^2+n_2^2}}$$

dan foydalanib topamiz:

$$\cos\varphi=\frac{1\cdot 2+(-4)\cdot(-2)+1\cdot(-1)}{\sqrt{1^2+(-4)^2+1^2}\cdot\sqrt{2^2+(-2)^2+(-1)^2}}=\frac{1}{\sqrt{2}}.$$

Demak, $\varphi=\frac{\pi}{4}$.▶

3-misol. Ushbu:

$$\begin{cases}x-2y+3z+1=0,\\ 2x-y-4z-8=0\end{cases}\tag{3}$$

to‘g‘ri chiziq tenglamasini kanonik ko‘rinishga keltiring.

◀Bu masalani hal etish uchun to‘g‘ri chiziqda bitta nuqtani va to‘g‘ri chiziqning yo‘naltiruvchi vektorini bilish kerak bo‘ladi.

Aytaylik, $z=-1$ bo‘lsin. Unda (3) to‘g‘ri chiziq tenglamasi

$$\begin{cases}x-2y=2,\\ 2x+y=4\end{cases}$$

ko‘rinishga keladi. Bu sistemani yechib, $x=2$, $y=0$ bo‘lishini topamiz.

Demak, $M(2, 0, -1)$ nuqta to‘g‘ri chiziq nuqtasi bo‘ladi.

Yuqoridagi (2) formuladan foydalanib, $\vec{N_1}=\{1, -2, 3\}$, $\vec{N_2}=\{2, 1, -4\}$ bo‘lishini e’tiborga olib, to‘g‘ri chiziqning yo‘naltiruvchi vektori $\vec{S}$ ni topamiz:

$$\vec{S}=\left[\vec{N_1},\ \vec{N_2}\right]=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ A_1 & B_1 & C_1\\ A_2 & B_2 & C_2\end{vmatrix}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & 3\\ 2 & 1 & 1\end{vmatrix}=5\vec{i}+10\vec{j}+5\vec{k}$$

Demak, $\vec{S}=5\vec{i}+10\vec{j}+5\vec{k}$.

To‘g‘ri chiziqning izlanayotgan kanonik tenglamasi:

$$\frac{x-2}{5}=\frac{y-0}{10}=\frac{z+1}{5}\ \text{ yoki }\ \frac{x-2}{1}=\frac{y}{2}=\frac{z+1}{1}.$$ ▶

Quyidagi masalalarni yeching

1431. $M(1, -2, 10)$ nuqtadan o‘tuvchi, normal vektori $\vec{N}=\{5, 1, -3\}$ bo‘lgan tekislik tenglamasini toping.

1432. Koordinata boshidan $p=3$ birlik masofada hamda $\vec{n}=\{3, 4, 12\}$ vektorga perpendikulyar bo‘lgan tekislik tenglamasini toping.

1433. $M_1(2, 0, -1)$, $M_2(-3, 1, 3)$ nuqtalardan o‘tuvchi hamda $\vec{S}=\{1, 2, -1\}$ vektorga parallel bo‘lgan tekislik tenglamasini toping.

1434. Ushbu:

$$\begin{cases}3x-4y-2z=0,\\ 2x+y-2z=0\end{cases}, \quad \begin{cases}4x+y-6z-2=0,\\ y-3z-2=0\end{cases}$$

to'g'ri chiziqlar orasidagi burchakning kosinusini toping.

1435. $M(-1,\ 0,\ 5)$ nuqtadan o'tuvchi, yo'naltiruvchi $\vec{S}$ vektor koordinata o'qlari $OX,\ OY,\ OZ$ lar bilan mos ravishda

$$\alpha=\frac{\pi}{4}, \beta=\frac{\pi}{4}, \gamma=\frac{2\pi}{3}$$

burchaklar tashkil etuvchi to'g'ri chiziqning kanonik tenglamasini toping.

1436. $M(1,\ -3,\ 5)$ nuqtadan o'tuvchi, hamda ushbu:

$$\begin{cases}3x-y+2z-7=0,\\ x+3y-2z+3=0\end{cases}$$

to'g'ri chiziqqa parallel bo'lgan to'g'ri chiziqning kanonik tenglamasini toping.

1437. Ushbu $M(2,\ 3,\ 1)$ nuqtadan

$$\frac{x+1}{1}=\frac{y-3}{-3}=\frac{z-3}{-2}$$

to'g'ri chiziqqacha bo'lgan masofani toping

1438. Ushbu:

$$\begin{cases}x-2y+3z-4=0,\\ 3x-2y+z=0\end{cases}$$

to'g'ri chiziqning yo'naltiruvchi kosinuslarini toping.

4-§. Vektor-funksiya, uning limiti va hosilasi

1⁰. Vektor-funksiya tushunchasi. Faraz qilaylik, t o'zgaruvchi (haqiqiy sonni qabul qiladigan o'zgaruvchi) $(\alpha,\ \beta)$ da o'zgarsin.

Agar t o'zgaruvchining $(\alpha,\ \beta)$ dan olingan har bir qiymatiga biror qoidaga ko'ra bitta aniq (yo'nalishi va uzunligi aniq) $\vec{a}$ vektor mos qo'yilsa, $\vec{a}$ vektor t o'zgaruvchining vektor-funksiyasi deyiladi. U $\vec{a}=\vec{a}(t)$ kabi yoziladi.

$\vec{a}(t)$ vektor-funksiyaning koordinatalari $a_x,\ a_y,\ a_z$ lar ham t ga bog'liq bo'ladi:

$$a_x=a_x(t), a_y=a_y(t), a_z=a_z(t).$$

Vektor hisobining asosiy formulasiga ko'ra,

$$\vec{a}(t)=a_x(t)\cdot\vec{i}+a_y(t)\cdot\vec{j}+a_z(t)\cdot\vec{k}$$

bo'ladi.

Demak,

$$\vec{a}=\vec{a}(t)$$

vektor-funksiyaning berilishi uchta:

$$a_x=a_x(t), a_y=a_y(t), a_z=a_z(t)$$

funksiyaning (skalyar funksiyani) berilishiga teng kuchli.

1-misol. Nuqta $y=\sqrt{x^2+1}$ egri chiziq bo'ylab o'ng tomonga harakatlanadi. $t=\frac{1}{4}$ vaqtda nuqta koordinatasi $(0,1)$ holatda bo'ladi. Nuqtaning chiziq bo'ylab boshlang'ich holatdan ko'chishi o'tgan t vaqtga to'g'ri proporsional. Berilgan harakatni ifodalovchi vektor funksiyani aniqlang.

◄ $\vec{a}(t)$ vektor funksiyani quyidagicha ifodalaymiz

$$\vec{a}(t)=x(t)\cdot\vec{i}+y(t)\cdot\vec{j},$$

bu yerda: x va y har bir $t\geq\frac{1}{4}$ uchun

$$y(t)=\sqrt{x^2(t)+1}$$

tenglamani qanoatlantiradi.

Nuqtaning chiziq bo'ylab boshlang'ich holatdan ko'chishi o'tgan vaqtga to'g'ri proporsional:

$$\sqrt{x^2(t)+y^2(t)}=kt.$$

Xususan,

$$\sqrt{\left[x\left(\tfrac{1}{4}\right)\right]^2+\left[y\left(\tfrac{1}{4}\right)\right]^2}=\frac{1}{4}k$$

yoki $\sqrt{0+1}=\frac{k}{4}$, bunda $k=4$. Demak,

$$\sqrt{x^2(t)+y^2(t)}=4t.$$

Oxirgi tenglikdan $y(t)$ ni $\sqrt{x^2(t)+1}$ ga almashtirib, tenglikning ikkala tomonini kvadratga oshirsak, quyidagini hosil qilamiz $2x^2(t)+1=16t^2$ yoki $x(t)=t\sqrt{8t^2-\frac{1}{2}}$. Manfiy ishorani musbat ishoraga almashtirsak bo'ladi, chunki nuqta birinchi chorakda joylashgan. Bundan kelib chiqadiki,

$$x(t)=\sqrt{8t^2-\frac{1}{2}} \text{ i } y(t)=\sqrt{x^2(t)+1}=\sqrt{8t^2+\frac{1}{2}}. ►$$

2⁰. Vektor-funksiyaning limiti va xossalari. Agar $\vec{\alpha}(t)$ vektor-funksiya uzunligi $|\vec{\alpha}(t)|$ning $t\to c$ dagi limiti nol bo'lsa;

$$\lim_{t\to c}|\vec{\alpha}(t)|=0$$

$\vec{\alpha}(t)$ cheksiz kichik vektor-funksiya deyiladi.

Aytaylik, $\vec{\alpha}(t)$ vektor-funksiya hamda o'zgarmas $\vec{a_0}$ vektor berilgan bo'lsin.

Agar

$$\lim_{t\to c}|\vec{\alpha}(t)-\vec{a_0}|=0$$

bo'lsa, $\vec{a_0}$ vektor $\vec{\alpha}(t)$ vektor-funksiyaning limiti deyiladi va

$$\lim_{t\to c}\vec{a}(t)=\vec{a_0} \text{ yoki } \vec{a}(t)\to\vec{a_0}$$

kabi yoziladi.

Agar

$$\lim_{t \to c} \vec{a}(t) = \vec{a_0}$$

bo'lib,

$$\vec{a}(t) = a_x(t)\cdot \vec{i} + a_y(t)\cdot \vec{j} + a_z(t)\cdot \vec{k} \quad \text{va} \quad \vec{a_0} = a_x^0 \cdot \vec{i} + a_y^0 \cdot \vec{j} + a_z^0 \cdot \vec{k}$$

bo'lsa, u holda

$$\lim_{t \to c} a_x(t) = a_x^0, \ \lim_{t \to c} a_y(t) = a_y^0, \lim_{t \to c} a_z(t) = a_z^0$$

bo'ladi va aksincha.

Aytaylik, $\vec{a}(t)$ va $\vec{b}(t)$ vektor-funksiyalar berilgan bo'lib,

$$\lim_{t \to c} \vec{a}(t) = \vec{a_0},$$

$$\lim_{t \to c} \vec{b}(t) = \vec{b_0}$$

bo'lsin. U holda:

$$\lim_{t \to c} (\vec{a}(t), \ \vec{b}(t)) = (\vec{a_0}, \vec{b_0})$$

$$\lim_{t \to c} [\vec{a}(t), \ \vec{b}(t)] = [\vec{a_0}, \vec{b_0}]$$

bo'ladi.

Ushbu:

$$\lim_{\Delta t \to c} \frac{\vec{a}(t+\Delta t) - \vec{a}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \vec{a}(t)}{\Delta t}$$

limit $\vec{a}(t)$ vektor-funksiyaning hosilasi deyiladi va u $\frac{d\vec{a}(t)}{dt}$ kabi belgilanadi:

$$\frac{d\vec{a}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\vec{a}(t+\Delta t) - \vec{a}(t)}{\Delta t}$$

Agar $\vec{a}(t)$ vektor-funksiya

$$\vec{a}(t) = a_x(x)\cdot \vec{i} + a_y(t)\vec{j} + a_z(t)\vec{k}$$

bo'lsa, u holda

$$\frac{d\vec{a}(t)}{dt} = \frac{da_x(t)}{dt}\cdot \vec{i} + \frac{da_y(t)}{dt}\cdot \vec{j} + \frac{da_z(t)}{dt}\cdot \vec{k}$$

bo'lib, bu vektorning uzunligi

$$\left|\frac{d\vec{a}(t)}{dt}\right| = \sqrt{\left(\frac{da_x(t)}{dt}\right)^2 + \left(\frac{da_y(t)}{dt}\right)^2 + \left(\frac{da_z(t)}{dt}\right)^2}$$

bo'ladi.

Aytaylik, moddiy nuqta tenglamasi quyidagicha

$$\vec{r}(t) = x(t)\cdot \vec{i} + y(t)\cdot \vec{j} + z(t)\cdot \vec{k}$$

bo'lgan trayektoriya bo'yicha harakat qilsin, bunda t vaqt.

Bu vektor-funksiyaning hosilasi

$$\frac{d\vec{r}(t)}{dt} = \vec{v}(t)$$

harakatning tezligini ifodalaydi va u

$$\vec{v}(t) = \frac{dx(t)}{dt}\cdot \vec{i} + \frac{dy(t)}{dt}\cdot \vec{j} + \frac{dz(t)}{dt}\cdot \vec{k}$$

bo'ladi. Bu vektor egri chiziqqa o'tkazilgan urinma yo'nalishida bo'ladi.

Eslatma. Funksiya hosilalarini hisoblash qoidalari hamda hosilalar jadvali vektor-funksiya hosilalari uchun ham o'rinli bo'ladi.

2-misol. Ushbu:

$$x = t, y = t^2, z = t^3$$

egri chiziqqa $M(1, 1, 1)$ $(t = 1)$ nuqtada o'tkazilgan urinmaning tenglamasini toping.

◄Ravshanki, bu holda

$$\vec{r}(t) = t\cdot \vec{i} + t^2\cdot \vec{j} + t^3\cdot \vec{k}$$

bo'lib,

$$\frac{d\vec{r}(t)}{dt} = \vec{i} + 2t\cdot \vec{j} + 3t^2\cdot \vec{k}$$

bo'ladi.

Keyingi tenglamadan ko'rinadiki, egri chiziqqa $M(1, 1, 1)$ nuqta o'tkazilgan urinmaning yo'nalishi ushbu:

$$\left.\frac{d\vec{r}(t)}{dt}\right|_M = \vec{i} + 2\vec{j} + 3\vec{k}$$

vektor bilan aniqlanadi.

Demak, izlanayotgan urinmaning tenglamasi

$$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3}$$

bo'ladi.►

3-misol. Ushbu:

$$\vec{r}(t) = a\cos t\cdot \vec{i} + b\sin t\cdot \vec{j} \qquad (a > 0, \ b > 0)$$

vektorning hosilasi va uning qiymatini toping.

◄Ravshanki, bu holda

$$x(t) = a\cos t, \qquad y(t) = b\sin t$$

bo'lib, bu trayektoriya ushbu:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

ellipsdan iborat bo'ladi.

Berilgan $\vec{r}(t)$ vektorning hosilasi

$$\frac{d\vec{r}(t)}{dt} = -a\cdot \sin t\cdot \vec{i} + b\cdot \cos t\cdot \vec{j}$$

bo'lib, uning qiymati (vektorning uzunligi)

$$\left|\frac{d\vec{r}(t)}{dt}\right| = \sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2} = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$$

bo'ladi.►

Berilgan harakatni ifodalovchi vektor fuksiyani aniqlang

1439. Nuqta $y=x^2$ parabola chizig'i bo'ylab o'ng tomonga harakatlanadi. $t=0$ vaqtda nuqta koordinatasi $(1,1)$ holatda bo'ladi. $t=3$ da nuqta koordinatasi $(2,4)$ holatga yetib keladi. Nuqtaning gorizontal chiziq bo'ylab boshlang'ich holatdan ko'chishi o'tgan vaqtning kvadratiga to'g'ri proporsional.

1440. Nuqta $y=x^3+1$ chiziq bo'ylab o'ng tomonga harakatlanadi. $t=1$ vaqtda nuqta koordinatasi $(-1,0)$ holatda bo'ladi. $t=2$ da nuqta koordinatasi $(1,2)$ holatga yetib keladi. Nuqtaning gorizontal chiziq bo'ylab boshlang'ich holatdan ko'chishi o'tgan vaqtning kvadratiga to'g'ri proporsional.

1441. Nuqta $y=8x$ to'g'ri chiziq bo'ylab o'ng tomonga harakatlanadi. $t=1$ vaqtda nuqta koordinatasi $(0,0)$ holatda bo'ladi. $t=2$ da nuqta koordinatasi $(1,8)$ holatga yetib keladi. Aytaylik, koordinata boshida $y=8x$ to'g'ri chiziq bo'ylab o'ng tomonga harakatlanib, $0X$ o'qidagi proyeksiyasi bilan hosil qlingan to'g'ri burchakli uchburchak yuzasi vaqtning kubiga to'g'ri proporsional.

1442. Nuqta $3y-2x^{\frac{3}{2}}=0$ egri chiziq bo'ylab o'ng tomonga harakatlanadi. $t=2$ vaqtda nuqta koordinatasi $(0,0)$ holatda bo'ladi. $t=16$ da nuqtaning abssissasi 3 ga teng. Nuqtaning $3y-2x^{\frac{3}{2}}=0$ egri chiziq bo'ylab bosib o'tgan masofasi vaqtga to'g'ri proporsional.

Quyidagi vektorlarning hosilalari va ularning qiymatlarini toping

1443. $\vec{r}(t)=t\cdot\vec{i}+\frac{1}{2}t^2\cdot\vec{j}$.

1444. $\vec{r}(t)=\cos^2 t\cdot\vec{i}+\sin^2 t\cdot\vec{j}$ $\quad (0\le t\le\frac{\pi}{2})$.

1445. $\vec{r}(t)=\cos^2 t\cdot\vec{i}+\cos 2t\cdot\vec{j}$

1446. $\vec{r}(t)=\frac{1}{2}t^2\cdot\vec{i}+\frac{1}{3}(2t+1)^{\frac{3}{2}}\cdot\vec{j}$

1447. $\vec{r}(t)=e^t\cdot\vec{i}+e^{-t}\cdot\vec{j}$

1448. $\vec{r}(t)=t\cdot\vec{i}+\ln t\cdot\vec{j}$ $\quad (t>0)$

Nazorat savollari

1. Vektor hisobining asosiy formulasini izohlab bering.
2. Vektorning uzunligi va yo'nalishi qanday aniqlanadi?
3. Vektorlar ustida chiziqli amallarni izohlab bering.
4. Ikki vektor orasidagi burchak qanday aniqlanadi?
5. Vektorlarning skalyar ko'paytmasi va uning xossalarini izohlab bering.
6. Vektorlarning vektor ko'paytmasi va uning xossalarini izohlab bering.
7. Tekislikning vektor ko'rinishidagi tenglamasi qanday aniqlanadi?
8. To'g'ri chiziqning vektor ko'rinishidagi tenglamasi qanday aniqlanadi?
9. "Vektor-funksiya" tushunchasini izohlab bering.
10. Vektor-funksiyaning limiti va xossalarini misollar yordamida izohlab bering.

13-bob

Ko'p o'zgaruvchili funksiyalar va ularning differensial hisobi

1-§. Ikki o'zgaruvchili funksiya, uning limiti va uzluksizligi

1^0. "Ikki o'zgaruvchili" funksiya tushunchasi. Funksiyaning aniqlanish sohasi. Ma'lumki, Dekart koordinatalar sistemasida tekislikning har bir nuqtasi ikkita – x va y haqiqiy sonlardan tuzilgan (x, y) tartiblangan juftlik bilan aniqlanadi va aksincha.

XOY tekislik nuqtalaridan tashkil topgan biror E to'plam $(E=\{(x,y): x\in R,\ y\in R\})$ berilgan bo'lsin.

Agar E to'plamdan olingan har bir (x, y) nuqtaga biror qoidaga ko'ra bitta z son mos qo'yilgan bo'lsa, E to'plamda ikki o'zgaruvchili funksiya berilgan (aniqlangan) deyiladi va

$$z=f(x,y)$$

kabi belgilanadi. Bunda E to'plam funksiyaning aniqlanish sohasi, x va y lar (erkli o'zgaruvchilar) funksiya argumentlari, z esa x va y larning funksiyasi deyiladi.

Odatda, funksiyaning aniqlanish sohasi funksional bog'lanishning (formulaning) ma'noga ega bo'lishiga ko'ra topiladi.

Aytaylik, $z=f(x,y)$ funksiya E to'plamda aniqlangan bo'lib, (x_0, y_0) nuqta E to'plamga tegishli bo'lsin: $(x_0, y_0)\in E$. Bu nuqtaga mos qo'yilgan z_0 son $z=f(x,y)$ funksiyaning (x_0, y_0) nuqtadagi xususiy qiymati deyiladi va

$$z_0=f(x_0,y_0)$$

kabi yoziladi.

1-misol. Ushbu

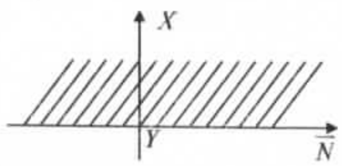
$$z=x+\sqrt{y}$$

funksiyaning aniqlanish sohasini toping.

◄Bu funksiyaning aniqlanish sohasi tekislikning shunday (x, y) nuqtalaridan iborat to'plam bo'lishi kerakki, bu to'plam nuqtalari uchun $x+\sqrt{y}$ ifoda ma'noga, ya'ni haqiqiy son qiymatga ega bo'lsin.

Ravshanki, buning uchun $y\ge 0$ bo'lishi lozim.

Demak, berilgan funksiyaning aniqlanish sohasi XOY tekisligining yuqori yarmidan iborat (1-chizma).►



1-chizma

2-misol. Agar

$$f(x,y)=xy+\frac{x}{y}$$

bo'lsa,

$$a)\ f(1,-1),\quad b)\ f\left(\frac{1}{2},3\right),\quad c)\ f(x-y,x+y)$$

larni toping.

◄$a)$ $f(1,-1)$ ni topish uchun $f(x, y)$ ning ifodasidagi x va y larning o'rniga mos ravishda 1 va –1 larni qo'yib $(x=1,\ y=-1)$, amallarni bajarib topamiz:

$$a)\ f(1,-1)=1\cdot(-1)+\frac{1}{(-1)}=-2\ .$$

$$b)\ f\left(\frac{1}{2},3\right)=\frac{1}{2}\cdot 3+\frac{\frac{1}{2}}{3}=\frac{3}{2}+\frac{1}{6}=\frac{10}{6}=\frac{5}{3}.$$

$$c)\ f(x-y,x+y)=(x-y)(x+y)+\frac{x-y}{x+y}=x^2-y^2+\frac{x-y}{x+y}.►$$

2^0. Funksiyaning grafigi. Sath chizig'i. Aytaylik, $z=f(x,y)$ funksiya tekislikdagi E to'plamda berilgan bo'lib, $(x_0, y_0)\in E$ bo'lsin. Funksiyaning shu nuqtadagi qiymatini z_0 deylik: $z_0=f(x_0,y_0)$. Ravshanki, (x_0, y_0, z_0) uchlik fazoda bitta nuqtani tasvirlaydi. Agar (x, y) nuqta E to'plamda o'zgara borsa, ularga mos ravishda $z=f(x,y)$ funksiya ham turli qiymatlarga ega bo'lib, fazoda $\{(x, y, z)\}$ nuqtalar to'plami hosil bo'ladi. Bunday to'plam, umuman aytganda, biror sirtni tasvirlaydi. Bu sirt $z=f(x,y)$ funksiyaning grafigi bo'ladi.

Ko'pincha ikki o'zgaruvchili $z=f(x,y)$ funksiya grafigini geometrik tasavvur etishda sath chizig'idan foydalaniladi.

Tekislikda shunday (x, y) nuqtalarni ko'rib chiqamizki, bu nuqtalardagi $z=f(x,y)$ funksiyaning qiymatlari bir xil o'zgarmasga teng bo'lsin:

$$z=f(x,y)=c\qquad (c=const).$$

Bunday (x, y) nuqtalar to'plami **sath chizig'i** deyiladi,

$$f(x,y)=c$$

esa sath chizig'ining tenglamasi bo'ladi.

3-misol. Ushbu

$$z=1-x-y$$

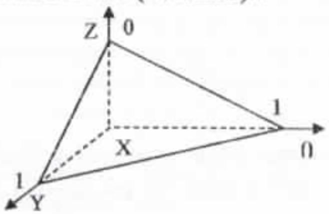
funksiyaning grafigini toping.

◄Bu funksiya tekislikning barcha nuqtalarida aniqlangan. Yuqoridagi tenglikni quyidagicha:

$$x+y+z-1=0\qquad (*)$$

yozib olamiz. Bizga ma'lumki, fazoda koordinatalari (*) tenglamani qanoatlantiruvchi (x, y, z) nuqtalar to'plami tekislikni ifodalaydi. Bu tekislik (0, 0, 1), (1, 0, 0), (0, 1, 0) nuqtalardan o'tadi.

Demak, berilgan funksiyaning grafigi (0, 0, 1), (1, 0, 0), (0, 1, 0) nuqtalardan o'tuvchi tekislik bo'ladi (2-chizma).▶



2-chizma

4-misol. Ushbu

$$z=\sqrt{1-x^2-y^2}$$

funksiyaning sath chizig'ini toping.

◀Bu funksiya tekislikning shunday (x, y) nuqtalarida aniqlanganki, ular uchun

$$1-x^2-y^2\ge 0,\text{ya'ni } x^2+y^2\le 1$$

bo'ladi. Demak, berilgan funksiya, markazi $(0,0)$ nuqtada, radiusi 1 ga teng bo'lgan doirada aniqlangan.

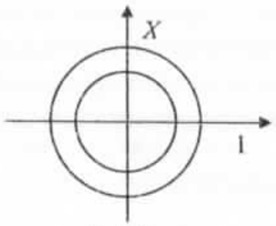
Uning sath chizig'i ushbu

$$\sqrt{1-x^2-y^2}=c \qquad (c=const)$$

tenglama bilan aniqlanadi. Bu tenglamani quyidagicha

$$1-x^2-y^2=c^2,\text{ ya'ni } x^2+y^2=1-c^2 \quad (0\le c\le 1)$$

yozib olamiz. Demak, berilgan funksiyaning sath chizig'i markazi $(0,0)$ nuqtada, radiusi $r=1-c^2$ bo'lgan konsentrik aylanalardan iborat bo'ladi (3-chizma).▶



3-chizma

3⁰. Funksiyaning limiti va uzluksizligi. Aytaylik, tekislikda $\{(x_n,y_n)\}$ nuqtalar ketma-ketligi hamda (x_0,y_0) nuqta berilgan bo'lsin.

Agar

$$\lim_{n\to\infty}x_n=x_0,\ \lim_{n\to\infty}y_n=y_0$$

bo'lsa, $\{(x_n,y_n)\}$ ketma-ketlik (x_0,y_0) nuqtaga intiladi deyiladi:

$$(x_n,y_n)\to(x_0,y_0)$$

Ma'lumki, (x_n,y_n) va (x_0,y_0) nuqtalar orasidagi masofa

$$d_n=\sqrt{(x_n-x_0)^2+(y_n-y_0)^2}$$

bo'ladi.

Ravshanki, $n\to\infty$ da $(x_n,y_n)\to(x_0,y_0)$ bo'lsa, u holda $n\to\infty$ da $d_n\to 0$ bo'ladi va aksincha.

Faraz qilaylik, $M_0=M_0(x_0,y_0)$ nuqta va $\delta>0$ son berilgan bo'lsin. Koordinatalari x va y lar ushbu

$$(x-x_0)^2+(y-y_0)^2<\delta^2$$

tengsizlikni qanoatlantiruvchi nuqtalar to'plami (ochiq doira) $M_0(x_0,y_0)$ **nuqtaning atrofi** deyiladi va u $U_\delta(x_0,y_0)$ kabi belgilanadi:

$$U_\delta(x_0,y_0)=\{(x,y):\ (x-x_0)^2+(y-y_0)^2<\delta^2\}.$$

Aytaylik, $z=f(x,y)$ funksiya $U_\delta(x_0,y_0)$ to'plamda berilgan bo'lsin (bunda funksiya (x_0,y_0) nuqtaning o'zida aniqlangan bo'lmasligi mumkin).

Agar $U_\delta(x_0,y_0)$ atrofdan olingan va (x_0,y_0) nuqtaga intiluvchi har qanday $\{(x_n,y_n)\}$ ketma-ketlik $(x_n\to x_0,\ y_n\to y_0,\ x_n\ne x_0,\ y_n\ne y_0)$ olinganda ham $\{f(x_n,y_n)\}$ ketma-ketlik uchun

$$\lim_{n\to\infty}f(x_n,y_n)=A$$

bo'lsa, A $f(x,y)$ funksiyaning $x\to x_0,\ y\to y_0$ dagi limiti deyiladi va

$$\lim_{\substack{x\to x_0\\ y\to y_0}}f(x,y)=A$$

kabi belgilanadi.

Ikki o'zgaruvchili funksiyaning limiti ham, 4-bob 2-§ da o'rganilgan funksiya limiti xossalari kabi, jumladan, yig'indining, ayirmaning, ko'paytmaning va nisbatning limiti haqida tasdiqlar o'rinli bo'ladi.

Aytaylik, $z=f(x,y)$ funksiya $U_\delta(x_0,y_0)$ da aniqlangan bo'lsin. Agar

$$\lim_{\substack{x\to x_0\\ y\to y_0}}f(x,y)=f(x_0,y_0) \tag{1}$$

bo'lsa, $f(x,y)$ funksiya (x_0,y_0) nuqtada uzluksiz deyiladi.

Agar $z=f(x,y)$ funksiya tekislikdagi E to'plamda berilgan bo'lib, uning har bir nuqtasida uzluksiz bo'lsa, $z=f(x,y)$ funksiya E to'plamda uzluksiz deyiladi.

Ravshanki, (1) munosabatdan

$$\lim_{\substack{x\to x_0\\ y\to y_0}}f(x,y)=\lim_{\substack{x\to x_0\\ y\to y_0}}f(x_0,y_0),$$

ya'ni

$$\lim_{\substack{x\to x_0\\ y\to y_0}}[f(x,y)-f(x_0,y_0)]=0$$

bo'lishi kelib chiqadi.

Agar

$$\Delta x = x - x_0,\ \Delta y = y - y_0,\ \Delta z = f(x,y) - f(x_0,y_0)$$

deyilsa,

$$x = x_0 + \Delta x,\ y = y_0 + \Delta y,\ \Delta z = f(x_0 + \Delta x,\ y_0 + \Delta y) - f(x_0,y_0)$$

bo'lib, yuqoridagi tenglik ushbu

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \Delta z = 0 \qquad (2)$$

ko'rinishga keladi.

Odatda, Δx erkli o'zgaruvchi x ning, Δy erkli o'zgaruvchi y ning orttirmasi, Δz esa funksiyaning orttirmasi deyiladi.

(2) munosabat $z = f(x,y)$ funksiyaning (x_0,y_0) nuqtada uzluksizligi ta'rifi sifatida qaralishi mumkin.

Agar

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x,y) = f(x_0,y_0)$$

munosabat bajarilmasa, $f(x,y)$ funksiya (x_0,y_0) nuqtada uziladi deyiladi.

5-misol. Ushbu

$$\lim_{\substack{x \to 0 \\ y \to 1}} \frac{1-xy}{x^2+y^2}$$

limitni toping.

◀Bu limitni topish uchun $f(x,y) = \dfrac{1-xy}{x^2+y^2}$ funksiyadan x ning o'rniga 0 ni, y ning o'rniga 1 ni qo'yamiz:

$$\lim_{\substack{x \to 0 \\ y \to 1}} \frac{1-xy}{x^2+y^2} = \frac{1-0\cdot 1}{0^2+1^2} = 1. ▶$$

6-misol. Ushbu

$$\lim_{\substack{x \to 2 \\ y \to 0}} \frac{tg\, xy}{y}$$

limitni toping.

◀Limit ostidagi

$$f(x,y) = \frac{tg\, xy}{y}.$$

funksiyani quyidagicha yozib olamiz:

$$\frac{tg\, xy}{y} = \frac{\sin xy}{y \cos xy} = \frac{\sin xy}{xy} \cdot x \cdot \frac{1}{\cos xy}.$$

So'ng $x \to 2,\ y \to 0,$ da $xy \to 0$ bo'lishini e'tiborga olib topamiz:

$$\lim_{\substack{x \to 2 \\ y \to 0}} \frac{tg\, xy}{y} = \lim_{\substack{x \to 2 \\ y \to 0}} \left(\frac{\sin xy}{xy} \cdot x \cdot \frac{1}{\cos xy} \right) = \lim_{\substack{x \to 2 \\ y \to 0}} \frac{\sin xy}{xy} \cdot \lim_{\substack{x \to 2 \\ y \to 0}} x \cdot \lim_{\substack{x \to 2 \\ y \to 0}} \frac{1}{\cos xy} = 1 \cdot 2 \cdot 1 = 2$$

Demak, $\lim\limits_{\substack{x \to 2 \\ y \to 0}} \dfrac{tg\, xy}{y} = 2$.

7-misol. Ushbu

$$\lim_{\substack{x \to 0 \\ y \to 2}} \frac{e^{y(x+y-2)} - 1}{3(1+x)(x+y-2)}$$

limitni hisoblang.

◀Limit ostidagi

$$f(x,y) = \frac{e^{y(x+y-2)} - 1}{3(1+x)(x+y-2)}$$

funksiyani quyidagicha yozib olamiz:

$$\frac{e^{y(x+y-2)} - 1}{3(1+x)(x+y-2)} = \frac{e^{y(x+y-2)} - 1}{y(x+y-2)} \cdot \frac{y}{3(1+x)}$$

So'ng $x \to 0,\ \ y \to 2$ da $x + y - 2 \to 0$ va $y(x+y-2) \to 0$ bo'lishi hamda

$$\lim_{\alpha \to 0} \frac{e^{\alpha} - 1}{\alpha} = 1$$

formuladan foydalanib topamiz:

$$\lim_{\substack{x \to 0 \\ y \to 2}} \frac{e^{y(x+y-2)} - 1}{3(1+x)(x+y-2)} = \lim_{\substack{x \to 0 \\ y \to 2}} \frac{e^{y(x+y-2)} - 1}{y(x+y-2)} \cdot \frac{y}{3(1+x)} = \lim_{\substack{x \to 0 \\ y \to 2}} \frac{e^{y(x+y-2)} - 1}{y(x+y-2)} \cdot \lim_{\substack{x \to 0 \\ y \to 2}} \frac{y}{3(1+x)} = 1 \cdot \frac{1}{3} = \frac{2}{3}$$

Demak,

$$\lim_{\substack{x \to 0 \\ y \to 2}} \frac{e^{y(x+y-2)} - 1}{3(1+x)(x+y-2)} = \frac{2}{3}. ▶$$

8-misol. Ushbu

$$\lim_{\substack{x \to 0 \\ y \to 2}} \frac{x+y}{x}$$

limitni toping.

◀Bu

$$f(x,y) = \frac{x+y}{x}$$

funksiyaning limitini topishda limit ta'rifidan foydalanamiz.

Aytaylik, $x_n = \frac{1}{n},\ \ y_n = \frac{1}{n}$ bo'lsin. Unda $n \to \infty$ da $x_n \to 0,\ \ y_n \to 0$ bo'lib,

$$f(x_n, y_n) = \frac{x_n + y_n}{x_n} = \frac{\frac{1}{n} + \frac{1}{n}}{\frac{1}{n}} = 2$$

va demak,

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x+y}{x} = 2$$

bo'ladi.

Aytaylik, $x_n = \frac{1}{n},\ \ y_n = \frac{2}{n}$ bo'lsin. Unda $n \to \infty$ da $x_n \to 0,\ \ y_n \to 0$ bo'lib,

$$f(x_n, y_n) = \frac{x_n + y_n}{x_n} = \frac{\frac{1}{n} + \frac{2}{n}}{\frac{1}{n}} = 3$$

va demak,

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x+y}{x} = 3$$

bo'ladi.

Demak, (0, 0) nuqtaga intiluvchi $\left(\frac{1}{n}, \frac{1}{n}\right)$ va $\left(\frac{1}{n}, \frac{2}{n}\right)$ ketma-ketliklar uchun

$$f\left(\frac{1}{n}, \frac{1}{n}\right) \to 2, f\left(\frac{1}{n}, \frac{2}{n}\right) \to 3 \quad (2 \neq 3)$$

bo'ldi. Bu holda funksiyaning limiti mavjud bo'lmaydi.
Qaralayotgan limit mavjud emas.▶

Quyidagi masalalarni yeching

1449. Aytaylik, tekislikdagi M nuqtaning (o'zgaruvchi nuqtaning) koordinatasi x, y bo'lsin:

$$M = M(x, y) \quad (x \in R, y \in R).$$

Koordinatalari quyidagi tengsizliklarni qanoatlantiruvchi tekislik nuqtalari to'plami (tekislikdagi soha) toping:

a) $a \le x \le b, \quad c \le y \le d;$ d) $-1 < x < 1, \quad 0 < y < 1 - |x|$.

b) $x \ge 0, \quad y \ge 0, \quad x + y \le a \quad (a > 0)$. e) $x^2 + y^2 \le 1$.

c) $x \ge y$. f) $x^2 + y^2 \le 4, \quad x^2 + y^2 \ge 1$.

1450. Quyidagi funksiyalarning aniqlanish sohalarini toping

1) $z = x^2 + 2y$. 6) $z = \sqrt{1 - (x^2 + y^2)}$.

2) $z = x + \sqrt{y}$. 7) $z = \ln(x + y)$.

3) $z = \frac{4}{x^2 + y^2}$. 8) $z = \sqrt{x - \sqrt{y}}$.

4) $z = \sqrt{xy}$. 9) $z = \ln(y^2 - 4x + 8)$.

5) $z = \sqrt{x - y}$. 10) $z = \arcsin \frac{y}{x}$.

1451. Agar $f(x, y) = x^2 + \frac{y}{x}$ bo'lsa, $f(1,0)$, $f(1,1)$, $f(2,1)$ni toping.

1452. Agar $f(x, y) = \frac{x^2 + y^2}{2xy}$ bo'lsa, $f(2,-3)$ni toping.

1453. Agar $f(x + 2y, x - 2y) = xy$ bo'lsa $f(x, y)$ni toping.

1454. Agar $f(x, y) = \frac{x + y}{2x - y}$ bo'lsa, bu funksiyaning a) $(1,2)$, b) $(2,1)$, c) $(10,20)$, d) $(20,10)$ nuqtalardagi qiymatlari mavjudmi?

1455. Quyidagi funksiyalarning sath chiziqlarini toping

1) $z = x + y$. 6) $z = y - x^2$.

2) $z = x - y$. 7) $z = \sqrt{1 - x^2 - y^2}$.

3) $z = \frac{y}{x}$. 8) $z = x^2 - y^2$.

4) $z = xy$. 9) $z = 1 - |x| - |y|$.

5) $z = x^2 + y^2$. 10) $z = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$.

1456. Quyidagi funksiyalarning grafiklarini toping

1) $(x, y) = x + y$. 4) $(x, y) = \sqrt{xy}$.

2) $(x, y) = x^2 + y^2$. 5) $f(x, y) = \sqrt{1 - x^2 - y^2}$.

3) $(x, y) = x^2 - y^2$.

Quyidagi limitlarni toping

1457. a) $\lim (x^2 - y^2)$, **b)** $\lim \frac{1 - xy}{x^2 + y^2}$.

1458. a) $\lim \frac{1}{x^2 + y^2}$, **b)** $\lim \frac{1}{x^2 + y^2}$.

1459. a) $\lim \frac{x}{x + y}$, **b)** $\lim \frac{\sin xy}{x}$.

1460. a) $\lim \left(1 + \frac{y}{x}\right)^x$, **b)** $\lim_{\substack{x \to +\infty \\ y \to +\infty}} \frac{x + y + 2x^2 + 2y^2}{x^2 + y^2}$.

1461. a) $\lim_{\substack{x \to +0 \\ y \to +0}} \frac{x^2 y}{x^2 + y^2}$, **b)** $\lim (x^2 + y^2) \sin \frac{1}{x^2 + y^2}$.

Quyidagi funksiyalarni uzluksizlikka tekshiring

1462. a) $f(x, y) = \frac{x - y}{1 + x^2 + y^2}$, **b)** $f(x, y) = \frac{x - y}{x + y}$.

1463. a) $f(x, y) = \frac{2x - 3}{x^2 + y^2 - 4}$, **b)** $f(x, y) = \frac{1}{2y + x + 1}$.

1464. a) $f(x, y) = \frac{2}{x^2 + y^2}$, **b)** $f(x, y) = \frac{x - y^2}{x + y^2}$.

1465. a) $f(x, y) = \frac{x - y}{x^3 - y^3}$, **b)** $f(x, y) = \ln(9 - x^2 - y^2)$.

1466. a) $f(x, y) = \frac{x^2 + y^2}{x^2 - y^2}$, **b)** $f(x, y) = \sin \frac{1}{x + y}$.

1467. a) $f(x, y) = \frac{x}{|y|}$, **b)** $f(x, y) = \cos \frac{1}{x^2 + y^2 - 9}$.

1468. a) $f(x,y)=\frac{1}{xy}$, b) $f(x,y)=\frac{x+y}{\sqrt{x^2+y^2-4}}$.

1469. $f(x,y)=\ln\sqrt{x^2+y^2}$.

1470. Ushbu $z=\frac{1}{\sqrt{xy}}$ funksiyaning aniqlanish sohasini toping.

A) $E=\{(x,y):x>0,y>0\}$

B) $E=\{(x,y):x>0,y>0\}\cup\{(x,y):x<0,y<0\}$

C) $E=\{(x,y):x<0,y<0\}$

D) $E=\{(x,y):x^2+y^2\le 1\}$

1471. Ushbu $z=\ln x+\sqrt{y}$ funksiyaning aniqlanish sohasini toping.

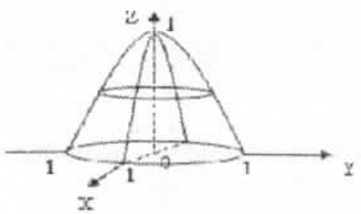
A) $E=\{(x,y):x>0,y>0\}$. *C)* $E=\{(x,y):x\ge 0,y>0\}$.

B) $E=\{(x,y):x>0,y\ge 0\}$. *D)* $E=\{(x,y):x\ge 0,y\ge 0\}$.

1472. Agar $f(x,y)=\frac{(x+y)^2}{2xy}$ bo'lsa, $f\left(\frac{1}{x},\frac{1}{y}\right)$ toping.

A) $f\left(x,\frac{1}{y}\right)$, B) $f\left(\frac{1}{x},y\right)$, C) $f(x,y)$, D) $-f(x,y)$.

1473. Ushbu chizmada tasvirlangan sirt qanday funksiyaning grafigi bo'ladi?



1-chizma

A) $z=x^2+y^2$. *C)* $z=xy$.

B) $z=\sqrt{1-x^2-y^2}$ *D)* $z=xy+1$.

1474. Ushbu $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{\sin xy}{x}$ limitni toping.

A) 1, B) 2, C) 3, D) 0.

1475. Ushbu $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{x-y^2}{\sqrt{4-x+y^2}-2}$ limitni toping.

A) 4, B) –4, C) 3, D) 5.

2-§. Ikki o'zgaruvchili funksiyaning hosila va differensiallari

1°. Funksiyaning xususiy hosilalari. Aytaylik, $z=f(x,y)$ funksiya biror (x,y) nuqtaning atrofi $U_\delta(x,y)$ da uzluksiz bo'lsin. Ushbu

$$\Delta_x z=f(x+\Delta x,y)-f(x,y)$$

ayirma berilgan funksiyaning x o'zgaruvchisi bo'yicha **xususiy orttirmasi** deyiladi.

Agar

$$\lim_{\Delta x\to 0}\frac{\Delta_x z}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

limit mavjud bo'lsa, unga $f(x,y)$ funksiyaning x o'zgaruvchisi bo'yicha xususiy hosilasi deyiladi. Bu xususiy hosila quyidagicha:

$$\frac{\partial z}{\partial x} \text{ yoki } f'_x(x,y)$$

belgilanadi. Demak,

$$f'_x(x,y)=\frac{\partial z}{\partial x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}.$$

Xuddi shunga o'xshash y o'zgaruvchi bo'yicha xususiy hosila ta'riflanadi:

$$f'_y(x,y)=\frac{\partial z}{\partial y}=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}.$$

Ikki o'zgaruvchili $f(x,y)$ funksiyaning x o'zgaruvchi bo'yicha xususiy hosila ta'rifida y ni o'zgarmas va y o'zgaruvchi bo'yicha xususiy hosila ta'rifida esa x o'zgarmas hisoblanadi.

Demak, ikki o'zgaruvchili funksiyaning xususiy hosilalarini hisoblashda bir o'zgaruvchili funksiyaning hosilasini hisoblashdagi ma'lum bo'lgan qoida va jadvallardan to'liq foydalanish mumkin.

1-misol. Ushbu

$$z=x^3+x^2y+y^3$$

funksiyaning xususiy hosilalarini toping.

◄Berilgan funksiyada y ni o'zgarmas deb qarab, x o'zgaruvchi bo'yicha hosilasini topamiz. Bunda ma'lum bo'lgan qoida va formulalardan foydalanamiz:

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}(x^3+x^2y+y^3)=3x^2+2xy+0=3x^2+2xy.$$

Xuddi shunga o'xshash x ni o'zgarmas deb hisoblab topamiz:

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}(x^3+x^2y+y^3)=0+x^2+3y^2=x^2+3y^2.$$ ►

2-misol. Ushbu

$$f(x,y)=\sqrt{x^2-y^2}$$

funksiyaning $(5;-3)$ nuqtadagi xususiy hosilalarini toping.

◀Avvalo, berilgan funksiyaning xususiy hosilalarini hisoblaymiz:

$$f'_x(x,y)=\frac{\partial}{\partial x}(\sqrt{x^2-y^2})=\frac{1}{2\sqrt{x^2-y^2}}\cdot 2x=\frac{x}{\sqrt{x^2-y^2}},$$

$$f'_y(x,y)=\frac{\partial}{\partial y}(\sqrt{x^2-y^2})=\frac{1}{2\sqrt{x^2-y^2}}\cdot(-2y)=\frac{-y}{\sqrt{x^2-y^2}}.$$

Endi, bu xususiy hosilalarning ko'rsatilgan $(5;-3)$ nuqtadagi qiymatlarini topamiz:

$$f'_x(5,-3)=\frac{5}{\sqrt{5^2-(-3)^2}}=\frac{5}{\sqrt{25-9}}=\frac{5}{4},$$

$$f'_y(5,-3)=\frac{-(-3)}{\sqrt{5^2-(-3)^2}}=\frac{3}{\sqrt{25-9}}=\frac{3}{4}$$

Demak, $f'_x(5;-3)=\frac{5}{4}$, $f'_y(5,-3)=\frac{3}{4}$.▶

2^0. Funksiyaning differensiali. Taqribiy formulalar. Aytaylik, $z=f(x,y)$ funksiya (x_0,y_0) nuqtaning biror $U_\delta(x,y)$ atrofida berilgan bo'lib, $(x_0+\Delta x,\ y_0+\Delta y)$ nuqta shu atrofga tegishli bo'lsin, bunda Δx va Δy lar argument orttirmalari.

Ushbu

$$\Delta z=\Delta f(x_0,y_0)=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)$$

ayirma $z=f(x,y)$ funksiyaning (x_0,y_0) nuqtadagi **to'liq orttirmasi** deyiladi.

Ravshanki, bu to'liq orttirma Δx va Δy larga bog'liq bo'ladi.

Agar funksiyaning to'liq orttirmasini quyidagi:

$$\Delta z=A\cdot\Delta x+B\cdot\Delta y+\alpha\cdot\Delta x+\beta\cdot\Delta y \qquad (1)$$

ko'rinishda ifodalash mumkin bo'lsa, $z=f(x,y)$ funksiya (x_0,y_0) nuqtada differensiallanuvchi deyiladi, bunda A va B lar Δx va Δy larga bog'liq bo'lmagan o'zgarmaslar, α va β lar esa Δx va Δy larga bog'liq va $\Delta x\to 0$, $\Delta y\to 0$ da $\alpha\to 0$, $\beta\to 0$:

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\alpha=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\beta=0$$

(1) tenglikdagi

$$A\cdot\Delta x+B\cdot\Delta y$$

ifoda $z=f(x,y)$ funksiyaning (x_0,y_0) nuqtadagi differensiali deyiladi va u dz kabi belgilanadi:

$$dz=A\cdot\Delta x+B\cdot\Delta y.$$

Agar $z=f(x,y)$ funksiya differensiallanuvchi bo'lsa, u holda

$$A=\frac{\partial z}{\partial x},\qquad B=\frac{\partial z}{\partial y}$$

bo'ladi. Unda funksiya differensiali ushbu

$$dz=\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y$$

ko'rinishga keladi. Bu tenglikda $\Delta x=dx$, $\Delta y=dy$ deyilsa, $z=f(x,y)$ funksiya differensiali uchun

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$$

bo'ladi.

3-misol. Ushbu

$$z=f(x,y)=x^2+xy$$

funksiyaning $(2;1)$ nuqtada $\Delta x=0{,}01$, $\Delta y=0{,}02$ bo'lgandagi to'liq orttirmasi hamda differensialini toping.

◀$z=f(x,y)$ funksiyaning to'liq orttirmasi hamda differensialini ta'riflardan foydalanib topamiz:

$$\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)=(x+\Delta x)^2+(x+\Delta x)(y+\Delta y)-(x^2+xy))=$$
$$=(2x+y)\Delta x+x\Delta y+\Delta x^2+\Delta x\Delta y,$$
$$dz=(2x+y)\cdot\Delta x+x\cdot\Delta y.$$

Bu ifodalarda x,y va $\Delta x,\Delta y$ larning o'rniga berilgan qiymatlari $x=2$, $y=1$, $\Delta x=0{,}01$, $\Delta y=0{,}02$ larni qo'yib topamiz:

$$\Delta z=(2\cdot 2+1)\cdot 0{,}01+2\cdot 0{,}02+(0{,}01)^2+0{,}01\cdot 0{,}02=0{,}0903$$
$$dz=(2\cdot 2+1)\cdot 0{,}01+2\cdot 0{,}02=0{,}09$$

Demak, $\Delta z=0{,}0903$, $dz==0{,}09$ bo'ladi.▶

4-misol. Ushbu

$$z=x^3+y^3+xy$$

funksiyaning differensialini toping.

◀Ma'lumki,

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy.$$

Berilgan funksiyaning xususiy hosilalari

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}(x^3+y^3+xy)=3x^2+0+y=3x^2+y,$$

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}(x^3+y^3+xy)=0+3y^2+x=3y^2+x$$

bo'ladi. Demak,

$$dz=(3x^2+y)dx+(3y^2+x)dy.$$▶

Aytaylik, $z=f(x,y)$ funksiya (x_0,y_0) nuqtaning biror atrofida berilgan bo'lib, bu nuqtada differensiallanuvchi bo'lsin. Bu funksiyaning to'liq orttirmasi

$$\Delta z=f(x_0+\Delta x,\ y_0+\Delta y)-f(x_0,\ y_0),$$

differensiali

$$dz=f'_x(x_0,\ y_0)\cdot\Delta x+f'_y(x_0,\ y_0)\cdot\Delta y$$

uchun Δx va Δy lar yetarlicha kichik bo'lganda

$$\Delta z\approx dz,$$

ya'ni

$$f(x_0+\Delta x,\ y_0+\Delta y)\approx f(x_0,\ y_0)+f'_x(x_0,y_0)\cdot\Delta x+f'_y(x_0,y_0)\cdot\Delta y \qquad (2)$$

bo'ladi. Bu taqribiy formuladan ko'p foydalaniladi.

5-misol. Tomonlari $x=6м$ va $y=8м$ bo'lgan to'g'ri to'rtburchak berilgan. Agar x tomoni 5sm ga oshirilsa, y tomoni esa 10sm ga kamaytirilsa, unda to'g'ri to'rtburchakning diagonali qanchagacha o'zgaradi?

◀To'g'ri to'rtburchakning diagonalini z bilan belgilaymiz. Unda

$$z=\sqrt{x^2+y^2}$$

bo'ladi. Bu funksiyaga

$$\Delta z\approx dz$$

taqribiy formulani qo'llab topamiz.

$$\Delta z\approx dz=\frac{\partial z}{\partial x}\cdot\Delta x+\frac{\partial z}{\partial y}\cdot\Delta y=\frac{2x}{2\sqrt{x^2+y^2}}\cdot\Delta x+\frac{2y}{2\sqrt{x^2+y^2}}\cdot\Delta y=\frac{x\cdot\Delta x+y\cdot\Delta y}{\sqrt{x^2+y^2}}$$

Keyingi munosabatda

$$x=6см,\quad \Delta x=0{,}05м$$
$$y=8см,\quad \Delta x=-0{,}10м$$

deb olamiz. Natijada,

$$\Delta z\approx\frac{6\cdot0{,}05+8\cdot(-0{,}10)}{\sqrt{36+64}}=-0{,}05м$$

bo'ladi.

Demak, to'g'ri to'rtburchakning diagonali taxminan 5sm ga kamayadi.▶

6-misol. Ushbu

$$\alpha=1{,}07^{3{,}97}$$

miqdorni taqribiy hisoblang.

◀Ravshanki, $\alpha=1{,}07^{3{,}97}$ son ushbu

$$f(x,y)=x^y$$

funksiyaning $x=1{,}07,\ y=3{,}97$ dagi xususiy qiymatidan iborat. Bu funksiya uchun $f(1,4)=1$ bo'lishini e'tiborga olib,

$$x_0=1,\quad y_0=4$$

deb olamiz. Unda

$$\Delta x=x-x_0=1{,}07-1=0{,}07,$$
$$\Delta y=y-y_0=3{,}97-4=-0{,}03$$

bo'ladi. Ushbu

$$f(x_0+\Delta x,\ y_0+\Delta y)\approx f(x_0,y_0)+f'_x(x_0,y_0)\cdot\Delta x+f'_y(x_0,y_0)\cdot\Delta y$$

formuladan foydalanib topamiz:

$$f(x_0+\Delta x,\ y_0+\Delta y)=f(1+0{,}7,\ 4-0{,}03)=f(1{,}7,\ 3{,}97)=1{,}7^{3{,}97}$$
$$f(x_0,y_0)=f(1{,}4)=1^4=1$$
$$f'_x(x_0,y_0)=\frac{\partial}{\partial x}(x^y)\Big|_{x_0,y_0}=y\cdot x^{y-1}\Big|_{\substack{x=1\\y=4}}=4\cdot1=4$$
$$f'_y(x_0,y_0)=\frac{\partial}{\partial y}(x^y)\Big|_{x_0,y_0}=x^y\ln x\Big|_{\substack{x=1\\y=4}}=1^4\cdot\ln1=0$$

$$1{,}7^{3{,}97}\approx1+4\cdot0{,}07+0\cdot(-0{,}03)=1+0{,}28=1{,}28$$

Demak,

$$\alpha=1{,}7^{3{,}97}\approx1{,}28.▶$$

3⁰. Murakkab funksiyaning hosilalari. Aytaylik,

$$z=f(x,\ y)$$

funksiyada

$$x=x(u,v),\qquad y=y(u,\ v)$$

bo'lib, ushbu

$$z=f(x(u,v),\ \ y(u,v))$$

murakkab funksiyaga ega bo'laylik. Bu murakkab funksiyaning xususiy hosilalari ushbu

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u},$$
$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} \qquad (3)$$

formulalar bo'yicha topiladi.

Xususan,

$$x=x(t),\qquad y=y(t)$$

bo'lsa, unda

$$z=f\big(x(t),\ y(t)\big)$$

murakkab funksiyaning xususiy hosilasi

$$\frac{dz}{dt}=\frac{dz}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \qquad (4)$$

bo'ladi.

Agar $z=f(x,\ y)$ va $y=y(x)$ bo'lsa, unda $z=f(x,\ y(x))$ murakkab funksiyaning hosilasi

$$\frac{dz}{dx}=\frac{dz}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dx} \qquad (5)$$

bo'ladi.

7-misol. Ushbu

$$z=e^{xy},\qquad x=u^2,\qquad y=u\cdot v$$

murakkab funksiyaning xususiy hosilalarini toping.

◀(3) formulalardan foydalanib topamiz:

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}=ye^{xy}\cdot2u+xe^{xy}v=3u^2ve^{u^3v}$$
$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}=ye^{xy}\cdot0+xe^{xy}u=u^3e^{u^3v}.▶$$

8-misol. Ushbu

$$z=x^2+xy+y^2,\qquad x=t^2,\quad y=t$$

murakkab funksiyaning hosilasini toping.

◀(4) formuladan foydalanib topamiz:

$$\frac{dz}{dt}=\frac{dz}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}=\frac{\partial}{\partial x}(x^2+xy+y^2)\cdot\frac{d}{dt}(t^2)+\frac{\partial}{\partial y}(x^2+xy+y^2)\cdot\frac{d}{dt}(t)=$$

$$=(2x+y)\cdot 2t+(x+2y)\cdot 1=(2t^2+1)\cdot 2t+t^2+2t=4t^3+3t^2+2t.\triangleright$$

4^0. Yuqori tartibli hosila va differensiallar. $z=f(x,y)$ funksiyaning xususiy hosilalari

$$\frac{\partial z}{\partial x} \text{ va } \frac{\partial z}{\partial y}$$

umuman aytganda, x va y o'zgaruvchilarning funksiyalari bo'ladi. Ularning xususiy hosilalari

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right),\quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right),\quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right),\quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

berilgan funksiyaning ikkinchi tartibli hosilalari deyiladi va ular mos ravishda

$$\frac{\partial^2 z}{\partial x^2},\quad \frac{\partial^2 z}{\partial x\partial y},\quad \frac{\partial^2 z}{\partial y\partial x},\quad \frac{\partial^2 z}{\partial y^2}$$

kabi belgilanadi:

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right),\quad \frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right),\quad \frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right),\quad \frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right).$$

Xuddi shunga o'xshash $z=f(x,y)$ funksiyaning uchinchi, to'rtinchi va h.k. tartibdagi hosilalari ta'riflanadi.

Umuman, $z=f(x,y)$ funksiyaning n-tartibli hosilasi

$$\frac{\partial^n z}{\partial x^m\partial y^p}\quad (m+p=n)$$

x va y o'zgaruvchi bo'yicha (mos ravishda m va p martadan) ketma-ket n marta hosila olish natijasida hosil bo'ladi.

9-misol. Agar

$$z=e^{xy}$$

bo'lsa, $\frac{\partial z}{\partial y\partial x^2}$ ni toping.

◀Avvalo, berilgan funksiyaning y bo'yicha xususiy hosilasini topamiz:

$$\frac{\partial z}{\partial y}=\frac{\partial}{y}(e^{xy})=x\cdot e^{xy}.$$

Bu funksiyaning x bo'yicha ketma-ket ikki marta hosilasini hisoblaymiz:

$$\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}(xe^{xy})=1\cdot e^{xy}+x\frac{\partial}{\partial x}(e^{xy})=e^{xy}+xye^{xy}=e^{xy}(1+xy),$$

$$\frac{\partial^3 z}{\partial y\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y\partial x}\right)=\frac{\partial}{\partial x}(e^{xy}(1+xy))=ye^{xy}(1+xy)+e^{xy}\cdot y=ye^{xy}(2+xy).\blacktriangleright$$

10-misol. Tebranib turgan tor z ning vaqtda boshlang'ich nuqtasidan x masofada joylashgan M dagi egilishi x va t ning z=z(x,t) funksiyasi

bo'lsin. $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial t}$ xususiy hosilalar qanday fizik (yoki geometrik) ma'noga ega? $\frac{\partial^2 z}{\partial t^2}$ ikkinchi tartibli xususiy hosilalar qanday fizik ma'noga ega? ◀$t=t_0$ tayin vaqtida tebranuvchi $z=z(x,t_0)$ chiziq bo'ylab joylashgan.

$\frac{\partial u}{\partial x}$ xususiy hosila $t=t_0$ da x nuqtadagi urinma burchak koeffitsiyentini aniqlaydi. Tayinlangan x_0 da $z(x_0,t)$ funksiya mos nuqtadagi tor muvozanatidan chetlanishini aniqlaydi. $\frac{\partial z}{\partial t}$ xususiy hosila t vaqtdagi nuqtaning x_0 koordinatasi bo'yicha oniy tezligiga teng. $\frac{\partial^2 z}{\partial t^2}$ esa nuqtaning tezlanishini aniqlaydi.▶

Aytaylik, $z=f(x,y)$ funksiya (x,y) nuqtaning atrofi $U_\delta(x,y)$ da berilgan bo'lsin.

Ma'lumki, bu funksiyaning differensiali

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$$

bo'ladi.

$z=f(x,y)$ funksiya differensiali dz ning differensiali $d(dz)$ berilgan funksiyaning ikkinchi tartibli differensiali deyiladi va d^2z kabi belgilanadi: $d^2z=d(dz)$. U quyidagicha:

$$d^2z=\frac{\partial^2 z}{\partial x^2}dx^2+2\frac{\partial^2 z}{\partial x\partial y}dxdy+\frac{\partial^2 z}{\partial y^2}dy^2 \qquad (6)$$

bo'ladi. Funksiyaning ikkinchi tartibli differensialini simvolik ravishda quyidagicha:

$$dz=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy\right)^2\cdot z$$

yozish mumkin. Bunda qavs ichidagi yig'indi kvadratga ko'tarilib, so'ng z ga "ko'paytiriladi". Keyin daraja ko'rsatkichlari xususiy hosilalar tartibi deb hisoblanadi.

$z=f(x,y)$ funksiyaning uchinchi, to'rtinchi va hokazo tartibli differensiallari ham xuddi yuqoridagidek ta'riflanadi. Masalan, n-tartibli differensial

$$d^nz=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy\right)^n\cdot z \qquad (7)$$

bo'ladi.

11-misol. Ushbu

$$z=2x^2y-2xy^2+3x-2y+5$$

funksiyaning ikkinchi tartibli differensialini toping.

◀Avvalo, berilgan funksiyaning xususiy hosilalarini topamiz:

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}(2x^2y-2xy^2+3x-2y+5)=4xy-2y^2+3,$$

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}(2x^2y-2xy^2+3x-2y+5)=2x^2-4xy-2$$

so'ng ikkinchi tartibli hosilalarni hisoblaymiz

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}(4xy-2y^2+3)=4y,$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}(4xy-2y^2+3)=4x-4y,$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}(2x^2-4xy-2)=-4x.$$

Unda (6) formulaga ko'ra,

$$d^2z=4ydx^2+8(x-y)dxdy-4xdy^2$$

bo'ladi.▶

Quyidagi funksiyalarning xususiy hosilalarini toping:

1476. $a)\, z=x^3+y^2-2xy,\quad b)\, z=5x^2+8xy^2+y^3.$

1477. $a)\, z=x^3+y^2-3axy,\quad b)\, z=x^2-2xy+y^2.$

1478. $a)\, z=\frac{xy}{x+y},\quad b)\, z=\frac{x-y}{x+y}.$

1479. $a)\, z=\frac{y}{x},\quad b)\, z=\sqrt{\frac{x}{y}}.$

1480. $a)\, z=\sqrt{x^2-y^2},\quad b)\, z=\sqrt{x+3y}.$

1481. $a)\, z=\frac{1}{\sqrt{x}-\sqrt{y}},\quad b)\, z=\frac{x}{\sqrt{x^2+y^2}}.$

1482. $a)\, z=y\sqrt{x}+\frac{x}{\sqrt{y}},\quad b)\, z=\ln\sin\frac{x+a}{\sqrt{y}}.$

1483. Agar $f(x,y)=\frac{1-xy}{1+xy}$ bo'lsa, $f'_x(0,1)$, $f'_y(0,1)$ni toping.

1484. Agar $f(x,y)=\sqrt{xy+\frac{x}{y}}$ bo'lsa, $f'_x(2,1)$, $f'_y(2,1)$ni toping.

1485. Agar $f(x,y)=\ln\frac{x+y}{x-y}$ bo'lsa, $f'_x(2e,e)$, $f'_y(2e,e)$ni toping.

1486. Agar $f(x,y)=y\sin x+\cos(x-y)$ bo'lsa, $f'_x(\pi,0)$, $f'_y(\pi,0)$ toping.

1487. Agar $f(x,y)=xye^{x+2y}$ bo'lsa, $f'_x(0,0)$, $f'_y(0,0)$ni toping.

1488. Agar $z=\ln(x^2+xy+y^2)$ bo'lsa, u holda ushbu $x\cdot\frac{\partial z}{\partial x}+y\cdot\frac{\partial y}{\partial x}=2$ tenglikning o'rinli bo'lishini isbotlang.

Quyidagi funksiyalarning differensiallarini toping:

1489. $z=x^2y-xy^2+3$. **1490.** $z=(x^2+y^2)^3$.

1491. $z=\sin^2 x+\cos^2 y$. **1492.** $z=arctg(xy)$. **1493.** $z=e^{12x+5y}$.

1494. $z=(\sin x)^{\cos y}$.

1495. $z=\ln\left(1+\frac{x}{y}\right)$. **1496.** $z=\frac{x}{y}e^{xy}$. **1497.** $x=arctg\sqrt{xy}$.

1498. Agar $f(x,y)=\frac{x}{y^2}$ bo'lsa, $df(1,1)$ni toping.

1499. Ushbu $f(x,y)=e^{xy}$ funksiya to'liq differensialini $x=1,\ y=2,\ dx=-0,1,\ dy=0,1$ bo'lgandagi qiymatini toping.

1500. Ushbu $f(x,y)=\frac{x}{x-y}$ funksiya to'liq differensialini $x=2,\ y=1,\ dx=-\frac{1}{3},\ dy=\frac{1}{2}$ bo'lgandagi qiymatini toping.

1501. Ushbu $f(x,y)=arctg\frac{x}{y}$ funksiya to'liq differensialini $x=1,\ y=3,\ dx=0,01,\ dy=-0,05$ bo'lgandagi qiymatini toping.

Quyidagi miqdorlarni taqribiy hisoblang:

1502. a) $1,08^{3,96}$, **b)** $1,94e^{0,12}$.

1503. a) $\sin 1,59\cdot tg 3,09$, **b)** $2,68e^{\sin 0,05}$.

Quyidagi murakkab funksiyalarning hosilalarini toping:

1504. Agar $z=e^{x^2+2y^2}$ bo'lib, $x=\sin t$, $y=\cos t$ bo'lsa, $\frac{dz}{dt}$ni toping.

1505. Agar $z=\frac{x}{y}$ bo'lib, $x=e^t$, $y=\ln t$ bo'lsa, $\frac{dz}{dt}$ni toping.

1506. Agar $z=\frac{x^2}{y}$ bo'lib, $x=u-2v$, $y=v+2v$ bo'lsa, $\frac{\partial z}{\partial u},\frac{\partial z}{\partial v}$larni toping.

1507. Agar $z=\ln\sin\frac{x}{\sqrt{y}}$ bo'lib, $x=3t^2$, $y=\sqrt{t^2+1}$ bo'lsa, $\frac{\partial z}{\partial t}$ni toping.

1508. Agar $z=x^2-y^2$ bo'lib, $x=u\cos v$, $y=u\sin v$ bo'lsa, $\frac{\partial z}{\partial u},\frac{\partial z}{\partial v}$larni toping.

1509. Agar $z=u^v$ bo'lib, $u=\sin x$, $v=\cos x$ bo'lsa, $\frac{\partial z}{\partial x}$ni toping.

1510. Agar $z=\ln\sqrt{\frac{u}{v}}$ bo'lib, $u=ax+by$, $v=ax-by$ bo'lsa, $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$ ni toping.

1511. Agar $z=e^{\frac{y^2}{x}}$ bo'lib, $x=x(u,v)$, $y=y(u,v)$ bo'lsa, $\frac{\partial z}{\partial u},\frac{\partial z}{\partial v}$ ni toping.

Quyidagi funksiyalarning ikkinchi tartibli hosilalarini toping:

1512. $z=x^4-4x^2y^2+y^4$. **1513.** $z=e^x\ln y$. **1514.** $z=\ln(x^2+y)$.

1515. $z=\sqrt{2xy+y^2}$. **1516.** $z=\frac{x^2}{2y-3}$. **1517.** $z=e^x\ln y+\sin y\ln x$.

1518. $z=e^{\frac{z}{x}}$. **1519.** $z=x\ln y+\sqrt{\sin x}$.

1520. Agar $z=arctg\frac{y}{x}$ bo'lsa, u holda ushbu $\frac{d^2z}{dx^2}+\frac{d^2z}{dy^2}=0$ tenglikning o'rinli bo'lishini isbotlang.

1521. Agar $z=\sin(x\cdot y)$ bo'lsa, $\frac{d^2z}{dxdy^2}$ ni toping.

1522. Agar $z=x^4-4x^2y^2+y^4$ bo'lsa, $\frac{d^2z}{dxdy^2}$ ni toping.

1523. Agar $z=\cos(x-y)$ bo'lsa, $\frac{d^2z}{dx^2dy}$ ni toping.

1524. Agar $z=\ln(x+y)$ bo'lsa, $\frac{d^2z}{dx^2dy}$ ni toping.

1525. Agar $z=arctg\frac{yx}{\sqrt{1+x^2+y^2}}$ bo'lsa, $\frac{d^2z}{dx^2dx^2}$ ni toping.

Quyidagi funksiyalarning ikkinchi tartibli differensiallarini toping:

1526. $z=2x^2-3xy-y^2$. **1527.** $z=e^{xy}$. **1528.** $z=\frac{x}{y}$. **1529.** $z=\ln\sqrt{x^2+y^2}$.

1530. Agar $f(x,y)=x^2+xy+y^2-4\ln x-10\ln y$ bo'lsa, $d^2f(1,2)$ toping.

1531. $z=\sin x\cos y$.

1532. Ushbu

$$z=3x^2+xy-y^2+1$$

funksiyaning $x_0=2$, $y_0=1$, $\Delta x=0{,}1$, $\Delta y=0{,}2$ bo'lgandagi to'liq orttirmasini toping.

a) 1,31, b) 1,33, c) 1,3, d) 1,34.

1533. Ushbu

$$z=e^{x^2+y^2}$$

funksiyaning xususiy hosilalarini toping.

a) $\frac{\partial z}{\partial x}=2xe^{x^2+y^2}$, $\frac{\partial z}{\partial y}=2ye^{x^2+y^2}$.

b) $\frac{\partial z}{\partial x}=2ye^{x^2+y^2}$, $\frac{\partial z}{\partial y}=2xe^{x^2+y^2}$.

c) $\frac{\partial z}{\partial x}=4xe^{x^2+y^2}$, $\frac{\partial z}{\partial y}=4ye^{x^2+y^2}$.

d) $\frac{\partial z}{\partial x}=xe^{x^2+y^2}$, $\frac{\partial z}{\partial y}=ye^{x^2+y^2}$.

1534. Ushbu

$$\alpha=1{,}04^{2{,}03}$$

miqdorning taqribiy qiymatini toping.

a) 1,01, b) 1,03, c) 1,05, d) 1,08.

1535. Ushbu

$$z=\ln(x^2+y^2)$$

funksiyaning differensialini toping.

a) $dz=\frac{xdx+ydy}{x^2+y^2}$. c) $dz=\frac{2(ydx+xdy)}{x^2+y^2}$.

b) $dz=\frac{2(xdx+ydy)}{x^2+y^2}$. d) $dz=\frac{ydx+xdy}{x^2+y^2}$.

3-§. Sirtga o'tkazilgan urinma tekislik va normal

$z=f(x,y)$ funksiya tekislikdagi E to'plamda berilgan bo'lib, uning grafigi fazoda biror Γ sirtni ifodalasin.

Aytaylik, (x_0, y_0, z_0) nuqta $(z_0=f(x_0,y_0))$ Γ sirtning nuqtasi bo'lsin. Unda sirtning (x_0,y_0,z_0) nuqtada o'tkazilgan urinma tekislik tenglamasi

$$z-z_0=\frac{\partial z(x_0,y_0)}{\partial x}(x-x_0)+\frac{\partial z(x_0,y_0)}{\partial y}(y-y_0) \quad (1)$$

normalning tenglamasi esa

$$\frac{x-x_0}{\frac{\partial z(x_0,y_0)}{\partial x}}=\frac{y-y_0}{\frac{\partial z(x_0,y_0)}{\partial y}}=\frac{z-z_0}{-1} \quad (2)$$

bo'ladi.

1-misol. Ushbu

$$z=x^2+y^2$$

tenglama bilan berilgan sirtga, uning (1; –1,2) nuqtasidan o'tuvchi urinma tekislik va normalning tenglamalarini toping.

◀Avvalo, $z=x^2+y^2$ funksiyalarning xususiy hosilalarini topamiz:

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}(x^2+y^2)=2x,\ \frac{\partial z}{\partial y}=\frac{\partial}{\partial y}(x^2+y^2)=2y$$

Bu xususiy hosilalarning $x=1,\quad y=-1$ nuqtadagi qiymatlari

$$\frac{\partial z(1,-1)}{\partial x}=2,\ \frac{\partial z(1,-1)}{\partial y}=-2$$

bo'ladi.

(1) va (2) formulalardan foydalanib, urinma tekislikning tenglamasi

$$z-2=2\cdot(x-1)-2\cdot(y+1)\ \text{ya'ni}\ 2x-2y-z-2=0$$

normalning tenglamasi

$$\frac{x-1}{2}=\frac{y+1}{-2}=\frac{z-2}{-1}$$

bo'lishini topamiz.▶

2-misol. Ushbu

$$z=e^{x\cos y}$$

tenglama bilan berilgan sirtning $(1,\pi,\frac{1}{e})$ nuqtadagi normalini toping.

◀Ma'lumki, $(1,\pi,\frac{1}{e})$ nuqtadan o'tuvchi normalning tenglamasi (2) formulaga ko'ra,

$$\frac{x-1}{\frac{\partial z(1,\pi)}{\partial x}}=\frac{y-\pi}{\frac{\partial z(1,\pi)}{\partial y}}=\frac{z-\frac{1}{e}}{-1}$$

bo'lib, bu to'g'ri chiziqning parametrik tenglamasi

$$x=1+\frac{\partial z(1,\pi)}{\partial x}t$$

$$y=\pi+\frac{\partial z(1,\pi)}{\partial y}t$$

$$z=\frac{1}{e}-t.$$

bo'ladi.

Ravshanki,

$$\frac{\partial z}{\partial x}=\cos y e^{x\cos y},\ \frac{\partial z}{\partial y}=-x\sin y e^{x\cos y}$$

bo'lib,

$$\frac{\partial z(1,\pi)}{\partial x}=\cos\pi\cdot e^{1\cos\pi}=-e^{-1}=-\frac{1}{e}$$

$$\frac{\partial z(1,\pi)}{\partial y}=\sin\pi\cdot e^{\cos\pi}=0\cdot e^{-1}=0$$

bo'ladi. Demak, normalning parametrik tenglamasi

$$x=1-\frac{t}{e},\quad y=\pi,\quad z=\frac{1}{e}-t$$

bo'ladi.▶

Quyidagi masalalarni yeching

1536. Ushbu

$$z=x^2+2y^2$$

tenglama bilan berilgan sirtga, uning (1, 1, 3) nuqtasida o'tkazilgan urinma tekislik tenglamasini toping.

1537. Ushbu

$$z=x^2-y^2$$

tenglama bilan beriigan sirtga, uning (2, 1, 3) nuqtasida o'tkazilgan urinma tekislik va normalning tenglamalarini toping.

1538. Ushbu

$$z=xy$$

tenglama bilan berilgan sirtga, uning (3, –2, –6) nuqtasida o'tkazilgan urinma tekislik va normalning tenglamalarini toping.

1539. Ushbu

$$z=x^3+y^3+3xy$$

tenglama bilan berilgan sirtga, uning (1, –1, –3) nuqtasida o'tkazilgan urinma tekislik va normalning tenglamalarini toping.

1540. Ushbu

$$z=\sqrt{1-x^2-y^2}$$

tenglama bilan berilgan sirtga, uning $\left(\frac{2}{3},\frac{2}{3},\frac{1}{3}\right)$ nuqtasida o'tkazilgan urinma tekislik va normalning tenglamalarini toping.

1541. Ushbu

$$x^2+y^2+z^2=2x$$

sferaga shunday urinma tekislik o'tkazilsinki, u quyidagi:

$$x-y-z=2\ \text{va}\ x-y-\frac{z}{2}=2$$

tekisliklarga perpendikulyar bo'lsin.

1542. Ushbu

$$z=x^4y-5xy^6$$

sirtga, uning (2, 1, 6) nuqtasida o'tkazilgan normal $x=0$ tekislikni qanday nuqtada kesadi?

4-§. Ikki o'zgaruvchili funksiyaning ekstremumi

1⁰. Ikki o'zgaruvchili funksiyaning Teylor formulasi. Aytaylik, $z=f(x,y)$ funksiya biror (x_0,y_0) nuqtaning $U_\delta(x,y)$ atrofida $(\delta>0)$ $n+1$-tartibgacha bo'lgan barcha uzluksiz xususiy hosilalarga ega bo'lsin. U holda

$$f(x,y)=f(x_0,y_0)+df(x_0,y_0)+\frac{1}{2!}d^2f(x_0,y_0)+...+\frac{1}{n!}d^nf(x_0,y_0)+\frac{1}{(n+1)!}d^{n+1}f(c_1,c_2)\quad (1)$$

bo'ladi, bunda c_1 son x_0 va x lar orasida, c_2 son esa y_0 va y lar orasida bo'ladi.

(1) formula $z = f(x, y)$ funksiyaning **Teylor formulasi** deyiladi.

1-misol. Ushbu

$$f(x, y) = x^y$$

funksiyaning (1,1) nuqta atrofida $n=3$ bo'lgan holda Teylor formulasini yozing.

◀Bu holda funksiyaning Teylor formulasi ushbu

$$f(x,y) = f(1,1) + \frac{df(1,1)}{1!} + \frac{d^2 f(1,1)}{2!} + \frac{d^3(1,1)}{3!} + R_3 \qquad (2)$$

ko'rinishda bo'ladi.

Berilgan funksiyaning 3-tartibligacha bo'lgan barcha xususiy hosilalarini topamiz:

$$f'_x(x,y) = \frac{\partial}{\partial x}(x^y) = yx^{y-1}, \; f'_y(x,y) = \frac{\partial}{\partial y}(x^y) = x^y \ln x;$$

$$f''_{x^2}(x,y) = \frac{\partial}{\partial x}(y \cdot x^{y-1}) = y \cdot (y-1) \cdot x^{y-2},$$

$$f''_{xy}(x,y) = \frac{\partial}{\partial y}(y \cdot x^{y-1}) = x^{y-1} + y \cdot x^{y-2} \ln x,$$

$$f''_{y^2}(x,y) = \frac{\partial}{\partial y}(x^y \ln x) = x^y (\ln x)^2,$$

$$f'''_{x^3}(x,y) = \frac{\partial}{\partial x}(y \cdot (y-1) \cdot x^{y-2}) = y(y-1)(y-2)x^{y-3},$$

$$f'''_{x^2 y}(x,y) = \frac{\partial}{\partial y}(y \cdot (y-1) \cdot x^{y-2}) = (2y-1)x^{y-2} + y(y-1)x^{y-2} \ln x,$$

$$f'''_{xy^2}(x,y) = \frac{\partial}{\partial y}(x^{y-1} + yx^{y-1} \ln x) = 2x^{y-1} \ln x + yx^{y-1} (\ln x)^2,$$

$$f'''_{y^3}(x,y) = \frac{\partial}{\partial y}(x^y (\ln x)^2 = x^y (\ln x)^3$$

Endi, berilgan funksiya va uning hosilalarining (1,1) nuqtadagi qiymatlarini hisoblaymiz:

$$f(1,1) = 1, \quad f'_x(1,1) = 1, \quad f'_y(1,1) = 0,$$

$$f''_{x^2}(1,1) = 0, \quad f''_{xy}(1,1) = 1, \quad f''_{y^2}(1,1) = 0,$$

$$f'''_{x^3}(1,1) = 0, \quad f'''_{x^2 y}(1,1) = 1, \quad f'''_{xy^2}(1,1) = 0, \quad f'''_{y^3}(1,1) = 0.$$

(2) formulada qatnashgan differensiallarni topamiz:

$$df(1,1) = f'_x(1,1)\Delta x + f'_y(1,1)\Delta y = \Delta x,$$

$$d^2 f(1,1) = f''_{x^2}(1,1)\Delta x^2 + 2f''_{xy}(1,1)\Delta x\Delta y + f''_{y^2}(1,1)\Delta y^2 = 2\Delta x\Delta y,$$

$$d^2 f(1,1) = f'''_{x^3}(1,1)\Delta x^3 + 3f'''_{x^2 y}(1,1)\Delta x^2 \Delta y + 3f'''_{xy^2}(1,1)\Delta x\Delta y^2 + f'''_{y^3}(1,1)\Delta y^3 = 3\Delta x^2 \Delta y$$

Bu differensiallarning (2) ga qo'ysak, unda

$$x^y = 1 + \Delta x + \Delta x\Delta y + \frac{1}{2}\Delta x^2 \Delta y + R_3$$

bo'ladi.▶

2⁰. Funksiyaning statsionar nuqtalari. $z = f(x, y)$ funksiya xususiy hosilalari $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ larni nolga aylantiradigan nuqtalar funksiyaning statsionar nuqtalari deyiladi. Statsionar nuqtalarning koordinatalari ushbu

$$\begin{cases} f'_x(x,y) = 0, \\ f'_y(x,y) = 0 \end{cases}$$

tenglamalar sistemasini yechib topiladi.

2-misol. Ushbu

$$z = f(x,y) = 4x^2 y + 24xy + y^2 + 32y - 6$$

funksiyaning statsionar nuqtalarini toping.

◀Berilgan funksiyaning xususiy hosilalarini hisoblaymiz:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(4x^2 y + 24xy + y^2 + 32y - 6) = 8xy + 24y,$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(4x^2 y + 24xy + y^2 + 32y - 6) = 4x^2 + 24x + 2y + 32.$$

So'ng ularni 0 ga tenglab, quyidagi:

$$\begin{cases} 8xy + 24y = 0, \\ 4x^2 + 24x + 2y + 32 = 0 \end{cases}$$

tenglamalar sistemasini hosil qilamiz.

Bu sistemani yechamiz: Ravshanki,

$$\begin{cases} y(x+3) = 0, \\ 2x^2 + 12x + y + 16 = 0 \end{cases}$$

Sistemaning birinchi tenglamasidan $y = 0$, $x = -3$, bo'lishi kelib chiqadi. Unda $(y = 0)$ da sistemaning ikkinchi tenglamasi

$$x^2 + 6x + 8 = 0$$

ko'rinishiga keladi. Bu kvadrat tenglamaning yechimlari $x_1 = -4$, $x_2 = -2$ bo'ladi.

Demak, berilgan funksiyaning statsionar nuqtalari

$$(-4, 0), (-2, 0), (-3, 2)$$

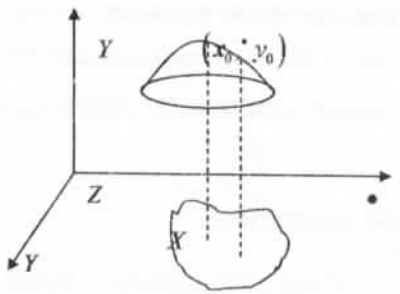
bo'ladi.▶

3⁰. Funksiyaning "ekstremum" tushunchasi. Ekstremumning zaruriy va yetarli shartlari. Aytaylik, $z = f(x, y)$ funksiya tekislikdagi E to'plamda berilgan bo'lib, $(x_0, y_0) \in E$ bo'lsin.

Agar (x_0, y_0) nuqtaning shunday $U_\delta(x_0, y_0)$ atrofi topilsaki, $U_\delta(x_0, y_0) \subset E$ bo'lib, ixtiyoriy $(x, y) \in U_\delta(x_0, y_0)$ uchun

$$f(x,y) \le f(x_0, y_0)$$

bo'lsa, $f(x,y)$ funksiya (x_0, y_0) nuqtada maksimumga erishadi deyiladi. Bunda (x_0, y_0) nuqta $f(x,y)$ funksiyaga maksimum qiymat beradigan nuqta, $f(x_0, y_0)$ ga esa funksiyaning maksimum qiymati deyiladi va $\max f(x,y) = f(x_0, y_0)$ kabi yoziladi (2-chizma)

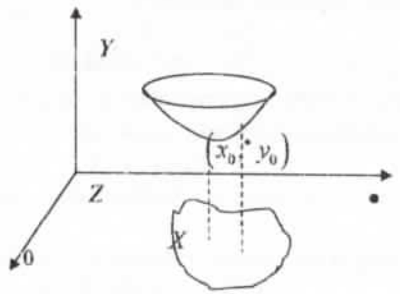


2-chizma

Agar (x_0, y_0) nuqtaning shunday $U_\delta(x_0, y_0)$ atrofi topilsaki, $U_\delta(x_0, y_0) \subset E$ bo‘lib, ixtiyoriy $(x,y) \in U_\delta(x_0, y_0)$ uchun

$$f(x,y) \geq f(x_0, y_0)$$

bo‘lsa, $f(x,y)$ funksiya (x_0, y_0) nuqtada minimumga erishadi deyiladi. Bunda (x_0, y_0) nuqta $f(x,y)$ funksiyaga minimum qiymat beradigan nuqta, $f(x_0, y_0)$ ga esa funksiyaning minimum qiymati deyiladi va $\min f(x,y) = f(x_0, y_0)$ kabi yoziladi (3-chizma)



3-chizma

Funksiyaning maksimumi va minimumi umumiy nom bilan uning ekstremumi deyiladi.

Agar $z = f(x, y)$ funksiya (x_0, y_0) nuqtada differensiallanuvchi bo‘lib, bu nuqtada ekstremumga erishsa u holda

$$f'_x(x_0, y_0) = 0, f'_y(x_0, y_0) = 0$$

bo‘ladi (ekstremumning zaruriy sharti).

Aytaylik, (x_0, y_0) nuqta $z = f(x, y)$ funksiyaning statsionar nuqtasi bo‘lsin. Ushbu belgilashlarni kiritamiz:

$$a = \frac{\partial^2 f(x_0, y_0)}{\partial x^2}, b = \frac{\partial^2 f(x_0, y_0)}{\partial x \partial y}, c = \frac{\partial^2 f(x_0, y_0)}{\partial y^2}$$

Agar

1) $ac - b^2 > 0$ va $a < 0$ bo‘lsa, $z = f(x, y)$ funksiya (x_0, y_0) nuqtada maksimumga erishadi;

2) $ac - b^2 > 0$ va $a > 0$ bo‘lsa, $z = f(x, y)$ funksiya (x_0, y_0) nuqtada minimumga erishadi;

3) $ac - b^2 < 0$ bo‘lsa, $z = f(x, y)$ funksiya (x_0, y_0) nuqtada ekstremumga ega bo‘lmaydi;

4) $ac - b^2 = 0$ bo‘lsa, $z = f(x, y)$ funksiya (x_0, y_0) nuqtada ekstremumga erishishi ham, erishmasligi ham mumkin. Bu holda qo‘shimcha tekshirish olib boriladi. (ekstremumning yetarli sharti).

3-misol. Ushbu

$$z = x^2 + xy + y^2 - 2x - 3y$$

funksiyani ekstremumga tekshiring.

◄Avvalo, berilgan funksiyaning statsionar nuqtalarini, ya’ni ekstremumning zaruriy shartining bajarilishini ko‘rsatamiz. Buning uchun funksiyaning xususiy hosilalarini hisoblab, ularni nolga tenglab, sistemani yechamiz:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2 + xy + y^2 - 2x - 3y) = 2x + y - 2,$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2 + xy + y^2 - 2x - 3y) = x + 2y - 3,$$

$$\begin{cases} 2x + y - 2 = 0, \\ x + 2y - 3 = 0 \end{cases}$$

Bu sistemaning yechimi $x = \frac{1}{3}$, $y = \frac{4}{3}$ bo‘ladi.

Demak, izlanayotgan statsionar nuqta $\left(\frac{1}{3}, \frac{4}{3}\right)$ bo‘ladi.

Endi funksiya ekstremumga erishishining yetarli shartlarining bajarilishini tekshiramiz.

Ravshanki,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(2x + y - 2) = 2,$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}(2x + y - 2) = 1,$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(x + 2y - 3) = 2$$

Demak,

$$a = 2, \quad b = 1, \quad c = 2$$

Unda $ac - b^2 = 2 \cdot 2 - 1 = 3 > 0$ bo‘ladi. Ayni paytda, $a = 2 > 0$ bo‘lgani uchun berilgan funksiya $\left(\frac{1}{3}, \frac{4}{3}\right)$ nuqta minimumga erishadi.►

4-misol. Ushbu

$$z = 3x^2y - x^3 - y^4$$

funksiyani ekstremumga tekshiring.

◀Ravshanki,

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(3x^2y - x - y^4) = 6xy - 3x^2,$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(3x^2y - x^3 - y^4) = 3x^2 - 4y^3$$

Quyidagi

$$\begin{cases} 6xy - 3x^2 = 0 \\ 3x^2 - 4y^3 = 0 \end{cases}$$

sistemani yechib, statsionar nuqtalarni topamiz.

Bu sistemaning yechimi $x_1 = 6,\ y_1 = 3$ va $x_2 = 0,\ y_2 = 0$ bo‘ladi.

Demak, $M_1(6,3)$, $M_2(0,0)$ statsionar nuqtalar bo‘ladi.

Berilgan funksiyaning ikkinchi tartibli xususiy hosilalarini hisoblaymiz:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(6xy - 3x^2) = 6y - 6x,$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(6xy - 3x^2) = 6x,$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial x}(3x^2 - 4y^3) = -12y$$

$M_1(6, 3)$ nuqtada

$$a = -18, \quad b = 36, \quad c = -108$$

bo‘lib,

$$ac - b^2 = -18 \cdot (-108) - 36^2 = 648 > 0$$

bo‘ladi. Ayni paytda, $a < 0$ bo‘lgani uchun berilgan funksiya $M_1(6, 3)$ nuqtada maksimumga erishadi. Uning maksimum qiymati

$$\max f(x, y) = f(6,3) = 3 \cdot 36 \cdot 3 - 6^3 - 3^4 = 324 - 216 - 81 = 27$$

bo‘ladi.

$M_2(0,0)$ nuqtada

$$a = 0, \quad b = 0, \quad c = 0$$

bo‘lib, $ac - b^2 = 0$ bo‘ladi. Demak, bu holda ekstremumga erishishini aniqlash uchun qo‘shimcha tekshirish qilish kerak.

Agar $x = 0,\ y \neq 0$ bo‘lsa, $z = -y^4$ bo‘lib, funksiya $(0, 0)$ nuqtaning atrofida manfiy qiymatga ega bo‘ladi, agar $x < 0,\ y = 0$ bo‘lsa, $z = -x^3$ bo‘lib, funksiya $(0, 0)$ nuqtaning atrofida musbat qiymatga ega bo‘ladi.

Demak, berilgan funksiya $M_2(0,0)$ nuqtaning atrofida ishora saqlamaydi. Binobarin, funksiya bu nuqtada ekstremumga ega bo‘lmaydi.▶

4^0. Funksiyaning eng katta va eng kichik qiymatlari. Aytaylik, $z = f(x, y)$ funksiya tekislikdagi chegaralangan yopiq E to‘plamda berilgan bo‘lsin. Ushbu

1. funksiyaning statsionar nuqtalardagi qiymatlari;
2. funksiyaning E to‘plamning chegarasidagi eng katta va eng kichik qiymatlari orasidagi eng katta qiymat (eng kichik qiymat) $z = f(x, y)$ funksiyaning E to‘plamdagi eng katta (eng kichik) qiymati bo‘ladi.

5-misol. Ushbu

$$z = x^2 - y^2$$

funksiyaning $E = \{(x, y) : x^2 + y^2 \leq 4\}$ – markazi (0, 0) nuqtada, radiusi $r = 2$ bo‘lgan yopiq doiradagi eng katta va eng kichik qiymatlarini toping.

◀Berilgan funksiyaning xususiy hosilalarini hisoblab, statsionar nuqtalarni topamiz:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2 - y^2) = 2x, \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2 - y^2) = -2y$$

$$\begin{cases} 2x = 0 \\ -2y = 0 \end{cases}$$

Demak, $M(0,0)$ statsionar nuqta bo‘ladi.

Ravshanki,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(2x) = 2, \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(2x) = 0, \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(-2x) = -2$$

bo‘lib, $ac - b^2 = 2 \cdot (-2) - 0 = -4 < 0$ bo‘ladi.

Demak, berilgan funksiya $M(0,0)$ nuqtada ekstremumga ega emas.

Endi, $z = x^2 - y^2$ funksiyani E ning chegarasi $x^2 + y^2 = 4$ aylanada qarab, hosil bo‘lgan funksiyaning eng katta va eng kichik qiymatlarini topamiz.

Agar chegarada $y^2 = 4 - x^2$ bo‘lishini e’tiborga olsak, u holda berilgan funksiya chegarada

$$z = x^2 - y^2 = x^2 - (4 - x^2) = 2x^2 - 4$$

bo‘ladi, bunda: $-2 \leq x \leq 2$

Ravshanki, $z' = 4x$ bo‘lib, $4x = 0,\ x_0 = 0$ nuqta $z = 2x^2 - 4$ funksiyaning statsionar nuqtasi bo‘ladi.

$z'' = 4 > 0$. Demak, $z = 2x^2 - 4$ funksiya $x_0 = 0$ nuqtada minimumga erishib, uning minimum qiymati -4 ga teng bo‘ladi.

Endi, $z = 2x^2 - 4$ funksiyani $x = -2,\ x = 2$ nuqtalardagi qiymatlarini topamiz:

$$z|_{x=2} = +4, \ z|_{x=-2} = 4$$

Shunday qilib, berilgan $z = x^2 - y^2$ funksiyani E to‘plamdagi eng katta qiymati 4, eng kichik qiymati -4 bo‘ladi.▶

Quyidagi funksiyalarni Teylor formulasi bo‘yicha yoying:

1543. Ushbu $f(x, y) = e^x \sin y$ funksiya $(0,0)$ nuqta atrofida uchinchi hadigacha Teylor formulasi bo‘yicha yoyilsin.

1544. Ushbu $f(x, y) = e^x \ln(1 + y)$ funksiya $(0,0)$ nuqta atrofida uchinchi hadigacha Teylor formulasi bo‘yicha yoyilsin.

1545. Ushbu $f(x,y)=\sqrt{1-x^2-y^2}$ funksiya $(0,0)$ nuqta atrofida uchinchi hadigacha Teylor formulasi bo'yicha yoyilsin.

Quyidagi funksiyalarning statsionar nuqtalarini toping:

1546. $z=x^2+xy+y^2-2x-3y$. **1547.** $z=x^3+8y^3-6xy+5$.
1548. $z=x^3+y^3-3xy$. **1549.** $z=x^2+3xy^2-15x-12y$.
1550. $z=x^2+xy+y^2-mx-ny$.

Quyidagi funksiyalarni ekstremumga tekshiring:

1551. $z=(x-1)^2+2y^2$. **1552.** $z=x^2+xy+y^2-6x-9y$.
1553. $z=x\sqrt{y}-x^2-y+6x+3$. **1554.** $z=xy(1-x-y)$.
1555. $z=x^2+y^2-9xy+27$. **1556.** $z=\frac{1}{2}xy+(47-x-y)\left(\frac{x}{3}+\frac{y}{4}\right)$.
1557. $z=x^3y^2(6-x-y)$. **1558.** $z=e^{-x^2-y^2}(x^3+2y^2)$.
1559. $z=e^{\frac{x}{2}}(x+y^2)$. **1560.** $z=3\ln\frac{x}{6}+2\ln y+\ln(12-x-y)$.

Quyidagi funksiyalarning ko'rsatilgan sohadagi eng katta qiymatlari va eng kichik qiymatlarini toping:

1561. $z=x-2y-3;\quad 0\le x\le 1, 0\le y\le 1, 0\le x+y\le 1$.
1562. $z=x^2+3y^3-x+18y-4;\quad 0\le x\le 1, 0\le y\le 1$.
1563. $z=x^2+y,\ x^2+y^2\le 1$.
1564. $z=x^3+y^3-3xy,\quad 0\le x\le 2, -1\le y\le 2$.
1565. $z=3xy,\ x^2+y^2\le 2$.
1566. $z=x^2+3y^2-x+18y-4,\quad 0\le x\le y\le 4$.

1567. Hajmi $V=32м^2$ bo'lgan usti ochiq qutining tomonlari qanday o'lchamda bo'lsa, uning sirti eng kichik bo'ladi?

1568. Silindr shaklidagi suv idishning material qalinligi d va sig'imi V bo'lganda, uni tayyorlash uchun ketadigan material o'lchovini aniqlang.

1569. Uzunligi l bo'lgan sim bo'lagidan hajmi eng katta bo'lgan to'g'riburchakli parallelepiped qanday bo'ladi?

5-§. Oshkormas funksiyalar

1°. "Oshkormas funksiya" tushunchasi. Aytaylik, ikki x va y o'zgaruvchilarning (argumentlarning) $F(x,y)$ funksiyasi

$$M=\{(x,y)\in R^2 : a<x<b, c<y<d\}$$

to'plamda (to'g'ri to'rtburchak shaklidagi sohada) berilgan bo'lsin. x va y larni noma'lumlar deb ushbu

$$F(x,y)=0 \tag{1}$$

tenglamani qaraylik. Biror x_0 sonni $(x_0\in(a,b))$ olib, uni (1) tenglamadagi x ning o'rniga qo'yamiz:

$$F(x_0,y)=0 \tag{2}$$

Ravshanki, (2) tenglama y ga bog'liq bo'ladi, ya'ni y ning tenglamasi bo'ladi.

Faraz qilaylik, (2) tenglama yagona $y=y_0$ yechimga ega bo'lsin:

$$F(x_0,y_0)\equiv 0.$$

(Shuni ta'kidlash lozimki, y_0 olingan x_0 ning qiymatiga bog'liq bo'ladi).

Endi x o'zgaruvchining qiymatlaridan iborat shunday M_x to'plamni (ravshanki, $M_x\subset(a,b)$) qaraylikki, bu to'plamdan olingan har bir x qiymatda

$$F(x,y)=0$$

tenglama yagona y yechimga ega bo'lsin.

Agar M_x to'plamdan olingan ixtiyoriy x songa unga $F(x,y)=0$ tenglamaning yechimi y son mos qo'yilsa, funksiya hosil bo'ladi. Bunda x va y o'zgaruvchilar orasidagi funksional bog'lanish

$$F(x,y)=0$$

tenglama yordamida bo'ladi:

$$x\to y:\ F(x,y)=0.$$

Bunday berilgan (aniqlangan) funksiya oshkormas ko'rinishda berilgan funksiya (yoki oshkormas funksiya) deyiladi.

Demak, oshkormas funksiya, erksiz o'zgaruvchi y ga nisbatan yechilmagan tenglama bilan aniqlanadi.

1-misol. Ushbu

$$F(x,y)=x-y+\frac{1}{2}\sin y=0 \tag{3}$$

tenglama y ni x ning oshkormas funksiyasi sifatida aniqlanishi ko'rsatilsin.

◄(3) tenglamani quyidagicha yozib olamiz:

$$x=y-\frac{1}{2}\sin y=\varphi(y)\qquad (y\in(-\infty,+\infty)).$$

Unda teoremaga ko'ra, $(0;1)$ nuqtaning biror atrofida (5) tenglama uzluksiz $y=\varphi(x)$ oshkormas funksiyani aniqlaydi.▶

4-misol. Ushbu

$$F(x,y)=y^2-2x^3y+x^6-x^4+x^2=0$$

tenglama $M(0;0)$ nuqtaning atrofida oshkormas funksiyani aniqlaydimi?

◀Yo'q, aniqlamaydi, chunki

$$\frac{\partial F(x,y)}{\partial y}=\frac{\partial}{\partial y}(y^2-2x^3y+x^6-x^4+x^2)=2y-2x^3$$

bo'lib,

$$\frac{\partial F(0,0)}{\partial y}=(2y-2x^3)_{\substack{x=0\\y=0}}=0$$

bo'ladi, teoremaning sharti bajarilmaydi.▶

3^0. Oshkormas funksiyaning hosilalari

Agar $F(x,y)$ funksiya yuqorida keltirilgan teoremaning barcha shartlarini bajarsa, u holda

$$F(x,y)=0$$

tenglama bilan aniqlanadigan $y=y(x)$ oshkormas funksiya $(x_0-\delta,\, x_0+\delta)$ da hosilaga ega va

$$y'_{x=x_0}=-\frac{F'_x(x_0,y_0)}{F'_y(x_0,y_0)}$$

bo'ladi.

Quyidagi tenglamalar ko'rsatilgan nuqta atrofida oshkormas funksiyani aniqlaydimi?

1570. $F(x,y)=e^y+y\sin x-x^2+7=0,\quad (2;0)$.

1571. $F(x,y)=x^3+y^3-3axy,\quad (a\sqrt[3]{4};\, a\sqrt[3]{2})$.

1572. $F(x,y)=y-xe^y+x=0,\quad (0;1)$.

1573. $F(x,y)=\sin(x+y)-y=0,\quad (0;0)$.

Quyidagi tenglamalar bilan aniqlanadigan oshkormas funksiyalarni oshkor $y=y(x)$ ko'rinishida yozib, ularning aniqlanish sohalarini toping

1574. $x^2-\arccos y-\pi=0$. **1575.** $10^x+10^y-10=0$.

1576. $x+|y|-2y=0$. **1577.** $e^{x^2+y^2}-x^6-5=0$.

Quyidagi tenglamalar bilan aniqlanadigan oshkormas funksiyalarning hosilalari toping

1578. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$. **1579.** $y-x-\ln y=0$. **1580.** $y-1-y^x=0$.

1581. $xe^{2y}-y\ln x-8=0$. **1582.** $\ln\frac{\sqrt{x^2+y^2}}{2}-arctg\frac{y}{x}=0$.

1583. $y^x-x^y=0$. **1584.** $1+xy-\ln(e^{xy}+e^{-xy})=0$.

Kompyuter yordamida aniq integrallarni hisoblash

Mapleda ko'p o'zgaruvchili funksiyaning hosilasini hisoblash uchun

```
>diff(expr,x1$n1,x2$n2,...);
```

buyrug'idan foydalanamiz, bu yerda `expr`- `x1, x2...` o'zgaruvchilarga bog'liq bo'lgan hosilasi hisoblanishi lozim bo'lgan kattalik, `n1, n2...`- hosila tartibi.

1-misol. $z=x^3+y^3\_2xy$ funksiyaning xususiy hosilalarini toping.

```
◀> z:=x^3+y^3-2*x*y:diff(z,x);diff(z,y);
```

javob: $3x^2-2y$

$3y^2-2x$ ▶

2-misol. $z=e^x\ln y+\sin y\ln x$ funksiyaning ikkinchi tartibli hosilalarini toping.

```
◀z:=exp(x)*ln(y)+sin(y)*ln(x):diff(z,x$2);diff(z,y$2);diff(z,x,y)
```

javob: $e^x\ln(y)-\frac{\sin(y)}{x^2}$

$-\frac{e^x}{y^2}-\sin(y)\ln(x)$

$\frac{e^x}{y}+\frac{\cos(y)}{x}$ ▶

Maple yordamida berilgan funksiyaning ekstremumlarini topish uchun quyidagi buyruq kiritiladi:

```
extrema(expr,constr,vars,nv),
```

bu yerda: `expr` – ekstremumi topilishi lozim bo'lgan kattalik, `constr` – chegara, `vars` – o'zgaruvchilar, `nv` – ekstremum erishadigan nuqtalar koordinatalari. Ekstremumga murojaat qilishdan oldin `readlib` buyrug'i chaqiriladi.

3-misol. $z=(x-1)^2+y^2$ funksiyani ekstremumga tekshiring

```
◀> readlib(extrema):
> extrema((x-1)^2+2*y^2,{},{x,y},'z');z;
```

$\{0\}$

$\{\{y=0, x=1\}\}$ ▶

4-misol. $z=e^{-x^2-y^2}(x^3+2y^2)$ funksiyani ekstremumga tekshiring

```
◀> readlib(extrema):
> extrema(exp(-x^2-y^2)*(x^3+2*y^2),{}.{x,y},'z');z;
```

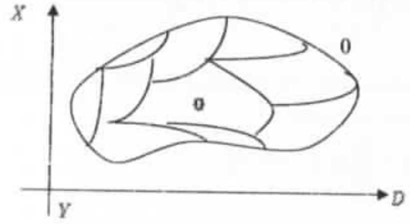
$\{\{y=0, x=1\}\}$ ▶

Nazorat savollari

1. Sath chizig'ini izohlab bering.
2. Funksiyaning limiti qanday topiladi?
3. Funksiyaning uzluksizligi qanday aniqlanadi?
4. Funksiyaning xususiy hosilalari qanday topiladi?
5. Funksiyaning differensialini izohlab bering.
6. Murakkab funksiyaning hosilalari qanday topiladi?
7. Yuqori tartibli hosila va differensiallar qanday topiladi?
8. Ikki o'zgaruvchili funksiyaning Teylor formulasi qanday topiladi?
9. Funksiyaning statsionar nuqtalari qanday topiladi?
10. Funksiyaning "ekstremum" tushunchasini izohlab bering.
11. Ekstremumning zaruriy va yetarli shartlari qanday topiladi?
12. Funksiyaning eng katta va eng kichik qiymatlari qanday topiladi?
13. Oshkormas funksiyani izohlab bering.
14. Oshkormas funksiyaning mavjudligi haqidagi teoremani keltring.
15. Oshkormas funksiyaning hosilalari qanday topiladi?

14-bob
Ko'p o'zgaruvchili funksiyaning integral hisobi
1-§. Ikki karrali integrallar. Integralning xossalari va hisoblash usullari
1^0. "Ikki karrali integral" tushunchasi. Integralning xossalari

Aytaylik, tekislikdagi D to'plam chegaralangan hamda yuzaga ega bo'lgan tekis shaklni ifodalasin (1-chizma).



1-chizma

Bu to'plamda $z = f(x,y)$ funksiya aniqlangan va uzluksiz bo'lsin.

$f(x,y)$ funksiyaning D to'plam bo'yicha ikki karrali integrali quyidagicha kiritiladi:

1. D to'plam (shakl) yuzaga ega bo'lgan

$$D_1, D_2, ..., D_n$$

bo'lakchalarga ajratiladi (1-chizma). Bunda D_k bo'lakchaning yuzini S_k bilan belgilaymiz, $k = 1,2,3,...,n$.

2. Har bir D_k bo'lakchada ixtiyoriy (ξ_k, η_k) nuqtani olib, funksiyaning shu nuqtadagi qiymati $f(\xi_k, \eta_k)$ ni S_k ga ko'paytiriladi:

$$f(\xi_k, \eta_k) \cdot S_k \qquad (k = 1,2,3,...,n)$$

3. Bu ko'paytmalardan quyidagi

$$\sigma = \sum_{k=1}^{n} f(\xi_k, \eta_k) \cdot S_k \qquad (1)$$

yig'indi tuziladi. Bu yig'indi **integral yig'indi** deyiladi.

4. Har bir bo'lakcha diametrining eng kattasi nolga intilganda, σ yig'indining limiti qaraladi:

$$\lim \sigma = \lim \sum_{k=1}^{n} f(\xi_k, \eta_k) \cdot S_k$$

Agar bu limit mavjud bo'lsa, ($f(x,y)$ funksiya D da uzluksiz bo'lsa, limit mavjud bo'ladi) u $f(x,y)$ funksiyaning D to'plam bo'yicha ikki karrali integrali deyiladi va

$$\iint_D f(x,y)dxdy$$

kabi belgilanadi. Demak,

$$\iint_D f(x,y)dxdy = \lim_{\lambda \to 0} \sum_{k=1}^{n} f(\xi_k, \eta_k) \cdot S_k$$

bunda: $\lambda=\max\{d_k\}$ ($d_k - D_k$ bo'lakchaning diametri. U D_k sohaning ikki nuqtasi orasidagi masofalarning eng kattasi).

Ikki karrali integral ham aniq integral xossalari kabi xossalarga ega bo'ladi.

1. Agar $f(x,y)$ va $g(x,y)$ funksiyalar D uzluksiz bo'lib, α va β lar o'zgarmas sonlar bo'lsa, u holda

$$\iint_D(\alpha f(x,y)\pm\beta g(x,y))dxdy=\alpha\iint_D f(x,y)dxdy\pm\beta\iint_D g(x,y)dxdy$$

bo'ladi.

2. Agar $f(x,y)$ va $g(x,y)$ funksiyalar D da uzluksiz bo'lib, ixtiyoriy $(x,y)\in D$ da $f(x,y)\le g(x,y)$ bo'lsa, u holda

$$\iint_D f(x,y)dxdy\le\iint_D g(x,y)dxdy$$

bo'ladi.

3. Agar $f(x,y)$ funksiya D da uzluksiz bo'lsa, u holda shunday $M_0(x_0,y_0)\in D$ nuqta topiladiki,

$$\iint_D f(x,y)dxdy=f(x_0,y_0)\cdot S\text{, ya'ni }\frac{1}{S}\iint_D f(x,y)dxdy=f(x_0,y_0)$$

bo'ladi.

4. Agar D to'plam $D=D_1\cup D_2$ (bunda $D_1\cap D_2=\varnothing$) bo'lsa, u holda

$$\iint_D f(x,y)dxdy=\iint_{D_1} f(x,y)dxdy+\iint_{D_2} f(x,y)dxdy$$

bo'ladi.

5. Agar $f(x,y)$ funksiya D da uzluksiz bo'lsa,

$$\left|\iint_D f(x,y)dxdy\right|\le\iint_D|f(x,y)|dxdy$$

bo'ladi.

1-misol. Ushbu

$$f(x,y)=c-const$$

funksiyaning D to'plam bo'yicha ikki karrali integralini toping.

◀(1) formulaga ko'ra, bu funksiyaning integral yig'indisi

$$\sigma=\sum_{k=1}^{n}f(\xi_k,\eta_k)\cdot S_k=\sum_{k=1}^{n}c\cdot S_k=c\sum_{k=1}^{n}S_k=cS$$

bo'ladi. Ravshanki,

$$\lim_{\lambda\to0}\sigma=\lim_{\lambda\to0}cS=cS.$$

Demak,

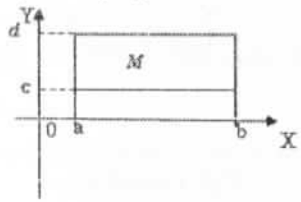
$$\iint_D cdxdy=cS$$

bo'ladi.

2⁰. Ikki karrali integrallarni hisoblash: a) to'g'ri to'rtburchak soha (to'plam) bo'yicha ikki karrali integrallarni hisoblash. Aytaylik, $z=f(x,y)$ funksiya tekislikdagi

$$M=\{(x,y):a\le x\le b,\quad c\le y\le d\}$$

to'plamda (to'g'ri to'rtburchak sohada) berilgan va uzluksiz bo'lsin (2-chizma).



2-chizma

Bu holda

$$\iint_M f(x,y)dxdy=\int_a^b\int_c^d f(x,y)dxdy=$$

$$=\int_c^d\left[\int_a^b f(x,y)dx\right]dy=\int_a^b\left[\int_c^d f(x,y)dy\right]dx \quad (2)$$

bo'ladi.

Bu munosabatlar yordamida ikki karrali integrallar takrorlab integrallash yo'li bilan hisoblanadi.

2-misol. Ushbu

$$J=\iint_M(x^2+y^2)dxdy$$

ikki karrali integralni hisoblang, bunda

$$M=\{(x,y):0\le x\le1,\quad 0\le y\le1\}$$

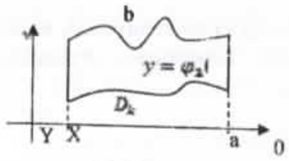
to'g'ri to'rtburchak (kvadrat) dan iborat.

◀Integrallash chegaralarini qo'yib, so'ng (2) munosabatdan foydalanib topamiz:

$$J=\int_0^1\int_0^1(x^2+y^2)dxdy=\int_0^1\left[\int_0^1(x^2+y^2)dy\right]dx=\int_0^1\left(x^2y+\frac{y^3}{3}\right)\Bigg|_{y=0}^{y=1}dx=$$

$$=\int_0^1\left(x^2+\frac{1}{3}\right)dx=\left(\frac{x^3}{3}+\frac{1}{3}x\right)\Bigg|_{x=0}^{x=1}=\frac{1}{3}+\frac{1}{3}=\frac{2}{3}\blacktriangleright$$

b) egri chiziqli trapetsiya soha bo'yicha ikki karrali integrallarni hisoblash. Tekislikda yuqoridan $\varphi_2(x)$ funksiya grafigi, pastdan $\varphi_1(x)$ funksiya grafigi, yon tomonlardan $x=a, x=b$ vertikal chiziqlar bilan chegaralangan sohani (to'plamni) qaraylik, bunda $\varphi_1(x)$ va $\varphi_2(x)$ funksiyalar $[a,b]$ da uzluksiz va unda $\varphi_1(x)\le\varphi_2(x)$. Odatda, bu soha egri chiziqli trapetsiya deyiladi. Uni M_1 bilan belgilaymiz (3-chizma).



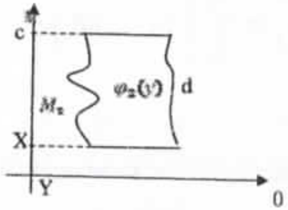
3-chizma

$$M_1=\{(x,y): a\le x\le b,\quad \varphi_1(x)\le y\le \varphi_2(x)\}$$

Aytaylik, $z=f(x,y)$ funksiya M_1 to'plamda uzluksiz bo'lsin. Unda

$$\iint_{M_1} f(x,y)dxdy=\int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy\right]dx \qquad (3)$$

bo'ladi.



4-chizma

Aytaylik, $z=f(x,y)$ funksiya

$$M_2=\{(x,y): \psi_1(y)\le x\le \psi_2(y), c\le y\le d\}$$

sohada (to'plamda) uzluksiz bo'lsin, unda

$$\iint_{M_2} f(x,y)dxdy=\int_c^d\left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx\right]dy \qquad (4)$$

bo'ladi (4-chizma).

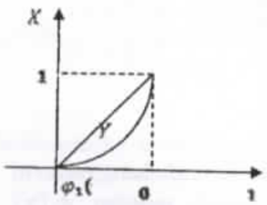
3-misol. Ushbu

$$J=\iint_{M_2} xy^2dxdy$$

ikki karrali integralni hisoblang, bunda:

$$M_2=\{(x,y): 0\le x\le 1,\quad x^2\le y\le x\}$$

to'plamdan iborat (5-chizma)



5-chizma

◀Ravshanki, bu hol uchun

$$a=0,\quad b=1,\quad \varphi_1(x)=x^2,\quad \varphi_2(x)=x$$

bo'lib, qaralayotgan integral (3) formulaga ko'ra,

$$J=\iint_{M_1} xy^2dxdy=\int_0^1\left[\int_{x^2}^{x} xy^2dy\right]dx$$

bo'ladi.

Avvalo, x ni o'zgarmas deb

$$\int_{x^2}^{x} xy^2dy$$

integralni hisoblaymiz:

$$\int_{x^2}^{x} xy^2dy=x\int_{x^2}^{x} y^2dy=x\cdot\left(\frac{y^3}{3}\right)\Bigg|_{y=x^2}^{y=x}=x\left(\frac{x^3}{3}-\frac{x^6}{3}\right)=\frac{x^4}{3}-\frac{x^7}{3}.$$

Unda

$$J=\int_0^1\frac{1}{3}\left(x^4-x^7\right)dx=\frac{1}{3}\left(\frac{x^5}{5}-\frac{x^8}{8}\right)\Bigg|_{x=0}^{x=1}=\frac{1}{3}\left(\frac{1}{5}-\frac{1}{8}\right)=\frac{1}{40}$$

bo'ladi.▶

4-misol. Ushbu

$$J=\iint_{M_2} e^{-y^2}dxdy$$

ikki karrali integralni hisoblang, bunda:

$$M_2=\{(x,y): 0\le x\le y,\quad 0\le y\le 1\}$$

◀Ravshanki, bu holda

$$\psi_1(y)=0,\quad \psi_2(y)=y,\quad c=0, d=1$$

bo'lib, qaralayotgan integral (4) formulaga ko'ra

$$J=\iint_{M_2} e^{-y^2}dxdy=\int_0^1\left[\int_0^y e^{-y^2}dx\right]dy$$

bo'ladi.

Keyingi integralni hisoblaymiz:

$$\int_0^1\left[\int_0^y e^{-y^2}dx\right]dy=\int_0^1 e^{-y^2}\left[\int_0^y dx\right]dy=\int_0^1 e^{-y^2}\cdot(x)\Big|_{x=0}^{x=y}dy=\int_0^1 ye^{-y^2}dy=$$

$$=-\frac{1}{2}e^{-y^2}\Bigg|_{y=0}^{y=1}=\frac{1}{2}\left(1-\frac{1}{e}\right).$$

Demak,

$$\iint_{M_2} e^{-y^2}dxdy=\frac{1}{2}\left(1-\frac{1}{e}\right);\blacktriangleright$$

c) sodda cohalarga ajraladigan soha bo'yicha ikki karrali integrallarni hisoblash. Tekislikdagi P shunday soha (to'plam) bo'lsinki, uni koordinata o'qlariga parallel to'g'ri chiziqlar bilan bo'lakchalarga ajratilish natijasida hosil bo'lgan bo'lakchalar yuqorida qaralgan M, M_1, M_2 ko'rinishdagi sohalar bo'lsin.

Masalan, $z=f(x,y)$ funksiya tekislikdagi P sohada uzluksiz bo'lib, bu soha yuqorida aytilgan ko'rinishdagi M, M_1, M_2 sohalarga ajralsin. Unda

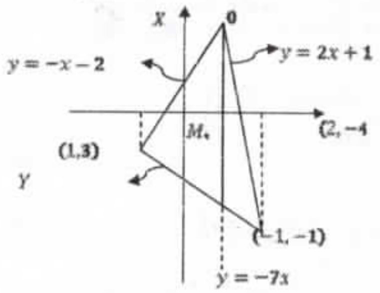
$$\iint_P f(x,y)dxdy=\iint_M f(x,y)dxdy+\iint_{M_1} f(x,y)dxdy+\iint_{M_2} f(x,y)dxdy$$

bo'ladi.

5-misol. Ushbu

$$J=\iint_D (2x+3y+1)\,dxdy$$

ikki karrali integralni hisoblang, bunda P to'plam tekislikda (–1, –1), (2, –4), (1, 3) nuqtalarni birlashtirishdan hosil bo'lgan uchburchak (6-chizma).



6-chizma

◀Avvalo, ikki nuqtadan o'tuvchi to'g'ri chiziq tenglamasi

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

dan foydalanib, uchburchak tomonlarining tenglamalarini topamiz. Ular 6-chizmada ko'rsatilgan.

So'ng $x=1$ to'g'ri chiziq yordamida P to'plamni P_1 va P_2 to'plamlarga ajratamiz, bunda

$$P_1=\{(x,y):\quad -1\le x\le 1,\quad -x-2\le y\le 2x+1\}$$

$$P_2=\{(x,y):\quad 1\le x\le 2,\quad -x-2\le y\le -7x+10\}$$

bo'ladi. Integral xossasiga ko'ra,

$$\iint_P (2x+3y+1)dxdy=\iint_{P_1}(2x+3y+1)dxdy+\iint_{P_2}(2x+3y+1)dxdy$$

bo'ladi.

Endi bu tenglikning o'ng tomonidagi integrallarni hisoblaymiz:

$$\iint_{P_1}(2x+3y+1)dxdy=\int_{-1}^{1}\left[\int_{-x-2}^{2x+1}(2x+3y+1)\,dy\right]dx=$$

$$=\int_{-1}^{1}\left(2xy+\frac{3}{2}y^2+y\right)\Bigg|_{y=-x-2}^{y=2x+1}dx=\int_{-1}^{1}\left(\frac{21}{2}x^2+9x-\frac{3}{2}\right)dx=$$

$$=\left(\frac{21}{2}\cdot\frac{x^3}{3}+9\cdot\frac{x^2}{2}-\frac{3}{2}x\right)\Bigg|_{x=-1}^{x=1}=4,$$

$$\iint_{P_2}(2x+3y+1)dxdy=\int_{1}^{2}\left[\int_{-x-2}^{-7x+10}(2x+3y+1)\,dy\right]dx=$$

$$=\int_{1}^{2}\left(2xy+\frac{3}{2}y^2+y\right)\Bigg|_{y=-x-2}^{y=-7x+10}dx=-1$$

Demak,

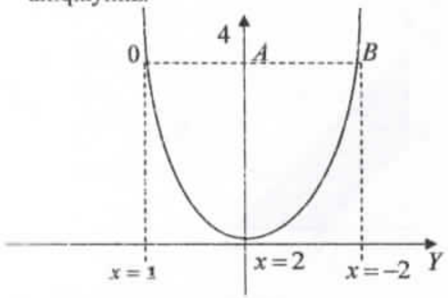
$$\iint_P (2x+3y+1)\,dxdy=4+(-1)=3.\ ▶$$

6-misol. Ushbu

$$\int_{-2}^{2}dx\int_{x^2}^{4}f(x,y)\,dy$$

integralda integrallash tartibini o'zgartiring.

◀Avvalo, integrallash chegaralariga ko'ra integrallash sohasini aniqlaymiz.



7-chizma

Buning uchun x o'zgaruvchini x bo'yicha integrallash chegaralariga tenglab, y o'zgaruvchini y bo'yicha integrallash chegaralariga tenglab olamiz:

$$x=-2,\quad x=2,\quad y=x^2,\quad y=4.$$

Bu chiziqlarni tekislikda tasvirlab, integrallash sohasini topamiz (7-chizma)

Endi integrallashni boshqa tartibda, avvalo, x bo'yicha, so'ng y bo'yicha bajaramiz. $y=x^2$ni x ga nisbatan yechamiz. Unda $x=-\sqrt{y}$, $x=\sqrt{y}$ bo'ladi. Demak, $-\sqrt{y}\le x\le\sqrt{y}$. Ravshanki, $0\le y\le 4$ bo'ladi. Demak,

$$\int_{-2}^{2}dx\int_{x^2}^{4}f(x,y)dy=\int_{0}^{4}dy\int_{-\sqrt{y}}^{\sqrt{y}}f(x,y)dx$$

bo'ladi▶

d) o'zgaruvchilarini almashtirish bilan ikki karrali integrallarni hisoblash. Ba'zan, integrallarda o'zgaruvchi x va y larni almashtirish natijasida integrallanadigan funksiya ham, integrallash to'plami ham soddaroq ko'rinishga keladi va ularni hisoblash osonroq bo'ladi.

Aytaylik, ushbu

$$\iint_D f(x,y)dxdy$$

integralni hisoblash kerak bo'lsin.

Bu integralda

$$x=r\cos\varphi,$$

$$y=r\sin\varphi$$

almashtirish bajaramiz (qaralsin, [1])

Natijada,

$$\iint_D f(x,y)dxdy=\iint_M f(r\cos\varphi,r\sin\varphi)\cdot r\,d\varphi\,dr \qquad (5)$$

bo'ladi (qaralsin, [1]). Agar

$$M=\{(\varphi,r):\alpha\le\varphi\le\beta,\quad r_1(\varphi)\le r\le r_2(\varphi)\}$$

bo'lsa,

$$\iint_D f(x,y)dxdy=\int_\alpha^\beta\left[\int_{r_1(\varphi)}^{r_2(\varphi)} f(r\cos\varphi,r\sin\varphi)\cdot rdr\right]d\varphi \qquad (6)$$

bo'ladi.

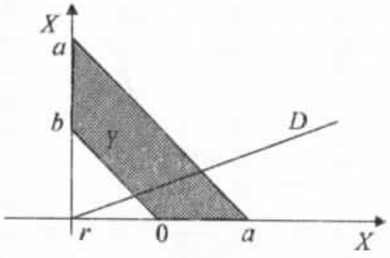
7-misol. Ushbu

$$J=\iint_D \frac{1}{\left(x^2+y^2\right)^{\frac{3}{2}}}dxdy$$

integralni hisoblang, bunda D – quyidagi $x=0,\quad y=0,$

$x+y=a,\quad x+y=b\quad (0<a<b)$

chiziqlar bilan chegaralangan soha (8-chizma).



(8-chizma)

◀Berilgan integralda

$$x=r\cos\varphi \qquad y=r\sin\varphi$$

almashtirish bajaramiz. Unda:

$$x^2+y^2=r^2\cos^2\varphi+r^2\sin^2\varphi=r^2$$

bo'lib

$$J=\iint_D\frac{rdrd\varphi}{r^3}=\iint_D\frac{drd\varphi}{r^2}$$

bo'ladi. Bu integralni takror integral ko'rinishida yozamiz:

$$J=\int_{\alpha_1}^{\alpha_2}d\varphi\int_{r_1}^{r_2}\frac{dr}{r^2}.$$

Endi α_1, α_2 va r_1, r_2 larni topamiz.

8-chizmadan ko'rinadiki, φ burchak 0 bilan $\frac{\pi}{2}$ orasida o'zgaradi.

Demak, $\alpha_1=0,\quad \alpha_2=\frac{\pi}{2}.$

Koordinatalar boshidan OX o'qi bilan φ burchak tashkil etadigan r nurni o'tkazamiz. Bu nurning D sohaga kirishi r_* va undan chiqib ketishi r^* larni topamiz.

Ravshanki, $x+y=a$ to'g'ri chiziqda

$$r\cos\varphi+r\sin\varphi=a$$

bo'ladi. Demak, nur D sohaga $r_*=\dfrac{b}{\cos\varphi+\sin\varphi}$ nuqtada kiradi.

Xuddi shunga o'xshash nur D sohadan

$$r^*=\frac{a}{\cos\varphi+\sin\varphi}$$

nuqtada chiqadi. Demak,

$$J=\int_0^{\frac{\pi}{2}}d\varphi\int_{r_*}^{r^*}\frac{dr}{r^2}$$

Ravshanki,

$$\int_{r_*}^{r^*}\frac{dr}{r^2}=\left(-\frac{1}{r}\right)\Big|_{r_*}^{r^*}=\frac{\cos\varphi+\sin\varphi}{a}-\frac{\cos\varphi+\sin\varphi}{b}=\frac{b-a}{ab}(\cos\varphi+\sin\varphi).$$

Natijada

$$J=\frac{b-a}{ab}\int_0^{\frac{\pi}{2}}(\sin\varphi+\cos\varphi)d\varphi=2\frac{b-a}{ab}$$

bo'ladi.▶

Quyidagi takroriy integrallarni hisoblang:

1585. $\int_1^3 dy\int_2^5 x^2ydx.$ **1586.** $\int_0^1 dx\int_2^5\frac{x}{y^2}dy.$

1587. $\int_{-1}^0 dx\int_0^1 e^{x-y}dy.$ **1588.** $\int_3^4 dx\int_1^2\frac{dy}{(x+y)^2}.$

1589. $\int_0^{2\pi} dx\int_0^1 y\sin^2 xdy.$ **1590.** $\int_0^1 dx\int_0^1\frac{x^2}{1+y^2}dy.$

1591. $\int_0^1 dx\int_{x^2}^x (x-2y)dy.$ **1592.** $\int_0^1 dy\int_y^{2-y}(x+y)dx.$

1593. $\int_0^1 dx\int_1^3(2x+y)dy.$ **1594.** $\int_1^2 dx\int_{\frac{1}{x}}^x\frac{x^2}{y^2}dy.$

1595. $\int_{-2}^4 dy\int_0^y\frac{y^3}{x^2+y^2}dx.$ **1596.** $\int_0^1 dx\int_{x^2}^x\left(1-\frac{x+y}{2}\right)dy.$

Quyidagi integrallarda integrallash tartibini o'zgartiring:

1597. $\int_1^2 dx\int_2^4 f(x,y)dy.$ **1598.** $\int_0^1 dx\int_0^x f(x,y)dy.$ **1599.** $\int_0^1 dx\int_0^{x^2} f(x,y)dy$

1600. $\int_0^1 dx\int_{x^3}^{\sqrt{x}} f(x,y)dy.$ **1601.** $\int_0^1 dy\int_{e^y}^e f(x,y)dx.$ **1602.** $\int_1^2 dy\int_{\ln y}^y f(x,y)dx.$

1603. $\int\limits_{-6}^{2} dy \int\limits_{\frac{y^2-4}{4}}^{2-y} f(x,y)dx$. **1604.** $\int\limits_{0}^{1} dy \int_{-\sqrt{1-y^2}}^{1-y} f(x,y)dx$.

1605. $\int\limits_{0}^{\frac{\pi}{4}} dx \int_{\sin x}^{\cos x} f(x,y)dy$.

Quyidagi ikki karrali integrallarni hisoblang:

1606. $\iint\limits_{(D)} xy^2 dxdy$, bunda $(D)=\{0\le x\le 1, -2\le y\le 3\}$

1607. $\iint\limits_{(D)} xydxdy$, bunda (D) soha $x=0,\ x=2,\ y=2x$,

$y=6-x$ to'g'ri chiziqlar bilan chegaralangan soha.

1608. $\iint\limits_{(D)} \cos^2 ydxdy$, bunda $(D)=\left\{0\le x\le \frac{\pi}{2}, 0\le y\le \frac{\pi}{2}\right\}$

1609. $\iint\limits_{(D)} (x^2-y^2)dxdy$, bunda (D) soha $y=x,\ x=2$ to'g'ri chiziqlar hamda $y=2-x^2$ parabola bilan chegaralangan soha.

1610. $\iint\limits_{(D)} \sqrt{4+x+y}dxdy$, bunda $(D)=\{0\le x\le 5-y, 0\le y\le 5\}$

1611. $\iint\limits_{(D)} \frac{x^2}{y^2}dxdy$, bunda (D) soha $y=x,\ x=2$ to'g'ri chiziqlar hamda $xy=1$ giperbola bilan chegaralandan soha.

1612. $\iint\limits_{(D)} (x+2y)dxdy$, bunda $(D)=\{y^2-4\le x\le 5, -3\le y\le 3\}$

1613. $\iint\limits_{(D)} \sin(x+y)dxdy$, bunda (D) soha $y=x,\ y=0,\ x+y=\frac{\pi}{2}$ to'g'ri chiziqlar bilan chegaralangan soha.

1614. $\iint\limits_{(D)} x^2(y-x)dxdy$, bunda (D) soha $y=x^2,\ x=y^2$ parabolalar bilan chegaralangan soha.

1615. $\iint\limits_{(D)} xdxdy$, bunda (D) soha uchlari $O(0,0)$, $A(1,1)$ va $B(0,1)$ nuqtalarda bo'lgan uchburchak soha.

1616. $\iint\limits_{(D)} e^{\frac{x}{y}}dxdy$, bunda (D) soha $x=0,\ y=1$ to'g'ri chiziqlar hamda $y^2=x$ parabola bilan chegaralangan soha.

1617. $\iint\limits_{(D)} \sqrt{\cos^2 y+x^2\sin^2 y}dxdy$, bunda $(D)=\left\{0\le x\le 1, 0\le y\le \frac{\pi}{2}\right\}$

1618. $\iint\limits_{(D)} \frac{dxdy}{\sqrt{2-x}}$, bunda (D) soha radiusi 1 ga teng, birinchi kvadrantda joylashgan, koordinata o'qlariga urinuvchi doiraviy soha.

1619. $\iint\limits_{(D)} (x^2+y^2+1)dxdy$, bunda (D) soha $y=0,\ y=x$ to'g'ri chiziqlar hamda $x^2+y^2=1,\ x^2+y^2=4$ aylanalar bilan chegaralangan soha.

Qutb koordinatalariga o'tib, quyidagi ikki karrali integrallarni hisoblang:

1620. $\iint\limits_{(D)} (x+y)dxdy$, bunda (D) soha $x^2+y^2=1,\ x^2+y^2=4$ aylanalar hamda $y=0$ to'g'ri chiziq bilan chegaralangan soha. (D) da $y>0$ deb qaraladi.

1621. $\iint\limits_{(D)} \frac{1}{1+x^2+y^2}dxdy$, bunda (D) soha markazi $(0,0)$ nuqtada, radiusi 1 ga teng bo'lgan, $x^2+y^2\le 1$ doiraviy soha.

1622. $\iint\limits_{(D)} e^{x^2+y^2}dxdy$, bunda $(D)=\{x^2+y^2\le 1\}$ doira.

1623. $\iint\limits_{(D)} \sqrt{1+x^2+y^2}dxdy$, bunda (D) soha $x^2+y^2\le 1$ doiraning birinchi kvadrantdagi qismi.

1624. $\iint\limits_{(D)} \frac{1}{\sqrt{x^2+y^2}}dxdy$, bunda (D) soha $x^2+y^2\le 1$ doiraning birinchi kvadrantdagi qismi.

1625. $\iint\limits_{(D)} \sin\left(\sqrt{x^2+y^2}\right)dxdy$, bunda $(D)=\{x^2+y^2\le \pi^2\}$.

1626. $\iint\limits_{(D)} ydxdy$, bunda (D) soha, markazi $M\left(\frac{a}{2},0\right)$ diametri a bo'lgan yarim doira $(y>0)$.

1627. $\iint\limits_{(D)} \sqrt{a^2-x^2-y^2}dxdy$, bunda $(D)=\{x^2+y^2\le a^2, y>0\}$.

1628. $\iint\limits_{(D)} \frac{1}{\sqrt{x^2+y^2}}dxdy$, bunda $(D)=\{1\le x^2+y^2\le 4\}$.

1629. $\iint\limits_{(D)} xy^2dxdy$, bunda (D) soha, $x^2+(y-1)^2=1$ va $x^2+y^2=4y$ aylanalar bilan chegaralalangan soha-halqa.

1630. $\iint\limits_{(D)} \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}dxdy$, bunda $(D)=\{x^2+y^2\le 1, x\ge 0, y\ge 0\}$.

1631. $\iint\limits_{(D)} arctg\frac{y}{x}dxdy$, bunda $(D)=\{x^2+y^2\le 1, x\ge 0, y\ge 0\}$.

1632. $\iint\limits_{(D)} (x^2+y^2)dxdy$, bunda $(D)=\{x^2+y^2\le 2x\}$.

1633. $\iint\limits_{(D)} \frac{y^2}{x^2+y^2}dxdy$, bunda (D), $y=x$, $y=-x$, $y=1$ to'g'ri chiziqlar bilan chegaralangan uchburchak soha.

2-§. Ikki karrali integrallarning tatbiqlari

1⁰.Ikki karrali integrallarning tatbiqlari:

a) tekis shaklning yuzi. Agar D tekislikdagi chegaralangan soha bo'lib, $S-D$ to'plam tasvirlagan shaklning yuzi bo'lsa,

$$S=\iint\limits_D dxdy \qquad (7)$$

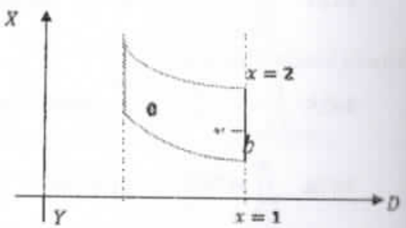
bo'ladi.

1-misol. Tekislikda, ushbu

$$x=1,\quad x=2,\quad y=\frac{a^2}{x},\quad y=\frac{2a^2}{x}\qquad (a>0)$$

chiziqlar bilan chegaralangan shaklning yuzini toping.

◄Ravshanki, $x=1, x=2$ vertikal to'g'ri chiziqlar, $y=\frac{a^2}{x}$, $y=\frac{2a^2}{x}$ lar esa giperbolalar bo'lib, ular bilan chegaralangan shakl 9-chizmada tasvirlangan:



9-chizma

Bu shaklning yuzini (7) formuladan foydalanib topamiz:

$$S=\iint\limits_D dxdy=\int\limits_1^2\left[\int\limits_{\frac{a^2}{x}}^{\frac{2a^2}{x}}dy\right]dx=\int\limits_1^2\left(\frac{2a^2}{x}-\frac{a^2}{x}\right)dx=$$

$$=\int\limits_1^2\frac{a^2}{x}dx=a^2\cdot(\ln x)\Big|_{x=1}^{x=2}=a^2(\ln 2-\ln 1)=a^2\cdot\ln 2;\ ►$$

b) jismning hajmi. Aytaylik, $z=f(x,y)$ funksiya tekislikdagi D sohada berilgan va uzluksiz bo'lib, $f(x,y)\ge 0$ bo'lsin.

Yuqoridan $z=f(x,y)$ funksiya grafigi bo'lgan S – sirt bilan, yon tomondan yasovchilari OZ o'qiga parallel yo'naltiruvchisi D ning chegarasi bo'lgan silindrik sirt hamda pastdan XOY tekislikdagi D soha bilan chegaralangan (V) jismning hajmi

$$V=\iint\limits_{(D)} f(x,y)dxdy \qquad (8)$$

bo'ladi.

2-misol. Ushbu

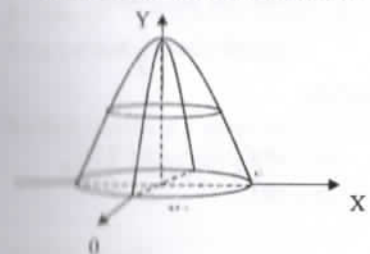
$$x^2+y^2+z-4=0$$

sirt (paraboloid) hamda XOY tekislik bilan chegaralangan jismning hajmini toping.

◄Bu holda

$$z=f(x,y)=4-x^2-y^2$$

bo'lib, u 10-chizmada tasvirlangan:



10-chizma

Izlanayotgan hajm (8) formulaga ko'ra

$$V=\iint\limits_{(D)}(4-x^2-y^2)dxdy$$

bo'ladi, bunda

$$D=\{(x,y): x^2+y^2\le 4\}$$

doiradan iborat.

Ikki karrali integralni hisoblash uchun unda:

$$x=r\cos\varphi,$$

$$y=r\sin\varphi$$

almashtirish bajaramiz. Unda

$$4-x^2-y^2=4-r^2,\qquad x^2+y^2\le 4\Rightarrow 0\le\varphi\le 2\pi,\quad 0\le r\le 2$$

bo'lib, 2-§da keltirilgan (6) formulaga ko'ra,

$$\iint_D (4-x^2-y^2)dxdy = \int_0^{2\pi}\left[\int_0^2 (4-r^2)rdr\right]d\varphi$$

bo'ladi.

Endi bu tenglikning o'ng tomonidagi takroriy integralni hisoblaymiz:

$$\int_0^{2\pi}\left[\int_0^2 (4-r^2)rdr\right]d\varphi = \int_0^{2\pi}\left(4\cdot\frac{r^2}{2}-\frac{r^4}{4}\right)\Bigg|_{r=0}^{r=2}d\varphi =$$

$$= \int_0^{2\pi}\left(4\cdot\frac{4}{2}-\frac{16}{4}-0\right)d\varphi = \int_0^{2\pi}4d\varphi = 8\pi.$$

Demak, izlanayotgan hajm $V = 8\pi$ bo'ladi;▶

c) sirtning yuzi. Aytaylik, $z = f(x,y)$ funksiya tekislikdagi D to'plamda aniqlangan, uzluksiz va uzluksiz xususiy hosilalarga ega bo'lib, uning grafigi fazoda (S) sirtni tasvirlasin. Bu sirt yuzi:

$$S = \iint_D \sqrt{1+f_x'^2(x,y)+f_y'^2(x,y)}dxdy \qquad (9)$$

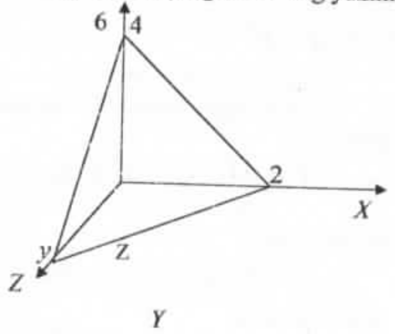
bo'ladi.

3-misol. Ushbu

$$6x+3y+2z=12$$

tekislikning birinchi oktantada joylashgan qismining yuzini toping.

◀Agar izlanayotgan sirtning yuzini S desak, unda (9) formulaga ko'ra,



$$S = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dxdy$$

(10) bo'ladi. Bu integralda

$$z = 6-3x-\frac{3}{2}y$$

bo'lib,

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(6-3x-\frac{3}{2}y\right) = -3,$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(6-3x-\frac{3}{2}y\right) = -\frac{3}{2},$$

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1+9+\frac{9}{4}} = \frac{7}{2}$$

bo'ladi.

Berilgan tekislikning XOY tekislikdagi proyeksiyasi OX, OY koordinata o'qlari va $6x+3y=12$ to'g'ri chiziq bilan chegaralangan uchburchakdan iborat. Shunig uchun izlanayotgan sirtning yuzi:

$$S = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dxdy = \int_0^2\left[\int_0^{4-2x}\frac{7}{2}dy\right]dx = \frac{7}{2}\int_0^2 y\Bigg|_{y=0}^{y=4-2x}dx =$$

$$= \frac{7}{2}\int_0^2 (4-2x)dx = \frac{7}{2}\left(4x-2\frac{x^2}{2}\right)\Bigg|_{x=0}^{x=2} = 14$$

bo'ladi. Demak, $S = 14$.▶

2^0. Ikki karrali integralning fizik va mexanik tatbiqlari:

a) tekislikdagi shaklning (plastinkaning) massasi. Aytaylik, tekislikda biror to'plam (shakl) berilgan bo'lib, bu shakl bo'yicha massa tarqatilgan bo'lsin. Uni plastinka deb, uning zichligini esa $\rho(x,y)$ deb qaraymiz. Bu $\rho(x,y)$ funksiya D da uzluksiz. Shu plastinkaning m massasi

$$m = \iint_D \rho(x,y)dxdy \qquad (11)$$

bo'ladi.

4-misol. Radiusi R ga teng bo'lgan doiraviy plastinkaning har bir (x,y) nuqtasidagi zichligi $\rho(x,y)$ qaralayotgan nuqtadan doira markazigacha bo'lgan masofaga proporsional bo'lsa, doiraviy plastinkaning massasini toping.

◀Tekislikda dekart koordinatalari sistemasining boshini doira markaziga joylashtiramiz. Unda doiraviy plastinkaning har bir nuqtasi uchun

$$x^2+y^2 \le R^2$$

bo'lib, (x,y) nuqtadan koordinatalar boshigacha bo'lgan masofa $\sqrt{x^2+y^2}$ ga teng bo'ladi. Demak, zichlik

$$\rho(x,y) = k\cdot\sqrt{x^2+y^2}$$

bo'ladi, bunda k – proporsionallik koeffitsiyenti.

(11) formulaga ko'ra,

$$m = \iint_D k\cdot\sqrt{x^2+y^2}dxdy$$

bo'ladi, bunda

$$D = \{(x,y): x^2+y^2 \le R^2\}.$$

Ikki karrali integralda

$$x = r\cos\varphi, \qquad y = r\sin\varphi$$

almashtirish bajaramiz. Unda

$$\sqrt{x^2+y^2} = r$$

bo'lib, integral quyidagi ko'rinishga keladi:

$$\iint_D k\cdot\sqrt{x^2+y^2}\,dxdy = k\int_0^{2\pi}\left[\int_0^R r\cdot rdr\right]d\varphi.$$

Keyingi integrallarni hisoblaymiz:

$$k\int_0^{2\pi}\left[\int_0^R r^2dr\right]d\varphi = k\int_0^{2\pi}\left(\frac{r^3}{3}\right)\Big|_0^R d\varphi = \frac{k\cdot R^3}{3}\int_0^{2\pi}d\varphi = \frac{2}{3}k\pi\cdot R^3.$$

Demak, plastinkaning massasi

$$m = \frac{2}{3}k\pi\cdot R^3$$

ga teng;▶

b) tekislikdagi shaklning (plastinkaning) og'irlik markazini (og'irlik markazinining koordinatalarini) topish. Agar tekis plastinkaning massasi m, zichligi $\rho(x,y)$ bo'lsa, plastinka og'irlik markazi (x_0, y_0) ning koordinatalari

$$x_0 = \frac{\iint_D x\cdot\rho(x,y)dxdy}{\iint_D \rho(x,y)dxdy} = \frac{1}{m}\iint_D x\rho(x,y)dxdy \qquad (12)$$

$$y_0 = \frac{\iint_A y\cdot\rho(x,y)dxdy}{\iint_A \rho(x,y)dxdy} = \frac{1}{m}\iint_A y\rho(x,y)dxdy \qquad (13)$$

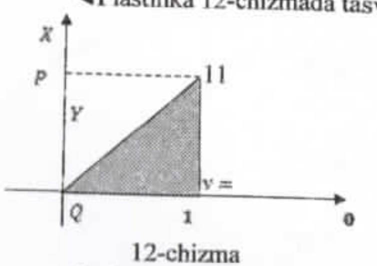
bo'ladi.

5-misol. Uchlari $O(0;0)$, $P(1;0)$, $Q(1,1)$ nuqtalarda bo'lgan uchburchak shakldagi plastinkaning zichligi

$$\rho(x,y) = x^2$$

bo'lsa, uning og'irlik markazi koordinatalarini toping.

◀Plastinka 12-chizmada tasvirlangan:



12-chizma

Bu holda

$$D = \{(x,y): 0\le x\le 1, \quad 0\le y\le x\}$$

bo'ladi. Avvalo, (11) formuladan foydalanib, plastinkaning massasini topamiz:

$$m = \iint_D x^2dxdy = \int_0^1\left[\int_0^x x^2dy\right]dx = \int_0^1 x^3dx = \frac{1}{4}$$

Unda (12), (13) formulalarga ko'ra

$$x_0 = \frac{1}{m}\iint_D x\cdot x^2dxdy = 4\int_0^1\left[\int_0^x x^3dy\right]dx = 4\int_0^1 x^4dx = 4\cdot\frac{x^5}{5}\Big|_{x=0}^{x=1} = \frac{4}{5},$$

$$y_0 = \frac{1}{m}\iint_D x^2\cdot ydxdy = 4\int_0^1\left[x^2\int_0^x ydy\right]dx =$$

$$= 4\int_0^1 x^2\cdot\left(\frac{y^2}{2}\right)\Big|_{y=0}^{y=x}dx = 4\cdot\int_0^1 x^4dx = 4\cdot\frac{1}{2}\cdot\left(\frac{x^5}{5}\right)\Big|_{x=0}^{x=1} = \frac{2}{5}$$

bo'ladi.

Demak, plastinkaning og'irlik markazi koordinatalari

$$x_0 = \frac{4}{5}, \quad y_0 = \frac{2}{5}$$

bo'ladi;▶

d) tekislikdagi shaklning (plastinkaning) statik momentlarini topish. Agar tekis plastinkaning zichligi $\rho(x,y)$ bo'lsa, plastinkaning koordinata o'qlari OX va OY larga nisbatan statik momentlari mos ravishda

$$M_x = \iint_D y\cdot\rho(x,y)dxdy, \qquad M_y = \iint_D x\cdot\rho(x,y)dxdy \qquad (14)$$

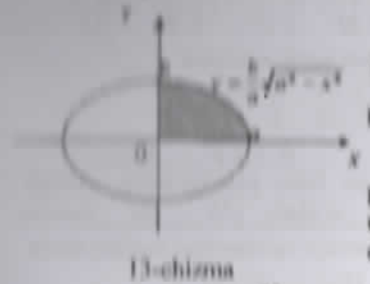
bo'ladi.

6-misol. Ushbu

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

ellipsning birinchi chorakdagi qismi va koordinata o'qlari bilan chegaralangan shakl plastinkaning statik momentlarini toping, bunda zichlik $\rho(x,y)=kxy$, $k = const$

◀Bu plastinka 13-chizmada tasvirlangan:



13-chizma

Bu holda

$$D = \left\{(x,y): 0\le x\le a, \quad 0\le y\le\frac{b}{a}\sqrt{a^2-x^2}\right\}$$

bo'ladi.

Shartga ko'ra, $\rho(x,y) = k\cdot xy$ (k-proporsionallik koeffitsiyenti) bo'lib, bu hol uchun (14) formulalar quyidagi ko'rinishga ega:

$$M_x = \iint_D kxy\cdot ydxdy, \qquad M_y = \iint_D kxy\cdot xdxdy.$$

Endi bu integrallarni hisoblaymiz:

$$M_x = \iint_D kxy^2 \cdot dxdy = \int_0^a \left[\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} kxy^2 dy \right] dx = \int_0^a \left[kx\left(\frac{y^3}{3}\right) \Bigg|_{y=0}^{y=\frac{b}{a}\sqrt{a^2-x^2}} \right] dx =$$

$$= \int_0^a \left[\frac{kx}{3} \cdot \frac{b^3}{a^3} \left(\sqrt{a^2-x^2}\right)^3 \right] dx = \frac{kb^3}{3a^3} \int_0^a x\left(a^2-x^2\right)^{\frac{3}{2}} dx = \frac{kb^3}{3a^3} \int_0^a \left(a^3-x^2\right)^{\frac{3}{2}} \square$$

$$\square \left[-\frac{1}{2} d\left(a^2-x^2\right) \right] = -\frac{kb^3}{6a^3} \cdot \frac{\left(a^2-x^2\right)^{5/2}}{\frac{5}{2}} \Bigg|_{x=0}^{x=a} = \frac{kb^3a^2}{15}.$$

Demak, plastinkaning OX o‘qiga nisbatan statik momenti

$$M_x = \frac{kb^3 \cdot a^2}{15}$$

bo‘ladi.

Yuqoridagidek, quyidagi

$$M_y = \iint_D kxy \cdot xdxdy$$

integral hisoblanadi:

$$M_y = \iint_D kyx^2 \cdot dxdy = \int_0^a \left[\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} kyx^2 dy \right] dx =$$

$$\int_0^a kx^2 \left(\frac{y^2}{2}\right) \Bigg|_{y=0}^{y=\frac{b}{a}\sqrt{a^2-x^2}} dx = \int_0^a \frac{kx^2}{2} \cdot \frac{b^2}{a^2}\left(a^2-x^2\right) dx = \frac{kb^2}{2}\int_0^a x^2 dx -$$

$$-\frac{kb^2}{2a^2}\int_0^a x^4 dx = \frac{kb^2}{2} \cdot \left(\frac{x^3}{3}\right)\Bigg|_{x=0}^{x=a} - \frac{kb^2}{2a^2} \cdot \frac{x^5}{5}\Bigg|_0^a = \frac{kb^2a^3}{2 \cdot 3} - \frac{kb^2a^3}{2 \cdot 5} = \frac{kb^2a^3}{15}$$

Demak, plastinkaning OY o‘qiga nisbatan statik momenti

$$M_y = \frac{kb^2 \cdot a^3}{15}$$

bo‘ladi;▶

e) inersiya momentlarni topish. Agar tekis plastinkaning zichligi $\rho(x,y)$ bo‘lsa, plastinkaning koordinata o‘qlari OX, OY hamda koordinatalar boshi $O(0,0)$ nuqtaga nisbatan inersiya momentlari mos ravishda

$$J_x = \iint_D y^2 \cdot \rho(x,y) dxdy, \quad J_y = \iint_D x^2 \cdot \rho(x,y) dxdy$$

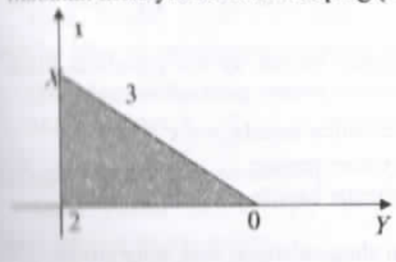
$$J_0 = J_x + J_y = \iint_D \left(x^2+y^2\right) \cdot \rho(x,y) dxdy \qquad (15)$$

bo‘ladi.

7-misol. Ushbu

$$x=0, \quad y=0, \quad 2x+3y=6$$

chiziqlar bilan chegaralangan bir jinsli plastinkaning koordinatalar boshiga nisbatan inersiya momentini toping (14-chizma).



14-chizma

◀ Plastinka bir jinsli bo‘lgani uchun uning zichligi $\rho(x,y)=1$ bo‘ladi.

Endi, $2x+3y=6$ tenglamani y ga nisbatan yechib topamiz:

$$y = 2 - \frac{2}{3}x.$$

Demak, bu holda

$$D = \left\{ (x,y): 0 \le x \le 3, \quad 0 \le y \le 2 - \frac{2}{3}x \right\}$$

bo‘ladi.

Yuqorida keltirilgan (15) formuladan foydalanib topamiz:

$$J_0 = \iint_D \left(x^2+y^2\right) \cdot 1 \cdot dxdy = \int_0^3 \left[\int_0^{2-\frac{2}{3}x} \left(x^2+y^2\right) dy \right] dx.$$

Bu tenglikning o‘ng tomonidagi takroriy integralni hisoblaymiz:

$$\int_0^3 \left[\int_0^{2-\frac{2}{3}x} \left(x^2+y^2\right) dy \right] dx = \int_0^3 \left(x^2y + \frac{y^3}{3} \right) \Bigg|_{y=0}^{y=2-\frac{2}{3}x} dx =$$

$$= \int_0^3 \left(x^2\left(2-\frac{2}{3}x\right) + \frac{\left(2-\frac{2}{3}x\right)^3}{3} \right) dx = \int_0^3 \left(\frac{26}{9}x^2 - \frac{62}{81}x^3 - \frac{8}{3}x + \frac{8}{3} \right) dx =$$

$$\left(\frac{26}{9}\frac{x^3}{3} - \frac{62}{81}\frac{x^4}{4} - \frac{8}{3}\frac{x^2}{2} + \frac{8}{3}x \right)\Bigg|_{x=0}^{x=3} = 6,5.$$

Demak, $J_0 = 6,5$ ▶

Quyidagi tekis shakllarning yuzasini hisoblang:

1634. Ushbu

$x+y=2, \quad x=0, \quad y=0$

chiziqlar bilan chegaralangan shakllar yuzasini toping.

1635. Ushbu

$y=0, \quad y=x, \quad y=2x-2$

chiziqlar bilan chegaralangan shaklning yuzini toping.

1636. Ushbu $y=2x$ to'g'ri chiziq hamda $y^2=\frac{9}{2}x$ parabola bilan chegaralangan shaklning yuzini toping.

1637. Ushbu $x+y-5=0$ to'g'ri chiziq hamda $xy=4$ giperbola bilan chegaralangan shaklning yuzini toping.

1638. Ushbu $x+y-5=0$ to'g'ri chiziq hamda $xy=4$ giperbola bilan chegaralangan shaklning yuzini toping.

1639. Ushbu $x=2$, $x=-2$ to'g'ri chiziqlar hamda $y=x^2$, $4y=x^2$ parabolalar bilan chegaralangan shaklning yuzini toping.

1640. Ushbu $x=1$, $x=2$ to'g'ri chiziqlar hamda $y=\frac{a^2}{x}$, $y=\frac{2a^2}{x}(a>0)$ giperbolalar bilan chegaralangan shaklning yuzini toping.

1641. Ushbu $y=x^2$ va $y^2=x$ parabolalar bilan chegaralangan shaklning yuzini toping.

1642. Ushbu $y^2=10x+25$, $y^2=-6x+9$ parabolalar bilan chegaralangan shaklning yuzini toping.

1643. Ushbu $(y-x)^2+x^2=1$ ellipsning yuzini toping.

1644. Ushbu $y=e^x$, $y=e^{2x}$, $x=1$ chiziqlar bilan chegaralangan shaklning yuzini toping.

1645. Ushbu $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellipsning yuzini toping.

1646. Ushbu $y=x^2$, $y=2-x^2$ parabolalar bilan chegaralangan shaklning yuzini toping.

Quyidagi sirtlar bilan chegaralangan jismning hajmini toping:

1647. $x+y+z=1$, $x=0$, $y=0$, $z=0$.

1648. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, $x=0$, $y=0$, $z=0$.

1649. $x=a$, $y=b$, $z=mx$, $y=0$, $z=0$.

1650. $z=x+y$, $x+y=1$, $x=0$, $y=0$, $z=0$.

1651. $y=x^2$, $y=1$, $x+y+z=4$, $z=0$.

1652. $z=x^2$, $z=0$, $y=0$, $x=0$, $x+y=1$.

1653. $z=2-x-y$, $z=0$, $x^2+y^2=1$, $x=0$, $y=0$.

1654. $z=\sqrt{x^2+y^2}$, $z=0$, $x^2+y^2=2x$.

1655. $x^2+y^2+z^2=r^2$, $z=a$, $z=b(r>b>a>0)$.

1656. Ushbu $x^2+y^2+z^2=r^2$ sfera bilan chegaralangan shar hajmini toping.

1657. $\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$, $y=\frac{b}{a}x$, $y=0$, $z=0$.

1658. Ushbu $z=x^2+y^2$ giperboloid, $y=x^2$ silindr va $y=1$, $z=0$ tekisliklar bilan chegaralangan jismning hajmini toping.

Quyidagi sirtlarning yuzini toping:

1659. Ushbu $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, tekislikning koordinata tekisliklari orasida joylashgan qismining yuzini toping.

1660. Ushbu $2z=x^2-y^2$, $x^2+y^2=1$ sirtlarning kesishishidan hosil bo'lgan sirtning yuzini toping.

1661. Ushbu $z=y^2-x^2+2xy$, $x^2+y^2=1$ sirtlarning kesishishidan hosil bo'lgan sirtning yuzini toping.

1662. Ushbu $x^2+y^2+z^2=2z$, $x+z=2$ sirtlarning kesishishidan hosil bo'lgan sirtning yuzini toping.

1663. Ushbu $x^2+y^2=z^2$, $x^2+y^2=4$ sirtlarning kesishishidan hosil bo'lgan sirtning yuzini toping.

Karrali integrallarni fizik masalalarni yechishga tatbiq etish

1664. Doiraviy plastinkaning har bir nuqtasidagi zichligi shu nuqtadan doira markazigacha bo'lgan masofaning kvadratiga teng. Shu plastinkaning massasini toping.

1665. Tomonlari a va b bo'lgan to'g'ri to'rtburchakning har bir nuqtasidagi zichligi shu nuqtadan bir tomonigacha bo'lgan masofaning kvadratiga teng. Shu to'g'ri to'rtburchak plastinkaning massasini toping.

1666. Ushbu $y^2=4x$, $x=4$ chiziqlar bilan chegaralangan tekis shaklning og'irlik markazi koordinatalarini toping.

1667. Ushbu $y=\sin x$, $y=0(0\le x\le\pi)$ chiziqlar bilan chegaralangan tekis shaklning og'irlik markazi koordinatalarini toping.

1668. Ushbu $y^2=8x$, $y=0$, $x+y=6$ chiziqlar bilan chegaralangan tekis shaklning og'irlik markazi koordinatalarini toping.

1669. Ushbu $x=a$, $y=b$ to'g'ri chiziqlar hamda koordinata o'qlari bilan chegaralangan to'g'ri to'rtburchakning koordinata boshiga nisbatan inersiya momentini toping.

1670. Koordinatalari a va b bo'lgan to'g'ri to'rtburchakning:
a) to'g'ri burchak uchiga nisbatan; b) a katetiga nisbatan inersiya momentlarini toping.

1671. Ushbu $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellipsning boshiga nisbatan inersiya momentini toping.

3-§. Uch o'zgaruvchili funksiya, uning differensiallari va integrallari (uch karrali integrallari). Integralning tatbiqlari

1^0. Funksiyaning xususiy hosilalari va differensiallari. Taqribiy formulalar. Aytaylik, $u=f(x,y,z)$ funksiya fazodagi biror V to'plamda (sohada) berilgan bo'lib, $(x_0,y_0,z_0)\in V$ bo'lsin. Odatda

$$\Delta f(x_0,y_0,z_0)=f(x_0+\Delta x,y_0+\Delta y,z_0+\Delta z)-f(x_0,y_0,z_0)$$

ayirma $u=f(x,y,z)$ funksiyaning (x_0,y_0,z_0) nuqtadagi to'liq orttirmasi, quyidagi

$$\Delta_x f(x_0,y_0,z_0)=f(x_0+\Delta x,y_0,z_0)-f(x_0,y_0,z_0)$$
$$\Delta_y f(x_0,y_0,z_0)=f(x_0,y_0+\Delta y,z_0)-f(x_0,y_0,z_0)$$
$$\Delta_z f(x_0,y_0,z_0)=f(x_0,y_0,z_0+\Delta z)-f(x_0,y_0,z_0)$$

ayirmalar esa $u=f(x,y,z)$ funksiyaning (x_0,y_0,z_0) nuqtadagi xususiy orttirmalari (mos ravishda x o'zgaruvchilari bo'yicha, y o'zgaruvchilari bo'yicha, z o'zgaruvchilari bo'yicha orttirma) deyiladi.

Agar

$$\lim_{\Delta x\to 0}\frac{\Delta_x f(x_0,y_0,z_0)}{\Delta x}$$

limit mavjud bo'lsa, uni $u=f(x,y,z)$ funksiyaning x o'zgaruvchi bo'yicha xususiy hosilasi deyiladi va

$$\frac{\partial u}{\partial x}\text{ yoki } f'_x(x_0,y_0,z_0)$$

kabi belgilanadi:

$$f'_x(x_0,y_0,z_0)=\frac{\partial u}{\partial x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0,z_0)-f(x_0,y_0,z_0)}{\Delta x}.$$

Xuddi shunga o'xshash $u=f(x,y,z)$ funksiyaning y va z o'zgaruvchilari bo'yicha xususiy hosilalari ta'riflanadi:

$$f'_y(x_0,y_0,z_0)=\frac{\partial u}{\partial y}=\lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y,z_0)-f(x_0,y_0,z_0)}{\Delta y},$$

$$f'_z(x_0,y_0,z_0)=\frac{\partial u}{\partial z}=\lim_{\Delta z\to 0}\frac{f(x_0,y_0,z_0+\Delta z)-f(x_0,y_0,z_0)}{\Delta z}$$

1-misol. Ushbu

$$u=\frac{1}{\sqrt{x^2+y^2+z^2}}$$

funksiyaning xususiy hosilalarini toping.

◀Bu funksiyaning xususiy hosilalari quyidagicha topiladi:

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)=\frac{-\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x}{x^2+y^2+z^2}=-\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}},$$

Xuddi shunga o'xshash

$$\frac{\partial u}{\partial y}=-\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}},\quad \frac{\partial u}{\partial z}=-\frac{z}{(x^2+y^2+z^2)^{3/2}}$$

bo'ladi.▶

Agar $u=f(x,y,z)$ funksiyaning to'liq orttirmasi Δu ni quyidagicha

$$\Delta u=f'_x(x,y,z)\Delta x+f'_y(x,y,z)\Delta y+f'_z(x,y,z)\Delta z+\alpha\cdot\Delta x+\beta\cdot\Delta y+\gamma\cdot\Delta z \quad (1)$$

yozish mumkin bo'lib,

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}}\alpha=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}}\beta=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}}\gamma=0$$

bo'lsa, $u=f(x,y,z)$ funksiya (x,y,z) nuqtada differensiallanuvchi deyiladi. Ushbu

$$f'_x(x,y,z)\Delta x+f'_y(x,y,z)\Delta y+f'_z(x,y,z)\Delta z$$

yig'indi $u=f(x,y,z)$ funksiyaning differensiali deyiladi. U du kabi belgilanadi. Agar

$$\Delta x=dx,\quad \Delta y=dy,\quad \Delta z=dz$$

deyilsa, unda

$$du=f'_x(x,y,z)dx+f'_y(x,y,z)dy+f'_z(x,y,z)dz \quad (2)$$

bo'ladi.

2-misol. Ushbu

$$u=e^{xyz}$$

funksiyaning differensialini toping.

◀Berilgan funksiyaning xususiy hosilalarini topamiz:

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}(e^{xyz})=e^{xyz}\cdot yz,\quad \frac{\partial u}{\partial y}=\frac{\partial}{\partial x}(e^{xyz})=e^{xyz}\cdot xz,\quad \frac{\partial u}{\partial z}=\frac{\partial}{\partial z}(e^{xyz})=e^{xyz}\cdot xy.$$

Unda (2) formulaga ko'ra berilgan funksiyaning differensiali

$$du=e^{xyz}yzdx+e^{xyz}xzdy+e^{xyz}xydz=e^{xyz}(yzdx+xzdy+xydz)$$

bo'ladi.▶

$u=f(x,y,z)$ funksiyaning argument orttirmalari: Δx, Δy, Δz lar yetarlicha kichik bo'lganda

$$\Delta u\approx du$$

bo‘lib, ushbu

$$f(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z) \approx f(x_0,y_0,z_0)+f'_x(x_0,y_0,z_0)\Delta x + $$
$$+f'_y(x_0,y_0,z_0)\Delta y + f'_z(x_0,y_0,z_0)\Delta z \qquad (3)$$

taqribiy formula hosil bo‘ladi.

3-misol. Ushbu

$$\alpha = 1{,}002\cdot 2{,}003^2\cdot 3{,}004^3$$

miqdorni taqribiy hisoblang.

◀Bu miqdorni taqribiy hisoblashda (3) formuladan foydalanamiz. Bu holda $u=f(x,y,z)$ funksiya sifatida

$$f(x,y,z)=x\cdot y^2\cdot z^3 \text{ ni}$$

olamiz. Unda bu funksiya uchun (3) taqribiy formula quyidagi

$$(x_0+\Delta x)\cdot(y_0+\Delta y)^2\cdot(z_0+\Delta z)^3 \approx x_0y_0^2z_0^2+f'_x(x_0,y_0,z_0)\Delta x+$$
$$+f'_y(x_0,y_0,z_0)\Delta y+f'_z(x_0,y_0,z_0)\Delta z = x_0y_0^2z_0^2+y_0^2z_0^3\cdot\Delta x+x_0\cdot$$
$$\cdot 2y_0\cdot z_0^3\cdot\Delta y+x_0\cdot 2y_0\cdot z_0^3\cdot\Delta y+x_0\cdot y_0^2\cdot 3z_0^2\cdot\Delta z$$

ko‘rinishga keladi.

Agar keyingi munosabatda

$$x_0=1,\ y_0=2,\ z_0=3,\ \Delta x=0{,}002,\ \Delta y=0{,}003,\ \Delta z=0{,}004$$

deyilsa, u holda

$$1{,}002\cdot 2{,}003^2\cdot 3{,}004^3 \approx 108+0{,}216+0{,}324+0{,}432=108{,}972$$

bo‘ladi. Demak,

$$\alpha \approx 108{,}972.$$ ▶

2⁰. Uch karrali integral va ularni hisoblash. Aytaylik, $u=f(x,y,z)$ funksiya fazodagi biror (V) to‘plamda (sohada) berilgan bo‘lsin. Bu sohaning (jismning) hajmini esa V deylik.

(V) ni (sirtlar yordamida) n ta bo‘lakka

$$(V_1),(V_2),\dots,(V_n)$$

bo‘lamiz. Ularning diametrlari

$$d_1,d_2,\dots,d_n$$

bo‘lib, hajmi esa

$$V_1,V_2,\dots,V_n$$

bo‘lsin.

Har bir (V_k) da ixtiyoriy (x_k,y_k,z_k) nuqtani olib, $u=f(x,y,z)$ funksiyaning shu nuqtadagi qiymati $f(x_k,y_k,z_k)$ ni (V_k) bo‘lakchaning hajmi V_k ga ko‘paytiramiz:

$$f(x_k,y_k,z_k)\cdot V_k \qquad (k=1,2,3,\dots,n).$$

Bu ko‘paytmadan quyidagi

$$\sigma=\sum_{k=1}^{n} f(x_k,y_k,z_k)\cdot V_k$$

yig‘indini tuzamiz. U **integral yig‘indi** deyiladi.

Agar $\max_k d_k \to 0$ da σ yig‘indi chekli limitga ega bo‘lsa, bu limit $f(x,y,z)$ funksiyaning (V) bo‘yicha uch karrali integrali deyiladi va

$$\iiint_{(V)} f(x,y,z)dxdydz$$

kabi belgilanadi.

Demak,

$$\iiint_{(V)} f(x,y,z)dxdydz = \lim_{\max_k d_k\to 0}\sum_{k=1}^{n} f(x_k,y_k,z_k)\cdot V_k.$$

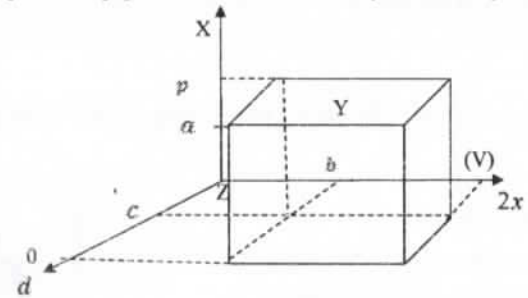
Agar $u=f(x,y,z)$ funksiya (V) da uzluksiz bo‘lsa, unda bu funksiyaning uch karrali integrali mavjud bo‘ladi.

Ko‘p hollarda uch karrali integral takrorlab integrallash yordamida hisoblanadi.

a) aytaylik, integrallash sohasi (V) quyidagi

$$(V)=\{(x,y,z):\ a\le x\le b,\ c\le y\le d,\ p\le z\le q\}$$

to‘plamdan – parallelepipeddan iborat bo‘lsin (15-chizma).



15-chizma

Bu holda $u=f(x,y,z)$ funksiyaning (V) bo‘yicha uch karrali integral

$$\iiint_{(V)} f(x,y,z)dxdydz=\int_a^b\left[\int_c^d\left(\int_p^q f(x,y,z)dz\right)dy\right]dx \qquad (4)$$

bo‘ladi.

4-misol. Ushbu

$$J=\iiint\limits_{(V)}(x+y+z)dxdydz$$

integralni hisoblang, bunda

$$(V)=\{(x,y,z):\ 0\le x\le 1,\ 0\le y\le 3,\ 0\le z\le 2\}$$

parallelopipeddan iborat.

◄Berilgan uch karrali integral (4) formulaga ko'ra

$$\iiint\limits_{(V)}(x+y+z)dxdydz=\int_0^1\left[\int_0^3\left(\int_0^2(x+y+z)dz\right)dy\right]dx$$

bo'ladi. Uni takrorlab, integrallash yordamida hisoblaymiz.

Avvalo,

$$\int_0^2(x+y+z)dz$$

integralni hisoblaymiz, bunda x va y lar o'zgarmas deb qaraladi.

$$\int_0^2(x+y+z)dz=(xz+yz+\frac{z^2}{2})\Big|_{z=0}^{z=2}=2x+2y+2=2(x+y+1)$$

Unda

$$\int_0^3\left(\int_0^2(x+y+z)dz\right)dy=\int_0^3 2(x+y+1)dy=2(xy+\frac{y^2}{2}+y)\Big|_{y=0}^{y=3}=$$

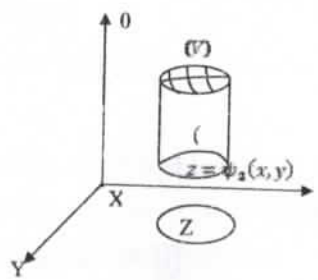
$$=2(3x+\frac{9}{2}+3)=6x+15$$

va nihoyat

$$\int_0^1\left[\int_0^3\left(\int_0^2(x+y+z)dz\right)dy\right]dx=\int_0^1(6x+15)dx=(3x^2+15x)\Big|_0^1=18$$

bo'ladi.►

Aytaylik, fazoda (V) soha (to'plam) pastdan $z=\Psi_1(x,y)$, yuqoridan $z=\Psi_2(x,y)$ sirtlar bilan, yon tomondan esa OZ o'qiga parallel silindrik sirt bilan chegaralangan jism bo'lib, uning XOY tekislikdagi proyeksiyasi D bo'lsin. (16- chizma)



16-chizma

Agar $u=f(x,y,z)$ funksiya (V) da, $z=\Psi_1(x,y)$, $z=\Psi_2(x,y)$ funksiyalar esa D uzluksiz bo'lsa,

$$\iiint\limits_{(V)}f(x,y,z)dxdydz=\iint\limits_D\left[\int_{\Psi_1(x,y)}^{\Psi_2(x,y)}f(x,y,z)dz\right]dxdy \qquad (5)$$

bo'ladi.

Agar

$$D=\{(x,y):\ a\le x\le b,\ \varphi_1(x)\le y\le\varphi_2(x)\}$$

bo'lib, $\varphi_1(x)$ va $\varphi_2(x)$ funksiyalar $[a,b]$ da uzluksiz bo'lsa, u holda

$$\iiint\limits_{(V)}f(x,y,z)dxdydz=\int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)}\left(\int_{\Psi_1(x,y)}^{\Psi_2(x,y)}f(x,y,z)dz\right)dy\right]dx \qquad (6)$$

bo'ladi.

5-misol. Ushbu

$$J=\iiint\limits_{(V)}\frac{1}{(1+x+y+z)^4}dxdydz$$

integralni hisoblang, bunda

$$(V)=\{(x,y,z):\ o\le x\le 1,\ 0\le y\le 1-x,\ o\le z\le 1-x-y\}$$

◄Bu integralni (6) formuladan foydalanib hisoblaymiz. (4) formulaga ko'ra,

$$\iiint\limits_{(V)}\frac{dxdydz}{(1+x+y+z)^4}=\int_0^1\left[\int_0^{1-x}\left(\int_0^{1-x-y}\frac{dz}{(1+x+y+z)^4}\right)dy\right]dx$$

bo'ladi.

Ravshanki,

$$\int_0^{1-x-y}\frac{1}{(1+x+y+z)^4}dz=\int_0^{1-x-y}(1+x+y+z)^{-4}d(1+x+y+z)=$$

$$=\frac{(1+x+y+z)^{-3}}{-3}\Bigg|_{z=0}^{z=1-x-y}=\frac{1}{3}\left[\frac{1}{(1+x+y)^3}-\frac{1}{8}\right];$$

$$\int_0^{1-x}\frac{1}{3}\left[\frac{1}{(1+x+y)^3}-\frac{1}{8}\right]dy=\frac{1}{3}\left[-\frac{1}{2(1+x+y)^2}-\frac{1}{8}y\right]\Bigg|_{y=0}^{y=1-x}=$$

$$=\frac{1}{3}\left[-\frac{1}{2\cdot 2^2}-\frac{1-x}{8}+\frac{1}{2(1+x)^2}\right]=\frac{1}{3}\left[\frac{1}{2(1+x)^2}-\frac{2-x}{8}\right].$$

Unda

$$\int_0^1\left[\int_0^{1-x}\left(\int_0^{1-x-y}\frac{dz}{(1+x+y+z)^4}\right)dy\right]dx=\int_0^1\frac{1}{3}\left[\frac{1}{2(1+x)^2}-\frac{2-x}{8}\right]dx=$$

$$=\frac{1}{6}\int_0^1\frac{dx}{(1+x)^2}-\frac{1}{12}\int_0^1 dx+\frac{1}{24}\int_0^1 xdx=-\frac{1}{6}\cdot\frac{1}{(1+x)}\Big|_0^1-\frac{x}{12}\Big|_0^1+\frac{x^2}{48}\Big|_0^1=$$

$$=-\frac{1}{12}+\frac{1}{6}-\frac{1}{12}+\frac{1}{48}=\frac{1}{48}$$

bo'ladi. Demak,

$$J=\frac{1}{48};\blacktriangleright$$

b) o'zgaruvchilarni almashtirish bilan uch karrali integrallarni hisoblash:

1) agar

$$V=\iiint_{(V)}dxdydz$$

uch karrali integralda o'zgaruvchilar quyidagicha

$$x=r\cos\varphi,$$
$$y=r\sin\varphi,$$
$$z=z$$

almashtirilsa, (silindrik koordinatalarga o'tilsa) u holda

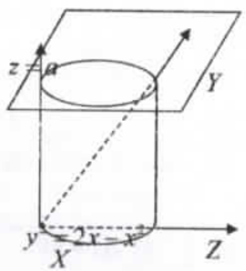
$$\iiint_{(V)}f(x,y,z)dxdydz=\iiint_{P}f(r\cos\varphi,r\sin\varphi,z)\cdot zdrd\varphi dz \qquad (7)$$

bo'ladi, bunda $0\le r<+\infty,\quad 0\le\varphi<2\pi,\quad -\infty<z<+\infty$

6-misol. Ushbu

$$\iiint_{(V)}z\sqrt{x^2+y^2}\,dzdxdy$$

uch karrali integralni hisoblang, bunda (V) soha $z=0,\quad z=a$ tekisliklar hamda $y^2=2x-x^2$ silindrik sirt bilan chegaralangan jism (17-chizma)



17-chizma

◀Bu integralni hisoblash uchun o'zgaruvchilarini quyidagicha:

$$x=r\cos\varphi,$$
$$y=r\sin\varphi,$$
$$z=z$$

almashtiramiz. Bu holda φ o'zgaruvchi $-\frac{\pi}{2}$ dan $\frac{\pi}{2}$ gacha o'zgaradi; $-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$, r o'zgaruvchining o'zgarish oralig'ini topish uchun

$$y^2=2x-x^2,\ \text{ya'ni}\ x^2+y^2=2x$$

da $x=r\cos\varphi$, $y=r\sin\varphi$ deymiz:

$$r^2\cos^2\varphi+r^2\sin^2\varphi=2r\cos\varphi,\qquad r^2=2r\cos\varphi$$

Keyingi tenglikdan $r=0,\quad r=2\cos\varphi$ bo'lishi kelib chiqadi. Demak, r o'zgaruvchi 0 bilan $2\cos\varphi$ oralig'ida o'zgaradi. Ravshanki, $0\le z\le a$

Unda (7) formulasiga ko'ra,

$$\iiint_{(V)}z\sqrt{x^2+y^2}\,dxdydz=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^{2\cos\varphi}\left(\int_0^a z\cdot\sqrt{z^2}\,rdz\right)dr\right]d\varphi$$

bo'ladi.

Endi bu tenglikdagi takroriy integrallarni hisoblaymiz:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^{2\cos\varphi}\left(\int_0^a z\cdot\sqrt{z^2}\,rdz\right)dr\right]d\varphi=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_0^{2\cos\varphi}r^2\left(\frac{z^2}{2}\right)\Big|_{z=0}^{z=a}dr\right)d\varphi=\frac{a^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{z^3}{3}\right)\Big|_{r=0}^{r=2\cos\varphi}d\varphi=$$

$$=\frac{8a^2}{6}\int_{-\frac{\pi}{2}}^{\pi/2}\cos^3\varphi d\varphi=\frac{4}{3}a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\sin^2\varphi)\cos\varphi d\varphi=\frac{4}{3}a^2\left(\sin\varphi-\frac{\sin^3\varphi}{3}\right)\Bigg|_{\varphi=-\frac{\pi}{2}}^{\varphi=\frac{\pi}{2}}=$$

$$=\frac{4}{3}a^2\left(1-\frac{1}{3}+1-\frac{1}{3}\right)=\frac{4}{3}a^2\left(2-\frac{2}{3}\right)=\frac{16}{9}a^2$$

Demak,

$$\iiint_{(V)}z\sqrt{x^2+y^2}\,dxdydz=\frac{16}{9}a^2.\blacktriangleright$$

2) Agar

$$\iiint_{(V)}f(x,y,z)dxdydz$$

uch karrali integralda o'zgaruvchilar quyidagicha:

$$x = r\cos\varphi \cdot \sin\theta,$$
$$y = r\sin\varphi \cdot \sin\theta,$$
$$z = r\cos\theta$$

almashtirilsa (sferik koordinatalarga o‘tilsa) u holda

$$\iiint\limits_{(V)} f(x,y,z)dxdydz = \iiint\limits_{(D)} f(r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta)r^2\sin\theta drd\theta d\varphi \quad (8)$$

bo‘ladi, bunda

$$0 \le r < +\infty, \quad 0 \le \varphi < 2\pi, \quad 0 \le \theta \le \pi$$

deb qaraladi.

7-misol. Ushbu

$$J = \iiint\limits_{(V)} dxdydz$$

integralni hisoblang, bunda (V) soha fazoda quyidagi

$$(x^2+y^2+z^2)^3 = a^2z^4$$

sirt bilan chegaralangan jismning birinchi oktandagi qismi.

◄Bu integralda

$$x = r\cos\varphi \cdot \sin\theta,$$
$$y = r\sin\varphi \cdot \sin\theta,$$
$$z = r\cos\theta$$

almashtirishni bajaramiz. Bunda sirt tenglamasi

$$r^6 = a^2r^4\cos^4\theta$$

bo‘lib, undan

$$r = 0, \quad r^2 = a^2\cos^4\theta, \quad r = a\cos^2\theta$$

bo‘lishi kelib chiqadi.

Sirtni birinchi oktantda qaralayotgani uchun

$$0 \le \varphi \le \frac{\pi}{2}, \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le a\cos^2\theta$$

bo‘ladi. Demak, qaralayotgan integral (8) formulaga ko‘ra

$$J = \int_0^{\frac{\pi}{2}}\left[\int_0^{\frac{\pi}{2}}\left(\int_0^{a\cos^2\theta} \sin\theta \cdot r^2dr\right)d\theta\right]d\varphi$$

bo‘ladi.

Endi takroriy integrallarni hisoblaymiz:

$$\int_0^{\frac{\pi}{2}}\left[\int_0^{\frac{\pi}{2}}\left(\int_0^{a\cos^2\theta} \sin\theta \cdot r^2dr\right)d\theta\right]d\varphi = \int_0^{\frac{\pi}{2}}\left(\int_0^{\frac{\pi}{2}}\sin\theta\cdot\left(\frac{r^3}{3}\right)\Bigg|_{r=0}^{r=a\cos\theta} d\theta\right)d\varphi =$$

$$= \int_0^{\frac{\pi}{2}}\left[\int_0^{\frac{\pi}{2}} a^3\frac{\cos^6\theta\sin\theta}{3}d\theta\right]d\varphi = \frac{a^3}{3}\int_0^{\frac{\pi}{2}}\left(-\frac{\cos^7\theta}{7}\right)\Bigg|_{\theta=0}^{\theta=\frac{\pi}{2}}d\varphi = \frac{a^3}{21}\int_0^{\frac{\pi}{2}}d\varphi = \frac{a^3}{21}\cdot\frac{\pi}{2}$$

Demak, $J = \frac{a^3\pi}{42}$. ►

3⁰. Uch karrali integrallarning tatbiqlari. Uch karrali integrallar yordamida fazodagi jismning hajmi, massasi, og‘irlik markazining koordinatalari, inersiya momentlari kabi ko‘pgina miqdorlar topiladi:

a) fazodagi jismning hajmi

$$V = \iiint\limits_{(V)} dxdydz \quad (9)$$

bo‘ladi;

b) jismning massasi

$$m = \iiint\limits_{(V)} \rho(x,y,z)dxdydz \quad (10)$$

bo‘ladi, bunda $\rho(x,y,z)$ - zichlik;

c) jismning og‘irlik markazi koordinatalari

$$x_0 = \frac{1}{m}\iiint\limits_{(V)} \rho(x,y,z)\cdot xdxdydz,$$
$$y_0 = \frac{1}{m}\iiint\limits_{(V)} \rho(x,y,z)\cdot ydxdydz,$$
$$z_0 = \frac{1}{m}\iiint\limits_{(V)} \rho(x,y,z)\cdot zdxdydz,$$

bo‘ladi;

d) koordinata tekisliklarga nisbatan jismning inersiya momentlari

$$J_{XOY} = \iiint\limits_{(V)} \rho(x,y,z)\cdot z^2dxdydz,$$
$$J_{XOZ} = \iiint\limits_{(V)} \rho(x,y,z)\cdot y^2dxdydz,$$
$$J_{YOZ} = \iiint\limits_{(V)} \rho(x,y,z)\cdot x^2dxdydz,$$

bo‘ladi.

8-misol. Fazoda ushbu

$$y=\frac{x^2+z^2}{b}, \quad y=b \qquad (b>0)$$

sirtlar bilan chegaralangan jismning hajmini toping.

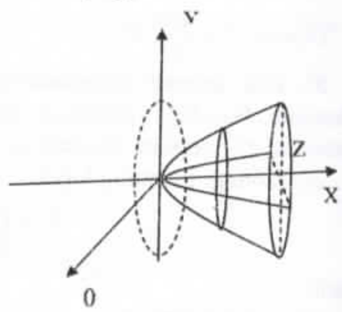
◀Bu jism chap tomondan

$$y=\frac{x^2+z^2}{b}$$

paraboloid bilan, o'ng tomondan esa $y=b$ tekislik bilan chegaralangan bo'lib, uning XOZ tekisligidagi proeksiyasi

$$x^2+z^2\le b^2$$

doira (uning radiusi b ga teng) bo'ladi (18-chizma).



18-chizma

(9) formulaga ko'ra, jismning hajmi

$$V=\iiint\limits_{(V)} dxdydz$$

bo'ladi, bunda

$$(V)=\left\{(x,y,z):\ -b\le x\le b,\ -\sqrt{b^2-x^2}\le z\le\sqrt{b^2-x^2},\ \frac{x^2+z^2}{b}\le y\le b\right\},$$

Bu holda yuqoridagi uch karrali integral quyidagicha ifodalanadi:

$$\iiint\limits_{(V)} dxdydz=\int_{-b}^{b}\left[\int_{-\sqrt{b^2-x^2}}^{\sqrt{b^2-x^2}}\left(\int_{\frac{x^2+z^2}{b}}^{b} dy\right)dz\right]dx$$

Ravshanki,

$$\int_{\frac{x^2+z^2}{b}}^{b} dy=y\Big|_{y=\frac{x^2+z^2}{b}}^{b}=b-\frac{x^2+z^2}{b}=\frac{1}{b}(b^2-x^2-z^2),$$

$$\int_{-\sqrt{b^2-x^2}}^{\sqrt{b^2-x^2}}\frac{1}{b}\left(b^2-x^2-z^2\right)dz=\frac{1}{b}\left[(b^2-x^2)z\frac{z^2}{3}\right]\Bigg|_{z=-\sqrt{b^2-x^2}}^{z=\sqrt{b^2-x^2}}=\frac{4}{3b}\left(\sqrt{b^2-x^2}\right)^3$$

Unda

$$V=\int_{-b}^{b}\frac{4}{3b}\left(\sqrt{b^2-x^2}\right)^3dx=\frac{4}{3b}\int_{-b}^{b}\left(\sqrt{b^2-x^2}\right)^3 dx$$

bo'ladi.

Endi keyingi integralni hisoblaymiz:

$$\int_{-b}^{b}\left(\sqrt{b^2-x^2}\right)^3dx=\left[\begin{array}{c} x=b\sin t,\ dx=b\cos tdt \\ x=-b\ \partial a\ t=-\frac{\pi}{2};\ x=b\ \partial a\ t=\frac{\pi}{2}\end{array}\right]=$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(b^2-b^2\sin^2 t\right)b\cos dt=b^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4 tdt=b^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{1+\cos 2t}{2}\right)^2dx=$$

$$=b^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1+2\cos 2t+\cos^2 2t\right)dt=\frac{b^4}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1+2\cos 2t+\frac{1+\cos 4t}{2}\right)dt=$$

$$=\frac{b^4}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(3+4\cos 2t+\cos 4t\right)dt=\frac{b^4}{8}\left(3t+2\sin 2t+\frac{1}{4}\sin 4t\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{b^4}{8}\cdot 3\pi$$

Natijada

$$V=\frac{4}{3b}\cdot\frac{b^4}{8}\cdot 3\pi=\frac{\pi b^3}{2}$$

bo'ladi.▶

9-misol. Ushbu

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

bir jinsli ellipsoidning uning markaziga nisbatan inersiya momentini hisoblang.

◀Bu holda $\rho(x,y,z)\equiv 1$ bo'lib, ellipsoidning markazga nisbatan inersiya momenti .

$$J_0=\iiint\limits_{(V)} r^2 dzdydx$$

bo'ladi. Ma'lumki,

$$r^2=x^2+y^2+z^2.$$

Unda

$$J_0=\iiint\limits_{(V)}\left(x^2+y^2+z^2\right)dzdydx=\iiint\limits_{(V)} x^2dzdydx+\iiint\limits_{(V)} y^2dzdydx+\iiint\limits_{(V)} z^2dzdydx=$$

$$=J_{yoz}+J_{xoz}+J_{xoy}$$

Endi yuqoridagi tenglikning o‘ng tomonidagi J_{yoz}, J_{xoz}, J_{xoy} uch karrali integrallarni – ellipsoidning koordinata tekisliklariga nisbatan inersiya momentlarini hisoblaymiz:

$$J_{yoz}=\iiint_{(V)}x^2dzdydx=\int_{-a}^{a}x^2\left(\iint_{(Q)}dzdy\right)dx,$$

bunda Q – x nuqta OX o‘qiga perpendikulyar bo‘lgan tekislik bilan ellipsoidning kesishgan kesimi.

Bu kesim ushbu

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1-\frac{x^2}{a^2}$$

ya’ni

$$\frac{y^2}{b^2\left(1-\frac{x^2}{a^2}\right)}+\frac{z^2}{c^2\left(1-\frac{x^2}{a^2}\right)}=1$$

ellipsdan iborat bo‘lib, uning yarim o‘qlari

$$b_1=b\sqrt{1-\frac{x^2}{a^2}},\quad c_1=c\sqrt{1-\frac{x^2}{a^2}}$$

bo‘ladi. Shuning uchun

$$\iint_{(Q)}dzdy=\pi bc\left(1-\frac{x^2}{a^2}\right).$$

Demak,

$$J_{yoz}=\int_{-a}^{a}\pi bcx^2\left(1-\frac{x^2}{a^2}\right)dx=\pi bc\int_{-a}^{a}\pi bcx^2\left(x^2-\frac{x^4}{a^2}\right)dx=\frac{4}{15}\pi a^3bc$$

bo‘ladi.

Xuddi shunga o‘xshash

$$J_{xoz}=\frac{4}{15}\pi ab^3c,\quad J_{xoy}=\frac{4}{15}\pi abc^3$$

bo‘lishi topiladi.

Demak,

$$J_0=J_{yoz}+J_{xoz}+J_{xoy}=\frac{4}{15}\pi abc\left(a^2+b^2+c^2\right).$$ ►

Quyidagi uch karrali integrallarni hisoblang:

1672. $\int_0^c dz\int_0^d dy\int_0^a\left(x^2+y^2+z^2\right)dx$. **1673.** $\int_0^a ydy\int_0^h dx\int_0^{a-y}dz$.

1674. $\int_0^1 dx\int_0^x dy\int_0^y xyzdz$. **1675.** $\int_0^2 dx\int_0^3 dy\int_0^4(x+y+z)dz$.

1676. $\int_0^a dx\int_0^x dy\int_0^{xy}x^3y^2zdz$. **1677.** $\int_0^1 dx\int_0^{\sqrt{x}} ydy\int_0^{2-2x}dz$.

1678. Ushbu $\iiint_{(V)}\frac{dxdydz}{(x+y+z)^3}$ integral hisoblansin, bunda (V) soha koordinata tekisliklari hamda $x+y+z=1$ tekislik bilan chegaralangan soha.

1679. Ushbu $\iiint_{(V)}\left(x^2+y^2\right)dxdydz$ integral hisoblansin, bunda (V) soha quyidagi $x^2+y^2=2z$, $z=2$ sirtlar bilan chegaralangan soha.

1680. Ushbu $\iiint_{(V)}xyzdxdydz$ integral hisoblansin, bunda (V) soha quyidagi $x=0$, $y=0$, $z=0$, $x+y+z=1$ tekisliklar bilan chegaralangan soha – piramida.

1681. Ushbu $\iiint_{(V)}(2x+3y-z)dxdydz$ integral hisoblansin, bunda (V) soha quyidagi $x=0$, $y=0$, $z=0$, $z=3$, $x+y=2$ tekisliklar bilan chegaralangan soha-prizma.

Silindrik hamda sferik koordinatalarga o‘tib quyidagi uch karrali integrallarni hisoblang

1682. $\int_0^1 dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dy\int_0^a dz$. **1683.** $\int_0^2 dx\int_0^{\sqrt{2x-x^2}}dy\int_0^a z\sqrt{x^2+y^2}dz$.

1684. $\int_0^{2r}dx\int_{-\sqrt{2rx-x^2}}^{\sqrt{2rx-x^2}}dy\int_0^{\sqrt{4r^2-x^2-y^2}}dz$.

1685. Ushbu $\iiint_{(V)}z\sqrt{x^2+y^2}dxdydz$, bunda (V) soha $z=0$, $z=a$ tekisliklar hamda $y^2=2x-x^2$ silindrik sirt bilan chegaralangan soha.

1686. Ushbu $x^2+y^2+z^2=a^2$ sharning hajmini toping.

1687. Ushbu $z=0$, $x^2+y^2=4az$, $x^2+y^2=2cx$ sirtlar bilan chegaralangan tekislik hajmini toping.

1688. Ushbu $z=2$, $z=3$ tekislik hamda $z^2=x^2+y^2$ sirt bilan chegaralangan tekislik hajmini toping.

1689. Ushbu $x^2+y^2+z^2=4$ yarim sferaning og‘irlik markazining koordinatalarini toping.

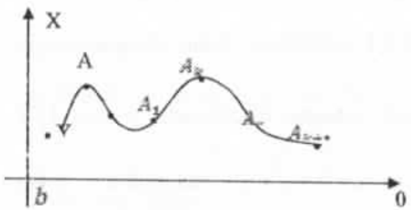
1690. Ushbu $z=4$ tekislik hamda $x^2+y^2=z$ paraboloidlar bilan chegaralangan tekislikning og'irlik markazining koordinatalarini toping.

1691. Ushbu $z=1(0\le z\le 1)$ tekislik hamda $x^2+y^2+z^2=2z$ sirt bilan chegaralangan jismning koordinata boshiga nisbatan inersiya mommentini toping.

1692. Koordinata tekisliklari hamda $x=2,\ \ y=3,\ \ z=4$ tekisliklar bilan chegaralangan jismning OZ o'qiga nisbatan inersiya momentini toping.

4-§. Egri chiziqli integrallar

1⁰. "Birinchi tur egri chiziqli integral" tushunchasi va uni hisoblash



19-chizma

Tekislikdagi biror $A\breve{B}$ egri chiziqda (egri chiziq nuqtalari to'plamida) $f(x,y)$ funksiyani qaraylik (19-chizma).

$A\breve{B}$ egri chiziqni (yoyni)

$$A_0, A_1, A_2, \ldots A_{n-1}, A_n \qquad (A_0=A, \quad A_n=B)$$

nuqtalari yordamida n ta

$$A_{k-1}\breve{A}_k$$

bo'lakka bo'lamiz. So'ng

$$A_{k-1}\breve{A}_k$$

yoyning uzunligini ΔS_k bilan belgilaymiz. Bu $A_{k-1}\breve{A}_k$ yoyda ixtiyoriy (x_k, y_k) nuqtani olib, funksiyaning shu nuqtadagi qiymati $f(x_k,y_k)$ ni ΔS_k ga ko'paytirib quyidagi:

$$\sigma=\sum_{k=1}^{n} f(x_k,y_k)\cdot\Delta S_k \qquad (1)$$

yig'indini tuzamiz. U **integral yig'indi** deyiladi.

Aytaylik, $\max\limits_k\{\Delta S_k\}=\lambda$ bo'lsin.

Agar $\lambda\to 0$ da σ yig'indi chekli limitga ega bo'isa, bu limit $f(x,y)$ funksiyaning $A\breve{B}$ egri chiziq bo'yicha birinchi tur egri chiziqli integrali deyiladi va

$$\int_{A\breve{B}} f(x,y)dS$$

kabi belgilanadi:

$$\int_{A\breve{B}} f(x,y)dS=\lim_{\lambda\to 0}\sum_{k=1}^{n} f(x_k,y_k)\cdot\Delta S_k .$$

Aytaylik, $f(x,y)$ funksiya $A\breve{B}$ egri chiziqda berilgan va uzluksiz bo'lsin:

a) agar $A\breve{B}$ egri chiziq ushbu

$$\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases} \qquad \alpha\le t\le\beta$$

sistema bilan (parametrik ko'rinishda) berilgan bo'lib, $\varphi(t)$ va $\psi(t)$ funksiyalar $[\alpha,\beta]$ da uzluksiz $\varphi'(t)$ va $\psi'(t)$ hosilalarga ega bo'lsa,

$$\int_{A\breve{B}} f(x,y)dS=\int_{\alpha}^{\beta} f(\varphi(t),\psi(t))\cdot\sqrt{\varphi'^2(t)+\psi'^2(t)}dt \qquad (2)$$

bo'ladi;

b) agar $A\breve{B}$ egri chiziq ushbu

$$y=y(x) \qquad (a\le x\le b)$$

tenglama bilan aniqlangan bo'lib, $y(x)$ funksiya $[a,b]$ uzluksiz $y'(x)$ hosilaga ega bo'lsa, u holda

$$\int_{A\breve{B}} f(x,y)dS=\int_{a}^{b} f(x,y(x))\cdot\sqrt{1+y'^2(x)}dx \qquad (3)$$

bo'ladi;

c) agar $A\breve{B}$ egri chiziq ushbu

$$r=r(\varphi) \qquad (\alpha\le\varphi\le\beta)$$

tenglama (qutb tenglama) bilan aniqlangan bo'lsa, u holda

$$\int_{A\breve{B}} f(x,y)dS=\int_{\alpha}^{\beta} f(r\cos\varphi, r\sin\varphi)\cdot\sqrt{r^2+r'^2}d\varphi \qquad (4)$$

bo'ladi.

Birinchi tur egri chiziqli integrallar (2), (3) va (4) formulalar yordamida hisoblanadi.

1-misol. Ushbu

$$\int_{\breve{AB}} x^2 dS$$

integralni hisoblang, bunda $\breve{AB}$ egri chiziq ushbu

$$y = \ln x \qquad (1 \le x \le 3)$$

tenglama bilan aniqlangan.

◀Bu integralni (3) foydalanib hisoblaymiz. Ravshanki,

$$y' = \frac{1}{x}$$

bo'lib, (3) formulaga ko'ra

$$\int_{\breve{AB}} x^2 dS = \int_1^3 x^2 \sqrt{1+\frac{1}{x^2}}dx$$

bo'ladi. Keyingi integralni hisoblaymiz:

$$\int_1^3 x^2\sqrt{1+\frac{1}{x^2}}dx = \int_1^3 x\sqrt{x^2+1}dx = \frac{1}{2}\int_1^3 (x^2+1)^{\frac{1}{2}} d(x^2+1) = \frac{1}{3}(x^2+1)\Big|_1^3 = \frac{2}{3}(5\sqrt{10}-\sqrt{2})$$

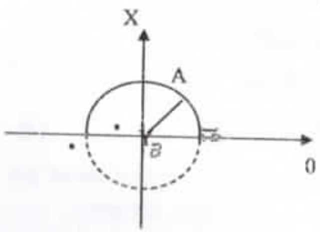
Demak,

$$\int_{\breve{AB}} x^2 dS = \frac{2}{3}(5\sqrt{10}-\sqrt{2}).$$ ▶

2-misol. Ushbu

$$\int_{\breve{AB}} \sqrt{x^2+y^2}dS$$

birinchi tur egri chiziqli integralni hisoblang, bunda $\breve{AB}$ egri chiziq markazi koordinata boshida, radiusi $r(r>0)$ ga teng aylananing yuqori yarim tekislikdagi qismi (20- chizma).



20-chizma

◀Ma'lumki, $\breve{AB}$ egri chiziq quyidagi

$$\begin{cases} x = r\cos t, \\ y = r\sin t \end{cases} \qquad (0 \le t \le \pi)$$

sistema bilan aniqlanadi.

$\breve{AB}$ egri chiziqda

$$f(x,y) = \sqrt{x^2+y^2} = \sqrt{(r\cos t)^2+(r\sin t)^2} = r$$

bo'lib, (2) formulaga ko'ra

$$\int_{\breve{AB}} \sqrt{x^2+y^2}dS = \int_0^\pi r\cdot\sqrt{(r\cos t)+(r\sin t)^2}dt = r^2\int_0^\pi 1\cdot dt = \pi r^2$$

bo'ladi. ▶

2⁰. Birinchi tur egri chiziqli integrallarning tatbiq etilishi:

a) yoy **uzunligini topish.** Agar $\breve{AB}$ egri chiziqning uzunligini ℓ desak, unda

$$\ell = \int_{\breve{AB}} ds \qquad (5)$$

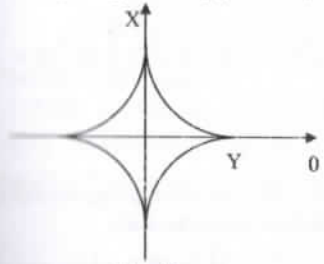
bo'ladi.

3-misol. Ushbu

$$\begin{cases} x = \varphi(t) = a\cos^3 t \\ y = \psi(t) = a\sin^3 t \end{cases} \qquad (0 \le t \le 2\pi)$$

sistema bilan aniqlanadigan $\breve{AB}$ egri chiziqning uzunligini toping.

◀ $\breve{AB}$ egri chiziq yopiq chiziq (ya'ni $A = B$) bo'lib, u 21-chizmada tasvirlangan egri chiziq (astroida) bo'ladi.



21-chizma

(5) formulaga ko'ra, yopiq $\breve{AB}$ egri chiziqning uzunligi

$$\ell = \int_{\breve{AB}} ds$$

bo'ladi. Bu yopiq chiziq (astroida) ning koordinata o'qlariga nisbatan simmetrik joylashgan bo'lishini e'tiborga olib, (2) formuladan foydalanib topamiz:

$$\int_{\breve{AB}} f(x,y)ds = 4\int_0^{\frac{\pi}{2}} \sqrt{\varphi'^2(t)+\psi'^2(t)}dt = 4\int_0^{\frac{\pi}{2}} \sqrt{(-3a\cos^2 t\cdot\sin t)^2+(3a\sin^2 t\cdot\cos t)^2}dt =$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{\frac{9a^2}{4}\sin^2 2t}dt = 6a\left(-\frac{\cos 2t}{2}\right)\Big|_0^{\frac{\pi}{2}} = 6a;$$ ▶

b) **massani topish.** Agar $\breve{AB}$ egri chiziq bo'yicha zichligi $\rho(x,y)$ bo'lgan massa tarqatilgan bo'lsa, bu $\breve{AB}$ massali egri chiziqning massasi

$$m = \int_{\breve{AB}} \rho(x,y)ds \qquad (6)$$

bo'ladi.

4-misol. Birjinsli sikloida

$$\begin{cases} x = a(t-\sin t), \\ y = a(1-\cos t), \end{cases} \qquad (0 \le t \le 2\pi)$$

yoyining massasini toping.

◀Sikloida bir jinsli bo‘lgani uchun $\rho(x,y)=1$ bo‘lib, uning massasi (6) formulaga ko‘ra,

$$m=\int_{\breve{AB}} ds$$

bo‘ladi.

Endi,

$$\begin{cases} x'(t)=\left(a(t-\sin t)\right)'=a(1-\cos t), \\ y'(t)=\left(a(1-\cos t)\right)'=a\sin t \end{cases}$$

bo‘lishini e’tiborga olib, (2) formuladan foydalanib topamiz:

$$m=\int_0^{2\pi}\sqrt{x'^2(t)+y'^2(t)}dt=\int_0^{2\pi}\sqrt{a^2\left(1-2\cos t+\cos^2 t\right)+a^2\sin^2 t}\,dt=$$

$$=\int_0^{2\pi}a\sqrt{1-2\cos t+\left(\cos^2 t+\sin^2 t\right)}dt=a\int_0^{2\pi}\sqrt{2(1-\cos t)}dt=2a\int_0^{2\pi}\sin\frac{t}{2}dt=$$

$$=-4a\cos\frac{t}{2}\Big|_0^{2\pi}=8a;$$ ▶

c) **statik momentlarini topish.** Massali $\breve{AB}$ egri chiziqning koordinata o‘qlari OX va OY larga nisbatan statik momentlari mos ravishda

$$M_x=\iint_{\breve{AB}} y\cdot\rho(x,y)ds,\quad M_y=\iint_{\breve{AB}} x\cdot\rho(x,y)ds \qquad (7)$$

bo‘ladi.

d) **og‘irlik markazini (og‘irlik markazinining koordinatalarini) topish.** Massali $\breve{AB}$ egri chiziqning og‘irlik markazi koordinatalari

$$x_0=\frac{1}{m}\iint_{\breve{AB}} x\rho(x,y)ds,\quad y_0=\frac{1}{m}\iint_{\breve{AB}} y\rho(x,y)ds \qquad (8)$$

bo‘ladi.

e) **inersiya momentlarini topish.** Massali $\breve{AB}$ egri chiziqning koordinata o‘qlari OX va OY hamda koordinatalar boshi $O(0,0)$ nuqtaga nisbatan inersiya momentlari mos ravishda

$$J_x=\iint_{\breve{AB}} y^2\cdot\rho(x,y)ds,\quad J_y=\iint_{\breve{AB}} x^2\cdot\rho(x,y)ds,$$

$$J_0=J_x+J_y=\iint_{\breve{AB}}\left(x^2+y^2\right)\cdot\rho(x,y)ds \qquad (9)$$

bo‘ladi.

3^0. **“Ikkinchi tur egri chiziqli integrallar” tushunchasi.** Tekislikda $\breve{AB}$ egri chiziq va unda $f(x,y)$ funksiya berilgan bo‘lsin.

$\breve{AB}$ egri chiziqni (yoyni)

$$A_0, A_1, A_2, \ldots, A_{n-1}, A_n \qquad (A_0=A,\quad A_n=B)$$

nuqtalar yordamida n ta

$$A_k\breve{A}_{k+1} \qquad (k=1,2,3,\ldots,n)$$

bo‘lakka bo‘lamiz. Bu $A_{k-1}\breve{A}_k$ yoyning OX o‘qidagi proeksiyasini Δx_k, OY o‘qidagi proeksiyasi Δy_k deylik. Har bir $A_{k-1}\breve{A}_k$ yoyda ixtiyoriy (x_k,y_k) nuqta olib, bu nuqtadagi funksiyaning $f(x_k,y_k)$ qiymatini mos ravishda Δx_k va Δy_k ko‘paytirib, ushbu

$$\sigma_1=\sum_{k=1}^n f(x_k,y_k)\cdot\Delta x_k,\qquad \sigma_2=\sum_{k=1}^n f(x_k,y_k)\cdot\Delta y_k$$

yig‘indilarni hosil qilamiz.

Agar $\lambda\to 0$ da bu yig‘indilar chekli limitga ega bo‘lsa, bu limitlar $f(x,y)$ funksiyaning $\breve{AB}$ egri chizig‘i bo‘yicha ikkinchi tur egri chiziqli integrallari deyiladi va mos ravishda

$$\int_{\breve{AB}} f(x,y)dx \qquad \int_{\breve{AB}} f(x,y)dy$$

kabi belgilanadi. Demak,

$$\int_{\breve{AB}} f(x,y)dx=\lim_{\lambda\to 0}\sum_{k=1}^n f(x_k,y_k)\cdot\Delta x_k,$$

$$\int_{\breve{AB}} f(x,y)dy=\lim_{\lambda\to 0}\sum_{k=1}^n f(x_k,y_k)\cdot\Delta y_k.$$

Eslatma. Ikkinchi tur egri chiziqli integrallar $\breve{AB}$ egri chiziqning yo‘nalishiga bog‘liq bo‘lib,

$$\int_{\breve{BA}} f(x,y)dx=-\int_{\breve{AB}} f(x,y)dx,$$

$$\int_{\breve{BA}} f(x,y)dy=-\int_{\breve{AB}} f(x,y)dy$$

bo‘ladi.

Agar $\breve{AB}$ egri chiziq OX o‘qiga perpendikulyar bo‘lgan to‘g‘ri chiziq kesmasidan iborat bo‘lsa,

$$\int_{\breve{AB}} f(x,y)dx=0,$$

OY o‘qiga perpendikulyar bo‘lgan to‘g‘ri chiziq kesmasidan iborat bo‘lsa,

$$\int_{\breve{AB}} f(x,y)dy=0$$

bo‘ladi.

Faraz qilaylik, $\breve{AB}$ egri chiziqda ikkita $P(x,y)$ va $Q(x,y)$ funksiyalar berilgan bo'lib,

$$\int_{\breve{AB}} P(x,y)dx, \text{ va } \int_{\breve{AB}} Q(x,y)dy$$

ularning ikkinchi tur egri chiziqli integrallari bo'lsin. Ushbu

$$\int_{\breve{AB}} P(x,y)dx + \int_{\breve{AB}} Q(x,y)dy$$

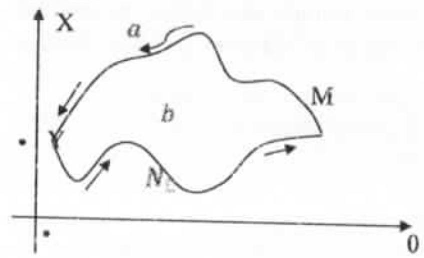
yig'indi ikkinchi tur egri chiziqli integralning umumiy ko'rinishi deyiladi va

$$\int_{\breve{AB}} P(x,y)dx + Q(x,y)dy$$

kabi yoziladi. Demak,

$$\int_{\breve{AB}} P(x,y)dx + Q(x,y)dy = \int_{\breve{AB}} P(x,y)dx + \int_{\breve{AB}} Q(x,y)dy.$$

Aytaylik, K yopiq egri chiziqda $f(x,y)$ funksiya berilgan bo'lsin (22-chizma).



22-chizma

Bu yopiq K chiziqda yo'nalish quyidagicha aniqlanadi, shunday yo'nalish musbat deb qabul qilinadiki, kuzatuvchi yopiq chiziq bo'ylab harakat qilganda, yopiq chiziq bilan chegaralangan soha unga nisbatan har doim chap tomonda yotadi.

Ushbu

$$\int_{Ma\breve{N}} f(x,y)dx + \int_{Nb\breve{M}} f(x,y)dx$$

yig'indi $f(x,y)$ funksiyaning K yopiq chiziq bo'yicha ikkinchi tur egri chiziqli integrali deyiladi va

$$\oint_K f(x,y)\,dx$$

kabi belgilanadi. Demak,

$$\oint_K f(x,y)\,dx = \int_{Ma\breve{N}} f(x,y)dx + \int_{Nb\breve{M}} f(x,y)dx.$$

Xuddi shunga o'xshash

$$\oint_K f(x,y)\,dy \quad \oint_K P(x,y)dx + Q(x,y)dy$$

integrallar ta'riflanadi.

4^0. Ikkinchi tur egri chiziqli integrallarni hisoblash.

Aytaylik, $f(x,y)$ funksiya $\breve{AB}$ egri chiziqda berilgan va uzluksiz bo'lsin:

a) agar $\breve{AB}$ egri chiziq ushbu

$$\begin{cases} x=\varphi(t), \\ y=\psi(t) \end{cases} \qquad (\alpha \le t \le \beta)$$

sistema bilan berilgan bo'lib, $\varphi(t)$ va $\psi(t)$ funksiyalar $[\alpha,\beta]$ da uzluksiz $\varphi'(t), \psi'(t)$ hosilalarga ega bo'lsa, u holda

$$\int_{\breve{AB}} f(x,y)dx = \int_\alpha^\beta f(\varphi(t),\psi(t))\cdot\varphi'(t)dt, \tag{10}$$

$$\int_{\breve{AB}} f(x,y)dy = \int_\alpha^\beta f(\varphi(t),\psi(t))\cdot\psi'(t)dt \tag{11}$$

$$\int_{\breve{AB}} P(x,y)dx + Q(x,y)dy = \int_\alpha^\beta \left[P(\varphi(t),\psi(t))\cdot\varphi'(t) + Q(\varphi(t),\psi(t))\cdot\psi'(t)\right]dt, \tag{12}$$

bo'ladi;

b) agar $\breve{AB}$ egri chiziq ushbu

$$y=y(x) \qquad (a\le x\le b)$$

tenglama bilan aniqlangan bo'lib, $y=y(x)$ funksiya $[a,b]$ da uzluksiz $y'(x)$ hosilalarga ega bo'lsa, u holda

$$\int_{\breve{AB}} f(x,y)dx = \int_a^b f(x,y(x))dx,$$

$$\int_{\breve{AB}} P(x,y)dx + Q(x,y)dy = \int_a^b \left[P(x,y(x)) + Q(x,y(x))\cdot y'(x)\right]dx \tag{13}$$

bo'ladi.

c) agar $\breve{AB}$ egri chiziq ushbu

$$x=x(y) \qquad (c\le y\le d)$$

tenglama bilan aniqlangan bo'lib, $x=x(y)$ funksiya $[c,d]$ da uzluksiz $x'(y)$ hosilalarga ega bo'lsa, u holda

$$\int_{\breve{AB}} f(x,y)dy = \int_c^d f(x(y),y)dy,$$

$$\int_{\breve{AB}} P(x,y)dx + Q(x,y)dy = \int_c^d \left[P(x(y),y)x'(y) + Q(x(y),y)\right]dy \tag{14}$$

bo'ladi.

Ikkinchi tur egri chiziqli integrallar yuqorida keltirilgan (12), (13) va (14) formulalardan foydalanib hisoblanadi.

5-misol. Ushbu

$$\int_{\breve{AB}} y^2dx + x^2dy$$

ikkinchi tur egri chiziqli integralni hisoblang, bunda $\breve{AB}$ egri chiziq

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

ellipsning yuqori yarim tekislikdagi qismi.

◀Ma'lumki, ellipsning parametrik tenglamasi

$$\begin{cases} x=\varphi(t)=a\cos t, \\ y=\Psi(t)=b\sin t \end{cases} \qquad (o \le t \le \pi)$$

bo'ladi. $A=A(a,0)$ nuqtaga parametr t ning $t=0$ qiymati, $B=B(-a,0)$ nuqtaga esa $t=\pi$ qiymati mos kelib, t parametr 0 dan π gacha o'zgarganda (x,y) nuqta A dan B ga qarab $\breve{AB}$ ni chizib chiqadi.

Bu holda

$$P(x,y)=y^2, \quad Q(x,y)=x^2$$

bo'lishini e'tiborga olib, (14) formuladan foydalanib topamiz.

$$\int_{AB} y^2dx + x^2dy = \int_0^{\pi}\left[b^2\sin^2 t(-a\sin t)+a^2\cos^2 t\cdot(b\cos t)\right]dt =$$

$$= ab\int_0^{\pi}\left(a\cos^3 t - b\sin^3 t\right)dt = -\frac{4}{3}ab^2$$ (qaralsin [2]▶

6-misol. Ushbu

$$\int_{\breve{AB}} (4x+y)^2\,dx + 5yx^2dy$$

integralni hisoblang, bunda $\breve{AB}$ egri chiziq quyidagi

$$y=3x^2$$

parabolaning $A=A(0,o)$, $B=B(1,3)$ nuqtalari orasidagi qismi.

◀Bu integralni (14) formuladan foydalanib hisoblaymiz:

$$\int_{\breve{AB}} (4x+y)^2\,dx + 5yx^2dy = \int_0^1\left[4x-3x^2+5x^2\cdot 3x^2\cdot 6x\right]dx =$$

$$= \int_0^1\left(4x-3x^2+90x^5\right)dx = \left(4\cdot\frac{x^2}{2}-3\cdot\frac{x^3}{3}+90\cdot\frac{x^6}{6}\right)\Bigg|_0^1 = 16. \blacktriangleright$$

5⁰. Ikkinchi tur egri chiziqli integrallarni tatbiq etish:

a) **tekis shaklning yuzini topish.** Aytaylik, tekislikda yuzaga ega bo'lgan D shakil berilgan bo'lsa, uning chegarasi (konturi – yopiq egri chiziq) ∂D bo'lsin. Bu shaklning S yuzi

$$S=\frac{1}{2}\oint_{\partial D} xdy - ydx \tag{15}$$

bo'ladi.

7-misol. Ushbu

$$\begin{cases} x=a\cos t, \\ y=b\sin t \end{cases} \qquad (0\le t\le 2\pi)$$

ellips bilan chegaralangan shaklning yuzini toping.

◀Bu shaklning yuzini (15) formuladan foydalanib topamiz:

$$S=\frac{1}{2}\oint_{\partial D} xdy-ydx=\frac{1}{2}\int_0^{2\pi}(a\cos t\cdot b\cos t+b\sin t\cdot a\sin t)dt=\frac{1}{2}ab\int_0^{2\pi}dt=\pi ab. \blacktriangleright$$

b) **bajarilgan ishni topish.** Aytaylik, o'zgaruvchi kuch

$$\vec{F}=P(x,y)\cdot\vec{i}+Q(x,y)\cdot\vec{j}$$

tekislikdagi $\breve{AB}$ egri chizig'i bo'yicha ish bajarsin, bunda $P(x,y)$ va $Q(x,y)$ uzluksiz funksiyalar bo'lib, ular $\vec{F}$ kuchning koordinata o'qlardagi proeksiyalari. Unda bu kuchning bajargan W

$$W=\int_{\breve{AB}} P(x,y)dx+Q(x,y)dy \tag{16}$$

bo'ladi.

8-misol. Ushbu

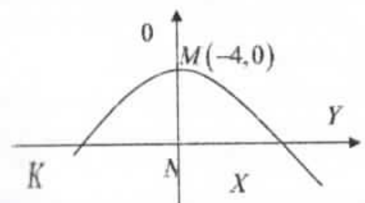
$$\vec{F}=(x^2+2y)\cdot\vec{i}+(y^2-2x)\cdot\vec{j}$$

kuchning

a) MN kesma bo'yicha, $M=M(-4,0)$, $N=N(0,2)$,

b) MON sniq chiziq bo'yicha,

c) MN yoy, ushbu $y=2-\frac{x^2}{8}$ parabola yoyi bo'yicha bajargan ishini hisoblang (23-chizma).



23-chizma

◄Kuchning bajargan ishini (16) formuladan foydalanib hisoblaymiz:

a) MN **kesma bo'yicha:**

Ravshanki, MN to'g'ri chiziqning tenglamasi

$$\frac{x}{-4}+\frac{y}{2}=1$$

ya'ni

$$y=\frac{1}{2}x+2$$

bo'lib,

$$dy=\frac{1}{2}dx$$

bo'ladi. Unda (16) formulaga ko'ra,

$$W=\int_{MN}(x^2+2y)dx+(y^2-2x)dy=\int_{-4}^{0}\left[x^2+2\left(\frac{1}{2}x+2\right)+\frac{1}{2}\left(\left(\frac{1}{2}x+2\right)^2-2x\right)\right]dx=$$

$$=\left(\frac{9}{8\cdot 3}x^3+\frac{x^2}{2}+6x\right)\Bigg|_{-4}^{0}=40$$

bo'ladi;

b) MON **siniq chiziq bo'yicha:**

Izlanayotgan ish ushbu formula yordamida hisoblanadi:

$$W=\int_{MON}(x^2+2y)dx+(y^2-2x)dy=\int_{M0}(x^2+2y)dx+(y^2-2x)dy+$$

$$+\int_{0N}(x^2+2y)dx+(y^2-2x)dy.$$

Agar $M0$ kesmada $y=0$ $(dy=0)$, $0N$ kesmada $x=0$ va $dx=0$ bo'lishini e'tiborga olsak, unda

$$W=\int_{M0}x^2dx+\int_{0N}y^2dy=\int_{-4}^{0}x^2dx+\int_{0}^{2}y^2dy=24.$$

bo'lishini topamiz:

c) MN **yoy, ushbu** $y=2-\frac{x^2}{8}$ **parabola yoyi bo'yicha:**

bu holda $y=2-\frac{x^2}{8}$ va $dy=-\frac{x}{4}dx$ bo'lib,

$$W=\int_{MN}(x^2+2y)dx+(y^2-2x)dy=\int_{-4}^{0}\left[x^2+4-\frac{x^2}{4}\right]dx+\left[\left(2-\frac{x^2}{8}\right)^2-2x\right]\left(-\frac{x}{4}dx\right)=$$

$$=\int_{-4}^{0}\left[-\frac{x^5}{256}+\frac{x^3}{8}+\frac{5}{4}x^2-x+4\right]dx=\left(-\frac{x^6}{256\cdot 6}+\frac{x^4}{8\cdot 4}+\frac{5}{4\cdot 3}x^3-\frac{x^2}{2}+4x\right)\Bigg|_{-4}^{0}=45\frac{1}{3}$$

bo'ladi.►

Quyidagi birinchi tur egri chiziqli integrallarni hisoblang:

1693. $\int_C xy^2dS$, bunda C, $A(0,0)$ va $B(4,3)$ nuqtalarni birlashtiruvchi to'g'ri chiziq kesmasi.

1694. $\int_C xdS$, bunda C, $y=x^2+1$ parabolaning $A(0,1)$, $B(1,2)$ nuqtalari orasidagi kesma.

1695. $\int_C (x+y)dS$, bunda C, $y=2x-1$, $(-1\le x\le 2)$ to'g'ri chiziq kesmasi.

1696. $\int_C x^2dS$, bunda C, $y=\ln x$, $(1\le x\le 3)$ egri chiziq qismi.

1697. $\int_C \frac{x^3}{y^2}dS$, bunda $C: xy=1$ egri chiziqning $A(1,1)$, $B\left(2,\frac{1}{2}\right)$ nuqtalar orasidagi qismi.

1698. $\int_C ydS$, bunda $C: y=x^3$ egri chiziqning $A(0,0)$, $B(1,1)$ nuqtalar orasidagi qismi.

1699. $\int_C (2x+y)dS$, bunda C: uchlari $A(1,0)$, $B(0,2)$, $C(0,0)$ nuqtalarida bo'lgan ABC uchburchakning qolgan kvadratidagi qismi.

1700. $\int_C \frac{\cos^2 x}{\sqrt{1+\cos^2 x}}dS$, bunda C: $y=\sin x$, $0\le x\le \pi$ sinusoiddan iborat.

1701. $\int_C xdS$, bunda C: markazi $(0,0)$ nuqtada radiusi R ga teng bo'lgan aylananing qolgan kvadratidagi qismi.

1702. $\int_C y^2dS$, bunda C: markazi $(0,0)$ nuqtada radiusi R ga teng bo'lgan aylananing yuqori yarim tekislikdagi qismi.

1703. Birinchi tur egri chiziqli integraldan foydalanib, ushbu $x=2a\cos t-a\cos 2t$, $y=2a\sin t-a\sin 2t$ egri chiziqning uzunligini toping.

1704. Birinchi tur egri chiziqli integraldan foydalanib ushbu massasi egri chiziq $2y=x^2$ ning $A(0,0)$, $B\left(1,\frac{1}{2}\right)$ nuqtalari orasidagi qismining massasini toping, bunda har bir nuqtadagi zichlik shu nuqtaning absissasi x ga proporsional.

1705. Ushbu bir jinsli $x^2+y^2=R^2$, $y\geq 0$ yarim aylananing og'irlik markazining koordinatalarini toping.

1706. Ushbu yarimkubik parabola $y=x^{\frac{3}{2}}$ $\left(0\leq x\leq\frac{4}{3}\right)$ ning OY o'qiga nisbatan inersiya momentini toping.

Quyidagi ikkinchi tur egri chiziqli integrallarni hisoblang:

1707. $\int\limits_C x^2dx+xy^2dy$, bunda C: $A(0,1)$, $B(1,2)$ nuqtalarni birlashtiruvchi to'g'ri chiziq kesmasi.

1708. $\int\limits_C (x^2+y)dx+(x+y^2)dy$, bunda C: $A(1,1)$, $B(3,1)$, $D(3,5)$ nuqtalarni birlashtiruvchi ABD siniq chiziq.

1709. $\int\limits_C (x+y)dx+(x-y)dy$ bunda $C: y=x^2$ parabolaning $A(-1,1)$, $B(1,1)$ nuqtalari orasidagi qismi.

1710. $\int\limits_C y^2dx-x^2dy$, bunda $C: y=1-x^2$ parabolaning $A(-1,0)$, $B(1,0)$ nuqtalarni birlashtiruvchi to'g'ri chiziq kesmasi.

1711. $\int\limits_C x^2dx+\frac{1}{y^2}dy$, bunda $C: x=\frac{1}{y}$ egri chiziqning $A(1,1)$, $B\left(4,\frac{1}{4}\right)$ nuqtalari orasidagi qismi.

1712. $\int\limits_C (x+y)dx+(x-y)dy$, bunda C: markazi koordinata bo'lgan, radiusi R ga teng bo'lgan aylananing birinchi chorakdagi qismi: $x=R\cos t$, $y=R\sin t$, $0\leq t\leq\frac{\pi}{2}$.

1713. $\int\limits_C y^2dx+xydy$, bunda $C: x=a\cos t$, $y=b\sin t$ ellipsning birinchi chorakdagi qismi: $0\leq t\leq\frac{\pi}{2}$.

1714. $\int\limits_C (ydx+xdy)$, bunda C: quyidagi $x=a\cos^3 t$, $y=a\sin^3 t$ $0\leq t\leq\frac{\pi}{4}$ astrondaning yoyi.

1715. 1) $\int\limits_C \frac{xdy-ydx}{x^2+y^2}$, bunda C: quyidagi $x=a\cos t$, $y=b\sin t$ $0\leq t\leq 2\pi$ aylana yoyidan iborat.

2) $\int\limits_C y(ydx+xdy)$, bunda C: uchlari $O(0,0)$, $A(2,1)$, $B(1,2)$ nuqtalarda bo'lgan OAB uchburchak konturidan iborat.

3) Ushbu $x=a\cos^3 t$, $y=a\sin^3 t$ $(0\leq t\leq 2\pi)$ chiziq bilan (astronda chizig'i bilan) chegaralangan shaklning yuzini toping.

4) Ushbu $x=\frac{3at}{1+t^3}$, $y=\frac{3at^2}{1+t^3}$ $(0\leq t\leq\infty)$ chiziq bilan (Dekart yaprog'i deb ataluvchi tekis shakl) chegaralangan shaklning yuzini toping.

5) Ushbu $y=x^3$ egri chiziqning har bir nuqtasiga qo'yilgan $\vec{F}=4x^6\cdot\vec{i}+xy\cdot\vec{j}$ kuchning shu chiziqning $O(0,0)$ nuqtasi $B(1,1)$ nuqtasiga o'tkazilgan sarflagan ishini toping.

5-§ Sirt integrallari

1°. Birinchi tur sirt integrallari va ularni hisoblash. Aytaylik,

$$z = z(x, y)$$

tenglama fazoda biror (S) sirtni tasvirlasin.

Bu sirtda $f(x, y, z)$ funksiya aniqlangan. (S) sirtni undagi chiziqlar yordamida n ta

$$(S_1), (S_2), ..., (S_n)$$

bo'lakka ajratamiz. So'ng (S_k) bo'lakchaning yuzini ΔS_k bilan belgilab, bu bo'lakchada ixtiyoriy (x_k, y_k, z_k) nuqtani olamiz. $f(x, y, z)$ funksiyaning shu nuqtadagi qiymati $f(x_k, y_k, z_k)$ ni ΔS_k ga ko'paytirib bu ko'paytmalardan ushbu

$$\sigma = \sum_{k=1}^{n} f(x_k, y_k, z_k) \cdot \Delta S_k \qquad (1)$$

yig'indini tuzamiz. (1) yig'indi $f(x, y, z)$ funksiyaning integral yig'indisi deyiladi.

(S_k) $(n = 1, 2, 3, ..., n)$ bo'lakchalar diametrining eng kattasini λ deylik.

Agar $\lambda \to 0$ da σ yig'indi chekli limitga ega bo'lsa, bu limit $f(x, y, z)$ funksiya (S) bo'yicha birinchi tur sirt integrali deyiladi va

$$\iint_{(S)} f(x, y, z)dS$$

kabi belgilanadi.

$$\iint_{(S)} f(x, y, z)dS = \lim_{\lambda \to 0} \sum_{k=1}^{n} f(x_k, y_k, z_k) \cdot \Delta S_k$$

Faraz qilaylik, (S) sirtni ifodalovchi

$$z = z(x, y)$$

funksiya tekislikdagi D sohada aniqlangan bo'lib, unda uzluksiz $z'_x(x, y)$, $z'_y(x, y)$ xususiy hosilalarga ega bo'lsin.

$f(x, y, z)$ funksiya esa (S) sirt bo'yicha uzluksiz bo'lsin.

U holda

$$\iint_{(S)} f(x, y, z)dS = \iint_{D} f(x, y, z(x, y))\sqrt{1 + z'^2_x(x, y) + z'^2_y(x, y)}dxdy \qquad (2)$$

bo'ladi. Bu formula yordamida birinchi tur sirt integrallari hisoblanadi.

1-misol. Ushbu

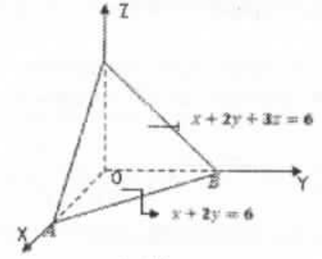
$$\iint_{(S)} (6x + 4y + 3z)ds$$

birinchi tur sirt integrali hisoblansin, bunda (S) sirt quyidagi

$$x + 2y + 3z = 6$$

tekislikning birinchi oktantdagi qismi.

◄ (S) sirt 1-chizmada tasvirlangan bo'lib, uning XOY tekislikdagi proyeksiyasi $D - ABO$ uchburchakdan iborat.



1-chizma

Ravshanki,

$$z = \frac{1}{3}(6 - x - 2y),$$

$$z'_x = \frac{\partial}{\partial x}\left(\frac{1}{3}(6 - x - 2y)\right) = -\frac{1}{3}, z'_y = \frac{\partial}{\partial y}\left(\frac{1}{3}(6 - x - 2y)\right) = -\frac{2}{3}$$

$$1 + z'^2_x(x, y) + z'^2_y(x, y) = 1 + \frac{1}{9} + \frac{4}{9} = \frac{14}{9}$$

(2) formuladan foydalanib topamiz:

$$\iint_{(S)} (6x + 4y + 3z)ds = \iint_{D} (6x + 4y + 3 \cdot \frac{1}{3}(6 - x - 2y)) \cdot \sqrt{\frac{14}{9}}dxdy = \frac{\sqrt{14}}{3}\iint_{D}(5x + 2y + 6)dxdy$$

Keyingi ikki karrali integral quyidagicha hisoblanadi:

$$\iint_{D} (5x + 2y + 6)dxdy = \int_0^3\left[\int_0^{6-2y}(5x + 2y + 6)dx\right]dy = \int_0^3\left(\frac{5}{2}x^2 + 2xy + 6x\right)\Bigg|_{x=0}^{x=6-2y} dy =$$

$$6\int_0^3 (y^2 - 10y + 2)dy = 6\left(\frac{y^3}{3} - 5y^2 + 21y\right)\Bigg|_{y=0}^{y=3} = 162$$

Demak,

$$\iint_{(S)} (6x + 4y + 3z)ds = \frac{\sqrt{14}}{3} \cdot 162 = 54 \cdot \sqrt{14}\ .►$$

2°. Birinchi tur sirt integralining tatbiq etilishi. Birinchi tur sirt integrallari yordamida sirtlarning yuzini, massalali sirtning massasini, og'irlik markazining koordinatalarini, inersiya momentlarini topish mumkin:

a) **sirtning yuzini topish**. (S) sirtning yuzi S ushbu

$$S = \iint_{(S)} ds$$

formula bilan topiladi.

2-misol. Ushbu

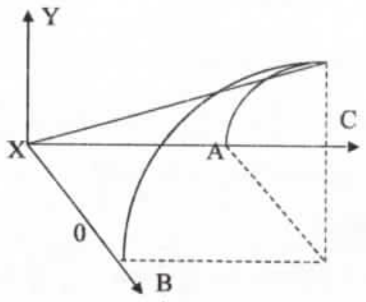
$$z^2 = 2xy$$

konusning (konus sirtning) birinchi oktantda $x = 2$, $y = 4$ tekisliklar orasidagi qismining yuzini toping.

◀Bu masalani yechishda (2) formuladan foydalanamiz. Konus tenglamasidan foydalanib, birinchi tur sirt integralini ikki karrali integralga keltiramiz:

$$S = \iint_{(S)} ds = \iint_{D} \sqrt{1 + z_x'^2 + z_y'^2}\, dxdy = \frac{1}{\sqrt{2}} \iint_{D} \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right) dxdy,$$

bunda $D-(S)$ sirtning XOY tekislikdagi proyeksiyasi – $OABC$ – to'g'ri to'rtburchak (2-chizma).



2-chizma

Endi ikki karrali integralni hisoblab topamiz:

$$S = \frac{1}{\sqrt{2}} \int_0^2 \left[\int_0^4 \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right) dy \right] dx = \frac{1}{\sqrt{2}} \int_0^2 \left(2\sqrt{xy} + \frac{2}{3}\sqrt{\frac{y^3}{x}} \right) \Bigg|_{y=0}^{y=4} dx =$$

$$2\sqrt{2} \int_0^2 \left(\sqrt{x} + \frac{4}{3\sqrt{x}} \right) dx = 2\sqrt{2} \left(\frac{2}{3}\sqrt{x^3} + \frac{8}{3}\sqrt{x} \right) \Bigg|_0^2 = 16$$

▶

b) materal sirtning massasini topish. Materal sirtning massasi m ushbu

$$m = \iint_{(S)} \rho(x, y, z) ds \qquad (3)$$

formula bilan topiladi, bunda $\rho(x, y, z)$ – zichlik.

3-misol. Agar ushbu

$$x^2 + y^2 + z^2 = a^2, \qquad z \geq 0$$

yarim sfera bo'yicha massa tarqatilgan bo'lib, uning har bir (x, y, z) nuqtasidagi zichlik.

$$\rho(x, y, z) = \frac{z}{a}$$

bo'lsa, massani toping.

◀Izlanayotgan massa (3) formulaga ko'ra,

$$m = \iint_{(S)} \frac{z}{a} ds = \frac{1}{a} \iint_{(S)} zds$$

bo'ladi, bunda (S) sirt

$$z = \sqrt{a^2 - x^2 - y^2}.$$

Sirt integralni ikki karrali integralga keltirib hisoblaymiz:

Ravshanki,

$$\iint_{(S)} zdS = \iint_{D} \sqrt{a^2 - x^2 - y^2} \cdot \sqrt{1 + z_x'^2(x, y) + z_y'^2(x, y)} dxdy$$

bunda D yarim sferaning XOY tekislikdagi proyeksiyasi bo'lib, u

$$x^2 + y^2 \leq a^2$$

doiradan iborat bo'ladi.

Endi,

$$z = \sqrt{a^2 - x^2 - y^2}$$

funksiyaning xususiy hosilalarini hisoblab,

$$\sqrt{1 + z_x'^2(x, y) + z_y'^2(x, y)}$$

ning qiymatini topamiz.

Ravshanki,

$$z_x'(x, y) = \frac{1}{2\sqrt{a^2 - x^2 - y^2}} \cdot (-2x) = -\frac{x}{\sqrt{a^2 - x^2 - y^2}},$$

$$z_y'(x, y) = \frac{1}{2\sqrt{a^2 - x^2 - y^2}} \cdot (-2x) = -\frac{y}{\sqrt{a^2 - x^2 - y^2}},$$

$$\sqrt{1 + z_x'^2(x, y) + z_y'^2(x, y)} = \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}$$

Natijada,

$$\iint_{(S)} zdS = \iint_{D} \sqrt{a^2 - x^2 - y^2} \cdot \frac{a}{\sqrt{a^2 - x^2 - y^2}} dxdy = a \iint_{x^2 + y^2 \leq a^2} dxdy = a \cdot \pi \cdot a^2$$

bo'lib,

$$m = \frac{1}{a} \iint_{(S)} zdS = \frac{1}{a} \cdot a \cdot \pi a^2 = \pi a^2$$

bo'ladi.▶

3°. Ikkinchi tur sirt integrallari va ularni hisoblash. Odatda, sirtlar bir va ikki tomonli bo'ladi. Masalan,

$$z = z(x, y)$$

tenglama bilan aniqlanadigan sirtning ustki hamda ostki tomoni, ushbu

$$x^2 + y^2 + z^2 = a^2$$

tenglama bilan aniqlangan sirtning (sferaning) tashqi va ichki tomoni bo'ladi.

Aytaylik, fazoda (S) sirt

$$z = z(x, y)$$

tenglama bilan aniqlangan bo'lib, bunda $z(x,y)$ funksiya XOY tekisligidagi (D) sohada uzluksiz hamda uzluksiz $z'_x(x,y)$, $z'_y(x,y)$ xususiy hosilalarga ega bo'lsin ((D) to'plam (S) sirtning XOY tekisligidagi proyeksiyasi). Bu ikki tomonli sirt bo'lib, uning har bir nuqtasida urinma tekislik mavjud.

(S) sirtda uning chegarasi bilan kesishmaydigan K yopiq chiziqni olaylik. Bu yopiq chiziqning XOY tekisligidagi proyeksiyasi K_I bo'lsin.

Agar (x_0,y_0,z_0) nuqta (S) sirtning K yopiq chiziq bilan chegaralangan qismiga tegishli bo'lib, bu nuqtadagi sirt normali OZ o'qi bilan o'tkir burchak tashkil etsa (bunda sirtning ustki tomoni qaralayotgan bo'ladi) K va K_I yopiq chiziqlarning yo'nalishlari musbat bo'lib, K_I bilan chegaralangan shaklning yuzi musbat ishora bilan olinadi.

Agar (x_0,y_0,z_0) nuqtadagi sirt normali OZ o'qi bilan o'tmas burchak tashkil etsa (bunda sirtning ustki tomoni qaralayotgan bo'ladi) K ning manfiy yo'nalishiga K_I ning musbat yo'nalishi mos kelib, K_I bilan chegaralangan shaklning yuzi manfiy ishora bilan olinadi.

Aytaylik yuqorida aytilgan

$$z = z(x,y)$$

tenglama bilan aniqlangan (S) sirtda $f(x,y,z)$ funksiya aniqlangan bo'lsin. Bu sirtning ikki tomonidan birini tanlaymiz.

(S) sirtni undagi chiziqlar yordamida n ta

$$(S_1), (S_2), ..., (S_n)$$

bo'laklarga ajaratamiz. Bu sirt bo'lakchasi (S_k) ning XOY tekisligidagi proyeksiyasi (D_k) ning yuzini D_k deylik.

Har bir (S_k) da ixtiyoriy (x_k,y_k,z_k) nuqta olib, bu nuqtadagi $f(x,y,z)$ funksiyaning qiymati $f(x_k,y_k,z_k)$ ni D_k ga ko'paytirib quyidagi

$$\sigma = \sum_{k=1}^{n} f(x_k,y_k,z_k)\cdot D_k \tag{4}$$

yig'indini tuzamiz. U integral yig'indi deyiladi.

Agar $\lambda \to 0$ da σ yig'indi chekli limitga ega bo'lsa, bu limit $f(x,y,z)$ funksiyaning (S) sirtning tanlangan tomoni bo'yicha ikkinchi tur sirt integrali deyiladi va

$$\iint_{(S)} f(x,y,z)dxdy$$

kabi belgilanadi:

$$\iint_{(S)} f(x,y,z)dxdy = \lim_{\lambda\to 0}\sum_{k=1}^{n} f(x_k,y_k,z_k)\cdot D_k$$

Yuqoridagidek, ushbu

$$\iint_{(S)} f(x,y,z)dydz,\ \iint_{(S)} f(x,y,z)dzdx$$

ikkinchi tur sirt integrallari ta'riflanadi.

Umumiy holda, (S) sirtda $P(x,y,z)$, $Q(x,y,z)$ va $R(x,y,z)$ funksiyalar berilgan bo'lib, ushbu

$$\iint_{(S)} P(x,y,z)dxdy,\ \iint_{(S)} Q(x,y,z)dydz,\ \iint_{(S)} R(x,y,z)dzdx$$

integrallar mavjud bo'lsa, u holda

$$\iint_{(S)} P(x,y,z)dxdy + \iint_{(S)} Q(x,y,z)dydz + \iint_{(S)} R(x,y,z)dzdx$$

yig'indi ikkinchi tur sirt integralning umumiy ko'rinishi deyiladi va u

$$\iint_{(S)} P(x,y,z)dxdy + Q(x,y,z)dydz + R(x,y,z)dzdx$$

kabi belgilanadi:

$$\iint_{(S)} P(x,y,z)dxdy + Q(x,y,z)dydz + R(x,y,z)dzdx =$$

$$\iint_{(S)} P(x,y,z)dxdy + \iint_{(S)} Q(x,y,z)dydz + \iint_{(S)} R(x,y,z)dzdy$$

Faraz qilaylik, fazoda (V) jism berilgan bo'lib, uni o'rab turgan yopiq silliq sirt (I) bo'lsin. Bu (V) da $f(x,y,z)$ funksiya aniqlangan. (V) jismni XOY tekisligiga parallel bo'lgan tekislik yordamida ikki qismga ajratamiz:

$$(V) = (V_1) + (V_2)$$

Natijada, uni o'rab turgan (I) sirt ham (I_1) ba (I_2) sirtlarga ajraladi.

Ushbu

$$\iint_{(I_1)} f(x,y,z)dxdy + \iint_{(I_2)} f(x,y,z)dxdy$$

yig'indi $f(x,y,z)$ funksiyaning (I) yopiq sirt bo'yicha ikkinchi tur sirt integrali deyiladi ba

$$\oiint_{(I)} f(x,y,z)dxdy$$

kabi belgilanadi:

$$\oiint_{(I)} f(x,y,z)dxdy = \iint_{(I)} f(x,y,z)dxdy + \iint_{(I)} f(x,y,z)dxdy$$

Xuddi yuqoridagidek

$$\oiint_{(I)} f(x,y,z)dydz,\ \oiint_{(I)} f(x,y,z)dzdx$$

hamda umumiy holda

$$\oiint_{(I)} P(x,y,z)dxdy + Q(x,y,z)dydz + R(x,y,z)dzdx$$

integrallar ta'riflanadi.

Eslatma. $f(x,y,z)$ funksiyaning (S) sirtning bir tomoni bo'yicha olingan sirt integrali, funksiyaning shu sirtning ikkinchi tomoni bo'yicha olingan sirt integralidan faqat ishora bilan farq qiladi.

Aytaylik, fazoda (S) sirt

$$z = z(x,y)$$

tenglama bilan aniqlangan bo'lib, $z(x,y)$ funksiya (S) sirtning XOY tekislikdagi proyeksiyasi (D) da berilgan ba tegishli shartlarni qanoatlantirsin.

Agar $f(x,y,z)$ funksiya (S) sirtda uzluksiz bo'lsa, u holda

$$\iint_{(S)} f(x,y,z)dxdy = \iint_{D} f(x,y,z(x,y))dxdy \qquad (5)$$

bo'ladi.

Xuddi yuqoridagidek, tegishli shartlarda

$$\iint_{(S)} f(x,y,z)dydz = \iint_{D} f(x,(y,z)y,z)dydz\,, \qquad (6)$$

$$\iint_{(S)} f(x,y,z)dzdx = \iint_{D} f(x,y(z,x),z)dzdx\,, \qquad (7)$$

bo'ladi.

Ikkinchi tur sirt integrallari (5), (6) va (7) formulalar yordamida hisoblanadi.

4-misol. Ushbu

$$\iint_{(S)} z^2 dxdy$$

ikkinchi tur sirt integralini hisoblang, bunda (S) sirt quyidagi

$$x^2+y^2+z^2=1, \qquad z\ge 0$$

sferaning tashqi tomoni.

◀Ravshanki, qaralayotgan yarim sfera

$$z=\sqrt{1-x^2-y^2}$$

tenglama bilan aniqlanadigan sirt bo'lib, uning XOY tekislikdagi proyeksiyasi

$$(D)=\{(x,y): x^2+y^2\le 1\}$$

doiradan iborat.

Sirtning tashqi tomoni sirt normalining OZ o'q bilan o'tkir burchak tashkil etilishi bilan aniqlanadi. (5) formuladan foydalanib topamiz:

$$\iint_{(S)} z^2 dxdy = \iint_{D} (1-x^2-y^2)dxdy$$

Endi ikki karrali integralni hisoblaymiz:

$$\iint_{D}(1-x^2-y^2)dxdy = \begin{bmatrix} x=r\cos\varphi, & \\ & 0\le r\le 1, \quad 0\le\varphi\le 2\pi \\ y=r\sin\varphi, & \end{bmatrix} = \int_0^{2\pi}\left[\int_0^1 (1-r^2)rdr\right]d\varphi =$$

$$\int_0^{2\pi}\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Bigg|_{r=0}^{r=1} d\varphi = \frac{1}{4}\int_0^{2\pi} d\varphi = \frac{1}{4}\cdot 2\pi = \frac{\pi}{2}$$

Demak,

$$\iint_{(S)} z^2 dxdy = \frac{\pi}{2}. ▶$$

Quyidagi birinchi tur sirt integrallarini hisoblang:

1716. 1. $\iint_{(S)} dS$, bunda (S): ushbu $x+y+z=0$ tekislikning birinchi oktantda joylashgan qismi.

2. $\iint_{(S)} xdS$, bunda (S): ushbu $z=\sqrt{1-x^2-y^2}$ yarim sferadan iborat.

3. $\iint_{(S)}(6x+4y+3z)dS$, bunda (S) ushbu $x+2y+3z=6$ tekislikning birinchi oktantda joylashgan qismi.

4. $\iint_{(S)}\left(y+z+\sqrt{a^2-x^2}\right)dS$, bunda (S) ushbu $x^2+y^2=a^2$ silindrik sirtini $z=0$, $z=h$ tekisliklar orasidagi qismi.

5. $\iint_{(S)}(x^2+y^2)dS$, bunda (S) ushbu $x^2+y^2=2z$ paraboloid sirtining $z=1$ tekislik ajratgan qismi.

6. $\iint_{(S)}(x^2+y^2)dS$, bunda (S) ushbu $z^2=x^2+y^2$ konus sirtining $z=0$, $z=1$ tekisliklar orasidagi qismi.

Quyidagi ikkinchi tur sirt integrallarini hisoblang:

1717. $\iint_{(S)}\frac{1}{2-z^2}dxdy$, bunda (S): markazi koordinata boshida, radiusi 1 ga teng bo'lgan sohaning XOY tekislikning yuqori qismida joylashgan yarim soha bo'lib, tashqi tomon bo'yicha olingan. ($z=\sqrt{1-x^2-y^2}$)

1718. 1) $\iint_{(S)} z^3 dxdy$, bunda (S): ushbu $x^2+y^2+z^2=R^2$ sohadan iborat bo'lib, uning tashqi tomoni olingan.

2) $\iint_{(S)} zdxdy$, bunda (S): ushbu $z^2=x^2+y^2$, $0\le z\le 1$ konus sirt bo'lib, uning tashqi tomoni olingan.

3) $\iint_{(S)} z^2 dxdy$, bunda (S) ushbu $x^2+y^2+2z^2=2$ ellipsoid sirt bo'lib, uning ichki tomoni olingan.

1719. $\iint_{(S)}\left(y^2+z^2\right)dxdz$, bunda (S) ushbu $x=a^2-y^2-z^2$ paraboloidning YOZ tekislikda ajratgan qismi bo'lib, uning tashqi tomoni olingan.

Kompyuter yordamida integrallarni hisoblash

Maple yordamida takroriy integralarni hisoblash uchun:

```
int(int(f(x),x=a..b),y=a..b);
```

buyruqni kiritib, Enter tugmasini bosish kifoya.

1-misol. $\int_1^3 dy \int_2^5 x^2 y dx$ takroriy integralni hisoblang.

```
> int(int(x^2*y,x=2..5),y=1..3);
```

Javob: 156.

2-misol. $\int_{-1}^0 dx \int_0^1 e^{(x-y)} y dy$ takroriy integralni hisoblang.

```
> int(int(exp(x-y),x=-1..0),y=0..1);
```

Javob: $-2e^{-1}+e^{-2}+1$.

3-misol. $\int_0^1 dy \int_y^{2-y} (x+y)dx$ takroriy integralni hisoblang.

```
> int(int(x+y,x=y..2-y),y=0..1);
```

Javob: $\frac{4}{3}$.

4-misol. $\int_0^1 dx \int_{x^2}^{x} \left(1-\frac{x+y}{2}\right) dy$ takroriy integralni hisoblang.

```
> int(int(1-((x+y)/(2)),y=x^2..x),x=0..1).
```

Javob: $\frac{11}{120}$.

Nazorat savollari

1. Ikki karrali integralga ta'rif bering?
2. Ikki karrali integralni xossalarini keltring.
3. Ikki karrali integrallarni hisoblash usullarini keltring.
4. Ikki karrali integralning fizik va mexanik tatbiqlarini izohlab bering.
5. Uch karrali integralga ta'rif bering?
6. Uch karrali integralni hisoblash usullarini keltring.
7. Uch karrali integrallarning tatbiqlarini izohlab bering.
8. "Birinchi tur egri chiziqli integral" tushunchasi va uni hisoblashni izohlab bering.
9. Birinchi tur egri chiziqli integrallarning tatbiqlarini keltring.
10. Ikkinchi tur egri chiziqli integrallarni izohlab bering.
11. Ikkinchi tur egri chiziqli integrallarni hisoblashni izohlab bering.
12. Ikkinchi tur egri chiziqli integrallarning tatbiqlarini keltring.
13. Birinchi tur sirt integraliga ta'rif bering va ular qanday hisoblanadi?
14. Ikkinchi tur sirt integraliga ta'rif bering va ular qanday hisoblanadi?

15-bob

Oddiy differensial tenglamalar

Bitta erkli x o'zgaruvchi, noma'lum funksiya (x ning funksiyasi $y=y(x)$) va uning turli tartibdagi hosilalari qatnashgan tenglama oddiy differensial tenglama deyiladi. U umumiy ko'rinishda quyidagicha

$$\Phi\left(x,y,y',y'',...,y^{(n)}\right)=0 \quad (1)$$

ifodalanadi.

Tenglamada qatnashgan noma'lum funksiya hosilasining yuqori tartibi differensial tenglamaning tartibi deyiladi.

Agar shunday $\varphi(x)$ funksiya topilsaki, (1) tenglamadagi y ning o'rniga $\varphi(x)$, y' ning o'rniga $\varphi'(x)$, y'' ning o'rniga $\varphi''(x)$,..., $y^{(n)}$ ning o'rniga $\varphi^{(n)}(x)$ qo'yilganda tenglama ayniyatga aylansa,

$$\Phi\left(x,\varphi(x),\varphi'(x),\varphi''(x),...,\varphi^{(n)}(x)\right)\equiv 0$$

$\varphi(x)$ funksiya (1) differensial tenglamaning yechimi deyiladi.

1-§. Birinchi tartibli differensial tenglamalar

1^0. Birinchi tartibli differensial tenglama hamda uning umumiy va xususiy yechimlari. Birinchi tartibli oddiy differensial tenglama, umumiy holda

$$\Phi(x,y,y')=0 \quad (2)$$

ko'rinishda bo'ladi, bunda x – erkli o'zgaruvchi, $y=y(x)$ – noma'lum funksiya, $y'=y'(x)$ esa noma'lum funksiyaning hosilasi.

Aytaylik, (2) tenglama y' ga nisbatan yechilgan bo'lsin:

$$y'=f(x,y). \quad (3)$$

Bu tenglama hosilaga nisbatan yechilgan differensial tenglama deyiladi.

Agar $\varphi(x)$ funksiya uchun

$$\varphi'(x)\equiv f(x,\varphi(x))$$

bo'lsa, $\varphi(x)$ funksiya (3) differensial tenglamaning yechimi bo'ladi.

(3) differensial tenglamaning barcha yechimlarini ixtiyoriy o'zgarmas C ga bog'liq bo'lgan

$$y=\varphi(x,C) \text{ yoki } F(x,y,C)=0$$

munosabat bilan umumiy ko'rinishda ifodalash mumkin. U differensial tenglamaning umumiy yechimi deyiladi. Bunda o'zgarmas C ning har bir qiymatida unga mos yechim hosil bo'ladi. Bunday yechimlar (3) differensial tenglamaning xususiy yechimlari deyiladi.

x argumentning, x_0 qiymatida funksiyaning qiymati y_0 deyilishi quyidagicha:

$$y\Big|_{x=x_0} = y_0$$

yozilib, u boshlang'ich shart deyiladi. Boshlang'ich shartdan foydalanib, differensial tenglamaning xususiy yechimi topiladi.

Differensial tenglamaning boshlang'ich shartni qanoatlantiruvchi yechimini topish masalasi Koshi masalasi deyiladi.

2^0. O'zgaruvchilari ajraladigan differensial tenglamalar. Agar

$$y' = f(x,y)$$

differensial tenglamada

$$f(x,y) = f_1(x)\cdot f_2(y)$$

bo'lsa, u **o'zgaruvchilari ajraladigan differensial tenglama** deyiladi. Bu differensial tenglamaning umumiy yechimi

$$\int \frac{dy}{f_2(y)} = \int f_1(x)dx + C \qquad (C = const)$$

dan topiladi.

1-misol. Ushbu

$$y' = xy + x + y + 1$$

differensial tenglamaning umumiy yechimini toping.

◀Berilgan differensial tenglamani quyidagicha

$$\frac{dy}{dx} = y(x+1)+(x+1) = (x+1)(y+1)$$

ya'ni

$$\frac{dy}{y+1} = (x+1)dx$$

ko'rinishda yozib, bu tenglikning har ikki tomonini integrallaymiz:

$$\int \frac{dy}{y+1} = \int (x+1)dx.$$

Ravshanki,

$$\int \frac{dy}{y+1} = \ln|y+1|,$$

$$\int (x+1)dx = \int (x+1)d(x+1) = \frac{(x+1)^2}{2}.$$

Demak,

$$\ln|y+1| = \frac{(x+1)^2}{2} + \ln C$$

Keyingi tenglikdan

$$\frac{y+1}{C} = e^{\frac{(x+1)^2}{2}}$$

ya'ni

$$y = Ce^{\frac{(x+1)^2}{2}} - 1$$

bo'lishi kelib chiqadi. Bu berilgan differensial tenglamaning umumiy yechimi bo'ladi.▶

2-misol. Ushbu

$$y' = g(x)$$

differensial tenglamaning

$$y\Big|_{x=x_0} = y_0$$

boshlang'ich shartni qanoatlantiradigan xususiy yechimini toping.

◀Avvalo, berilgan differensial tenglamani quyidagi

$$\frac{dy}{dx} = g(x) \text{ ya'ni } dy = g(x)dx$$

ko'rinishida yozib olamiz. So'ng bu tenglamani ikki tomonini integrallab,

$$\int dy = \int g(x)dx,$$

undan

$$y = \int g(x)dx = F(x) + C$$

bo'lishini topamiz, bunda

$$F'(x) = g(x).$$

Boshlang'ich shart

$$y\Big|_{x=x_0} = y_0$$

ga ko'ra

$$y_0 = F(x_0) + C$$

bo'lib, undan

$$C = y_0 - F(x_0)$$

bo'lishi kelib chiqadi. Natijada,

$$y = F(x) + C = F(x) + y_0 - F(x_0) = y_0 + [F(x) - F(x_0)] = y_0 + \int_{x_0}^{x} g(x)dx$$

bo'ladi.

Demak, berilgan differensial tenglamaning boshlang'ich shartni qanoatlantiruvchi xususiy yechimi

$$y = y_0 + \int_{x_0}^{x} g(x)dx$$

bo'ladi.▶

Eslatma. Agar

$$y' = f(x,y)$$

differensial tenglamada $f(x,y)$ funksiya quyidagi

$$f(tx,ty) = f(x,y)$$

shartni qanoatlantirsa, unda qaralayotgan differensial tenglama

$$\frac{y}{x} = u \qquad (u = u(x))$$

almashtirish yordamida o'zgaruvchilari ajraladigan differensial tenglamaga keladi.

3-misol. Ushbu

$$y' = \frac{x^2+y^2}{xy}$$

differensial tenglamaning umumiy yechimini toping.

◀Berilgan tenglamada

$$f(x,y) = \frac{x^2+y^2}{xy}$$

bo'lib, uning uchun

$$f(tx,ty) = \frac{(tx)^2+(ty)^2}{(tx)\cdot(ty)} = \frac{t^2(x^2+y^2)}{t^2xy} = \frac{x^2+y^2}{xy} = f(x,y)$$

bo'ladi.

Qaralayotgan tenglamada

$$\frac{y}{x} = u, \text{ ya'ni } y = ux$$

almashtirish bajaramiz. Unda

$$y' = (u\cdot x)' = u + x\cdot u'$$

bo'lib, tenglama ushbu

$$x\cdot u' + u = \frac{x^2+u^2x^2}{xux} = \frac{1+u^2}{u},$$

ya'ni

$$x\frac{du}{dx} = \frac{1+u^2}{u} - u = \frac{1}{u}$$

ko'rinishga keladi. Bu o'zgaruvchilari ajraladigan differensial tenglamadir. Uni yechamiz:

$$x\frac{du}{dx} = \frac{1}{u}, \quad udu = \frac{dx}{x}, \quad \int udu = \int\frac{dx}{x},$$

$$\frac{u^2}{2} = \ln|x| + \ln C, \quad u^2 = 2\ln|Cx|.$$

Keyingi tenglikdagi u ning o'rniga $\frac{y}{x}$ ni qo'yib topamiz:

$$y = |x|\sqrt{2\ln|Cx|}.▶$$

3^0. Chiziqli differensial tenglamalar. Noma'lum funksiya $y = y(x)$ va uning hosilasi $y' = y'(x)$ ga nisbatan chiziqli bo'lgan ushbu

$$y' + p(x)\cdot y = q(x) \tag{4}$$

tenglama birinchi tartibli chiziqli differensial tenglama deyiladi, bunda $p(x)$ va $q(x)$ lar uzluksiz funksiyalar.

Xususan, (4) tenglamada $q(x) = 0$ bo'lsin. Unda (4) tenglama

$$y' + p(x)\cdot y = 0 \tag{5}$$

bo'lib, bu tenglamaning umumiy yechimi

$$y = Ce^{-\int p(x)dx}$$

bo'ladi.

Odatda, (5) bir jinsli chiziqli differensial tenglama deyiladi.

Yuqorida keltirilgan (4) chiziqli differensial tenglama quyidagicha yechiladi:

(5) bir jinsli tenglamaga

$$y' + p(x)\cdot y = 0$$

tenglamaning umumiy yechimi

$$y = Ce^{-\int p(x)dx}$$

dagi C ni x o'zgaruvchining funksiyasi bo'lsin deb qaraladi:

$$C = C(x),$$

(4) tenglamaning umumiy yechimi

$$y = C(x)\cdot e^{-\int p(x)dx} \tag{6}$$

ko'rinishda izlanadi.

Natijada, $C(x)$ ni topish uchun ushbu

$$\frac{dC(x)}{dx} = q(x)e^{\int p(x)dx}$$

differensial tenglama hosil bo'lib, uning yechimi

$$C(x)=\int q(x)\cdot e^{\int p(x)dx}dx+C_1, \qquad (C_1=const)$$

bo'ladi. U (6) tenglikka qo'yilsa, u

$$y=e^{-\int p(x)dx}\left[\int q(x)e^{\int p(x)dx}dx+C_1\right] \qquad (7)$$

ko'rinishga keladi. Bu

$$y'+p(x)\cdot y=q(x)$$

differensial tenglamaning umumiy yechimi bo'ladi.

4-misol. Ushbu

$$y'+xy=x^3$$

chiziqli differensial tenglamaning umumiy yechimini toping.

◀Bu tenglama uchun

$$p(x)=x, \quad q(x)=x^3$$

bo'ladi.

(7) formuladan foydalanib topamiz:

$$y=e^{-\int p(x)dx}\left[\int q(x)e^{\int p(x)dx}dx+C_1\right]=e^{-\int xdx}\left[\int x^3e^{\int xdx}dx+C_1\right]=$$

$$=e^{-\frac{x^2}{2}}\left[\int x^3e^{\frac{x^2}{2}}dx+C\right].$$

Endi bu tenglikning o'ng tomonidagi integralni hisoblaymiz:

$$\int x^3e^{\frac{x^2}{2}}dx=\left[\frac{x^2}{2}=t, xdx=dt, x^2=2t\right]=\int 2te^t dt=2\int te^t dt=$$

$$=\begin{bmatrix}u=t, & du=dt\\ dv=e^t dt, & v=e^t\end{bmatrix}=2\left[te^t-\int e^t dt\right]=2\left(\frac{x^2}{2}e^{\frac{x^2}{2}}-e^{\frac{x^2}{2}}\right)=x^2e^{\frac{x^2}{2}}-2e^{\frac{x^2}{2}}.$$

Demak,

$$y=e^{-\frac{x^2}{2}}\left(x^2e^{\frac{x^2}{2}}-2e^{\frac{x^2}{2}}+C_1\right)$$

berilgan differensial tenglamaning umumiy yechimi bo'ladi.▶

Eslatma. Ushbu

$$y'+p(x)\cdot y=q(x)\cdot y^m \qquad (8)$$

ko'rinishdagi differensial tenglama (**Bernulli tenglamasi**)

$$z=y^{1-m}$$

almashtirish natijasida chiziqli differensial tenglamaga keladi.

(8) tenglamaning umumiy yechimi quyidagicha:

$$y=\left\{e^{-\int(1-m)p(x)dx}\left[\int(1-m)q(x)\cdot e^{\int(1-m)p(x)dx}dx+C\right]\right\}^{\frac{1}{1-m}} \qquad (9)$$

bo'ladi.

5-misol. Ushbu

$$y'-\frac{1}{x}y=e^x\cdot y^2$$

differensial tenglamaning umumiy yechimini toping.

◀Bu differensial tenglama uchun

$$p(x)=-\frac{1}{x}, \quad q(x)=e^x, \quad m=2$$

bo'lib, uning umumiy yechimi (9) formulaga ko'ra quyidagicha bo'ladi:

$$y=\left\{e^{-\int(1-2)\left(-\frac{1}{x}\right)dx}\left[\int(1-2)\cdot e^x\cdot e^{\int(1-2)\left(-\frac{1}{x}\right)dx}dx+C\right]^{\frac{1}{1-2}}\right\}=$$

$$=\left\{e^{-\int\frac{1}{x}dx}\left[-\int e^x\cdot e^{\int\frac{1}{x}dx}dx+C\right]\right\}^{-1}.$$

Ravshanki,

$$e^{-\int\frac{1}{x}dx}=e^{-\ln x}=e^{\ln\frac{1}{x}}=\frac{1}{x},$$

$$e^{\int\frac{1}{x}dx}=e^{\ln x}=x,$$

$$\int e^x\cdot xdx=x\cdot e^x-e^x.$$

Demak,

$$y=\left\{\frac{1}{x}\left(xe^x-e^x+C\right)\right\}^{-1}=\frac{x}{e^x-xe^x+C}.$$ ▶

4⁰. To'liq differensialli tenglama

Agar

$$M(x,y)dx+N(x,y)dy=0 \qquad (10)$$

differensial tenglamaning chap tomonidagi ifoda biror $u=u(x,y)$ funksiyaning to'liq differensiali bo'lsa, (10) to'liq differensialli tenglama deyiladi va u quyidagicha:

$$du(x,y)=0$$

ko'rinishga keladi. Bu tenglikning har ikki tomonini integrallash natijasida

$$u(x,y)=C \qquad (C=const)$$

bo'lishi kelib chiqadi. Bu (10) differensial tenglamaning umumiy yechimi bo'ladi.

Agar $M(x,y)$ va $N(x,y)$ funksiyalar ushbu

$$\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}$$

shartni qanoatlantirsa, u holda

$$M(x,y)dx+N(x,y)dy$$

ifoda biror $u(x,y)$ funksiyaning to‘liq differensiali bo‘ladi:

$$du(x,y)=M(x,y)dx+N(x,y)dy.$$

Ayni paytda, ta’rifga ko‘ra,

$$du(x,y)=\frac{\partial u(x,y)}{\partial x}dx+\frac{\partial u(x,y)}{\partial y}dy$$

bo‘lib, keyingi tengliklardan

$$\frac{\partial u(x,y)}{\partial x}=M(x,y), \qquad \frac{\partial u(x,y)}{\partial y}=N(x,y)$$

bo‘lishi kelib chiqadi. Bu tengliklardan foydalanib, $u(x,y)$ yechim quyidagicha topiladi:

Ushbu

$$\frac{\partial u(x,y)}{\partial x}=M(x,y)$$

tenglikni, x bo‘yicha integrallaymiz (bunda y ni o‘zgarmas hisoblaymiz)

$$\int\frac{\partial u(x,y)}{\partial x}dx=\int M(x,y)dx$$

Natijada,

$$u(x,y)=\int M(x,y)dx+C(y) \qquad (11)$$

bo‘ladi, bunda $C(y)$ hosilaga ega bo‘lgan ixtiyoriy funksiya.

Endi (11) tenglikning ikki tomonini y bo‘yicha differensiallab topamiz:

$$\frac{\partial u(x,y)}{\partial y}=\frac{\partial}{\partial y}\left(\int M(x,y)dx\right)+C'(y).$$

Bu tenglikdan

$$C'(y)+\frac{\partial}{\partial y}\left(\int M(x,y)dx\right)=N(x,y)$$

bo‘lishi kelib chiqadi.

Keyingi tenglikdan $C(y)$ ni topib, uni (11) tenglikdagi $C(y)$ ning o‘rniga qo‘yish natijasida izlanayotgan $u(x,y)$ topiladi.

6-misol. Ushbu

$$\left(2xy+3y^2\right)dx+\left(x^2+6xy-3y^2\right)dy=0$$

differensial tenglamani yeching.

◄Bu tenglamada

$$M(x,y)=2xy+3y^2, \qquad N(x,y)=x^2+6xy-3y^2$$

bo‘lib,

$$\frac{\partial M(x,y)}{\partial y}=\frac{\partial}{\partial y}\left(2xy+3y^2\right)=2x+6y,$$

$$\frac{\partial N(x,y)}{\partial x}\frac{\partial}{\partial x}\left(x^2+6xy-3y^2\right)=2x+6y,$$

bo‘ladi. Demak,

$$\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}.$$

Unda

$$du(x,y)=\left(2xy+3y^2\right)dx+\left(x^2+6xy-3y^2\right)dy$$

bo‘ladi.

Ravshanki,

$$\frac{\partial u(x,y)}{\partial x}=2xy+3y^2, \qquad \frac{\partial u(x,y)}{\partial y}=x^2+6xy-3y^2.$$

Endi

$$\frac{\partial u(x,y)}{\partial x}=2xy+3y^2$$

ni integrallab topamiz:

$$u(x,y)=\int\left(2xy+3y^2\right)dx=x^2y+3xy^2+C(y) \qquad (12)$$

Bu funksiyaning y bo‘yicha xususiy hosilasini hisoblaymiz:

$$\frac{\partial u(x,y)}{\partial y}=\left(x^2y+3xy^2+C(y)\right)_y'=x^2+6xy+C'(y).$$

Demak,

$$x^2+6xy+C'(y)=x^2+6xy-3y^2.$$

Keyingi tenglikdan

$$C'(y)=-3y^2$$

ya’ni

$$\frac{dC(y)}{dy}=-3y^2$$

bo‘lishi kelib chiqadi. Bu differensial tenglamani yechamiz:

$$\frac{dC(y)}{dy}=-3y^2, \quad dC(y)=-3y^2dy,$$

$$C(y)=-3\int y^2dy=-y^3+C_1,$$

bunda C_1 – ixtiyoriy o'zgarmas. Topilgan, $C(y)$ ni (12) tenglikdagi $C(y)$ o'rniga qo'ysak, unda

$$u(x,y)=x^2y+3xy^2-y^3+C_1$$

bo'ladi.

Shunday qilib, berilgan differensial tenglamaning yechimi

$$u(x,y)=x^2y+3xy^2-y^3+C_1=C_2,$$

ya'ni

$$x^2y+3xy^2-y^3=C$$

ko'rinishida (oshkormas ko'rinishda) bo'ladi.▶

7-misol. (Bakteriya ko'payishining tezligi haqida) Bakteriyaning ko'payish tezligi uning soniga to'g'ri proporsional. Boshlang'ich $t=0$ vaqtda 100 ta bakteriya bo'lsin, 3 soatdan keyin ularning soni ikki barobar ko'payadi. Bakteriya sonining vaqtga bog'liqligini aniqlash kerak va 9 soatda bakteriya qancha marta ko'payadi?

◀Aytaylik, x bakteriyalar soni bo'lsin. Masala shartiga ko'ra,

$$\frac{dx}{dt}=kx$$

bu yerda: k – proporsionallik koyeffitsiyenti. Tenglamani o'zgaruvchilarga ajratib integrallasak, quyidagini hosil qilamiz:

$$x=Ce^{kt}$$

S ni aniqlash uchun $t=0$ va $x=100$ dan foydalanamiz. $C=100$ bo'ladi, demak,

$$x=100e^{kt}$$

k-proporsionallik koyeffitsiyentini $t=3$ va $x=200$ dan foydalanib topamiz:

$$200=100e^{3k}\text{ yoki } 2=e^{3k}$$

bundan kelib chiqadiki $e^k=2^{\frac{1}{3}}$. Shuning uchun qidirilayotgan funksiya

$$x=100\cdot 2^{\frac{t}{3}}$$

bundan $t=9$ da $x=800$ ekanligini topamiz. Demak, 9 soat ichida bakteriya 8 marta ko'payar ekan.▶

8-misol. (Aralashmaning konsentratsiyasi.) Tarkibida 100l suv va 10 kg tuz bo'lgan idishga 30 l/min tezlik bilan suv to'xtovsiz quyib turiladi va idishdan 20 l/min tezlik bilan aralashma oqib chiqadi. Faraz qilamiz, suv bilan tuz tez aralashib ketadi. t vaqt ichida idishda qancha tuz qolishini aniqlang.

◀Aytaylik, t vaqt ichida x-miqdorda tuz bor. dt vaqt ichida idishda dx-miqdorda tuz chiqib ketadi (minus ishorasi x– kamayuvchi funksiya ekanini bildiradi). t vaqtda idishda aralashma hajmi quyidagiga teng.

$$v=100+30t-20t=100+10t$$

shuning uchun tuz miqdori (bir litr aralashmada) t vaqtda

$$\frac{x}{100+10t}$$

ga teng. Bundan kelib chiqadiki, dt vaqt ichida tuz

$$\frac{x}{100+10t}\cdot 20t$$

ga kamayadi.

Bundan quyidagi differensial tenglamaga ega bo'lamiz.

$$-dx=\frac{20xdt}{100+10t}$$

yoki

$$-dx=\frac{2xdt}{10+t}$$

o'zgaruvchilarga ajratib integrallasak,

$$\frac{dx}{x}=-\frac{2dt}{10+t},$$

$$\ln x=-2\ln(10+t)+\ln C$$

bundan kelib chiqadiki

$$x=\frac{C}{(10+t)^2}.$$

Agar $t=0$, $x=10$ da $C=1000$ ga teng.

Shunday qilib, t vaqt ichida idishda tuzning kg hisobiga kamayish qonuni quyidagi formula bilan beriladi:

$$x=\frac{1000}{(10+t)^2} \qquad (1)$$

(1) formula orqali havzadagi tuz miqdorini bilgan holda yuqoridagi hodisaning boshlanganiga qancha vaqt o'tganini bilish mumkin. Mana shu fikr asosida dengiz va okean yoshi aniqlanadi.

9-misol. (Jismning sovishi.) Atrofdagi havo temperaturasi 20^0 ga teng bo'lsin. Jismning sovish tezligi jism temperaturasi va atrofdagi havo temperaturasi ayirmasiga to'g'ri proporsional. Ma'lumki, 20 daqiqa ichida jism 100^0C dan 60^0C gacha soviydi. Jism temperaturasi θ ning t vaqt ichida o'zgarish qonunini aniqlang.

◀Masala shartiga ko'ra, quyidagini yozamiz:

$$\frac{d\theta}{dt}=k(\theta-20)$$

bu yerda: k– proporsionallik koyeffitsiyenti. O'zgaruvchilarga ajratib integrallasak:

$$\frac{d\theta}{\theta-20}=kdt,$$

$$\ln(\theta-20)=kt+\ln c.$$

Bu ifodani potensirlasak,

$$\theta-20=ce^{kt}.$$

c ni aniqlash uchun boshlang'ich shartdan foydalanamiz:

$$t=0 \text{ da } \theta=100^0.$$

Bundan $c=80$ Shuning uchun

$$\theta=20+80e^{kt}.$$

Proporsionallik koyeffitsiyenti k ni qo'shimcha shartlar yordamida aniqlaymiz, $t=20$ $\theta=60^0$. Bundan:

$$60=20+80e^{20k}$$

yoki

$$e^{20k}=\frac{1}{2}.$$

Demak,

$$e^k=\left(\frac{1}{2}\right)^{\frac{1}{20}}.$$

Shunday qilib, natija quyidagicha:

$$\theta=20+80\left(\frac{1}{2}\right)^{\frac{t}{20}}$$ ►

10-misol. Ovqatlanish resursi juda yaxshi sharoitda bo'lgan mikroorganizmlar jamoasini qaraylik. Vaqt o'tishi bilan jamoaning ko'payishi va nobud bo'lishi o'zgarib turadi. Ana shu o'zgarish qonunini toping.

◄Aytaylik, $x=x(t)$ t vaqt ichidagi tirik organizmlarning soni bo'lsin, $x(t+\Delta t)$ esa – $t+\Delta t$ vaqtdagi soni. U holda ayirma

$$x(t+\Delta t)-x(t)=\Delta x$$

ni beradi. Δt vaqt ichida balog'atga yetganlarining bir qismi nasl qoldiradi, qolgan qismi nobud bo'lishi mumkin. Shunday qilib,

$$\Delta x=G-H$$

bu yerda: G t dan $t+\Delta t$ vaqt o'tganda tug'ilganlari soni, H shu vaqt ichida nobud bo'lganlar soni.

Tug'ilganlar soni G Δt vaqt oralig'iga bog'liq va nasl qoldiruvchi "ota-ona"larning soniga bog'liq, chunki ular qancha ko'p bo'lsa tug'ilish shuncha ko'p bo'ladi.

Shunday qilib,

$$G=\Phi(x,\Delta t)$$

bu yerda: $\Phi(x,\Delta t)$ funksiya x yoki Δt ning o'sishi bilan o'sadi yoki x yoki Δt larning biri nolga intilsa nolga teng bo'ladi.

Δt o'zgaruvchiga kelsak, eng oddiy tajriba shuni ko'rsatadiki, u chiziqli bo'lib agar kuzatishni ikki marta uzaytirsak, mikroorganizmlar nasli ham ikki marta oshadi. Shunday qilib,

$$\Phi(x,\Delta t)=f(x)\Delta t.$$

$f(x)$ funksiya xususiyati murakkabroq. Biz bilamizki, x o'sishi bilan $f(x)$ monoton o'sadi va $x=0$ bo'lsa nol bo'ladi. Ammo o'sish mikroorganizm turiga bog'liq. Biz nasl miqdorining "ota-ona" lar soniga to'g'ri proporsional bo'lgan holati bilan chegaralanamiz, ya'ni $f(x)=\alpha x$ $(\alpha=const)$. Shunday qilib,

$$G=\alpha x\Delta t.$$

Shunga o'xshash,

$$H=\beta x\cdot\Delta t$$

va bundan kelib chiqadiki,

$$\Delta x=\alpha x\Delta t-\beta x\Delta t$$

yoki

$$\Delta x=\gamma x\Delta t \quad (1)$$

bu yerda:

$$\gamma=\alpha-\beta$$

(1) da tenglamaning ikkala tomonini Δt ga bo'lib, limitga o'tamiz:

$$\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}=\frac{dx}{dt}$$

natijada, quyidagini hosil qilamiz

$$\frac{dx}{dt}=\gamma x \quad (2)$$

yoki

$$\frac{dx}{dt}-\gamma x=0.$$

Birinchi tartibli chiziqli bir jinsli tenglamaga kelamiz. Bu tenglamani yechib

$$x=Ce^{\gamma t} \quad (3)$$

ni hosil kilamiz.

$t=t_0$ da $x=x(t_0)$ (bu yerda t_0 boshlang'ich vaqtda $x(t_0)=x_0$ tirik mikroorganizmlar soni) boshlang'ich shart bilan S ni topamiz.

$$C=x_0e^{-\gamma t_0}$$

buni (3) ga qo'yib, vaqt davomida mikroorganizmlar o'zgarish qonunini topamiz.

$$x=x_0e^{\gamma(t-t_0)} \quad (4)$$

Ammo topgan bu qonuniyatimiz qanchalik haqiqiy hayotga to'g'ri kelish kelmasligini tajriba va kuzatishlar hal etadi. (4) formula shuni ko'rsatadiki, o'sish eksponensial darajada, lekin hayotda birorta ham tirik organizm bu darajada o'smaydi. Chunki biz faraz qilgan (2) tenglamada ovqatlanish sharoiti yaxshiligi va tashqi faktorlarning ta'siri yo'qligi bu haqiqatga ziddir. Shunday qilib, (2) tenglama yoki nazariy xarakterga ega (uzluksiz oziqlantirib turilganda va tashqi halaqit beruvchi kuchlar bo'lmasa, tirik organizmlar qanday ko'payishini ko'rish mumkin) yoki sun'iy ko'paytirishlar natijasini ko'rsatadi.

(2) tenglamani birinchi marta 1802-yil Maltus qo'llagan. Uning xatosi bu tenglamani nafaqat tabiatga, hatto insonlarga qo'llasa ham bo'ladi deb tushungan. Aslida, tenglama tor doirada qo'llaniladi.▶

Quyidagi keltirilgan differensial tenglamalar uchun ko'rsatilgan funksiyalar yechim bo'lishini isbotlang.

1720. $y'=3x, \quad y=\frac{3}{2}x^2.$

1721. $y'+y=0, \quad y=\cos x.$

1722. $y'-x^2y=0, \quad y=e^{\frac{x^3}{3}}.$

1723. $2yy'=1, \quad y=\sqrt{x}.$

1724. $y'+2y=0, \quad y=e^{-2x}.$

1725. $y''=x^2+y^2, \quad y=\frac{1}{x}.$

1726. $y''+y=0, \quad y=3\sin x-4\cos x.$

1727. $y''-2y'+y=0, \quad y=xe^x.$

1728. $y'''=\cos x, \quad y=-\sin x+x^2+x+1.$

1729. $y''-2y'+y=0, \quad y=c_1e^x+c_2xe^x$, c_1, c_2 – o'zgarmaslar.

1730. $y''+y=0, \quad y=c_1\sin x+c_2\cos x$, c_1, c_2 – o'zgarmaslar.

1731. Agar $y'-3y=0$ differensial tenglamaning umumiy yechimi $y=Ce^{3x}$ bo'lsa, uning $y(1)=e^3$ shartni qanoatlantiruvchi xususiy yechimini toping.

1732. Agar $xy'-2y=0$ differensial tenglamaning umumiy yechimi $y=Cx^2$ ekanligi ma'lum bo'lsa, uning $y(2)=12$ shartni qanoatlantiruvchi xususiy yechimini toping.

Quyidagi o'zgaruvchilari ajraladigan differensial tanlamalarning umumiy yechimlarini toping

1733. $\frac{dy}{dx}=\frac{y}{x}.$ **1734.** $dx=xdy.$ **1735.** $y'=2+y.$

1736. $y'=e^{x+y}.$ **1737.** $x+yy'=0.$ **1738.** $2x\left(1+y^2\right)+y'=0.$

1739. $xydx+(x+1)dy=0.$ **1740.** $\sqrt{y^2+1}dx-xydy=0.$

1741. $\sqrt{y}dx+x^2dy=0.$ **1742.** $(1-y)dx+(x+1)dy=0.$

1743. $x\sqrt{1-y^2}dx+\sqrt{1-x^2}dy=0.$ **1744.** $e^xdx+e^y\left(1+e^x\right)dy=0.$

1745. $\frac{dy}{dx}=\left(y^2+1\right)\cos x.$ **1746.** $e^x\left(y'+1\right)=1.$

1747. $\left(x^2+1\right)^2dy-\left(y^2-2\right)^2dx=0.$ **1748.** $\left(x^2+1\right)dy+2x(y-1)dx=0.$

1749. $1+(1+y')e^y=0.$ **1750.** $y'tgx=y.$

1751. $2x\sin ydx+\left(x^2+3\right)\cos ydy=0.$ **1752.** $\left(\sqrt{xy}+\sqrt{x}\right)y'-y=0.$

Quyidagi differensial tenglamalarning ko'rsatilgan shartni qanoatlantiruvchi xususiy yechimlarini toping

1753. $y'=3x, \quad y(2)=3.$

1754. $ydx+ctgxdy=0, \quad y\left(\frac{\pi}{3}\right)=-1.$

1755. $y^2+x^2y'=0, \quad y(-1)=1.$

1756. $y'\sin^2 x\ln y+y=0, \quad y\left(\frac{\pi}{4}\right)=1.$

1757. $y'=\frac{y^2-1}{x^2+1}, \quad y\left(\frac{\pi}{4}\right)=0.$

1758. $2\left(1+e^x\right)yy'=e^x, \quad y(0)=0.$

Quyidagi bir jinsli differensial tenglamalarning umumiy yechimlarini toping

1759. $\left(x^2+y^2\right)dx-xydy=0.$

1760. $y'=\frac{y}{x}\ln\frac{y}{x}.$

1761. $y'=e^{\frac{y}{x}}+\frac{y}{x}.$

1762. $(x+2y)dx-xdy=0.$

1763. $y^2+x^2y'=xyy'.$

1764. $y'=\frac{y}{x}+tg\frac{y}{x}.$

1765. $y - xy' = y\ln\frac{x}{y}$.

1766. $y - xy' = x + yy'$.

1767. $ydx + \left(2\sqrt{xy} - x\right)dy = 0$.

Quyidagi chiziqli differensial tenglamalarning umumiy yechimlarini toping

1768. $y' - \frac{y}{x} = x$.

1769. $y' + 2y = e^{-x}$.

1770. $y' + xy + x = 0$.

1771. $xy' = 2x\ln x - y$.

1772. $\left(y + e^{x}\right)dx - dy = 0$.

1773. $y' - 2tgxy = 0 \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$.

1774. $y' + 3x^2y = x^2$.

1775. $y' - 2tgxy = \sin x \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$.

1776. $\frac{dy}{dx} + \frac{2y}{x} = x^3$.

1777. $y' - yctgx = \sin x$.

1778. $\left(x^2 + 1\right)y' + 4xy = 3$.

1779. $y' + y = x\sqrt{y}$.

Quyidagi chiziqli differensial tenglamalarning ko'rsatilgan shartlarni qanoatlantiruvchi xususiy yechimlarini toping

1780. $(1-x)(y'+y) = e^{-x}, \quad y(2) = 0$.

1781. $(x+y)y' = 1, \quad y(-1) = 0$.

1782. $x^2y' + 5xy + 4 = 0, \quad y\left(\frac{1}{2}\right) = 62$.

1783. $x\frac{dy}{dx} + y = \cos x, \quad y(\pi) = \frac{1}{\pi}$.

1784. $\frac{dy}{dx} - \frac{xy}{x^2+1} = x, \quad y\left(2\sqrt{2}\right) = 3$.

1785. $x(x-1)y' + y = x^2(2x-1), \quad y(2) = 4$.

1786. $y' - ytgx = \frac{1}{\cos^2 x}, \quad y(0) = 0$.

1787. $ydx - (3x + 1 + \ln y)dy = 0, \quad y\left(-\frac{1}{3}\right) = 1$.

Quyidagi to'liq differensialli tenglamalarning umumiy yechimlarini toping

1788. $\left(3x^2y^2 + 7\right)dx + 2x^2ydy = 0$.

1789. $\left(e^{y} + ye^{x} + 3\right)dx = \left(2 - xe^{y} - e^{x}\right)dy$.

1790. $\cos(x-y)dx - \cos(y-x)dy = 0$.

1791. $(x+y)dx + \left(x + 2y - e^{y}\right)dy = 0$.

1792. $\left(3x^2 + 2xy - y^2\right)dx + \left(x^2 - 2xy - 3y^2\right)dy = 0$.

1793. $(2y-3)dx + \left(2x + 3y^2\right)dy = 0$.

1794. $\sin(x+y)dx + x\cos(x+y)(dx+dy) = 0$.

1795. $xdx + ydy = \frac{xdy - ydx}{x^2 + y^2}$.

1796. $\frac{2xdx}{y^3} - \frac{y^2 - 3x^2}{y^4}dy = 0$.

2-§. Ikkinchi tartibli differensial tenglamalar

1⁰. Ikkinchi tartibli differensial tenglamaning umumiy va xususiy yechimlari. Ikkinchi tartibli oddiy differensial tenglama umumiy holda ushbu

$$F\left(x, y, y', y''\right) = 0 \qquad (1)$$

ko'rinishga ega bo'ladi. Aytaylik, bu tenglama y'' ga nisbatan yechilgan bo'lsin:

$$y'' = F(x, y, y')$$

Bunday tenglamaning umumiy yechimi (agar u mavjud bo'lsa) ikkita ixtiyoriy o'zgarmaslarga bog'liq bo'lib,

$$y = \varphi(x, C_1, C_2) \text{ yoki } \psi(x, y, C_1, C_2) = 0$$

ko'rinishida ifodalanadi.

Bu yechimdan differensial tenglamaning xususiy yechimini keltirib chiqarish uchun izlanayotgan $y = y(x)$ va uning hosilasi $y' = y'(x)$, argument x ning x_0 qiymatida

$$y\Big|_{x = x_0} = y_0, \qquad y'\Big|_{x = x_0} = y'_0$$

bo'lishini bilish yetarli bo'ladi.

1-misol. Ushbu

$$y'' = \frac{1}{1+x^2} + 6$$

differensial tenglamaning

$$y\Big|_{x=0} = 5 \qquad y'\Big|_{x=0} = 2$$

shartlarni qanoatlantiruvchi xususiy yechimini toping.

◀Berilgan tenglamani integrallasak, unda

$$\int y''dx = \int\left(\frac{1}{1+x^2}+6\right)dx$$

bo‘lib,

$$y' = arctgx + 6x + C_1$$

bo‘ladi. Agar

$$y'\Big|_{x=0} = 2$$

bo‘lishini e’tiborga olsak, unda keyingi tenglik

$$y' = arctgx + 6x + 2$$

ko‘rinishga keladi.

Bu tenglamani integrallab:

$$\int y'dx = \int (arctgx + 6x + 2)dx$$

ya’ni

$$y = \int arctgxdx + 3x^2 + 2x + C_2$$

bo‘lishini topamiz.

Ravshanki,

$$\int arctgxdx = \left[\begin{matrix} u = arctgx, & du = \frac{1}{1+x^2}dx \\ dv = dx & v = x \end{matrix}\right] = x \cdot arctgx - \int x \cdot \frac{1}{1+x^2}dx =$$

$$= xarctgx - \frac{1}{2}\int \frac{d(1+x^2)}{1+x^2} = x \cdot arctgx - \frac{1}{2}\ln(1+x^2).$$

Natijada

$$y = xarctgx - \frac{1}{2}\ln(1+x^2) + 3x^2 + 2x + C_2$$

bo‘ladi. Yuqoridagi

$$y\Big|_{x=0} = 5$$

shartdan foydalansak, unda

$$5 = 0 \cdot arctg0 - \frac{1}{2}\ln(1+0) + 3 \cdot 0 + 2 \cdot 0 + C_2$$

ya’ni

$$C_2 = 5$$

bo‘ladi.

Shunday qilib, berilgan tenglamaning ko‘rsatilgan shartni qanoatlantiruvchi xususiy yechimi

$$y = xarctgx - \frac{1}{2}\ln(1+x^2) + 3x^2 + 2x + 5$$

bo‘ladi.▶

2^0. Ikkinchi tartibli differensial tenglamaning ba’zi xususiy hollari va ularni yechish: a) aytaylik,

$$F(x, y, y', y'') = 0 \qquad (1)$$

differensial tenglamada $y = y(x)$ funksiya qatnashmasin. Bu holda (1) tenglama quyidagi

$$F(x, y', y'') = 0 \qquad (2)$$

ko‘rinishga ega bo‘ladi.

Agar (2) tenglamada

$$y' = z \qquad (z = z(x))$$

deyilsa, unda $y'' = z'$ bo‘lib, (2) tenglama z ga nisbatan birinchi tartibli differensial tenglamaga keladi:

$$F(x, z, z') = 0$$

2-misol. Ushbu

$$y'' - \frac{1}{x}y' = xe^x$$

differensial tenglamani yeching.

◀Bu tenglamada

$$y' = z \qquad (z = z(x))$$

deyilsa, unda $y'' = z'$ bo‘lib, berilgan tenglama $z = z(x)$ ga nisbatan quyidagi

$$z' - \frac{1}{x}z = xe^x \qquad (3)$$

birinchi tartibli differensial tenglamaga keladi.

1-§ da keltirilgan (4) formuladan foydalanib, (bu holda $p(x) = -\frac{1}{x}$, $q(x) = xe^x$ bo‘ladi). (3) tenglamaning umumiy yechimi

$$z = (e^x + C_1)x$$

bo‘lishini topamiz. Demak,

$$y' = (e^x + C_1)x.$$

Keyingi tenglikni integrallash natijasida

$$y = \int y'dx = \int (e^x + C_1)xdx = xe^x - e^x + C_1\frac{x^2}{2} + C_2$$

bo‘lishi kelib chiqadi.

Shunday qilib, berilgan differensial tenglamaning umumiy yechimi

$$y = xe^x - e^x + C_1\frac{x^2}{2} + C_2$$

bo‘ladi.▶

b) aytaylik,

$$F(x, y, y', y'') = 0 \qquad (1)$$

differensial tenglamada x erkli o'zgaruvchi qatnashmasin. Bu holda (1) tenglama quyidagicha

$$F(y, y', y'') = 0 \qquad (4)$$

ko'rinishga ega bo'ladi.

Bu holda $z = y'$ ni yangi noma'lum funksiya, y ni esa yangi erkli o'zgaruvchi deyilsa, (4) tenglama ushbu

$$F\left(y, z, z \cdot \frac{dz}{dy}\right) = 0$$

birinchi tartibli differensial tenglamaga keladi.

3-misol. Ushbu

$$y \cdot y'' - 2y'^2 = 0$$

differensial tenglamaning umumiy yechimini toping.

◀Bu tenglamada

$$y' = z$$

deyilsa, unda

$$y'' = \frac{dy'}{dy} \cdot \frac{dy}{dx} = z \cdot \frac{dz}{dy}$$

bo'lib, berilgan tenglama quyidagi:

$$y \cdot z\frac{dz}{dy} - 2z^2 = 0$$

ya'ni

$$z\left(y\frac{dz}{dy} - 2z\right) = 0$$

ko'rinishga keladi. Keyingi tenglikdan

$$z = 0,$$

$$y \cdot \frac{dz}{dy} - 2z = 0$$

bo'lishi kelib chiqadi. Birinchi tenglikdan

$$z = y' = 0, \text{ ya'ni } y = C \qquad (C - const),$$

ikkinchi tenglikdan esa

$$\frac{dz}{z} = \frac{2dy}{y}$$

bo'lishini topamiz. Bu differensial tenglamani integrallasak, unda

$$\ln z = 2\ln y + \ln C_1$$

ya'ni

$$z = C_1 y^2$$

bo'ladi. Ma'lumki,

$$z = \frac{dy}{dx}.$$

Demak,

$$\frac{dy}{dx} = C_1 y^2.$$

Keyingi tenglamani yechamiz:

$$\frac{dy}{y^2} = C_1 dx, \quad \int \frac{dy}{y^2} = C_1 \int dx,$$

$$-\frac{1}{y} = C_1 x + C_2,$$

$$y = -\frac{1}{C_1 x + C_2}. ▶$$

c) aytaylik,

$$F(x, y, y', y'') = 0 \qquad (1)$$

differensial tenglama ushbu

$$y'' = F(y)$$

ko'rinishda bo'lsin. Bu tenglama yuqorida keltirilgan almashtirish natijasida birinchi tartibli differensial tenglamaga keladi va uni yechib berilgan tenglamaning umumiy yechimi topiladi.

4-misol. Ushbu

$$y'' = \frac{3}{2}y^2$$

differensial tenglamaning quyidagi

$$y\Big|_{x=3} = 1, \qquad y'\Big|_{x=3} = 1$$

shartlarni qanoatlantiruvchi yechimini toping.

◀Agar $y' = z, \quad y'' = z \cdot \frac{dz}{dy}$ bo'lishidan foydalansak, unda berilgan tenglama ushbu

$$2zdz = 3y^2 dy$$

ko'rinishga keladi. Bu tenglikning ikki tomonini integrallab

$$z^2 = y^3 + C_1,$$

ya'ni,

$$z = \pm\sqrt{y^3 + C_1}$$

bo'lishini topamiz. Shartga ko'ra $x = 3$ da

$$y=c_1y_1\int e^{-\int\frac{2y_1'+p(x)y_1}{y_1}dx}dx+y_1\int\left[e^{-\int\frac{2y_1'+p(x)y_1}{y_1}dx}\int\frac{f(x)}{y_1}e^{\int\frac{2y_1'+p(x)y_1}{y_1}dx}dx\right]dx+c_2y_1 \quad (9)$$

bo'ladi. ►

6-misol. Ushbu

$$y''-2xy'-2y=2 \quad (10)$$

differensial tenglamaning umumiy yechimini toping.

◄Bu tenglamaga mos bir jinsli differensial tenglama

$$y''-2xy'-2y=0$$

bo'lib, uning bitta yechimi

$$y_1(x)=e^{x^2}$$

bo'ladi. Haqiqatdan ham

$$y_1'=e^{x^2}\cdot 2x,\quad y_1''=e^{x^2}\cdot 4x^2+e^{x^2}\cdot 2=2e^{x^2}+4x^2e^{x^2}$$

va

$$y''-2xy'-2y=2e^{x^2}+4x^2e^{x^2}-2x\cdot 2x\cdot e^{x^2}-2e^{x^2}=0$$

Ravshanki, berilgan differensial tenglama uchun

$$p(x)=-2x,\quad q(x)=-2,\quad f(x)=2,\quad y_1=e^{x^2}$$

bo'ladi. (9) formuladan foydalanib, tenglamaning umumiy yechimini topamiz:

$$y=c_1\cdot e^{x^2}\cdot\int e^{-\int\frac{2\cdot 2xe^{x^2}-2xe^{x^2}}{e^{x^2}}dx}dx+e^{x^2}\int\left[e^{-\int\frac{2\cdot 2xe^{x^2}-2xe^{x^2}}{e^{x^2}}dx}\cdot\int\frac{2}{e^{x^2}}e^{\int\frac{2\cdot 2xe^{x^2}-2xe^{x^2}}{e^{x^2}}dx}dx\right]dx+$$

$$+c_2e^{x^2}=c_1\cdot e^{x^2}\cdot\int e^{-x^2}dx+e^{x^2}\int\left[e^{-x^2}\int 2dx\right]dx=e^{x^2}\left[c_1\int e^{-x^2}dx+c_2\right]-1.$$

Demak,

$$y''-2xy'-2y=2$$

differensial tenglamaning umumiy yechimi

$$y=e^{x^2}\left[c_1\int e^{-x^2}dx+c_2\right]-1$$

bo'ladi.►

Quyidagi ikkinchi tartibli differensial tenglamalarning umumiy yechimlarini toping

1797. $y''=1-x^2$. **1798.** $y''=\frac{1}{y^3}$. **1799.** $y''=\cos 2x$.

1800. $\frac{d^2y}{dx^2}=e^{-3x}+4$. **1801.** $xy''+y'=0$. **1802.** $y''=1-y'^2$.

1803. $yy''=y'^2$. **1804.** $\frac{d^2y}{dx^2}=\ln x$. **1805.** $y''=\sin x-1$.

1806. $2yy''=1+y'^2$. **1807.** $y''=x\sin x$. **1808.** $x(y''+1)+y'=0$.

1809. $y''=\sqrt{1-y'^2}$. **1810.** $y''-\frac{y'}{x}=xe^x$. **1811.** $yy''-2y'^2=0$.

Quyidagi ikkinchi tartibli differensial tenglamalarning ko'rsatilgan shartlarni qanoatlantiruvchi xususiy yechimlarini toping

1812. $y''=4x^3-2x+1,\quad y(1)=\frac{11}{30},\ y'(1)=2$.

1813. $y''y^2=1,\quad y\left(\frac{1}{2}\right)=1,\ y'\left(\frac{1}{2}\right)=1$.

1814. $yy'+y'^2+yy''=0,\quad y(0)=1,\ y'(-1)=0$.

1815. $xy'=2x-y'',\quad y(1)=\frac{1}{2},\ y'(1)=1$.

1816. $y''=\frac{3}{2}y^2,\quad y(3)=1,\ y'(3)=1$.

1817. $2y(y')^3+y''=0,\quad y(0)=0,\ y'(0)=-3$.

Quyidagi ikkinchi tartibli chiziqli differensial tenglamalarning umumiy yechimlarini toping

1818. $(x-1)y''-xy'+y=0$.

1819. $2y''-y'-y=4xe^{2x}$.

1820. $y''-2y'+y=xe^x$.

1821. $y''+y=x\sin x$.

1822. $y''-\frac{1}{x}y'+\frac{1}{x^2}y=\frac{1}{x}\quad (x>0)$.

1823. $y''+\frac{2x}{1-x^2}y'-\frac{2}{1-x^2}y=0$.

3-§. Ikkinchi tartibli chiziqli o'zgarmas koeffitsiyentli differensial tenglamalar

1⁰. Bir jinssiz hamda bir jinsli differensial tenglamalar. Ushbu

$$y''+ay'+by=f(x), \quad (1)$$

$$y''+ay'+by=0 \quad (2)$$

differensial tenglamalarni qaraymiz, bunda a,b o'zgarmas sonlar, $f(x)$ berilgan funksiya.

Odatda, (1) tenglama bir jinssiz chiziqli o'zgarmas koeffitsiyentli tenglama, (2) tenglama esa bir jinsli chiziqli o'zgarmas koeffitsiyentli differensial tenglama deyiladi.

Ushbu

$$k^2+ak+b=0 \qquad (3)$$

kvadrat tenglama (2) bir jinsli differensial tenglamaning xarakteristik tenglamasi deyiladi.

2^0. Bir jinsli differensial tenglamaning umumiy yechimlari. Aytaylik,

$$y''+ay'+by=0 \qquad (2)$$

bir jinsli differensial tenglama berilgan bo'lib,

$$k^2+ak+b=0$$

kvadrat tenglama uning xarakteristik tenglamasi bo'lsin. Bu xarakteristik tenglamaning ildizlarini k_1 va k_2 deylik:

1) agar $k_1=k_2$ bo'lsa, (2) differensial tenglamaning umumiy yechimi

$$y=e^{k_1x}(c_1+c_2x) \qquad (4)$$

bo'ladi;

2) agar $k_1\neq k_2$ bo'lsa, (2) differensial tenglamaning umumiy yechimi

$$y=c_1xe^{k_1x}+c_2e^{k_2x} \qquad (5)$$

bo'ladi;

3) agar $k_1=\alpha+i\beta$, $k_2=\alpha-i\beta$ (kompleks sonlar) bo'lsa, (2) differensial tenglamaning umumiy yechimi

$$y=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x) \qquad (6)$$

bo'ladi.

1-misol. Ushbu

$$y''-2y'-8y=0$$

bir jinsli differensial tenglamaning umumiy yechimi toping.

◀Bu differensial tenglamaning xarakteristik tenglamasi

$$k^2-2k-8=0$$

bo'lib, uning ildizlari $k_1=4$, $k_2=-2$ bo'ladi.

(5) formuladan foydalanib, berilgan differensial tenglamaning umumiy yechimi

$$y=c_1e^{4x}+c_2e^{-2x}$$

bo'lishini topamiz.▶

2-misol. Ushbu

$$y''-10y'+25y=0$$

bir jinsli differensial tenglamaning umumiy yechimini toping.

◀Berilgan differensial tenglamaning xarakteristik tenglamasi

$$k^2-10k+25=0$$

bo'lib, bu kvadrat tenglamaning ildizlari $k_1=k_2=5$ bo'ladi.

Demak, xarakteristik tenglama karrali ildizga ega. (4) formuladan foydalanib, berilgan differensial tenglamaning umumiy yechimi

$$y=c_1e^{5x}+c_2xe^{5x}=e^{5x}(c_1+c_2x)$$

bo'lishini topamiz.▶

3-misol. Ushbu

$$y''-6y'+13y=0$$

bir jinsli differensial tenglamaning umumiy yechimini toping.

◀Berilgan differensial tenglamaning xarakteristik tenglamasi

$$k^2-6k+13=0$$

bo'lib, bu kvadrat tenglamaning ildizlari

$$k_1=3+2i, \qquad k_2=3-2i$$

bo'ladi.

Endi $\alpha=3$, $\beta=2$ bo'lishini e'tiborga olib, (6) formuladan foydalanib, berilgan differensial tenglamaning umumiy yechimi

$$y=e^{3x}(c_1\cos 2x+c_2\sin 2x)$$

bo'lishini topamiz.▶

3^0. Bir jinssiz differensial tenglamaning umumiy yechimlari

Aytaylik,

$$y''+ay'+by=f(x) \qquad (1)$$

bir jinssiz differensial tenglama berilgan bo'lib, unga mos bir jinsli tenglama differensial tenglama

$$y''+ay'+by=0 \qquad (2)$$

ning xarakteristik tenglamasi

$$k^2+ak+b=0$$

bo'lsin.

1. Xarakteristik tenglamaning ildizi karrali bo'lsin: $k_1=k_2$. Bu holda (1) bir jinssiz differensial tenglamaning umumiy yechimi

$$y=c_1xe^{k_1x}+c_2e^{k_1x}+xe^{k_1x}\int f(x)e^{-k_1x}dx-e^{k_1x}\int xf(x)e^{-k_1x}dx \qquad (7)$$

bo'ladi.

2. Xarakteristik tenglamaning ildizi k_1 va k_2 turli va haqiqiy bo'lsin. Bu holda (1) bir jinssiz differensial tenglamaning umumiy yechimi

$$y=c_1e^{k_1x}+c_2e^{k_2x}+\frac{e^{k_2x}}{k_2-k_1}\int f(x)e^{-k_2x}dx+ \\ +\frac{e^{k_1x}}{k_1-k_2}\int f(x)e^{-k_1x}dx \qquad (8)$$

bo'ladi

3. Xarakteristik tenglamaning ildizlari kompleks sonlar bo‘lsin: $k_1=\alpha+i\beta$, $k_2=\alpha-i\beta$

Bu holda (1) bir jinssiz differensial tenglamaning umumiy yechimi

$$y=e^{\alpha x}(a\cos\beta x+b\sin\beta x)+\frac{e^{k_2x}}{k_2-k_1}\int f(x)e^{-k_2x}dx+\frac{e^{k_1x}}{k_1-k_2}\int f(x)e^{-k_1x}dx \quad (9)$$

bo‘ladi.

4-misol. Ushbu

$$y''-3y'+2y=4x^2$$

bir jinssiz differensial tenglamaning umumiy yechimini toping.

◀Avvalo, bu differensial tenglamaga mos bir jinsli

$$y''-3y'+2y=0$$

differensial tenglamaning umumiy yechimini topamiz. Ravshanki, bu tenglamaning xarakteristik tenglamasi

$$k^2-3k+2=0$$

bo‘lib, uning ildizlari $k_1=1$, $k_2=2$ bo‘ladi. Demak, bir jinsli differensial tenglamaning umumiy yechimi (5) formulaga ko‘ra,

$$y=c_1e^x+c_2e^{2x}$$

ga teng.

Endi, (9) formuladan foydalanib, unda $k_1=1$, $k_2=2$ va $f(x)=4x^2$ ekanligini e’tiborga olib, bir jinssiz differensial tenglamaning umumiy yechimi

$$y=c_1e^x+c_2e^{2x}+\frac{e^{2x}}{2-1}\int 4x^2e^{-2x}dx+\frac{e^x}{1-2}\int 4x^2e^{-x}dx$$

bo‘lishini topamiz.

Bo‘laklab integrallash usulidan foydalanib, integrallarni hisoblaymiz:

$$\int e^{-2x}\cdot 4x^2dx=e^{-2x}\left(\frac{4x^2}{-2}-\frac{8x}{4}+\frac{8}{-8}\right)=e^{-2x}(-2x^2-2x-1),$$

$$\int e^{-x}\cdot 4x^2dx=e^{-x}\left(\frac{4x^2}{-1}-\frac{8x}{1}+\frac{8}{-1}\right)=e^{-x}(-4x^2-8x-8)$$

Demak, berilgan bir jinssiz differensial tenglamaning umumiy yechimi

$$y=c_1e^x+c_2e^{2x}+(-2x^2-2x-1)-(-4x^2-8x-8)=$$
$$=c_1e^x+c_2e^{2x}+2x^2+6x+7$$

bo‘ladi.▶

5-misol. Ushbu

$$y''+6y'+9y=xe^{-3x}$$

bir jinssiz differensial tenglamaning umumiy yechimini toping.

◀Bu bir jinsciz differensial tenglamaga mos bo‘lgan bir jinsli

$$y''+6y'+9y=0 \quad (10)$$

differensial tenglamaning umumiy yechimini topamiz. Ravshanki, (10) tenglamaning xarakteristik tenglamasi

$$k^2+6k+9=0$$

bo‘lib, uning ildizlari $k_1=k_2=3$ bo‘ladi. Unda bir jinsli tenglamaning umumiy yechimi

$$y=e^{-3x}(c_1+c_2x)$$

bo‘ladi.

Endi (8) formuladan foydalanib, unda $k_1=-3$, $f(x)=x\cdot e^{-3x}$ ekanligini e’tiborga olib, berilgan bir jinssiz differensial tenglamaning umumiy yechimi

$$y=e^{-3x}(c_1+c_2x)+xe^{-3x}\int xe^{-3x}e^{3x}dx-e^{-3x}\int x\cdot xe^{-3x}e^{3x}dx=$$
$$=e^{-3x}(c_1+c_2x)+\frac{x^2}{2}e^{-3x}-\frac{x^3}{3}e^{-3x}=e^{-3x}\left(c_1+c_2x+\frac{x^3}{6}\right)$$

bo‘lishini topamiz.▶

4°. Populyatsiya miqdorining dinamikasi

Populyatsiya miqdorining dinamikasi (ya’ni, populyatsiya davrida tug‘ilish va o‘lish natijasida tirik mavjudotlar miqdorining o‘zgarishi) ekologiyaning muhim masalalaridan biri. Birinchi tartibli differensial tenglamalarni o‘rganishda yuqoridagi masalaning eng oddiy holatini qaradik. Oziq-ovqat bilan ta’minlangan va tashqi muhit ta’siridan chegaralangan holda populyatsiya dinamikasi quyidagi differensial tenglama bilan berildi:

$$\frac{dx}{dt}=\gamma x \quad (1)$$

Bu yerda $x=x(t)-t$ vaqtdagi populyatsiya miqdori. Bu tenglamaning yechimi quyidagicha:

$$x=x_0e^{\gamma(t-t_0)}$$

ekanligi kelib chiqdi.

Bu yerda x_0, t_0‘ boshlang‘ich vaqtdagi populyatsiya miqdori. (1) tenglama yoki nazariy ahamiyatga ega yoki sun’iy sharoitdagi mavjudotlarning populyatsiyasini aniqlaydi.

Populyatsiyaning rivojlanishini 1845-yilda Ferxyulsta-Perla tenglamasi aniqroq yoritib beradi. Bu tenglama mavjudotlarning ichki qarama-qarshiliklarini hisobga oladi, bu esa populyatsiya miqdorining tezligini sekinlashtiradi. Bu qarama-qarshiliklarga oziq-ovqat uchun kurash, jips yashaganda infeksiya tarqalishi va h.k. kiradi. Yuqoridagi faktlarni hisobga olib, Δx o‘sishni hisoblashda $\gamma x\Delta t$ qiymatdan $h(x,\Delta t)$ qiymatni ayiramiz:

$$\Delta x=\gamma x\Delta t-h(x,\Delta t)$$

$h(x,\Delta t)$ funksiya o‘rniga ko‘pchilik hollarda $\delta x^2 \Delta t$ populyatsiyani qaraymiz:

$$h(x,\Delta t)=\delta x^2 \Delta t$$

bu yerda: δ – koeffitsiyent ichki qarama-qarshiliklar.

$h(x,\Delta t)$ qiymat – bu ichki qarama-qarshiliklar evaziga populyatsiya miqdori tezligining kamayishini ifodalaydi. qarama-qarshiliklar qancha yuqori bo‘lsa, urug‘lanadiganlarning bir-biri bilan uchrashishi shuncha ko‘p, bu uchrashishlar soni $x \cdot x$ ko‘paytmaga to‘g‘ri proporsional, ya’ni x^2. Ikki xil mavjudotning uchrashishi xy ga to‘g‘ri proporsional. Bu turlar bir-birining joyini egallashi mumkin.

Shunday qilib,

$$\Delta x = \nu x \Delta t - \delta x^2 \Delta t \tag{2}$$

bu tenglikni Δt ga bo‘lamiz.

$$\frac{\Delta x}{\Delta t} = \gamma x - \delta x^2$$

va $\Delta t \to 0$ da limitga o‘tamiz.

$$\frac{dx}{dt} = \gamma x - \delta x^2 \tag{3}$$

bu tenglama **Ferxyulsta-Perla tenglamasi**.

Bu tenglamadan γx ni qavsdan chiqarib, quyidagicha yozamiz:

$$\frac{dx}{dt} = \gamma x\left(1-\frac{\delta}{\gamma}x\right)$$

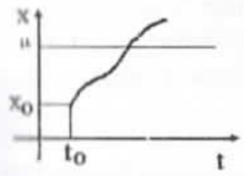
yoki

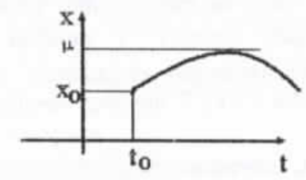
$$\frac{dx}{dt} = \gamma x \cdot \frac{\frac{\nu}{\delta}-x}{\frac{\gamma}{\delta}}$$

$\frac{\gamma}{\delta}=\mu$ deb belgilaymiz:

$$\frac{dx}{dt} = \gamma x \frac{\mu - x}{\mu} \tag{5}$$

Agar $x_0 < \mu$ bo‘lsa, u holda barcha $t > t_0$ vaqt uchun haqiqatan ham $x(t)$ differensiallanuvchi funksiya. (5) tenglamadan kelib chiqadiki, $x(t) < \mu$ da $\frac{dx}{dt}$ musbat bundan kelib chiqadiki, $x(t)$ o‘sadi. Bundan shuni xulosa qilish mumkinki, $x(t)$ μ ga teng qiymatni qabul qilsa o‘sadi. $x=\mu$ to‘g‘ri chiziqni kesib o‘tadi (1-chizma) yoki $x=\mu$ to‘g‘ri chiziqqa urinadi (2-chizma).





1- **2-chizma**

Birinchi holda: $x(t)>\mu$ va $x'(t)>0$. Bu (5) tenglamaga zid. Ikkinchi holda, $x(t)<\mu$ va $x'(t)<0$, bu ham (5)-tenglamaga zid. Shunday qilib, $x(t)$ μ ga teng bo‘lishi mumkin emas,agar $x_0<\mu$ bo‘lsa.

O‘zgaruvchilarni ajratib,

$$\frac{\mu dx}{x(\mu - x)} = \gamma dt$$

yoki

$$\frac{(\mu - x)+x}{x(\mu - x)}dx = \gamma dt$$

bundan

$$\left(\frac{1}{x}+\frac{1}{\mu - x}\right)dx = \gamma dt$$

Agar $x_0<\mu$ deb hisoblasak, quyidagiga ega bo‘lamiz:

$$\ln x - \ln(\mu - x) = \gamma t + \ln C$$

bundan

$$\frac{x}{\mu - x} Ce^{\gamma t} \tag{6}$$

qulaylik uchun $t_0=0$ va $x(0)=x_0<\mu$ deb olib, (6) ga qo‘ysak,

$$C=\frac{x_0}{\mu - x_0}$$

topilgan qiymatni (6) ga qo‘ysak, quyidagini hosil qilamiz:

$$\frac{x}{\mu - x} = \frac{x_0}{\mu - x_0}e^{\gamma t}.$$

Bundan qidiralayotgan Ferxyulsta-Perla modelini hosil qilamiz:

$$x=\frac{x_0 \mu e^{\gamma t}}{\mu - x_0 + x_0 e^{\gamma t}}$$

Epidemiya nazariyasida differensial tenglama

Epidemiyaning eng oddiy turini qaraymiz. Aytaylik, o‘rganilayotgan kasallik uzoq vaqt davom etadi, demak, infeksiya tarqalishi kasallanishga nisbatan tezroq tarqaladi. Biz infeksiya tarqalish holati bilan chegaralanamiz.

Aytaylik, a va n lar mos ravishda infeksiya yuqtirganlar va yuqtirmaganlar sonini aniqlasin. $x=x(t)$ t vaqtdagi sog'lom organizmlar, $y=y(t)$ t vaqtdagi infeksiyalangan organizmlar soni. Uncha katta bo'lmagan vaqt oralig'ida, ya'ni $0 \le t \le T$ quyidagi tenglik o'rinli:

$$x+y=n+a \tag{9}$$

Demak, uchrashish chog'ida infeksiya yuqtirilgan organizmdan sog'lom organizmga o'tadi, u holda sog'lom organizmlar soni vaqt o'tishi bilan kamayadi va u uchrashuvlar soniga proporsional bo'ladi (ya'ni, $x \cdot y$ ga proporsional). Bundan sog'lom organizmlar soni kamayadi va kamayish tezligi quyidagiga teng:

$$\frac{dx}{dt}=-\beta \cdot xy \tag{10}$$

bu yerda: β – proporsionallik koeffitsiyenti. Bundan y ni topib, (9) ga qo'ysak:

$$\frac{dx}{dt}=-\beta \cdot x(n+a-x).$$

O'zgaruvchilarga ajratib quyidagini topamiz:

$$\frac{dx}{x(n+a-x)}=-\beta \cdot dt$$

yoki

$$\frac{(n+a-x)+x}{x(n+a-x)}dx=-\beta(n+a)\cdot dt \cdot$$

Bundan

$$\frac{dx}{x}+\frac{dx}{(n-x+a)}=-\beta(n+a)\cdot dt \cdot$$

Integrallab, quyidagini hosil qilamiz:

$$\ln x-\ln(n-x+a)=-\beta(n+a)t+\ln C$$

yoki

$$\ln\frac{x}{(n-x+a)}=Ce^{-\beta(n+a)t}$$

C ni topish uchun quyidagi boshlang'ich shartdan foydalanamiz. Agar $t=0$ da sog'lom organizmlar soni n bo'lsa(ya'ni $x=n$). Bundan $C=\frac{n}{a}$ ekanligi kelib chiqadi va

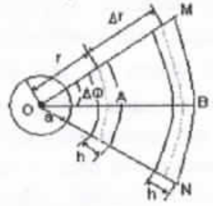
$$\frac{x}{n-x+a}=\frac{n}{a}e^{-\beta(n+a)t}.$$

Demak, qidirayotgan ifodamiz quyidagiga teng ekan:

$$x=\frac{n(n+a)}{n+ae^{\beta(n+a)t}}$$

Chumoli inidan tashqarida chumolining zichligi

Chumoli hayotining umumiy yashash belgilari ko'pchilikka ma'lum. Chumoli topilgan oziq-ovqatni yoki qurilish materiallarini topilgan joyidan uyasiga tashiydi. Shuning uchun chumoli soni uyasining yaqinida uyidan uzoqroqqa nisbatan ko'proq (3-chizma).



3-chizma

Oddiylik uchun chumoli uyasining markazi deb radiusi a ga teng doirani olamiz va bu doiradan tashqarida chumolilar uchun oziq-ovqat bir xilda taqsimlangan deb hisoblaymiz. Bu degani radiusi r ga teng aylanadagi muhit undan tashqarida ham bir xil. Shuning uchun radiusi r ga teng aylanada $(r>a)$ chumoli zichligi bir xil taqsimlangan deb faraz qilamiz (hasharotlar zichligi deganda ma'lum atrofdagi hasharotlar sonining shu atrof yuziga nisbatiga aytiladi). Bundan kelib chiqadiki, zichlik bu r masofaning funksiyasi bo'lib, nuqtalari bitta nurda yotgan nuqtalar bilan chegaralansak kifoya.

Soddalik uchun statsionar holatni, ya'ni vaqt o'tishi bilan zichlik har bir nuqtada o'zgarmas deb hisoblaymiz. Ammo bu degani, chumoli bir-biri bilan aralashib ketmaydi degani emas. Oziq-ovqat qidirishda chumoli bir joydan ikkinchi joyga ko'chib yuradi. Biz bu qidirishni tasodifiy deb hisoblaymiz, ya'ni birlik vaqt ichida ovqat qidirib biror atrofni tark etsa, xuddi shuncha sondagi chumolilar shu atrofga qo'shni atrofdan o'tadi. Shunday qilib, chumoli uyasi atrofidan ovqat izlab boshqa joyga ko'chishsa, yana shu sondagi chumolilar uyasidan chiqadi.

Bir nurda yotgan ikkita qo'shni nuqtani qaraymiz: A nuqta chumolilar uyasining markazidan r masofada joylashgan bo'lsin, B nuqta $r+\Delta r$ masofada bo'lsin. Bu nuqtalar atrofida chumolilarning ko'chishini kuzatamiz. Aytaylik, $Q(r+\Delta r)$ – B nuqtadagi chumolilar soni. Chumoli ovqat qidirayotganda birorta yo'nalish ikkinchisidan yaxshi bo'la olmaydi. Chunki atrof bir xil deb hisoblangan. Chumoli soni uning qaysi yo'nalishdan ovqat qidirishiga bog'liq emas. Bundan kelib chiqadiki, agarda A nuqta atrofidan B nuqtaga qarab $\alpha Q(r)$, $\alpha<1$ chumolilar chiqqan bo'lsa, u holda B nuqta atrofidan A nuqtaga $\alpha Q(r+\Delta r)$ sondagi chumolilar chiqishdi.

Bunda, eng muhimi, ikkala holatda ham α sonining bir xil ekanligidadir. α fiksirlangan yo'nalishdagi chumolilar qismini aniqlaydi. Ammo A nuqta atrofidan B nuqta atrofiga chiqqan chumolilarning barchasi ham B nuqtaga yetib bormaydi, chunki yo'lda ovqatni topganlari uyasiga qaytib ketadi. Bu toifadagi chumolilar A va B nuqtalar orasidagi masofa qancha katta bo'lsa shuncha ko'p bo'ladi. Shuning uchun A nuqta atrofidan chiqqan B nuqta atrofiga yetib boradigan chumolilar soni quyidagi ayirma bilan ifodalanadi:

$$Q_{AB} = \alpha Q(r) - \beta\alpha Q(r)\Delta r$$

Bu yerda β-proporsianallik koyeffitsiyenti (manzilga yetmasdan qaytgan chumolilarni aniqlaydi). Bu koeffitsiyent atrofga bog'liq, lekin atrofda sharoit bir xil bo'lganligi uchun β shu atrofda o'zgarmas. $\alpha Q(r+\Delta r)$ qiymatga B nuqta atrofidan A nuqta atrofiga emas, boshqa yo'nalishga chiqib, yo'ldan ovqat topib orqaga qaytganlari ham kiradi. Bunday chumolilarga OMN sektordan hali chiqib ulgurmaganlari ham A nuqta atrofiga kiradi. Δr qancha katta bo'lsa, ular shuncha ko'p bo'ladi. Shunday qilib, B nuqta atrofidan chiqib, A nuqta atrofiga tushadiganlari, quyidagi yig'indi orqali aniqlanadi.

$$Q_{AB} = \alpha Q(r+\Delta r) + \beta\alpha Q(r+\Delta r)\Delta r$$

bu yerda: β_1-chumoli uyasiga qaytgan chumolilar sonini aniqlaydigan proporsionallik koyeffitsiyenti. Biz statsionar holatda bo'lganimiz uchun, ya'ni nuqta atrofida o'zgarmas sonda qolgani uchun quyidagi tenglik bajariladi.

$$Q_{AB} = Q_{BA}$$

ya'ni,

$$\alpha Q(r) - \beta\alpha Q(r)\Delta r = \alpha Q(r+\Delta r) + \beta_1\alpha Q(r+\Delta r)\Delta r$$

Bundan biz differensial tenglamaga kelamiz. Chumolilar soni uning zichligining yuza ko'paytmasiga teng bo'lgani uchun (α ga qisqartirib) oxirgi tenglikni quyidagicha yozamiz:

$$n(r)S_A - \beta n(r)S_A\Delta r = n(r+\Delta r)S_B + \beta_1 n(r+\Delta r)S_B\Delta r \qquad (12)$$

bu yerda: $n(r)$ va $n(r+\Delta r)$ lar A va B nuqtalardagi mos ravishda zichligini belgilaydi; S_A va S_B shu nuqta atrofining yuzalari.

Yuzani qutb koordinatalar sistemasida hisoblab, quyidagini hosil qilamiz:

$$S_A \approx hr\Delta\Phi; \quad S_B \approx h(r+\Delta r)\Delta\Phi.$$

Buni (12) tenglikka qo'yamiz:

$$n(r)hr\Delta\Phi - \beta n(r)hr\Delta\Phi\Delta r = n(r+\Delta r)h(r+\Delta r)\Delta\Phi + \beta_1 n(r+\Delta r)h(r+\Delta r)\Delta\Phi\Delta r$$

buni $h\Delta\Phi$ ga qisqartirib gruppalasak, quyidagini hosil qilamiz:

$$n(r+\Delta r)(r+\Delta r) - n(r)r = -\left[\beta n(r)r + \beta_1 n(r+\Delta r)(r+\Delta r)\right]\Delta r.$$

Bu tenglikni Δr ga bo'lib, $\Delta r \to 0$ intiltirsak

$$\frac{d}{dr}(r\cdot n(r)) = -(\beta_1+\beta)\cdot rn(r) \qquad (13)$$

Qisqalik uchun $\beta_1 + \beta = \gamma$ deb belgilaymiz va

$$\frac{d(r\cdot n(r))}{r\cdot n(r)} = \gamma\cdot dr. \qquad (14)$$

Biz $n(r)$ zichlik uchun differensial tenglama hosil qildik.

Aytaylik, $n(a)$–chumoli uyasining chegarasidagi ($r=a$) zichlik qiymati – (14)ni integrallab, quyidagini hosil qilamiz:

$$\ln\left[r\cdot n(r)\right] = -\gamma r + C. \qquad (15)$$

Boshlang'ich shartdan foydalanib, $C = \ln\left[a\cdot n(a)\right] + \gamma a$ ni hosil qilamiz.

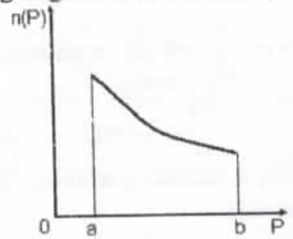
Buni (15) ga qo'yib,

$$\ln\frac{r\cdot n(r)}{a\cdot n(a)} = -\gamma(r-a)$$

Bundan

$$n(r) = \frac{a}{r}n(a)e^{-\gamma(r-a)}. \qquad (16)$$

Bu – qidirilayotgan egri chiziq tenglamasi. Agar a, $n(a)$ va γ larning qiymatlari ma'lum bo'lsa,grafigini chizish mumkin.(4-chizma).



4-chizma

r o'sishi bilan egri chiziq kamayadi. a va $n(a)$ larni tajriba yo'li bilan topish qiyin emas. Ammo γ koeffitsiyentni hisoblash qiyinroq. Agarda tuzilgan matematik modelni to'g'ri deb hisoblasak, bundan kelib chiqadiki, qandaydir γ ni berilgan deb (16) dan zichlikning masofaga bog'liqligidan γ ni hisoblasak bo'ladi. Agar (16) o'rinli bo'lsa, u holda $r > a$ uchun ham o'rinli, xususan, $r = b$ uchun

$$n(b) = \frac{a}{b}n(a)e^{-\gamma(b-a)},$$

bundan

$$\gamma = \frac{1}{b-a}\ln\frac{a\cdot n(a)}{b\cdot n(b)} \qquad (17)$$

γ ni hisoblash uchun n ni ham $n(a)$ kabi (16) formuladan hisoblashimiz kerak. Buning uchun quyidagilarni hisobga olishimiz kerak.

A) (16) formuladan ixtiyoriy katta r uchun $n(r) \neq 0$. Haqiqiy hayotda, albatta, bunday emas;

B) γ qiymat vaqtga, albatta, bog‘liq, chunki kyechasi va kunduzi quyosh aktiv vaqtda chumoli boshqa vaqtga nisbatan aktivligi kam bo‘ladi.

Ammo kunning har xil vaqtda $n(a)$ va $n(b)$ qiymatlarni (17) formulaga asoslanib hisoblab, γ ni (16) formulaga qo‘yib, har xil vaqtda o‘zining egri chizig‘ini topa olamiz.

O‘simlik bargining o‘sishi

Tuzilishi doira shaklida bo‘lgan yosh yaproq yuzasining o‘sish tezligi yaproq aylanasi uzunligi va unga tushgan quyosh nuri miqdoriga to‘g‘ri proporsional. Bu esa quyosh nuri va yaproq orasidagi burchak kosinusi va yaproq yuzasiga to‘g‘ri proporsional. Agar ertalab soat 6^{00} da yaproq yuzasi 1600 sm^2 va kech soat 18^{00} da shu kuni 2500 sm^2 bo‘lsa, yaproq yuzi S bilan t vaqt orasidagi bog‘lanishni toping.

Quyosh nuri va yaproq orasidagi burchakni, ya’ni soat 6^{00} va 18^{00} ni (ishorasini hisobga olmagan holda) 90^0 ga teng, kun yarmida 0^0 ga teng deb qabul qilamiz.

Aytaylik, t vaqt yarim tun 00 dan boshlansin. Agar yaproq yuzasi S o‘zgarsa, u holda yaproq o‘sishining tezligi

$$\frac{ds}{dt} = k_1 2\pi\gamma Q$$

bu yerda: $2\pi\gamma$ –yaproq aylanasining uzunligi, Q –yoruglik nurining soni, k_1 –proporsionallik koeffitsiyenti.

Yaproq yuzasi $S = \pi\gamma^2$ dan quyidagini yozib olamiz:

$$r = \sqrt{\frac{S}{\pi}}.$$

U holda

$$\frac{ds}{dt} = k_1 \frac{2\pi}{\sqrt{\pi}}\sqrt{S}Q \qquad (18)$$

Quyidagi shartdan:

$$Q = k_2 S\cos\alpha \qquad (19)$$

(bu yerda: α –nur va vertikal orasidagi burchak, k_2 –proporsionallik koeffitsiyenti) α burchak t argumentning chiziqli o‘suvchi funksiyasi ekanligi kelib chiqadi:

$$\alpha = k_3 t + b$$

k_3 va b parametrlarni qo‘shimcha shartlar asosida topamiz:

agar $t = 6$ bo‘lsa, $\alpha = -\frac{\pi}{2}$,

agar $t = 12$ bo‘lsa, $\alpha = 0$,

agar $t = 18$ bo‘lsa, $\alpha = \frac{\pi}{2}$.

Oxirgi ikkita shartdan quyidagiga ega bo‘lamiz:

$$\begin{cases} 0 = 12k_3 + b \\ \frac{\pi}{2} = 18k_3 + b \end{cases}$$

bu sistemani yechib,

$$k_3 = \frac{\pi}{12},\ b = -\pi.$$

ni topamiz. Bundan

$$\alpha = \frac{\pi}{12}(t-12)$$

buni (19) ga qo‘yamiz.

$$Q = k_2 S\cos\left[\frac{\pi}{12}(t-12)\right]$$

buni (18) ga qo‘yamiz.

$$\frac{ds}{dt} = k_1 k_2 \frac{2\pi}{\sqrt{\pi}} S\sqrt{S}\cos\left[\frac{\pi}{12}(t-12)\right]$$

$k_1 \cdot k_2 = k$ deb belgilasak. U holda o‘zgaruvchilarni ajratsak,

$$\frac{dS}{S\sqrt{S}} = k\frac{2\pi}{\sqrt{\pi}}\cos\left(\frac{\pi}{12}(t-12)\right)dt$$

buni integrallab,

$$-\frac{2}{\sqrt{S}} = \frac{24k}{\sqrt{\pi}}\sin\left[\frac{\pi}{12}(t-12) + C\right] \qquad (20)$$

$t = 6$ da $S = 1600$ va $t = 18$ da $S = 2500$ shartlar bilan

$$\begin{cases} -\frac{1}{20} = -\frac{24k}{\sqrt{\pi}} + C \\ -\frac{1}{25} = \frac{24k}{\sqrt{\pi}} + C \end{cases}$$

Bu sistemani yechib:

$$C = -\frac{9}{200},\ k = \frac{\sqrt{\pi}}{24\cdot 200}$$ ni

topamiz. Buni (20) ga qo‘yib, quyidagiga ega bo‘lamiz:

$$-\frac{2}{\sqrt{S}} = \frac{24\sqrt{\pi}}{24\cdot 200\sqrt{\pi}}\sin\left[\frac{\pi}{12}(t-12)\right] - \frac{9}{200}$$

Bundan esa

$$S = \frac{160000}{\left\{9 - \sin\left[\frac{\pi}{12}(t-12)\right]\right\}^2}$$

ni topamiz.

Daraxt o'sishini hisoblash haqidagi masala

Nega sharoit eng yaxshi bo'lganda ham daraxt ma'lum uzunlikdan oshmaydi? Nega daraxt turiga bog'liq bo'lmagan holda boshlang'ich vaqtda tez o'sadi, so'ngra o'sishi sekinlashib, asta-sekin o'sishi nolga teng bo'ladi?

Biroq biz bilamizki, daraxt tomirlarining o'sishi fotosintez yordamida energiyasi ko'payishiga olib keladi, ammo oziqlanish borgan sari qiyinlashib, bora-bora energiya yetishmay boshlaydi va daraxt o'sishdan to'xtaydi.

Shu fikrlarga asoslanib, energiya balansining taxminiy tenglamasini tuzamiz, ya'ni matematik modelini tuzamiz.

1. O'sayotgan daraxt o'sish davrida geometrik xususiyatini saqlaydi, ya'ni uzunligining daraxt diametriga nisbati o'zgarmaydi.

2. Erkin energiyani (yoki harakatdagi moddani) faqat fotosintez orqali oladi.

3. Erkin energiya tirik to'qima hosil qilishga va tuproqdan aralashmalarning ko'tarilishiga sarf bo'ladi. O'rtacha hisobda katta vaqt oralig'ida birlik sirt yuzasiga o'zgarmas miqdorda yorug'lik tushadi va tarkibidagi moddalardan bir qismini yutishi mumkin.

Aytaylik, x – daraxtning chiziqli o'lchami. Bu degani daraxt balandligini x orqali, yaproqning yuzasini x^2 orqali va nihoyat daraxt hajmini x^3 orqali belgilaymiz. x ning o'zgarishini t orqali, ya'ni $x = x(t)$ orqali ifodalashga harakat qilamiz.

Aytaylik, $x(t_0) = 0$ bo'lsin. Balans tenglamasini x bo'yicha ifodalasak, E erkin energiya daraxt tanasining yashil qismidan fotosintez orqali hosil bo'ladi, yashil qismi qanchalik ko'p bo'lsa, shuncha energiya ko'p bo'ladi. Shunday qilib, E x^2 ga to'g'ri proporsional

$$E = \alpha x^2$$

bu yerda: α – proporsionallik koeffitsiyenti (α yaproqning o'lchamiga va tuzilishiga hamda fotosintezga bog'liq). Bizning farazimizga ko'ra, boshqa energiya beruvchi omillar yo'q va biz energiyaning taqsimlanishini kuzatishimiz kerak. Energiya birinchidan, fotosintez sodir bo'lishi uchun sarflanadi. Bu sarflanish ham x^2 ga to'g'ri proporsional, ya'ni βx^2, bu yerda β proporsionallik koeffitsiyenti α dan kichik.

Energiya ozuqaning butun daraxt tanasiga tarqalishi uchun sarflanadi. Ma'lumki, u energiya qancha ko'p sarflansa, tana shuncha katta bo'ladi. Bundan tashqari, bu sarflanish og'irlik kuchini yengishga bog'liq va bundan kelib chiqadiki, agar ozuqani qancha balandga ko'tarib sarflasa, energiya shuncha ko'p sarflanadi. Shunday qilib, energiyaning bu sarfi hajmi x^3 ga va balandlik x ga to'g'ri proporsional, ya'ni $\gamma \cdot x^3 \cdot x$.

Nihoyat, energiya daraxtning massasini oshirishga sarflanadi (ya'ni o'sishiga). Bu sarflanish o'sish tezligiga to'g'ri proporsional, ya'ni $m = \rho x^3$

massadan vaqt bo'yicha hosila (ρ–daraxtning o'rtacha zichligi, x^3–hajmi). Shunday qilib, oxirgi energiya sarflanishi quyidagicha ifodalanadi.

$$\delta \frac{d}{dt}\left(\rho x^3\right)$$

Energiyaning saqlanish qonunidan energiya sarfi quyidagiga teng bo'ladi:

$$E = \beta x^2 + \gamma x^4 + \delta \frac{d}{dt}\left(\rho x^3\right)$$

yoki

$$\alpha x^2 = \beta x^2 + \gamma x^4 + 3\delta \rho x^2 \frac{dx}{dt} \tag{21}$$

bu esa biz qidirayotgan energiya balansining tenglamasi. Bu tenglamani $3\delta\rho x^2$ bo'lib quyidagicha yozamiz:

$$\frac{\alpha - \beta}{3\delta\rho} = a, \quad \frac{\gamma}{3\delta\rho} = b$$

bundan:

$$\frac{dx}{dt} a - bx^2, \quad a > 0, \quad b > 0 \tag{22}$$

kelib chiqadi.

Daraxt o'sayotgan ekan, u holda $\frac{dx}{dt} > 0$ bu demak, $a - bx^2 \geq 0$, bundan $x^2 < \frac{a}{b}$ kelib chiqadi. Shuning uchun (22) ni quyidagicha yozamiz:

$$-\frac{dx}{b\left(x^2 - \frac{a}{b}\right)} = dt$$

buni integrallab,

$$\frac{1}{2\sqrt{ab}} \ln \frac{\sqrt{\frac{a}{b}} + x}{\sqrt{\frac{a}{b}} - x} = t + c$$

boshlang'ich shart $x(t_0) = 0$ dan foydalanib, $c = -t_0$ ni hosil qilamiz.

$$\frac{1}{2\sqrt{ab}} \ln \frac{\sqrt{\frac{a}{b}} + x}{\sqrt{\frac{a}{b}} - x} = t + t_0 \tag{23}$$

bu tenglamani x ga nisbatan yechamiz:

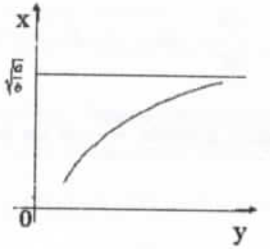
$$x=\sqrt{\frac{a}{b}}\cdot\frac{1-e^{-2\sqrt{ab}}(t-t_0)}{1+e^{-2\sqrt{ab}}(t-t_0)} \qquad (24)$$

Bu daraxt o'sishini aniqlaydigan formula.

Agar a,b va t_0 qiymatlari ma'lum bo'lsa, bu formula yordamida vaqtga nisbatan o'rtacha o'sishini hisoblash mumkin. (24) formula bilan berilgan egri chiziqni tekshirish mumkin. $\frac{dx}{dt}>0$ ligidan va (22) formuladan quyidagini hosil qilamiz:

$$\frac{d^2x}{dt^2}=-2bx$$

Shunday qilib, agar $t>t_0$, $x(t)>0$ (bu daraxt balandligi) bo'lsa, u holda oxirgi tenglikdan $\frac{d^2x}{dt^2}<0$ ga kelamiz. Demak, (24) chiziq o'suvchi qavariq chiziq. (24) formuladan $t\to+\infty$ da $x(t)\to\sqrt{\frac{a}{b}}$ hosil bo'ladi. Grafigi



5-chizma

quyidagicha bo'ladi:

$\sqrt{\frac{a}{b}}$ balandlik daraxt o'sishining chegarasi, chunki energiya daraxtning fotosintez va ozuqa bilan ta'minlanishiga sarflanadi. Daraxt shuning uchun bu ko'rsatkichdan yuqori o'smaydi.

(24) formula bilan berilgan egri chiziq daraxt o'sishini qanchalik to'g'ri ifodalaydi? Bu savolga javob berish uchun dub daraxtini olib ko'ramiz. Yoshi 40 yildan 220 yilgacha bo'lgan dub daraxti uzunligi (24) formulada tekshirildi. a va b o'zgaruvchili ikkita tenglamalar sistemasi hosil bo'ladi va bu o'zgaruvchilarni topish mumkin. Topilgan a va b qiymatga qarab aniq egri chiziq chizildi. Bu egri chiziq tajribadagi dub daraxt o'sishining egri chizig'i bilan ustma-ust tushdi. Boshqacha qilib aytganda, $(40:x(40))$ va $(220:x(220))$ nuqtadagi nazariy va tajribadagi egri chiziqlar ustma-ust tushdi. Demak, ko'rilgan matematik model ishonarli.

Quyidagi ikkinchi tartibli chiziqli o'zgarmas koeffitsiyentli differensial tenglamalarning umumiy yechimlarini toping

1824. $y''-9y=0$. **1825.** $y''+15y'=0$. **1826.** $y''+49y=0$.

1827. $y''+y'-2y=0$. **1828.** $y''+2y'+2y=0$.

1829. $2y''-3y'-2y=0$. **1830.** $y''-4y'+13y=0$.

1831. $y=y''+y'$. **1832.** $\frac{y'-y}{y''}=3$. **1833.** $y''+6y'+25y=0$.

1834. $y''+y'=\frac{1}{2}$. **1835.** $y''-5y'+6y=3$. **1836.** $y''+9y=2x$.

1837. $y''+2y=e^{-x}$. **1838.** $y''-y=e^{2x}$. **1839.** $y''-5y'+6y=x^2$.

1840. $y''+4y=\sin x$. **1841.** $y''-4y'+3y=10e^{3x}$.

1842. $y''-5y'=30x-11$. **1843.** $y''+y'-2y=8\sin 2x$.

1844. $y''+y'-6y=xe^{2x}$.

Quyidagi ikkinchi tartibli chiziqli differensial tenglamalarning ko'rsatilgan shartni qanoatlantiruvchi xususiy yechimlarini toping.

1845. $y''+3y'=0$, $y(0)=1$, $y'(0)=3$.

1846. $y''+4y'=0$, $y(0)=0$, $y'(0)=2$.

1847. $y''-y'+2y=0$, $y(0)=0$, $y'(0)=1$.

1848. $y''+4y'+29y=0$, $y(0)=0$, $y'(0)=15$.

1849. $y''-2y'+10y=0$, $y(0)=1$, $y'(0)=0$.

1850. $y''+4y'=0$, $y\left(\frac{\pi}{2}\right)=1$, $y'\left(\frac{\pi}{2}\right)=-1$.

1851. $y''-6y'+9y=0$, $y(0)=2$, $y'(0)=7$.

1852. $y''-2y'+2y=0$, $y(0)=0$, $y'(0)=1$.

1853. $y''-2y'+3y=0$, $y(0)=1$, $y'(0)=3$.

1854. $y''-2y'+2y=2x$, $y(0)=0$, $y'(0)=0$.

1855. $y''-9y'=2-x$, $y(0)=0$, $y'(0)=1$.

1856. $y''+4y'=2\cos 2x$, $y(0)=0$, $y'(0)=4$.

1857. $y''-2y'+10y=74\sin 3x$, $y(0)=6$, $y'(0)=3$.

1858. $y''-6y'=18e^{6x}$, $y(0)=1$, $y'(0)=-9$.

1859. $y''+9y'=15\sin 2x$, $y(0)=-7$, $y'(0)=0$.

1860. $y''+y'=-8\sin x-6\cos x$, $y\left(\frac{\pi}{2}\right)=-\frac{\pi}{2}$, $y'\left(\frac{\pi}{2}\right)=-2\pi$.

1861. $y''=\frac{y}{a^2}$, $y(0)=a$, $y'(0)=0$.

Darajali qatorlar yordamida quyidagi differensial tenglamalarning ko'rsatilgan shartlarni qanoatlantiruvchi xususiy yechimlarini toping.

1862. $y'' + y' - xy^2 = 0, \quad y(0)=2, \; y'(0)=1.$

1863. $y'' + xy = 0, \quad y(0)=1, \; y'(0)=0.$

1864. $y'' + y \cdot e^x = 0, \quad y(0)=2, \; y'(0)=1.$

1865. $y'' - xy = 0, \quad y(0)=0, \; y'(0)=1.$

1866. $y' = 2y^2, \quad y(0)=1.$

1867. $y'' + xy' + y = x\cos x, \quad y(0)=0, \; y'(0)=1.$

Nazorat savollari

1. Birinchi tartibli differensial tenglama va uning umumiy va xususiy yechimlariga ta'rif bering.
2. O'zgaruvchilari ajraladigan differensial tenglamalar deb nimaga aytiladi?
3. Chiziqli differensial tenglamalar deb nimaga aytiladi?
4. To'liq differensialli tenglama deb nimaga aytiladi?
5. Ikkinchi tartibli differensial tenglamaning umumiy va xususiy yechimlari deb nimaga aytiladi?
6. Ikkinchi tartibli differensial tenglamaning ba'zi xususiy hollari va ularni yechish usullarini keltring.
7. Ikkinchi tartibli chiziqli differensial tenglamalar deb nimaga aytiladi?
8. Bir jinssiz hamda bir jinsli differensial tenglamalarga izoh bering.
9. Bir jinsli differensial tenglamaning umumiy yechimi deb nimaga aytiladi?
10. Bir jinssiz differensial tenglamaning umumiy yechimi deb nimaga aytiladi?

16-bob
Maydon nazariyasi elementlari. Matematik fizikaning ba'zi bir tenglamalari

Maydon nazariyasi elementlari

Fazoda biror P to'plamni (fazo nuqtalari M lardan iborat to'plamni) qaraylik: $P=\{M\}$. Uni P soha deb ham yuritamiz.

Agar P sohadan olingan har bir M nuqtaga ma'lum qoidaga ko'ra biror u son (ko'p hollarda fizik ma'nosi bo'lgan u son) mos qo'yilgan bo'lsa,

$$M \to u$$

P sohada skalyar maydon berilgan deyiladi va

$$u = u(M)$$

kabi belgilanadi.

Masalan, fazoning (atmosferaning) har bir nuqtasiga shu nuqtadagi havo haroratini mos qo'yish bilan skalyar maydon hosil bo'ladi. U harorat maydoni deyiladi.

Agar P sohadan olingan har bir M nuqtaga ma'lum qoidaga ko'ra, $\vec{a}$ vektor (fizik ma'noga ega bo'lgan vektor) mos qo'yilgan bo'lsa,

$$M \to \vec{a}$$

P sohada vektor maydon berilgan deyiladi va

$$\vec{a} = \vec{a}(M)$$

kabi belgilanadi.

Masalan, fazoda, uzluksiz massa tarqatilgan materiya harakatida uning har bir nuqtasiga shu nuqtadagi tezlik (nuqta tezligi) mos qo'yilsa, vektor maydon hosil bo'ladi.

1-§. Skalyar maydonning sath sirti va gradiyenti

1^0. Skalyar maydonning sath sirti. Ushbu

$$u = u(M) = u(x,y,z)$$

funksiya bilan berilgan skalyar maydonning unda $u(x,y,z)$ funksiyaning qiymati bir xil (ya'ni o'zgarmas C ga teng) bo'lgan nuqtalarini qaraymiz:

$$u(x,y,z) = C \qquad (C = const) \tag{1}$$

Bu tenglama aniqlagan sirt skalyar maydonning **sath sirti** deyiladi, (1) tenglama esa **sath sirtning tenglamasi** deyiladi.

Har bir nuqta orqali bitta sath sirt o'tib, qaralayotgan sohani butunlay to'ldiradi va ular o'zaro kesishmaydi.

Ma'lumki, skalyar maydonni quyidagicha:

$$u = u(\vec{r})$$

ham yozish mumkin, bunda $\vec{r}$ vektor $M=M(x,y,z)$ nuqtaning radiusi-vektori. Bu holda maydon sath sirtining tenglamasi

$$u(\vec{r})=C \quad (C=const)$$

bo'ladi.

1-misol. Ushbu

$$u(x,y,z)=\sqrt{R^2-x^2-y^2-z^2}$$

skalyar maydonning sath sirtini toping.

◀Bu skalyar maydonning sath sirti

$$\sqrt{R^2-x^2-y^2-z^2}=C, \text{ ya'ni } x^2+y^2+z^2=R^2-C^2$$

bo'ladi. Keyingi tenglama markazi koordinatalar boshida bo'lgan konsentrik sferalar oilasini (to'plamini) ifodalaydi.

Xususan, $C=0$ bo'lganda bu sath sirti

$$x^2+y^2+z^2=R^2$$

sfera bo'lib, u maydonni chegaralab turadi.▶

2⁰. Skalyar maydonning gradiyenti. Skalyar maydon

$$u=u(M)=u(x,y,z)$$

ning aniqlanish sohasiga tegishli bo'lgan $M_0=M_0(x_0,y_0,z_0)$ nuqtani va shu nuqtadan o'tuvchi hamda shu sohaga tegishli yo'nalishga ega bo'lgan l chiziqni ($\vec{l}$ vektorni) olaylik. $\vec{l}$ vektor yo'nalishida $M_0(x_0,y_0,z_0)$ nuqtadan ρ masofada bo'lgan nuqtani $M=M(x,y,z)$ deylik: $M_0M=\rho$.

Agar α,β va γ lar $\vec{l}$ vektor bilan mos ravishda OX, OY va OZ koordinata o'qlarining musbat yo'nalishlari orasidagi burchaklar bo'lsa, u holda

$$\frac{x-x_0}{\rho}=\cos\alpha,\ \frac{y-y_0}{\rho}=\cos\beta,\ \frac{z-z_0}{\rho}=\cos\gamma \qquad (2)$$

bo'ladi.

Agar M nuqta l chiziq bo'ylab M_0 nuqtaga intilganda (bu holda $\rho\to0$) ushbu

$$\frac{u(M)-u(M_0)}{\rho}=\frac{u(x,y,z)-u(x_0,y_0,z_0)}{\rho}$$

nisbatning limiti mavjud bo'lsa, bu limit $u=u(x,y,z)$ funksiyaning M_0 nuqtadagi l yo'nalish bo'yicha hosilasi deyiladi va

$$\frac{\partial u(M_0)}{\partial l} \text{ yoki } \frac{\partial u(x_0,y_0,z_0)}{\partial l}$$

kabi belgilanadi. Demak,

$$\frac{\partial u(M_0)}{\partial l}=\lim_{\rho\to0}\frac{u(M)-u(M_0)}{\rho}.$$

Skalyar maydonning M_0 nuqtasidagi l yo'nalish bo'yicha hosilasi

$$\frac{\partial u(M_0)}{\partial l}$$

maydonning (maydon aniqlagan fizik holatning) shu yo'nalish bo'yicha o'zgarish tezligini ifodalaydi.

Agar $u=u(M)=u(x,y,z)$ funksiya $M_0(x_0,y_0,z_0)$ nuqtada differensiallanuvchi bo'lsa, u holda bu funksiya ixtiyoriy yo'nalish bo'yicha hosilaga ega bo'lib,

$$\frac{\partial u(M_0)}{\partial l}=\frac{\partial u(M_0)}{\partial x}\cos\alpha+\frac{\partial u(M_0)}{\partial y}\cos\beta+\frac{\partial u(M_0)}{\partial z}\cos\gamma \qquad (3)$$

bo'ladi.

Aytaylik,

$$u=u(x,y,z)$$

skalyar maydon berilgan bo'lib, $u(x,y,z)$ funksiya uzluksiz xususiy hosilalarga ega bo'lsin.

Ushbu

$$\frac{\partial u}{\partial x}\cdot\vec{i}+\frac{\partial u}{\partial y}\cdot\vec{j}+\frac{\partial u}{\partial z}\cdot\vec{k}$$

vektor skalyar maydonning gradiyenti deyiladi va *gradu* kabi yoziladi:

$$grad u=\frac{\partial u}{\partial x}\cdot\vec{i}+\frac{\partial u}{\partial y}\cdot\vec{j}+\frac{\partial u}{\partial z}\cdot\vec{k}.$$

Skalyar maydon $u=u(x,y,z)$ning gradiyenti *gradu* shunday vektorki, u sath sirtining normali $(\vec{n})$ bo'yicha u ning o'sish tomoniga qarab yo'nalgan bo'lib, qiymati (uzunligi)

$$|grad u|=\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2}$$

esa, shu yo'nalish bo'yicha o'zgarish tezligiga teng bo'ladi.

Gradiyentning koordinata o'qlaridagi proyeksiyalari

$$grad_{ox}u=\frac{\partial u}{\partial x},\quad grad_{oy}u=\frac{\partial u}{\partial y},\quad grad_{oz}u=\frac{\partial u}{\partial z} \qquad (4)$$

bo'ladi.

2-misol. Ushbu

$$u=xy+yz+1$$

funksiyaning $\vec{l}=\{12,-3,-4\}$ vektor yo'nalishi bo'yicha ixtiyoriy nuqtadagi hosilasini toping.

◀Ravshanki,

$$\frac{\partial u}{\partial x}=y,\quad \frac{\partial u}{\partial y}=x+z,\quad \frac{\partial u}{\partial z}=y;$$

$$\cos\alpha=\frac{12}{13},\quad \cos\beta=-\frac{3}{13},\quad \cos\gamma=-\frac{4}{13}$$

Unda (3) formulaga ko'ra berilgan funksiyaning $\vec{l}$ vektor yo'nalishi bo'yicha ixtiyoriy nuqtadagi hosilasi

$$\frac{\partial u}{\partial l}=\frac{12}{13}y-\frac{3}{13}(x+z)-\frac{4}{13}y=\frac{8y-3(x+z)}{13}$$

bo'ladi.▶

3-misol. Ushbu

$$u=\frac{e}{r}=\frac{e}{\sqrt{x^2+y^2+z^2}}$$

nuqtaviy zaryaddan hosil bo'lgan elektrostatik maydonning potensial gradiyentini toping.

◀(4) formuladan foydalanib, gradiyentning koordinata o'qlaridagi proyeksiyalarini topamiz. $u=\frac{l}{r}$ tenglikni x bo'yicha differensiyallab topamiz:

$$\frac{\partial u}{\partial x}=-\frac{e}{r^2}\frac{\partial r}{\partial x}.$$

Demak, $r^2=x^2+y^2+z^2$ bo'lsa, u holda

$$2r\frac{\partial r}{\partial x}=2x,\quad \frac{\partial r}{\partial x}=\frac{x}{r}$$

bo'ladi. Shunday qilib,

$$\frac{\partial u}{\partial x}=-\frac{ex}{r^3}.$$

Shunga o'xshash

$$\frac{\partial u}{\partial y}=-\frac{ey}{r^3},\quad \frac{\partial u}{\partial z}=-\frac{ez}{r^3}$$

(4) formuladan foydalanib topamiz:

$$grad\frac{e}{r}=-\frac{ex}{r^3}\cdot\vec{i}-\frac{ey}{r^3}\cdot\vec{j}-\frac{ez}{r^3}\cdot\vec{k}=-\frac{e}{r^3}\left(x\cdot\vec{i}+y\cdot\vec{j}+z\cdot\vec{k}\right)=-\frac{e}{r^3}\cdot\vec{r}.$$ ▶

4-misol. Ushbu

$$u(M)=u(x,y,z)=\frac{10}{x^2+y^2+z^2+1}$$

funksiya $M(x,y,z)$ nuqtadan $M_0(-1,2,-2)$ nuqtaga o'tish uchun qanday eng katta tezlik bilan o'sishiga erishishi mumkin?

Funksiya $u(M)$ eng katta tezlik bilan kamayishi uchun $M(x,y,z)$ nuqtadan $M_1(2,0,1)$ nuqtaga o'tayotganda qaysi yo'nalish bo'yicha harakatlanishi kerak?

◀$u(M)$ funksiya $M(x,y,z)$ nuqtadan P nuqtaga o'tayotganda tezligining eng katta absolyut qiymati qiymat jihatdan funksiyaning P nuqtadagi gradiyentning moduliga teng. Shu bilan birga, bu funksiya eng katta tezlik bilan o'sadi yoki kamayadi. Uning o'sishi yoki kamayishi $M(x,y,z)$ nuqta P nuqta orqali o'tayotganda funksiyaning P nuqtadagi gradiyenti yo'nalishi bo'yichami yoki qarama-qarshi yo'nalish bo'yicha o'tishiga bog'liq bo'ladi.

Yuqoridagilarni hisobga olib, $u(M)$ funksiyaning gradiyentini topamiz:

$$gradu=-\frac{20}{\left(x^2+y^2+z^2+1\right)^2}\left(x\cdot\vec{i}+y\cdot\vec{j}+z\cdot\vec{k}\right).$$

Bundan tashqari, quyidagilarni topamiz:

1) $gradu(M_0)=\frac{1}{5}\left(\vec{i}-2\vec{j}+2\vec{k}\right)$ ning moduli qiymat jihatdan $u(M)$ $M(x,y,z)$ nuqtadan $M_0(-1,2,-2)$ nuqtaga o'tayotganda funksiya o'sishi uchun qidirilayotgan eng katta tezlik quyidagiga teng:

$$|gradu(M_0)|=\sqrt{\left(\frac{1}{5}\right)^2+\left(-\frac{2}{5}\right)^2+\left(\frac{2}{5}\right)^2}=\frac{3}{5}.$$

2) $gradu(M_1)=\left(-\frac{10}{9}\vec{i}-\frac{5}{9}\vec{k}\right)$; qidirilayotgan qarama-qarshi yo'nalishga ega funksiya gradiyenti $-gradu(M_1)=\left(\frac{10}{9}\vec{i}+\frac{5}{9}\vec{k}\right)$ bo'ladi. $u(M)$ funksiya eng katta tezlik bilan kamayishi uchun $M_1(2,0,1)$ nuqtadan $M(x,y,z)$ nuqtaga o'tayotganda vektor yo'nalishi $-gradu(M_1)$ bo'ladi.▶

Quyidagi masalalarni yeching

1868. Ushbu

$$u=r,\quad \vec{r}=x\cdot\vec{i}+y\cdot\vec{j}+z\cdot\vec{k}\ \text{ va } r=|\vec{r}|$$

skalyar maydonning sath sirtini toping.

1869. Ushbu

$$u=\frac{e}{r}=\frac{e}{\sqrt{x^2+y^2+z^2}}$$

elektrostatik maydon potensialining sath sirtini toping.

1870. Ushbu

$$u = \arcsin \frac{z}{\sqrt{x^2 + y^2}}$$

skalyar maydonning sath sirtini toping.

1871. Ushbu

$$u = f(\vec{r})$$

skalyar maydonning sath sirtini toping, bunda $\vec{r} = x \cdot \vec{i} + y \cdot \vec{j} + z \cdot \vec{k}$ va $r = |\vec{r}|$ bo‘lib, $f(t)$ monoton funksiya. $t \geq 0$.

1872. Ushbu

$$u = xyz$$

funksiyaning $Q(1,-2,2)$ nuqtadagi ixtiyoriy yo‘nalishi bo‘yicha va Q nuqtaning radius-vektori yo‘nalishi bo‘yicha hosilasini toping.

Quyidagi $u = (x,y,z)$ **skalyar maydonning gradiyentini toping**

1873. $u = (x,y,z) = x^2 + y^2 - z^2$.

1874. $u = (x,y,z) = e^{xy} - yz^2$.

1875. $u = (x,y,z) = \ln(x^2 + y^2 + z^2)$.

1876. $u = (x,y,z) = 3x^2 - xy^3 + xz - z^2, \quad A(1,2,3)$.

1877. $u = (x,y,z) = z\sin(x-y), \ A\left(\frac{\pi}{2}, \frac{\pi}{6}, 1\right)$.

1878. $u = (x,y,z) = \frac{x+y}{z}, \ A(2,0,1)$.

1879. Ushbu

$$u = (x,y,z) = \frac{x}{y^2 + x^2}$$

skalyar maydonning $A(3,0,1)$ va $B(1,-1,0)$ nuqtalardagi gradiyentlari orasidagi burchakni toping.

1880. Ushbu

$$F(M) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$$

funksiya eng katta tezlik bilan o‘sishiga erishishi uchun $M(x,y,z)$ nuqta $M_0(-1,1,-1)$ nuqtadan o‘tayotganda qaysi yo‘nalish bo‘yicha harakatlanishi kerak?

1881. $u(M) = \ln(x^2 - y^2 + z^2)$ funksiya $M(x,y,z)$ nuqtadan $M_0(1,1,1)$ nuqtaga o‘tayotganda qanday eng katta tezlik bilan kamayishiga erishishi mumkin?

2-§. Vektor maydonning vektor chizig‘i va oqimi

1°. Vektor maydonning vektor chizig‘i. Biror

$$\vec{a} = a(\vec{r}) = \vec{a}(x,y,z)$$

vektor maydon berilgan bo‘lib, $\vec{a}$ vektorning koordinata o‘qlaridagi proyeksiyalari

$$a_x = a_x(x,y,z),$$
$$a_y = a_y(x,y,z),$$
$$a_z = a_z(x,y,z)$$

bo‘lsin. Unda

$$\vec{a} = a_x \cdot \vec{i} + a_y \cdot \vec{j} + a_z \cdot \vec{k}$$

bo‘ladi.

Agar vektor maydondagi chiziqning (egri chiziqning) har bir nuqtasidagi urinmasi maydonning shu nuqtasidagi vektor yo‘nalishi bilan ustma-ust tushsa, bunday chiziq vektor maydonning **vektor chizig‘i** deyiladi.

Vektor chiziqlari vektor maydonning har bir nuqtasidagi vektorning yo‘nalishini aniqlaydi.

Vektor maydon

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} \quad (\vec{a} \neq 0)$$

ning vektor chiziqlari haqida quyidagi tasdiqlar o‘rinli:

1) ixtiyoriy vektor chiziq uchun

$$\frac{dx}{a_x} = \frac{dy}{a_y} = \frac{dz}{a_z} \qquad (1)$$

bo‘ladi;

2) (1) sistemaning har qanday yechimi vektor chizig‘ini aniqlaydi;

3) $\vec{a} = \vec{a}(M)$ vektor maydonning har bir $M \ (M \neq 0)$ nuqtasi orqali faqat bitta vektor chizig‘i o‘tadi.

1-misol. Ushbu

$$\vec{a} = x \cdot \vec{i} + y \cdot \vec{j} - 2z \cdot \vec{k} \quad (\vec{a} \neq 0)$$

vektor maydonning vektor chiziqlarini toping.

◀Bu holda

$$a_x = x, \quad a_y = y, \quad a_z = -2z$$

bo‘ladi. Unda (1) sistema quyidagi

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{-2z}$$

ko‘rinishga keladi.

Ravshanki,

$$\frac{dy}{y}=\frac{dx}{x}, \qquad \frac{dz}{z}=-2\frac{dx}{x}$$

Bu tenglamalarni yechib topamiz:

$$y=C_1x, \qquad z=\frac{C_2}{x^2}$$

Demak, $y=C_1x$ va $z=\frac{C_2}{x^2}$ chiziqlar qaralayotgan vektor maydonning vektor chiziqlari bo‘ladi. ►

2^0. Vektor maydon oqimi. Vektor maydonning muhim tushunchalaridan biri "vektor maydon oqimi" tushunchasidir.

Aytaylik, fazoda harakatdagi materiyaning, masalan, suyuqlikning har bir $M=M(x,y,z)$ nuqtasidagi tezlik ushbu

$$\vec{a}(M)=\vec{a}(x,y,z)=a_x\cdot\vec{i}+a_y\cdot\vec{j}+a_z\cdot\vec{k}$$

vektor bilan berilgan bo‘lib,

$$a_x=a_x(x,y,z), \qquad a_y=a_y(x,y,z), \qquad a_z=a_z(x,y,z)$$

uzluksiz funksiyalar bo‘lsin. Bu fazoda, L chiziq bilan chegaralangan (S) sirt (bu ikki tomonli sirt bo‘lib, uning ma’lum tomoni olinadi) orqali vaqt birligi oralig‘ida oqib o‘tgan suyuqlikning miqdori

$$W=\iint\limits_{(S)} a_x dydz+a_y dxdz+a_z dydx \qquad (2)$$

bo‘iadi.

Eslatma. Umuman,

$$\vec{a}=\vec{a}(M)$$

vektor maydonning, uning fizik ma’nosidan qat’i nazar, ushbu

$$W=\iint\limits_{(S)}(\vec{a}(M),\vec{n}(M))ds$$

sirt integrali maydonning oqimi deyiladi.

Shuni ham aytish kerakki, vektor oqimi skalyar miqdor bo‘ladi.

Agar (S) yopiq sirt (fazoda biror jismni o‘rab turuvchi sirt) bo‘lsa, u holda (S) sirt orqali o‘tuvchi vektor oqimi

$$W=\oiint\limits_{(S)}(\vec{a}(M),\vec{n}(M))ds \qquad (3)$$

bo‘ladi (bu holda, tashqi normal olinsa, unda oqim (S) sirtning ichki oqimi deyiladi).

2-misol. Ushbu

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \text{(ellipsoid)}$$

sirt ichidan oqib o‘tuvchi

$$\vec{a}=x\cdot\vec{i}-y^2\cdot\vec{j}+\left(x^2+z^2-1\right)\vec{k}$$

vektor maydon oqimini toping.

◄(2) formulaga ko‘ra,

$$W=\oiint\limits_{+\sigma} xdydz-y^2dxdz+\left(x^2+z^2-1\right)dxdy.$$

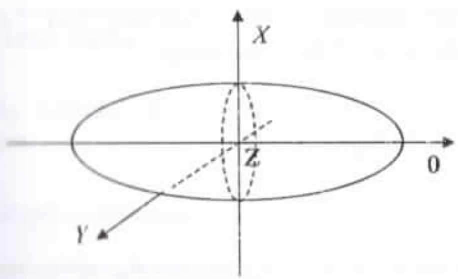
Bu sirt integralini quyidagi integral yig‘indi ko‘rinishida yozamiz:

$$W_1=\oiint\limits_{(+\sigma)} xdydz=\iint\limits_{\sigma_1} xdydz+\iint\limits_{\sigma_2} xdydz,$$

bunda σ_1 va σ_2 ellipsning qismlari $Y0Z$ tekislikning ikki tomonida joylashga bo‘lib quyidagi tenglamalar bilan ifodalanadi:

$$x_{\sigma_1}=-a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}},$$

$$x_{\sigma_2}=a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}$$



Karrali integraldan foydalanib, quyidagilarni yozib olamiz:

$$\iint\limits_{\sigma_1} xdydz=-\iint\limits_{(\sigma_1)_{yz}} -a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}},$$

σ_1 sirtning qismi $0X$ o‘qining manfiy qismida joylashgan,

$$\iint\limits_{\sigma_2} xdydz=\iint\limits_{(\sigma_2)_{yz}} a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}},$$

σ_2 sirtning qismi $0X$ o'qining musbat qismida joylashgan.

σ_1 va σ_2 sirtlarning $Y0Z$ tekisligidagi proyeksiyalari $(\sigma_1)_{YZ}$ va $(\sigma_2)_{YZ}$ bir xil

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$$

ellipsni ifodalaydi.

Shuning uchun

$$W_1=2a\iint\limits_{(\sigma_1)_{YZ}}\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}dydz=2a\int\limits_{-b}^{b}\left[\int\limits_{-z_1}^{z_1}\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}dz\right]dy,$$

bunda z_1

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

ellips tenglamasidagi z ning musbat qiymati. Ikki karrali integralni hisoblab quyidagini topamiz:

$$W_1=\frac{4}{3}\pi abc \qquad \left(a=\sqrt{1-\frac{y^2}{b^2}},\ t=\frac{z}{c}\right).$$

2) $W_2=\iint\limits_{(+\sigma)} y^2dxdz=\iint\limits_{\sigma_3} y^2dxdz+\iint\limits_{\sigma_4} y^2dxdz,$

bunda σ_3 va σ_4 ellipsning qismlari $X0Z$ tekislikning ikki tomonida joylashgan bo'lib, quyidagi tenglamalar bilan ifodalanadi:

$$y_{\sigma_3}=-b\sqrt{1-\frac{x^2}{a^2}-\frac{z^2}{c^2}},$$

$$y_{\sigma_4}=b\sqrt{1-\frac{x^2}{a^2}-\frac{z^2}{c^2}}.$$

Yuqoridagi kabi bu sirt integralini quyidagi integral yig'indi ko'rinishida yozamiz:

$$W_2=-\iint\limits_{(\sigma_3)_{XZ}} b^2\left(1-\frac{x^2}{a^2}-\frac{z^2}{c^2}\right)dxdz+\iint\limits_{(\sigma_4)_{XZ}} b^2\left(1-\frac{x^2}{a^2}-\frac{z^2}{c^2}\right)dxdz=0,$$

demak, σ_3 va σ_4 sirtlarning $X0Z$ tekisligidagi proyeksiyalari $(\sigma_3)_{XZ}$ va $(\sigma_4)_{XZ}$ lar bir xil;

3) xuddi shu sababga ko'ra sirt integralining integral ostidagi funksiya va σ sirtning $X0Z$ tekislikka nisbatan simmetrikligidan quyidagini keltrib chiqaramiz:

$$W_3=\iint\limits_{(+\sigma)}\left(x^2+z^2-1\right)dxdy=0.$$

Demak,

$$W=W_1-W_2+W_3=\frac{4}{3}\pi abc. ▶$$

Quyidagi masalalarni yeching

1882. I kuchga ega bo'lgan doimiy elektr tokining cheksiz to'g'ri chiziqli simdan o'tishidan hosil bo'lgan vektor maydon chizig'ini aniqlang.

1883. Ushbu

$$\vec{a}=x\cdot\vec{i}+y\cdot\vec{j}+z\cdot\vec{k}$$

vektor maydonning (S) sirt orqali o'tgan oqimini toping bunda (S) sirt asosining radiusi – a, balandligi – h bo'lgan silindrning yon sirti.

1884. Uchi koordinatalar boshida,balandligi H va asos radiusi R bo'lgan konusning tashqi sirtidan o'tadigan nuqtadagi $\vec{r}$ radius-vektor oqimini toping.

1885. Ushbu

$$-a\le x\le a,\quad -a\le y\le a,\quad -a\le z\le a$$

to'la kub sirtining ichidan o'tadigan

$$\vec{r}=x\cdot\vec{i}+y\cdot\vec{j}+z\cdot\vec{k}$$

radius-vektor oqimini toping.

1886. Ushbu

$$x^2+y^2+z^2=1$$

sferaning birinchi oktantidagi qismidan uning tashqi normali bo'yicha yo'nalgan

$$\vec{P}=xy\cdot\vec{i}+yz\cdot\vec{j}+xz\cdot\vec{k}$$

vektor maydon oqimini toping.

3-§. Vektor maydonning divergensiyasi va rotori

1⁰. Ostrogradskiy-Gauss formulasi. Fazoda, pastdan $z=z_1(x,y)$ tenglama bilan aniqlangan (S_1) sirt, yuqoridan $z=z_2(x,y)$ tenglama bilan aniqlangan (S_2) sirt, yon tomondan yo'naltiruvchilari XOY tekisligidagi (D) sohaning chegarasi $\partial(D)$ ((D) soha $z_1(z,y)$, $z_2(x,y)$ larning XOY tekislikdagi proyeksiyasi), yasovchilari esa OZ o'qiga parallel bo'lgan silindrik (S_3) sirt bilan chegaralangan soha (jism)ni (V), bu jismni o'rab turgan sirtni (S) deylik.

Faraz qilaylik, (V)da $P=P(x,y,z)$, $Q=Q(x,y,z)$, $R=R(x,y,z)$ funksiyalar uzluksiz bo'lib, ular uzluksiz

$$\frac{\partial P}{\partial x}=\frac{\partial P(x,y,z)}{\partial x},\quad \frac{\partial Q}{\partial y}=\frac{\partial Q(x,y,z)}{\partial y},\quad \frac{\partial R}{\partial z}=\frac{\partial R(x,y,z)}{\partial z}$$

xususiy hosilalarga ega bo'lsin. U holda

$$\iiint\limits_{(V)}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\,dy\,dz=\iint\limits_{(S)}P\,dy\,dz+Q\,dz\,dx+R\,dx\,dy \qquad (1)$$

bo'ladi (bunda sirt integral (S) sirtning tashqi tomoni bo'yicha olingan).

(1) formula **Ostrogradskiy-Gauss formulasi** deyiladi.

2⁰. Vektor maydonning divergensiyasi. Aytaylik,

$$\vec{a}(M)=\vec{a}(x,y,z)=a_x\cdot\vec{i}+a_y\cdot\vec{j}+a_z\cdot\vec{k}$$

biror vektor maydonni ifodalasin, bunda

$$a_x=a_x(x,y,z),\quad a_y=a_y(x,y,z),\quad a_z=a_z(x,y,z)$$

funksiyalar uzluksiz

$$\frac{\partial a_x}{\partial x},\ \frac{\partial a_y}{\partial y},\ \frac{\partial a_z}{\partial z}$$

xususiy hosilalarga ega bo'lsin. Ushbu

$$\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}$$

miqdor $\vec{a}(M)$ vektor maydonning divergensiyasi deyiladi va $div\,\vec{a}(M)$ kabi belgilanadi:

$$div\,\vec{a}=\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}. \qquad (2)$$

Agar $div\,\vec{a}(M_0)>0$ bo'lsa,u holda M_0 nuqta manba deyiladi, agar $div\,\vec{a}(M_0)<0$ bo'lsa, u holda M_0 nuqta yig'uvchi deyiladi yoki birinchi holda M_0 nuqtani o'z ichiga olgan ixtiyoriy cheksiz kichik sohadagi suyuqlik paydo bo'ladi, ikkinchi holda suyuqlik yo'qoladi.

$div\,\vec{a}(M_0)$ ning absolyut qiymati manba va yig'uvchining quvvatini xarakterlaydi.

Ostrogradskiy-Gauss formulasiga ko'ra, vektor maydonning oqimi va divergensiyasi quyidagi formula bilan bog'langan:

$$\iint\limits_{+\sigma}a_x dydz+a_y dxdz+a_z dxdy=\iiint\limits_{G}\left(\frac{a_x}{\partial x}+\frac{a_y}{\partial y}+\frac{a_z}{\partial z}\right)dx\,dy\,dz, \qquad (3)$$

buning ma'nosi: (σ) yopiq sirtdan oqib o'tuvchi vektor maydon oqimi (G) sohada chegaralangan divergensiya oqimi bo'yicha olingan uch karrali integralga teng.

1-misol. Quyidagi vektor maydon divergensiyalarini toping:

$$a)\ \vec{r}=x\cdot\vec{i}+y\cdot\vec{j}+z\cdot\vec{k};$$

$$b)\ \vec{p}=\frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt[3]{(x+y+z)^2}};$$

$$c)\ \vec{q}=e^{xy}\left(y\cdot\vec{j}-x\cdot\vec{i}+x\cdot y\cdot\vec{k}\right).$$

◄(2) formulani qo'llab quyidagilarni aniqlaymiz:

$$a)\ div\,\vec{r}(M)=\frac{\partial r_x}{\partial x}+\frac{\partial r_y}{\partial y}+\frac{\partial r_z}{\partial z}=1+1+1=3;$$

buning ma'nosi quyidagicha: maydondagi har bir nuqtaning radius vektori $\vec{r}$ doimiy quvvatli manbani izohlaydi.

$$b)\ P_x=P_y=P_z=(x+y+z)^{-\frac{2}{3}};$$

$$\frac{\partial P_x}{\partial x}=\frac{\partial P_y}{\partial y}=\frac{\partial P_z}{\partial z}=-\frac{2}{3\sqrt[3]{(x+y+z)^5}};$$

$$div\,\vec{P}(M)=-2(x+y+z)^{-\frac{5}{3}}$$

buning ma'nosi quyidagicha, $\vec{p}$ vektor maydon M nuqtasining koordinatalariga qarab yoki manba yoki yig'uvchi bo'lishi mumkin.

$$c)\ q_x=xe^{xy};\ q_y=ye^{xy};\ q_z=xye^{xy},$$

$$\frac{\partial q_x}{\partial x}=-e^{xy}(1+xy)=-\frac{\partial q_y}{\partial y},\quad \frac{\partial q_z}{\partial z}=0,$$

$$div\,\vec{q}(M)=0.$$

$\vec{q}$ vektor maydonda manba ham yoki yig'uvchi ham yo'q.►

3⁰. Vektor maydonning sirkulyatsiyasi va rotori. Aytaylik, biror

$$\vec{a}=\vec{a}(M)$$

vektor maydon berilgan bo'lib, uning koordinata o'qlaridagi proyeksiyalari

$$P=P(x,y,z),\ Q=Q(x,y,z),\ R=R(x,y,z)$$

bo'lsin:

$$\vec{a}=P\cdot\vec{i}+Q\cdot\vec{j}+R\cdot\vec{k}.$$

Bu vektor maydonda biror yopiq chiziqni olaylik. Uni L deylik. Ushbu

$$C=\oint\limits_{L}Pdx+Qdy+Rdz \qquad (4)$$

integral $\vec{a}=\vec{a}(M)$ **vektor maydonning sirkulyatsiyasi** deyiladi.

Kuch ta'sirida bo'lgan vektor maydonning L chiziq bo'yicha sirkulyatsiyasi massali nuqtaning (zaryadning) bir joydan ikkinchi joyga ko'chirishda bajarilgan ishni bildiradi.

2-misol. Ushbu

$$\vec{a} = x \cdot \vec{j}$$

vektor maydonning quyidagi

$$\begin{cases} x = a\cos t, \\ y = a\sin t \end{cases} \qquad (0 \le t \le 2\pi)$$

aylana bo'yicha sirkulyatsiyasini toping.

◄Bu holda

$$P = 0, \quad Q = x, \quad R = 0$$

bo'ladi. (4) formuladan foydalanib topamiz:

$$C = \oint_L x dy = \int_0^{2\pi} a\cos t \cdot (a\sin t)' dt =$$

$$= a^2 \int_0^{2\pi} \cos^2 t dt = \frac{a^2}{2}\int_0^{2\pi}(1+\cos 2t)dt = \frac{a^2}{2}\left(t + \frac{1}{2}\sin 2t\right)\Big|_0^{2\pi} = \pi \cdot a^2. ►$$

Ko'pincha vektor maydonlar turli holatlar, jumladan, fizik holatlar bilan bog'langan bo'ladi. Bunday vektor maydonlarda aylanma harakatning sodir etilishi maydonning muhim xususiyatlaridan hisoblanadi. Maydonning bunday xususiyatga ega bo'lishi quyida keltiriladigan maxsus vektor yordamida aniqlanadi.

Aytaylik,

$$\vec{a} = \vec{a}(M)$$

vektor maydon berilgan bo'lib, uning koordinata o'qlaridagi proyeksiyalari

$$a_x = P = P(x,y,z),$$

$$a_y = Q = Q(x,y,z),$$

$$a_z = R = R(x,y,z)$$

bo'lsin, bunda $P(x,y,z)$, $Q(x,y,z)$ va $R(x,y,z)$ barcha o'zgaruvchilari bo'yicha uzluksiz xususiy hosilalarga ega bo'lgan funksiyalar.

Ushbu

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cdot\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\cdot\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cdot\vec{k}$$

vektor $\vec{a} = \vec{a}(M)$ vektor maydonning rotori deyiladi va $rot\vec{a}$ kabi belgilanadi:

$$rot\vec{a} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cdot\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\cdot\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cdot\vec{k}. \qquad (5)$$

Vektor maydonning rotori quyidagi uchinchi tartibli determinant yordamida simvolik yozilishi mumkin:

$$rot\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Agar $\vec{a}$ vektor maydonning M nuqtasi orqali birlik normal vektori $\vec{n}$ bo'lgan T tekislik o'tkazilsa, unda

$$\left(rot\vec{a}(M), \vec{n}\right)$$

skalyar ko'paytma qaralayotgan maydonning M nuqtadagi aylanma (kuchini) xarakterlaydi. U M nuqtaning koordinatalari hamda T tekislikka bog'liq bo'lib, T tekislik $rot\vec{a}$ vektorga perpendikulyar bo'lganda eng katta qiymatga ega va u

$$\left|rot\vec{a}(M)\right| = \sqrt{\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)^2 + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)^2 + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)^2} \qquad (6)$$

ga teng bo'ladi.

3-misol. Ushbu

$$\vec{a} = \vec{a}\left(xz, -yz^2, xy\right)$$

vektor maydonning $\left(0; -a, a^2\right)$ nuqtadagi rotorini toping.

◄Bu vektor maydon uchun

$$P(x,y,z) = xz, \quad Q(x,y,z) = -yz^2, \quad R(x,y,z) = xy$$

bo'lib,

$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}(xz) = x, \qquad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(xz) = 0,$$

$$\frac{\partial Q}{\partial z} = \frac{\partial}{\partial z}\left(-yz^2\right) = -2yz, \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(-yz^2\right) = 0,$$

$$\frac{\partial R}{\partial y} = \frac{\partial}{\partial y}(xy) = x, \qquad \frac{\partial R}{\partial x} = \frac{\partial}{\partial x}(xy) = y$$

bo'ladi. (5) formuladan foydalanib topamiz:

$$rot\vec{a} = (x + 2yz)\cdot\vec{i} + (x - y)\cdot\vec{j}.$$

Bu vektor $\left(0; -a, a^2\right)$ nuqtada

$$rot\vec{a}\left(0, -a, a^2\right) = -2a^3\cdot\vec{i} + a\cdot\vec{j}$$

bo'ladi.►

Ma'lumki, ushbu

$$\int Pdx + Qdy + Rdz = \iint_{(S)}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dzdx \qquad (7)$$

formula **Stoks formulasi** deyiladi.

Stoks formulasi sirt bo'yicha olingan sirt integralini shu sirtning chegarasi bo'yicha olingan egri chiziqli integralni o'zaro bog'lovchi formuladir.

Aytaylik,

$$\vec{a}=\vec{a}(M)=P(x,y,z)\cdot\vec{i}+Q(x,y,z)\cdot\vec{j}+R(x,y,z)\cdot\vec{k}$$

vektor maydon berilgan bo'lsin. Tegishli shartlarda

$$\int_L P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz=$$

$$=\int_L np_x\vec{a}dx+np_y\vec{a}dy+np_z\vec{a}dz$$

miqdor $\vec{a}(M)$ vektor maydonning sirkulyatsiyasi,

$$\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\cdot\vec{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\cdot\vec{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\cdot\vec{k}=$$

$$=np_x rot\vec{a}\cdot\vec{i}+np_y rot\vec{a}\cdot\vec{j}+np_z rot\vec{a}\cdot\vec{k}$$

vektor esa **maydonning rotori** bo'ladi.

Bu munosabatlar yordamida Stoks formulasi quyidagicha yoziladi:

$$\int_L np_x\vec{a}dx+np_y\vec{a}dy+np_z\vec{a}dz=\iint_{(S)} np_x rot\vec{a}dydz+np_y rot\vec{a}dzdx+np_z rot\vec{a}dxdy \quad (8)$$

Demak, $\vec{a}=\vec{a}(M)$ vektor maydonning yopiq egri chiziq L bo'yicha sirkulyatsiyasi, shu maydonning yopiq chiziq bilan chegaralangan (S) sirti bo'yicha rotor oqimiga teng bo'ladi. Bu Stoks formulasining fizik ma'nosini anglatadi.

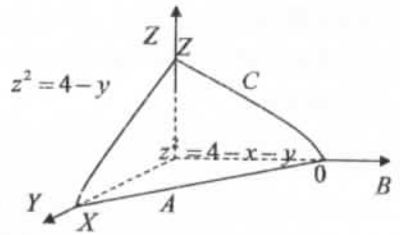
4-misol. Ushbu

$$z^2=4-x-y$$

sirtning koordinata tekisliklari bilan kesishishidan hosil bo'lgan $ACBA$ konturdagi

$$\vec{a}=x\cdot\vec{i}+xz\cdot\vec{j}+z\cdot\vec{k}$$

vektor maydonning sirkulyatsiyasini Stoks formulasidan foydalanib hisoblang.



◀Quyidagi formulaga ko'ra,

$$rot\vec{a}=\left(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z}\right)\cdot\vec{i}+\left(\frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x}\right)\cdot\vec{j}+\left(\frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y}\right)\cdot\vec{k}$$

berilgan maydonda vektor uyurmasini topamiz:

$$rot\vec{a}=z\cdot\vec{k}-x\cdot\vec{i}.$$

Proyeksiyasini (8) formulaga qo'yib quyidagini hosil qilamiz:

$$C=\iint_\sigma z\,dx\,dy-x\,dy\,dz=\iint_\sigma z\,dx\,dy-\iint_\sigma x\,dy\,dz,$$

bu yerda

$$C=\int_{ACBA} xdx+xzdy+zdz.$$

σ sirt sifatida ushbu sirtning birinchi oktantdagi kontur bilan chegaralangan qismini olamiz. Integralni konturning soat strelkasiga qarama-qarshi yo'nalish bo'yicha olamiz

$$C_1=\iint_\sigma z\,dx\,dy=-\iint_{\sigma_{XY}}\sqrt{4-x-y}dxdy=\int_4^0\left[\int_0^{4-x}(4-x-y)^{\frac{1}{2}}dy\right]dx=$$

$$=\frac{2}{3}\int_0^4(4-x-y)^{\frac{3}{2}}\Bigg|_{y=0}^{y=4-x}dx=\frac{2}{3}\int_4^0(4-x)^{\frac{3}{2}}dx=\frac{4}{15}(4-x)^{\frac{5}{2}}\Bigg|_0^4=-\frac{128}{15}$$

(bunda $\sigma_{XY}-0AB$ uchburchak);

$$C_2=\iint_\sigma x\,dy\,dz=-\iint_{\sigma_{YZ}}(4-y-z^2)dydz=\int_4^0\left[\int_0^{\sqrt{4-x}}(4-y-z^2)dz\right]dy=$$

$$=\int_4^0\left[(4-y)z-\frac{z^3}{3}\right]\Bigg|_0^{\sqrt{4-x}}dy=\frac{2}{3}\int_4^0(4-y)^{\frac{3}{2}}dy=\frac{4}{15}(4-y)^{\frac{5}{2}}\Bigg|_0^4=-\frac{128}{15}$$

(bunda σ_{YZ} – egri chiziqli $0BC$ uchburchak).

Demak, qidirilayotgan sirkulyasiya $C=C_1-C_2=0$ bo'ladi.▶

Quyidagi masalalarni yeching

1887. Ushbu

$$\vec{a}=xy^2\cdot\vec{i}+x^2y\cdot\vec{j}+z^3\cdot\vec{k}$$

vektor maydonning $A(1;-1;3)$ nuqtadagi divergensiyasini toping.

1888. Ushbu

$$\vec{a}=(x^2+y^2)\cdot\vec{i}+(y^2+z^2)\cdot\vec{j}+(z^2+x^2)\cdot\vec{k}$$

vektor maydonning ixtiyoriy nuqtadagi divergensiyasini toping.

1889. Ushbu $\vec{a}=\vec{c}$ vektor maydonning divergensiyasini toping, bunda $\vec{c}$ – o'zgarmas vektor.

1890. Ushbu

$$\vec{a} = x \cdot \vec{i} + y \cdot \vec{j} + z \cdot \vec{k}$$

vektor maydonning divergensiyasini toping.

1891. Ushbu

$$\vec{a} = x^2 y \cdot \vec{i} + y^2 z \cdot \vec{j} + z^2 x \cdot \vec{k}$$

vektor maydonning divergensiyasini toping.

1892. Hisoblang:

$$div\frac{\vec{r}}{r},$$

bunda $\vec{r} = \vec{r}(x, y, z)$, $r = |\vec{r}|$.

1893. Ushbu

$$\vec{a} = x^2 y^3 \cdot \vec{i} + \vec{j} + z \cdot \vec{k}$$

vektor maydonning $x^2 + y^2 = a^2$, $z = 0$ aylana bo'yicha sirkulyatsiyasini toping.

1894. Ushbu

$$\vec{a} = (x - 2z) \cdot \vec{i} + (x + 3y + z) \cdot \vec{j} + (5x + y) \cdot \vec{k}$$

vektor maydonning uchlari $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$ nuqtalarda bo'lgan ACB uchburchak perimetri bo'yicha sirkulyatsiyasini toping.

1895. Ushbu

$$\vec{a} = y \cdot \vec{i} + x \cdot \vec{j} + x \cdot \vec{k}$$

vektor maydonning rotorini toping.

1896. Ushbu

$$\vec{a} = xy \cdot \vec{i} + yz \cdot \vec{j} + zx \cdot \vec{k}$$

vektor maydonning rotorini toping.

1897. Ushbu

$$\vec{a} = \frac{zy}{x} \cdot \vec{i} + \frac{xz}{y} \cdot \vec{j} + \frac{xy}{z} \cdot \vec{k}$$

vektor maydonning rotorini toping.

1898. Fazoning qanday nuqtalarida ushbu

$$\vec{a} = (y^2 + z^2) \cdot \vec{i} + (z^2 + x^2) \cdot \vec{j} + (x^2 + y^2) \cdot \vec{k}$$

vektor maydonning rotori OX koordinata o'qiga perpendikulyar bo'ladi.

4-§. Matematik fizikaning ba'zi-bir tenglamalari

Fizikaning ko'p masalalarini yechishda u yoki bu funksional bog'liqlikni topish talab etiladi. Masalan, qandaydir fizik kattalikning vaqt bilan, yoki nuqtaning koordinatalari va shu kabilar bilan bog'liqliklarini topish talab etiladi. Tug'ridan-to'g'ri qidirilayotgan bog'liqlikni topish qiyin yoki mumkin emas, bunday hollarda qidirilayotgan bog'liqlikni topish masalasi qo'yiladi: qidirilayotgan funksiya bilan uning hosilasi orasidagi bog'lanishni topish, ya'ni funksiyani qanoatlantiradigan differensial tenglama tuzish.

Agar qidirilayotgan funksiya faqat bitta erksiz o'zgaruvchiga bog'liq bo'lsa ($y = f(x)$), u holda bu funksiyani qanoatlantiradigan differensial tenglama oddiy differensial tenglama deyiladi.

Agar qidirilayotgan funksiya ikki va undan ortiq erksiz o'zgaruvchilarga bog'liq bo'lib, hamda erksiz o'zgaruvchilarning hususiy hosilalarini ham o'z ichiga olsa, bunday tenglamaga xususiy hosilali **differensial tenglama** deyiladi.

Xususiy hosilali differensial tenglama noma'lum funksiya va barcha uning xususiy hosilalariga nisbatan chiziqli bo'lsa, chiziqli deyiladi

Fizikaning ko'p masalalari ikkinchi tartibli xususiy hosilasi chiziqli differensial tenglamalarga keladi. Shuning uchun ham ular **matematik fizika tenglamalari** deyiladi.

Quyidagi ikkinchi tartibli xususiy hosilali chiziqli differensial tenglamani qaraylik:

$$a \cdot \frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \qquad (1)$$

bu yerda a, b, c lar x va y ning funksiyalari.

(1) tenglama qaralayotgan sohada giperbolik tipdagi tenglama deyiladi, agarda shu sohada quyidagi shart bajarilsa, $b^2 - ac > 0$. Unga misol to'lqin (tebranish) tenglamasi

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (u = u(x,t)).$$

Agarda shu sohada (1) tenglama uchun $b^2 - ac = 0$ shart bajarilsa, tenglama **parabolik tipdagi tenglama** deyiladi. Unga misol issiqlik tarqalish tenglamasi

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (u = u(x,t)).$$

Agarda shu sohada (1) tenglama uchun $b^2 - ac < 0$ shart bajarilsa, tenglama **elliptik tenglama** deyiladi. Unga misol **Laplas tenglamasi**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = 0 \qquad (u = u(x,y)).$$

Ushbu

$$\frac{\partial^2 u}{\partial x \partial y} = F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right),$$

$$\frac{\partial^2 u}{\partial y^2} = F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right),$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

tenglamalar mos ravishda giperbolik, parabolik va elliptik tipdagi kanonik tenglamalar deyiladi.

Ushbu

$$a \cdot dy^2 - 2b dx dy + c \cdot dx^2 = 0 \qquad (2)$$

tenglama (1) tenglamaning xarakteristik tenglamasi deyiladi.

Agar (1) tenglama giperbolik tipdagi tenglama bo'lsa,u holda (2) tenglama ikkita integralga ega bo'ladi:

$$\varphi(x, y) = C_1, \qquad \varphi(x, y) = C_2.$$

(1) tenglamada $\xi = \varphi(x, y)$, $\eta = \psi(x, y)$ o'zgaruvchilarni almashtirish orqali uni kanonik ko'rinishga keltiramiz.

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = \Phi\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right)$$

Agar (1) tenglama parabolik tipdagi tenglama bo'lsa,u holda (2) tenglama bitta integralga ega bo'ladi:

$$\varphi(x, y) = C.$$

Bunday holda $\xi = \varphi(x, y)$, $\eta = \psi(x, y)$ o'zgaruvchi almashtiriladi. Bu yerda $\psi(x, y) - \frac{\partial \xi}{\partial x} \cdot \frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y} \cdot \frac{\partial \eta}{\partial x} \neq 0$ shart o'rinli bo'lgan qandaydir funksiya.

O'zgaruvchi almashtirilgandan so'ng quyidagi tenglamaga kelamiz:

$$\frac{\partial^2 u}{\partial \eta^2} = \Phi\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right).$$

Elliptik tipdagi tenglama uchun xarakteristik tenglama integrali quyidagicha

$$\varphi(x, y) \pm i\psi(x, y) = C_{1,2},$$

bu yerda $\varphi(x, y)$ va $\psi(x, y)$ haqiqiy o'zgaruvchili funksiyalar. (1) tenglamada $\xi = \varphi(x, y)$, $\eta = \psi(x, y)$ o'zgaruvchilarni almashtirsak, u holda tenglama

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = \Phi\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right)$$

ko'rinishga ega bo'ladi.

1-misol. Ushbu

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0$$

tenglamani kanonik ko'rinishga keltiring.

◀Tenglamada $a = 1, b = -1, c = 2$ bo'lganligidan $b^2 - ac = -1 < 0$ ekanligi kelib chiqadi, demak, tenglama elliptik tipda, uning xarakteristik tenglamasi

$$dy^2 + 2dxdy + 2dx^2 = 0 \text{ yoki } y'^2 + 2y' + 2 = 0$$

bo'ladi. Bunda $y' = -1 \pm i$ dan

$$y + x - ix = C_1 \text{ va } y + x + ix = C_2$$

kelib chiqadi.

O'zgaruvchilarni $\xi = y + x$, $\eta = x$ deb almashtirsak,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta};$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi};$$

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial^2 u}{\partial \xi^2} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \xi \partial \eta} \cdot \frac{\partial \eta}{\partial x}\right) + \left(\frac{\partial^2 u}{\partial \eta \partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial x}\right) = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2};$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial \xi^2} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \xi \partial \eta};$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta} \cdot \frac{\partial \eta}{\partial y} = \frac{\partial^2 u}{\partial \xi^2}.$$

Natijani tenglamaga qo'yib, quyidagini hosil qilamiz:

$$\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} - 2\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + 2\frac{\partial^2 u}{\partial \xi^2} = 0$$

ya'ni

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = 0. ▶$$

Quyida tebranish hamda issiqlik tarqalish tenglamalarini Fure usuli yordamida yechilishiga doir misollar keltiramiz.

1⁰. Torning tebranishi tenglamasi va uning yechimi. Odatda, egiluvchan hamda og'irligi hisobga olinmaydigan **ip tor** deyiladi. Aytaylik, bunday torning uzunligi l ga teng bo'lib, uning uchlari OX o'qining $x=0$, $x=l$ nuqtalariga tarang tortilib mustahkamlangan.

Agar tashqi kuch ta'sirida tor muvozanat holatidan qo'zg'atilsa, unda torning tebranma harakati sodir bo'ladi. Tebranish jarayonida tor nuqtasining OX o'qidan uzoqlanishi (chetlanishi) u shu nuqtaning abssissasi x hamda t vaqtga bog'liq bo'ladi.

Shunday qilib, torning nuqtasida, ixtiyoriy vaqtdagi holatini bilish uchun u ning x va t orqali bog'lanishini aniqlash, ya'ni $u=u(x,t)$ funksiyani (tor harakat qonunini) bilish kerak bo'ladi. Bu funksiya ushbu

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2} \tag{1}$$

tenglamadan topiladi.

(1) ikkinchi tartibli xususiy hosilali differensial tenglama **tor tebranishining tenglamasi** deyiladi.

Tor tebranishi tenglamasining ushbu boshlang'ich

$$u\big|_{t=0}=f(x), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0}=\varphi(x)$$

hamda chegaraviy

$$u\big|_{x=0}=0, \qquad u\big|_{x=l}=0$$

shartlarni qanoatlantiruvchi yechimi

$$u(x,t)=\sum_{k=1}^{\infty}u_k(x,t)=\sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi}{l}t+b_k\sin\frac{k\pi}{l}t\right)\sin\frac{k\pi}{l}x.$$

bo'ladi, bunda

$$a_k=\frac{2}{l}\int_0^1 f(x)\sin\frac{k\pi}{l}xdx, \qquad (k=1,2,3,...)$$

$$b_k=\frac{2}{k\pi a}\int_0^1\varphi(x)\sin\frac{k\pi}{l}xdx.$$

2-misol. Uzunligi ℓ ga teng bo'lgan simning ikki uchi mahkamlangan. Vaqtning boshlang'ich vaqtida simning $x=\frac{\ell}{2}$ nuqtasini $\frac{p}{10}$ masofaga tarang tortib tebrantirmasdan qo'yib yuboriladi. Sim $u(x,t)$ nuqtasining ixtiyoriy vaqtdagi chetlanishi aniqlansin.

◀Qaralayotgan masalada ikki uchi mahkamlangan simning erkin tebranishi qaralmoqda va u quyidagi boshlang'ich va chegaraviy shartlar bilan berilgan, ya'ni boshlang'ich shartlari

a) $u(x;0)=f(x)=\begin{cases}\frac{x}{5} & 0\le x\le\frac{l}{2},\\ -\frac{1}{5}(x-l) & \frac{l}{2}\le x\le l\end{cases}$

b) $\frac{\partial u(x,0)}{\partial t}=\varphi(x)=0$

(sim tarang tortib tebrantirmasdan qo'yib yuboriladi, demak, uning boshlang'ich tezligi nolga teng).

Chegaraviy sharti

$$u(0,t)=0, \quad u(\ell,t)=0$$

buning fizik ma'nosi shuki, $x=0$ va $x=\ell$ nuqtalarga mahkamlangan.

Shu shartlarda berilgan

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$$

tenglamaning yechimini aniqlash talab etilmoqda (bu yerda: $a^2=\frac{T}{\rho}$, $T-$ simning tarangligi, $\rho-$simning zichligi).

Tenglamaning umumiy yechimini quyidagi qator ko'rinishida qidiramiz:

$$u(x,t)=\sum_{k=1}^{\infty}\left(a_k\cos\frac{\pi akt}{l}+b_k\sin\frac{\pi akt}{l}\right)\sin\frac{\pi kx}{l}, \tag{2}$$

bu yerda:

$$a_k=\frac{2}{l}\int_0^l f(x)\sin\frac{\pi kx}{l}dx$$

$$b_k=\frac{2}{k\pi a}\int_0^l\varphi(x)\sin\frac{\pi kx}{l}dx \tag{3}$$

Qaralayotgan masala $\varphi(x)=0$. Demak, $b_k=0$ $(k=1,2,3,...)$. a_k ni hisoblaymiz:

$$a_k=\frac{2}{l}\int_0^l f(x)\sin\frac{\pi kx}{l}dx=\frac{2}{5l}\left[\int_0^{\frac{l}{2}}x\sin\frac{\pi kx}{l}dx+\int_{\frac{l}{2}}^{l}(l-x)\sin\frac{\pi kx}{l}dx\right]=$$

$$=\frac{4}{5l}\cdot\frac{l^2}{\pi^2k^2}\sin\frac{\pi k}{2}.$$

Shunday qilib,

$$a_k=\frac{4l}{5\pi^2k^2}\sin\frac{\pi k}{2} \qquad (k=1,2,...).$$

k juft bo'lganda $a_k=0$, chunki

$$\sin\frac{\pi k}{2}=\sin\frac{2m\pi}{2}=0.$$

$k=2m-1$ toq bo'lganda

$$\sin\frac{\pi k}{2}=\sin\frac{(2m-1)\pi}{2}=(-1)^{m-1} \qquad (m=1,2,...).$$

Umumiy holda a_k uchun quyidagi formulani hosil qilamiz:

$$a_{2k-1}=(-1)^{k-1}\frac{4l}{5\pi^2(2k-1)^2} \qquad (k=1,2,...)\cdot$$

Quyilgan masalaning yechimi quyidagi ko'rinishda yoziladi:

$$u(x,t)=\sum_{k=1}^{\infty}\left(a_k\cos\frac{\pi akt}{l}\sin\frac{\pi kx}{l}\right)=\frac{4l}{5\pi^2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{(2k-1)^2}\cos\frac{\pi akt}{l}\sin\frac{\pi kx}{l}. ▶$$

2⁰. Issiqlik tarqalish tenglamasi va uning yechimi. OX o'qi bo'yicha joylashgan sterjen bir xilda isitilmagan (ya'ni notekis qizdirilgan) bo'lib, tashqi muhitdan (issiqlik tarqalishidan) saqlangan bo'lsin. Bu holda sterjen bo'yicha issiqlikning tenglashish hodisasi sodir bo'ladi, ya'ni qattiqroq qizigan qismi bilan kamroq qizigan qismi orasida issiqlik almashishi (issiqlik tarqalishi) ro'y beradi.

t momentda sterjenning x nuqtasidagi harorat x va t larga bog'liq bo'ladi. Uni

$$u=u(x,t)$$

deylik. Bu funksiya ushbu

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2} \tag{4}$$

tenglama yordamida topiladi. (4) ikkinchi tartibli xususiy hosilali differensial tenglama **issiqlik tarqalish tenglamasi** deyiladi.

OX o'qida joylashgan uchlari $x=0$, $x=l$ nuqtalarda bo'lgan sterjenning issiqlik tarqalish tenglamasi

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$$

ning ushbu

$$u(x,0)=u\big|_{t=0}=f(x)$$

boshlang'ich shartni, quyidagi

$$u(o,t)=u\big|_{x=0}=0, \qquad u(l,t)=u\big|_{x=l}=0$$

chegaraviy shartlarni qanoatlantiruvchi yechimi

$$u(x,t)=\sum_{k=1}^{\infty}\left(\frac{2}{l}\int_0^l f(x)\sin\frac{k\pi}{l}xdx\right)\cdot\sin\frac{k\pi}{l}xl^{-a^2\frac{k^2\pi^2}{l^2}} \tag{5}$$

bo'ladi.

3-misol. Quyidagi chegaraviy

$$u(o,t)=0, \qquad u(l,t)=0$$

va boshlang'ich shartlar $u(x,o)=\begin{cases}x, & если \quad 0\le x<\frac{l}{2},\\ l-x, & если \quad \frac{l}{2}\le x\le l.\end{cases}$

bilan berilgan

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$$

issiqlik tarqalish tenglamasini yeching.

◀Yechim quyidagi formula bilan aniqlanadi

$$u(x,t)=\sum_{n=1}^{\infty}c_n e^{-\frac{a^2\pi^2n^2t}{l^2}}\cdot\sin\frac{\pi nx}{l}$$

bunda c_n

$$c_n=\frac{2}{l}\left[\int_0^{\frac{l}{2}}x\sin\frac{\pi nx}{l}dx+\int_{\frac{l}{2}}^{l}(l-x)\sin\frac{\pi nx}{l}dx\right].$$

bo'ladi.

Integrallarni hisoblaymiz:

$$\int_0^{\frac{l}{2}}x\sin\frac{\pi nx}{l}dx=-\frac{l^2}{2\pi n}\cos\frac{\pi n}{2}+\frac{l^2}{\pi^2n^2}\sin\frac{\pi n}{2},$$

$$\int_{\frac{l}{2}}^{l}(l-x)\sin\frac{\pi nx}{l}dx=\frac{l^2}{2\pi n}\cos\frac{\pi n}{2}+\frac{l^2}{\pi^2n^2}\sin\frac{\pi n}{2}.$$

Olingan natijalarni jamlab topamiz:

$$c_n=\frac{4l}{\pi^2}\frac{\sin\frac{\pi n}{2}}{n^2}.$$

Shunday qilib, $\sin\pi n=0$ bo'lsa, u holda $c_{2n}=0$. Bundan tashqari,

$$c_{2n+1}=\frac{4l}{\pi^2}\cdot\frac{(-1)^{n-1}}{(2n-1)^2}.$$

Masala yechimi quyidagicha yoziladi:

$$u(x,t)=\frac{4l}{\pi^2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)^2}e^{-\frac{\pi^2a^2(2n-1)^2}{l^2}t}\cdot\sin\frac{\pi(2n-1)}{l}x. ▶$$

Quyidagi masalalarni yeching

1899. $u=u(x,y)$ uchun quyidagi

$$\frac{\partial^2 u}{\partial y^2}=12y$$

tenglamani yeching.

1900. Quyidagi

$$\frac{\partial^2 z}{\partial x\partial y}=1$$

tenglamaning umumiy yechimini toping.

Quyidagi tenglamalarni kanonik ko'rinishga keltiring

1901. $x^2\cdot\frac{\partial^2 u}{\partial x^2}-y^2\cdot\frac{\partial^2 u}{\partial y^2}=0$.

1902. $\frac{\partial^2 u}{\partial x^2}\cdot\sin^2 x-2y\sin x\cdot\frac{\partial^2 u}{\partial x\partial y}+y^2\cdot\frac{\partial^2 u}{\partial y^2}=0$.

1903. $x^2\cdot\frac{\partial^2 u}{\partial x^2}+2xy\cdot\frac{\partial^2 u}{\partial x\partial u}+y^2\cdot\frac{\partial^2 u}{\partial y^2}=0$.

1904. $\frac{\partial^2 u}{\partial x^2}-4\cdot\frac{\partial^2 u}{\partial x\partial u}-3\frac{\partial^2 u}{\partial y^2}-2\cdot\frac{\partial u}{\partial x}+6\cdot\frac{\partial u}{\partial y}=0$.

1905. $\frac{1}{x^2}\cdot\frac{\partial^2 u}{\partial x^2}+\frac{1}{y^2}\cdot\frac{\partial^2 u}{\partial y^2}=0$.

1906. Ushbu

$$u\big|_{t=0}=x^2,\qquad \frac{\partial u}{\partial t}\bigg|_{t=0}=0$$

boshlang'ich shart bilan berilgan

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}$$

tenglamani yeching.

1907. Ushbu

$$u\big|_{t=0}=0,\quad \frac{\partial u}{\partial t}\bigg|_{t=0}=x$$

boshlang'ich shart bilan berilgan

$$\frac{\partial^2 u}{\partial t^2}=4\cdot\frac{\partial^2 u}{\partial x^2}$$

tenglamani yeching.

1908. Ushbu

$$u\big|_{t=0}=\sin x,\quad \frac{\partial u}{\partial t}\bigg|_{t=0}=1$$

boshlang'ich shart bilan berilgan

$$\frac{\partial^2 u}{\partial t^2}=a^2\cdot\frac{\partial^2 u}{\partial x^2}$$

tenglamaning $t=\frac{\pi}{2a}$ vaqtdagi torning holatini aniqlang.

1909. Ushbu

$$u\big|_{t=0}=x,\quad \frac{\partial u}{\partial t}\bigg|_{t=0}=-x,$$

boshlang'ich shart bilan berilgan

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}$$

tenglamani yeching.

1910. Ushbu

$$u\big|_{t=0}=0,\quad \frac{\partial u}{\partial t}\bigg|_{t=0}=\cos x$$

boshlang'ich shart bilan berilgan

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$$

tenglamani yeching.

1911. Ushbu

$$u\big|_{t=0}=\sin x,\quad \frac{\partial u}{\partial t}\bigg|_{t=0}=\cos x$$

boshlang'ich shart bilan berilgan

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}$$

tenglamaning $t=\pi$ vaqtdagi torning holatini aniqlang.

1912. Uchlari mahkamlangan ℓ uzunlikdagi tor $u=x(\ell-x)$ parabola formasida egilgan va boshlang'ich tezliksiz qo'yib yuborilgan. Torning tebranish qonunini toping.

1913. Uchlari mahkamlangan ℓ uzunlikdagi turg'un holatda turgan torga bolg'acha bilan urildi. Urish natijasida torning $[c-h,\ c+h]$ oraliqda joylashgan nuqtalari υ_0 boshlang'ich tezlik oldilar. Torning tebranish qonunini toping.

1914. Torning $x=0$ va $x=\ell$ uchlari qo'zg'almas qilib mahkamlangan, boshlang'ich chetlanish quyidagi ko'rinishga ega:

$$u(x,0)=A\sin\frac{\pi x}{\ell},\quad 0\le x\le\ell.$$

Boshlang'ich tezliklar nolga teng. $u(x,t)$ siljishni toping.

1915. $x=0$ va $x=\ell$ uchlari mahkamlangan tor $x=\frac{1}{3}\ell$ nuqtada muvozanat holatidan h masofaga tortilgan, so'ngra uning nuqtalariga boshlang'ich tezlik berilmagan holda qo'yib yuborilgan. $u(x,t)$ siljishni toping.

1916. Ushbu boshlang'ich

$$u(x,0)=\sin\frac{4\pi x}{3}, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0}=0$$

hamda chegaraviy

$$u(0,t)=0, \qquad u(3,t)=0$$

shartlarni qanoatlantiruvchi

$$\frac{\partial^2 u}{\partial t^2}=4\frac{\partial^2 u}{\partial x^2}$$

tor tebranishi tenglamasi Furye usuli yordamida $u(x,t)$ yechimini toping.

1917. Uzunligi l sterjenning uchlari nol haroratda deb faraz qilamiz. Boshlang'ich temprattura quyidagi:

$$u(x,0)=5\sin\frac{\pi x}{l}-2\sin\frac{3\pi x}{l}$$

formula bilan aniqlansa, ixtiyoriy t vaqtdagi sterjin tempraturasini aniqlang.

1918. Sterjinning boshlang'ich tempraturasi quyidagi

$$\left.u(x,t)\right|_{t=0}=f(x)=\begin{cases} u_0 \text{ агар } x_1<x<x_2 \\ 0, \text{ агар } x<x_1 \text{ ёки } x>x_2 \end{cases}$$

formula bilan aniqlansa, quyidagi tenglamani yeching:

$$\frac{\partial u}{\partial t}=a^2\cdot\frac{\partial^2 u}{\partial x^2}.$$

Nazorat savollari

1. Skalyar maydonning sath sirtini izohlab bering.
2. Skalyar maydonning gradiyenti deb nimaga aytiladi?
3. Vektor maydonning vektor chizig'iga ta'rif bering.
4. Vektor maydon oqimi deb nimaga aytiladi?
5. Ostrogradskiy-Gauss formulasini izohlab bering.
6. Vektor maydonning divergensiyasi deb nimaga aytiladi?
7. Vektor maydonning sirkulyatsiyasi va rotori deb nimaga aytiladi?
8. Stoks formulasini izohlab bering.
9. Ikkinchi tartibli xususiy hosilasi chiziqli differensial tenglama qachon giperbolik tipdagi differensial tenglama deyiladi?
10. Ikkinchi tartibli xususiy hosilasi chiziqli differensial tenglama qachon parabolik tipdagi differensial tenglama deyiladi?
11. Ikkinchi tartibli xususiy hosilasi chiziqli differensial tenglama qachon elleptik tipdagi differensial tenglama deyiladi?
12. Torning tebranishi tenglamasi va uning yechimini izohlab bering.
13. Issiqlik tarqatish tenglamasi va uning yechimini izohlab bering.

17-bob
Ehtimollar nazariyasi va matematik statistika asoslari
1-§. Ehtimollar nazariyasining asosiy tushunchalari va teoremalari

1^0. Tasodifiy hodisalar. Hodisalar ustida amallar. Hodisa deganda, kuzatish yoki tajriba natijasida yuzaga keladigan fakt (dalil) tushuniladi.

Odatda, hodisalar ma'lum shartlar bajarilganda yoki tajriba (sinov) natijasida sodir bo'ladi (ya'ni ro'y beradi). Hodisalar bosh harflar bilan belgilanadi.

Tajriba natijasida har doim sodir bo'ladigan hodisa **muqarrar hodisa** deyiladi va U harfi bilan belgilanadi.

Tajriba natijasida sodir bo'lmaydigan hodisa mumkin bo'lmagan hodisa deyiladi va V harf bilan belgilanadi.

Tajriba natijasida sodir bo'lishi ham, sodir bo'lmasligi ham mumkin bo'lgan hodisa **tasodifiy hodisa** bo'ladi.

Keyingi o'rinlarda **tasodifiy hodisa** deyish o'rniga **hodisa** deb ishlatamiz.

Tajribaning har bir natijasini ifodalovchi hodisa elementar hodisa deyiladi. Tajribadagi barcha elementar hodisalardan iborat to'plam **elementar hodisalar fazosi** deyiladi va Ω bilan belgilanadi.

Aytaylik, biror tajriba natijasida A va B hodisalar sodir bo'lishi mumkin bo'lsin.

Agar A hodisa sodir bo'lganda har doim B hodisa ham sodir bo'lsa, A hodisa B hodisani ergashtiradi deyiladi va $A\subset B$ kabi belgilanadi.

Bu holda A hodisaning sodir bo'lishi B hodisaning sodir bo'lishiga qulaylik tug'diradi deb ham yuritiladi.

Agar A va B hodisalari uchun

$$A\subset B, \qquad B\subset A$$

bo'lsa, A va B teng kuchli hodisalar deyiladi va $A=B$ kabi yoziladi.

A va B hodisalarning hech bo'lmaganda bittasining sodir bo'lishi natijasida sodir bo'ladigan hodisa A va B hodisalar yig'indisi deyiladi va $A+B$ kabi yoziladi.

A va B hodisalarning ikkalasini sodir bo'lishi natijasida sodir bo'ladigan hodisa, A va B hodisalar ko'paytmasi deyiladi va $A\cdot B$ kabi yoziladi.

Agar A hodisasining sodir bo'lishi B hodisasining ham sodir bo'lishini inkor etmasa, A va B birgalikda bo'lgan hodisalar deyiladi.

Agar A hodisasining sodir bo'lishi, B hodisasining sodir bo'lishini inkor etsa, A va B birgalikda bo'lmagan hodisalar deyiladi.

Agar tajriba natijasi A hodisani sodir bo'lishidan B hodisasining sodir bo'lmasligini ifodalasa, bunday hodisa A va B hodisalar ayirmasi deyiladi va $A-B$ kabi yoziladi.

Agar A va B hodisalar uchun

$$A+B=U, \qquad A\cdot B=V$$

bo'lsa, A va B o'zaro qarama-qarshi hodisalar deyiladi. A hodisaga qarama-qarshi hodisa $\overline{A}$ kabi belgilanadi.

1-misol. 3 ta talaba bir-biriga bog'liq bo'lmagan holda bitta topshiriqni bajarishmoqda. Ushbu hodisalarni toping.

1) $A=\{$barcha talabalar topshiriqni bajardi$\}$
2) $B=\{$topshiriqni faqat 1-talaba bajardi$\}$
3) $C=\{$topshiriqni hech bo'lmasa 1 ta talaba bajardi$\}$
4) $D=\{$topshiriqni faqat bitta talaba bajardi$\}$

◄1) A hodisa ro'y berishi uchun A_1, A_2 va A_3 hodisalarning barchasi ro'y berishi kerak:

$$A=A_1A_2A_3$$

2) bu holda A_1 hodisa ro'y berishi, A_2A_3 hodisalar esa ro'y bermasligi kerak, ya'ni $\overline{A}_2, \overline{A}_3$ lar ro'y berishi kerak:

$$B=A_1\overline{A}_2\overline{A}_3$$

3) C hodisa ro'y berishi uchun yoki A_1, yoki A_2, yoki A_3 ro'y berishi, yoki ularning ixtiyoriy 2 tasi yoki barchasi ro'y berishi kerak, shuning uchun

$$C=A_1+A_2+A_3$$

4) bu holda yoki faqat 1-talaba bajardi $(A_1\cdot\overline{A}_2\cdot\overline{A}_3)$, yoki 2-talaba bajardi $(\overline{A}_1\cdot A_2\cdot\overline{A}_3)$, yoki 3-talaba bajardi $(\overline{A}_1\cdot\overline{A}_2\cdot A_3)$, ya'ni

$$D=A_1\cdot\overline{A}_2\cdot\overline{A}_3+\overline{A}_1\cdot A_2\cdot\overline{A}_3+\overline{A}_1\cdot\overline{A}_2\cdot A_3.$$ ►

2°. "Tasodifiy hodisa ehtimoli" tushunchasi. Aytaylik, tajriba natijasida bir xil imkoniyat bilan

$$e_1, e_2, ..., e_n$$

hodisalar (elementar hodisalar) yuzaga kelgan bo'lsin.

Agar

1) $e_1+e_2+...+e_n=U$ (muqarrar hodisa)

2) $e_i\cdot e_j=V$ (mumkin bo'lmagan hodisa) $(i,j=1,2,3,...,n, \quad i\neq j)$

bo'lsa, $e_1, e_2, ..., e_n$ hodisalar juft-jufti bilan birgalikda bo'lmagan teng imkoniyatli hodisalarning to'la gruppasini tashkil etadi deyiladi.

A hodisa hamda hodisalarning to'la gruppasini tashkil etuvchi n ta $e_1, e_2, ..., e_n$ elementar hodisalarni qaraylik. Aytaylik, bu elementar hodisalardan m tasi $(m\leq n)$ A hodisaning sodir bo'lishiga qulaylik yaratsin.

Ushbu

$$\frac{m}{n}$$

son A hodisaning ehtimoli deyiladi va $P(A)$ kabi belgilanadi:

$$P(A)=\frac{m}{n} \qquad (1)$$

(Bu hodisa ehtimolining klassik ta'rifi deyiladi).

Hodisa ehtimoli quyidagi xossalarga ega:

1) muqarrar hodisa ehtimoli 1 ga teng:

$$P(U)=1.$$

2) mumkin bo'lmagan hodisaning ehtimoli nolga teng:

$$P(V)=0$$

3) A tasodifiy hodisa ehtimoli musbat bo'lib, u nol bilan bir orasida bo'ladi:

$$0<P(A)<1$$

4) A hodisaga qarama-qarshi bo'lgan $\overline{A}$ hodisaning ehtimoli

$$P(\overline{A})=1-P(A)$$

bo'ladi.

Aytaylik, N marta tajriba o'tkazilgan bo'lib, unda A hodisa μ marta sodir bo'lsin. Ushbu

$$w=\frac{\mu}{N}$$

nisbat A hodisaning **nisbiy chastotasi** deyiladi.

Agar N ning katta qiymatlarida A hodisaning nisbiy chastotasi p soni atrofida tebranib tursa, p soni A hodisaning ehtimoli deyiladi.

(Bu hodisa ehtimolining statistik ta'rifi deyiladi.)

2-misol. 5000 ta tavakkaliga tanlangan detaldan 32 tasi sifatsiz bo'lsa, partiyadagi sifatsiz detallar nisbiy chastotasini toping.

◄Bu masalada $A-$ detalning sifatsiz bo'lishi hodisasi deb olaylik. $N=5000$ ta tajribada A hodisa $\mu=32$ marta ro'y berdi. Shuning uchun

$$W=\frac{32}{5000}=0{,}0064.$$ ►

3-misol. Kitob 500 sahifadan iborat. Tavakkaliga ochilgan sahifa raqami 7 ga karrali bo'lishi ehtimolini toping.

◄$n=500$ umumiy tajribalar soni. Ulardan qulaylik tug'diradiganlari $m=71$ ta, chunki 7 ga karrali sahifalar soni $7k$ ta: $0<7k\leq 500$ $\left(k\leq\frac{500}{7}=71\frac{3}{7}\right)$.

Demak,

$$p=\frac{71}{500}=0{,}142.$$ ►

3^0. Ehtimollarni qo‘shish va ko‘paytirish teoremalari. A va B hodisalar birgalikda bo‘lmagan hodisalar $(A\cdot B=V)$ bo‘lib, $P(A)$ va $P(B)$ ularning ehtimollari bo‘lsa, u holda A va B hodisalar yig‘indisining ehtimoli bu hodisalar ehtimollari yig‘indisiga teng bo‘ladi:

$$P(A+B)=P(A)+P(B) \qquad (2)$$

Agar A va B hodisalar birgalikda bo‘lgan hodisalar bo‘lib, $P(A)$ va $P(B)$ ularning ehtimollari bo‘lsa, u holda

$$P(A+B)=P(A)+P(B)-P(A\cdot B) \qquad (3)$$

bo‘ladi, bunda $P(AB)$ – A va B hodisalar ko‘paytmasining ehtimoli.

4-misol. Usta 5 ta dastgohga xizmat ko‘rsatadi. Usta ish vaqtida 20% vaqti 1-dastgohda, 10% ini 2-dastgohda, 15% ini 3-dastgohda, 25% ini 4-dastgohda va 30% ini 5-dastgohda o‘tkazadi. Tavakkaliga tanlangan vaqtda usta:

1) 1 yoki 3-dastgoh yonida bo‘lishi;
2) 2 yoki 5-dastgoh yonida bo‘lishi;
3) 1 yoki 4-dastgoh yonida bo‘lishi;
4) 1 yoki 2 yoki 3-dastgoh yonida bo‘lishi;
5) 4 yoki 5-dastgoh yonida bo‘lishi ehtimollarini hisoblang.

◀Quyidagi belgilashlarni kritamiz, A,B,C,D,E hodisalar mos ravishda tavakkaliga olingan vaqtda ustaning 1–,2–,3–,4–,5–, dastgohining yonida bo‘lishi hodisalar bo‘lsin. Masala shartiga ko‘ra, A,B,C,D,E juft-juft bilan birgalikda bo‘lmagan hodisalar va

$P(A)=0,20,\quad P(B)=0,10,\quad P(C)=0,15,\quad P(D)=0,25,\quad P(E)=0,30,$

1) Ustaning 1 yoki 2-dastgoh yonida bo‘lishi hodisasi $A+C$ ekanligidan $P(A+C)$ (2) formulaga ko‘ra $P(A+C)=P(A)+P(C)=0,20+0,15=0,35$

2) $P(B+E)=P(B)+P(E)=0,10+0,30=0,40$

3) $P(A+D)=P(A)+P(D)=0,20+0,25=0,45$

4) $P(A+B+C)=P(A)+P(B)+P(C)=0,20+0,10+0,15=0,45$

5) $P(D+E)=P(D)+D(E)=0,25+0,30=0,55$.▶

5-misol. Tavakkalida tanlangan 2 xonali son yoki 2 ga, yoki 5 ga, yoki ikkalasiga bir vaqtda karrali bo‘lishi ehtimolini toping.

◀ A hodisa tavakkaliga tanlangan son 2 ga karrali bo‘lishi, B hodisa esa 5 ga karrali bo‘lishi hodisalari bo‘lsin. $P(A+B)$ni topish kerak. A va B hodisalar birgalikda bo‘lganligidan

$$P(A+B)=P(A)+P(B)-P(A\cdot B)$$

2 xonali sonlar 10, 11, ..., 98, 99 hammasi bo‘lib 90 ta. Ulardan 45 tasi 2 ga karrali, 18 tasi 5 ga karrali, 9 tasi ham 2 ga ham 5 ga karrali shuning uchun

$$P(A)=\frac{45}{90}=0,50;\quad P(B)=\frac{18}{90}=0,2;\quad P(AB)=\frac{9}{90}=0,1$$

va $P(A+B)=0,5+0,2-0,1=0,6$ bo‘ladi.▶

Agar A va B hodisalar har birining sodir bo‘lishi ehtimoli boshqasining sodir bo‘lishi yoki bo‘lmasligiga bog‘liq bo‘lmasa, A va B hodisalar erkli hodisalar, aks holda, A va B bog‘liq hodisalar deyiladi.

Agar A va B erkli hodisalar bo‘lib, $P(A)$ va $P(B)$ ularning ehtimollari bo‘lsa, u holda A va B hodisalar ko‘paytmasining ehtimoli bu hodisalar ehtimollarining ko‘paytmasiga teng bo‘ladi:

$$P(A\cdot B)=P(A)\cdot P(B) \qquad (4)$$

Ko‘p hollarda A hodisasining ehtimolini biror B hodisasi sodir bo‘lgan degan shartda hisoblashga to‘g‘ri keladi. A hodisasining bunday ehtimoli shartli ehtimol deyiladi va $P(A/B)$ kabi belgilanadi.

Agar, A va B erkli hodisalar bo‘lsa,

$$P(A/B)=P(A)$$

bo‘ladi.

Agar A va B bog‘liq hodisalar bo‘lsa, u holda

$$P(A\cdot B)=P(A)\cdot P(B/A)$$

bo‘ladi.

6-misol. Ustaxonada 2 ta motor bir-biriga bog‘liqsiz ravishda ishlamoqda. 1 soat mobaynida 1-motorga ustani kerak bo‘lmasligi ehtimoli 0,9, 2-motor uchun esa 0,85. 1 soat mobaynida birorta ham motor uchun ustaning kerak bo‘lmasligi ehtimolini toping.

◀ A hodisa 1 soat davomida 1-motor uchun ustaning kerak bo‘lmasligi hodisasi, B esa 2 motor uchun ustaning kerak bo‘lmasligi hodisasi bo‘lsin. $P(A\cdot B)$ ni topish kerak. A va B hodisalar bog‘liqsizligidan foydalanib topamiz:

$$P(A\cdot B)=P(A)\cdot P(B)=0,9\cdot 0,85=0,765 .▶$$

7-misol. Korxonada ishlab chiqarilgan mahsulotning 96% i yaroqli bo‘lib, yaroqli mahsulotlarning 100 tasidan 75 tasi birinchi navli. Korxonada ishlab chiqarilgan yaroqli mahsulotlarning birinchi navli bo‘lishi ehtimolini toping.

◀Aytaylik, ishlab chiqarilgan mahsulotning yaroqli bo‘lishi hodisasi A, ulardan birinchi navli bo‘lishi hodisasi esa B bo‘lsin.

Shartga ko‘ra,

$$P(A)=0,96,\qquad P(B/A)=0,75$$

bo‘ladi.

(4) formuladan foydalanib, yaroqli mahsulotlarning birinchi navli bo‘lishi ehtimolini topamiz:

$$P(A\cdot B)=0,96\cdot 0,75=0,72 .▶$$

4^0. **To'la ehtimol formulasi. Bayes formulasi.** Faraz qilaylik, A hodisa n ta juft-juft bilan birgalikda bo'lmagan (hodisalarni to'la gruppasini tashkil etuvchi)

$$H_1, H_2, ..., H_n$$

hodisalarning faqat bittasi bilangina sodir etilishi mumkin bo'lsin. Odatda $H_1, H_2, ..., H_n$ hodisalar A hodisaning gipotezalari deyiladi.

U holda

$$P(A) = P(H_1)\cdot P(A/H_1) + P(H_2)\cdot P(A/H_2) + \\ +...+ P(H_n)\cdot P(A/H_n) \quad (5)$$

bo'ladi.

(5) formula **to'la ehtimol formulasi** deyiladi.

Aytaylik, $H_1, H_2, ..., H_n$ hodisalar o'zaro birgalikda bo'lmagan gipotezalarning to'la gruppasidan iborat bo'lib, tajriba o'tkazilganga qadar ularning ehtimollari $P(H_i)$ $(i = 1,2,3,...,n)$ ma'lum bo'lsin.

Tajriba natijasida A hodisasi sodir bo'ldi degan shartda tajribadan so'ng H_i hodisalarning ehtimollari

$$P(H_i/A) = \frac{P(H_i)\cdot P(A/H_i)}{P(H_1)\cdot P(A/H_1) + ... + P(H_n)\cdot P(A/H_n)} \quad (6)$$

bo'ladi $(i = 1,2,3,...,n)$.

(6) **Bayes formulasi** deyiladi.

8-misol. Elektrolampalar 3 ta zavodda ishlab chiqariladi. 1-zavod 45% ini, 2-si 40% ini, 3-si esa 15% ini ishlab chiqaradi. 1-zavod ishlab chiqargan mahsulotning 70% i standart, 2-zavodni 80% i, 3-zavodning esa 81% i standart. Do'konlarda mahsulot uchala zavodlardan kelib tushadi. Do'kondan sotib olingan mahsulotning standart bo'lishi ehtimolini hisoblang.

◀Quyidagi belgilashlarni kiritamiz:

H_1 – sotib olingan lampa 1-zavodda ishlab chiqarilgan, H_2 – 2-zavodda ishlab chiqarilgan, H_3 – 3-zavodda ishlab chiqarilgan va A – lampa standart bo'lishi hodisalari bo'lsin. Masalani shartiga ko'ra:

$$P(H_1) = 0,45, \quad P(H_2) = 0,40, \quad P(H_3) = 0,15,$$

$$P(A/H_1) = 0,70, \quad P(A/H_2) = 0,80, \quad P(A/H_3) = 0,81$$

$$A = H_1A + H_2A + H_3A$$

ekanligidan hamda (5) formuladan foydalanib topamiz:

$$P(A) = P(H_1)\cdot P(A/H_1) + P(H_2)\cdot P(A/H_2) + P(H_3)\cdot P(A/H_3) =$$

$$= 0,45\cdot 0,70 + 0,40\cdot 0,80 + 0,15\cdot 0,81 = 0,7565. ▶$$

9-misol. Partiyadagi detallar 3 ta ishchi tomonidan ishlab chiqarilgan. Partiyadagi detallarning 25% ini 1-ishchi, 35% ini 2-ishchi, 40% ini esa 3-ishchi tayyorlagan. 1-ishchi ishlab chiqargan mahsulotning 5% i, 2-ishchi ishlab chiqargan mahsulotning 4% i, 3-ishchi ishlab chiqargan mahsulotning 2% i sifatsiz. Tekshirish uchun tavakkaliga olingan detal sifatsiz bo'lsa, bu detalni 2-ishchi ishlab chiqargan bo'lish ehtimolini hisoblang.

◀Quyidagi belgilashlar kiritamiz: A – tekshirish uchun tavakkaliga olingan detalning sifatsiz bo'lishi, H_1, H_2, H_3 lar mos ravishda 1-, 2- va 3-ishchilarning ishlab chiqargan detallari bo'lsin. Masalani shartiga ko'ra:

$$P(H_1) = 0,25, \quad P(H_2) = 0,35, \quad P(H_3) = 0,40,$$

$$P(A/H_1) = 0,05, \quad P(A/H_2) = 0,04, \quad P(A/H_3) = 0,02$$

ekanligidan hamda (6) formuladan foydalanib topamiz:

$$P(H_2/A) = \frac{0,35\cdot 0,04}{0,25\cdot 0,05 + 0,35\cdot 0,04 + 0,40\cdot 0,02} = \frac{28}{69}. ▶$$

Quyidagi masalalarni yeching

1919. Tavakkaliga olingan detal yoki 1-(A hodisa) yoki 2-(V hodisa) yoki 3 – navli (S hodisa) bo'lishi mumkin. Quyidagi hodisalar nimani anglatadi:

$$A + B, \quad \overline{A + C}, \quad A\cdot C, \quad AB + C.$$

1920. A va B hodisalar uchun qanday shartlar bajarilganda quyidagi tengliklar o'rinli:

1. $A + B = A\cdot B$, 2. $(A + B) - B = A$, 3. $A + \overline{A} = A$, 4. $A\cdot \overline{A} = A$?

1921. Quyidagi hodisalar berilgan bo'lsin:

A = {imtihon topshirildi}

B = {imtihon a'loga topshirildi}

1) $A - B$, 2) $\overline{A - B}$, 3) $A - \overline{B}$ hodisalar qanday elementar hodisalardan iborat.

1922. Qizil, sariq va oq rangli atirgullar solingan savatdan tavakkaliga 1 ta gul olinadi.

A = {qizil gul tanlangan}

B = {sariq gul tanlangan}

C = {oq gul tanlangan} hodisalar bo'lsa, quyidagi hodisalar nimani anglatadi:

1) $\overline{A}$, 2) $A + B$, 3) $A\cdot C$, 4) $\overline{A + B}$, 5) $\overline{A} + \overline{B}$, 6) $AB + C$

1923. Quyidagi xolatlarni o'z ichiga oluvchi hodisalar uchun ifodani toping:

1) faqat A hodisa ro'y berdi,
2) faqat 1 ta hodisa ro'y berdi,
3) faqat 2 hodisa ro'y berdi,
4) 3 la hodisa ham ro'y berdi.
5) kamida 1 ta hodisa ro'y berdi.
6) ko'pi bilan 2 ta hodisa ro'y berdi.

1924. 100 ta o'q uzishda nishonga 89 tasi tegdi. O'qning nishonga tegish hodisasining chastotasini toping.

1925. 1000 ta yangi tug'ilgan chaqaloqning 517 tasi o'g'il bola bo'lsa, o'g'il bola tug'ilish hodisasining chastotasini toping.

1926. Tavakkaliga tanlangan yilning yanvar oyida 4 ta yakshanba bo'lishi ehtimolini toping.

1927. 5 ta ayol va jami 25 kishidan iborat majlisdan tavakkaliga delegasiyaga 3 kishi tanlandi. Majlisdagi har bir kishi bir xil Internet bilan tanlanishi mumkin bo'lsa, delegatsiyaga 2 ta ayol va 1 ta erkak tanlangan bo'lishi ehtimolini hisoblang.

1928. Yosh oila kelajakda 3 ta farzand ko'rishni rejalashtirishdi. Bu farzandlarining uchalasi ham qiz yoki o'g'il bo'lish ehtimoli toping.

1929. Elektrostansiyadagi 15 ta navbatchi injenerlardan 3 tasi ayol kishi. 3 kishi navbatchilik bilan band. Ixtiyoriy tanlangan kunda 2 tadan kam bo'lmagan erkak kishi navbatchi bo'lishi ehtimolini topish.

1930. Ikki yashikning har birida 10 tadan detal bo'lib, birinchi yashikdagi detallardan 8 tasi standart, ikkinchisidagi 7 tasi standart. Har bir yashikdan tavakkaliga bittadan detal olingan. Olingan ikkala detalning standart bo'lish ehtimolini toping.

1931. 25 ta elektr lampochkaning 4 tasi nostandart ekanligi ma'lum. Bir vaqtda olingan ikki lampochkaning nostandart bo'lishi ehtimolini toping.

1932. Qurilma bir-biriga bog'liqsiz ravishda ishlaydigan elementlardan tashkil topgan. Har bir qurilmaning 1 kun davomida beto'xtov ishlashi ehtimoli mos ravishda 0,9, 0,95, 0,85 ga teng. Kamida 1 ta element ishdan chiqsa, qurilma ham ishlamaydi. Qurilmaning kun davomida beto'xtov ishlashi ehtimolini toping.

1933. Nishonga 2 ta quroldan o'q uzildi. 1-quroldan nishonga tekkizish ehtimoli 0,85 ga 2-quroldan esa 0,91 ga teng. Nishonga tegish ehtimolini toping.

2-§. O'zaro bog'liq bo'lmagan tajribalar ketma-ketligi. Bernulli formulasi

1^0. Bernulli tajribalari sxemasi. Aytaylik, n ta tajriba o'tkazilgan bo'lib, ular quyidagi shartlarni qanoatlantirsin:

1) tajribalar o'zaro bog'liq bo'lmasin;

2) har bir tajriba natijasida yoki A hodisasi, yoki unga qarama-qarshi $\overline{A}$ hodisalardan biri sodir bo'lsin;

3) har bir tajribada A hodisaning sodir bo'lishi ehtimoli o'zgarmas bo'lib, u p ga teng bo'lsin: $P(A)=p$.

Ravshanki, bunda $\overline{A}$ hodisasining sodir bo'lishi ehtimoli $P(\overline{A})=1-p$ bo'ladi. Uni q bilan belgilaylik: $q=P(\overline{A})$. Demak, $q=1-p$.

Odatda, bunday tajribalar ketma-ketligi Bernulli sxemasi deyiladi.

2^0. Bernulli formulasi. Bernulli sxemasini qaraylik. n ta tajribada A hodisasining k marta $(k\geq 0)$ sodir bo'lishi ehtimoli

$$P_n(k)=C_n^k p^k q^{n-k} \tag{1}$$

bo'ladi, bunda

$$C_n^k=\frac{n!}{k!(n-k)!}$$

(1) formula **Bernulli formulasi** deyiladi.

1-misol. Har bir mahsulotning yaroqli bo'lish (A hodisa) ehtimoli 0,8 ga teng. Tayyorlangan 5 ta mahsulotdan 3 tasining yaroqli bo'lish ehtimolini toping.

◀Masalaning shartidan

$$n=5, \quad k=3, \quad P(A)=p=0,8, \qquad P(\overline{A})=q=1-p=0,2$$

bo'lishini topamiz.

Bernulli formulasiga ko'ra,

$$P_5(3)=C_5^3 0,8^3\cdot 0,2^2=\frac{5!}{3!(5-3)!}0,8^3\cdot 0,2^2=0,2048$$

bo'ladi.▶

2-misol. Bug'doyning unib chiqishi ehtimoli 0,9 bo'lsa, ekilgan 7 ta bug'doydan 5 tasi unib chiqishi ehtimolini hisoblang.

◀Unib chiqish ehtimoli $p=0,9$ ekanligidan, unib chiqmasligi ehtimoli $q=1-p=0,1$ bo'lishi kelib chiqadi. Bernulli formulasiga ko'ra

$$P_7(5)=C_7^5 p^5 q^{7-5}=C_7^{7-5}p^5q^2=C_7^2(0,9)^5\cdot(0,1)^2=0,124 .$$▶

3^0. Laplasning lokal teoremasi. Bernulli sxemasida tajribalar soni n yetarlicha katta bo'lganda $P_n(k)$ ehtimolni Bernulli formulasi yordamida hisoblash katta qiyinchiliklar tug'diradi. Natijada

$$P_n(k)=C_n^k p^k(1-p)^{n-k}$$

ifodani o'ziga qaraganda soddaroq, ayni paytda, hisoblash uchun oson bo'lgan ifoda bilan taqribiy ifodalash zaruriyati yuzaga keladi. Bernulli sxemasida n yetarlicha katta bo'lib, har bir tajribada A hodisaning sodir bo'lish ehtimoli p o'zgarmas bo'lsa, $(0<p<1)$ u holda $P_n(k)$ ehtimol uchun

$$P_n(k)\approx\frac{1}{\sqrt{2\pi np(1-p)}}\cdot e^{-\frac{x^2}{2}} \tag{2}$$

bo'ladi, bunda

$$x=\frac{k-np}{\sqrt{np(1-p)}}.$$

Agar

$$\varphi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

deyilsa, (2) formula ushbu

$$P_n(k)\approx\frac{1}{\sqrt{np(1-p)}}\varphi\left(\frac{k-np}{\sqrt{np(1-p)}}\right) \qquad (3)$$

ko'rinishga keladi.

Ko'pincha masalalarni yechishda (3) formuladan foydalaniladi. Bunda $\varphi(x)$ juft funksiya bo'lib, x ning ma'lum qiymatlarida $\varphi(x)$ funksiyaning qiymatlari 1-ilovada keltirilgan.

3-misol. Har bir ekilgan chigitning unib chiqish (A hodisa) ehtimoli $P(A)=p=0,8$ ga teng bo'lsa, ekilgan 100 ta chigitdan 85 tasi unib chiqish ehtimolini toping.

◀Shartga ko'ra,

$$n=100, \quad P(A)=p=0,8, \quad 1-p=0,2, \quad k=85$$

bo'lib,

$$x=\frac{k-np}{\sqrt{np(1-p)}}=\frac{85-100\cdot 0,8}{\sqrt{100\cdot 0,8\cdot 0,2}}=\frac{85-80}{\sqrt{16}}=1,25$$

bo'ladi.

(3) formuladan foydalanib topamiz:

$$P_{100}(85)\approx\frac{1}{\sqrt{100\cdot 0,8\cdot 0,2}}\varphi(1,25)=\frac{1}{4}\varphi(1,25)$$

1-ilovada keltirilgan ma'lumotdan foydalanib,

$$\varphi(1,25)\approx 0,1826$$

bo'lishini aniqlaymiz. Demak,

$$P_{100}(85)\approx\frac{1}{4}\cdot 0,1826=0,0456.$$ ▶

4-misol. Stanokda ishlab chiqarilgan detalning oliy navli bo'lishi ehtimoli 0,4 ga teng. Tavakkaliga olingan 26 ta detaldan yarmi oliy navli bo'lishi ehtimolini hisoblang.

◀Laplas teoremasiga ko'ra,

$p=0,4;\quad np=26\cdot 0,4=10,4;\quad q=1-p=0,6;\quad npq=10,4\cdot 0,6=6,24;$

$n=26;\quad \sqrt{npq}=\sqrt{6,24}=2,50;\quad k=13;\quad k-np=13-10,4=2,6;$

$x=\frac{k-np}{\sqrt{npq}}=\frac{2,60}{2,50}=1,04;$

$\varphi(x)=\varphi(1,04)=0,2323;$

$P_{26}(13)\approx\frac{0,2323}{2,50}=0,093$ bo'ladi.▶

4⁰. Laplasning integral teoremasi. Bernulli sxemasini qaraylik. Bunda A hodisaning sodir bo'lish ehtimoli $P(A)=p$ bo'lib, $(0<p<1)$ tajribalar soni yetarlicha katta bo'lsin. Bu tajribada A hodisa k_1 martadan kam bo'lmagan, k_2 martadan ortiq bo'lmagan sonda sodir bo'lishi ehtimoli $P_n(k_1,k_2)$ ni aniqlash mumkin.

Bu ehtimol uchun

$$P_n(k_1,k_2)\approx\frac{1}{\sqrt{2\pi}}\int_{\frac{k_1-np}{\sqrt{np(1-p)}}}^{\frac{k_2-np}{\sqrt{np(1-p)}}}e^{-\frac{x^2}{2}}dx \qquad (4)$$

taqribiy formula o'rinli bo'ladi.

Odatda,

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{x^2}{2}}dx$$

funksiya **Laplas funksiyasi** (yoki ehtimol integrali) deyiladi. Bu funksiya yordamida (4) munosabat quyidagicha:

$$P_n(k_1,k_2)\approx\Phi\left(\frac{k_2-np}{\sqrt{np(1-p)}}\right)-\Phi\left(\frac{k_1-np}{\sqrt{np(1-p)}}\right) \qquad (5)$$

yoziladi. Bu (5) formula Laplas formulasi deyiladi.

$\Phi(x)$ toq funksiya bo'lib, x ning ma'lum qiymatlarida $\Phi(x)$ ning qiymatlari 2-ilovada keltirilgan.

5-misol. Korxonada ishlab chiqarilgan mahsulotning yaroqsiz chiqish ehtimoli 0,2 ga teng bo'lsin. 400 ta mahsulotdan 70 tadan 130 tagacha yaroqsiz bo'lishi ehtimolini toping.

◀Shartga ko'ra,

$$n=400, \quad k_1=70, \quad k_2=130, \quad p=0,2 \quad 1-p=0,8$$

bo'ladi.

(5) formuladan foydalanib topamiz:

$$P_{400}(70,130)\approx\Phi\left(\frac{130-400\cdot 0,2}{\sqrt{400\cdot 0,2\cdot 0,8}}\right)-\Phi\left(\frac{70-400\cdot 0,2}{\sqrt{400\cdot 0,2\cdot 0,8}}\right)=$$

$$=\Phi(6,25)-\Phi(-1,25).$$

2-ilovada keltirilgan

$$\Phi(-1,25)=-0,39435, \quad \Phi(6,25)=0,5$$

ma'lumotlarga binoan

$$\Phi(6,25)-\Phi(-1,25)=0,89435$$

bo'lib, izlanayotgan ehtimol

$P_{400}(70,130)\approx 0,89435$

bo'ladi.▶

Quyidagi masalalarni yeching

1934. Korxona ishlab chiqaradigan mahsulotning 30% i oliy navli. Tavakkaliga olingan 6 ta mahsulotdan 4 tasi oliy navli bo'lishi ehtimolini toping.

1935. O'g'il bola tug'ilishi ehtimoli 0,515 bo'lsa, 10 chaqaloqdan 4 tasi qiz bola bo'lishi ehtimolini toping.

1936. Ustaxonada 12 ta motor bor. Motorning muayan sharoitda ishlashi ehtimoli 0,8 ga teng. Muayan sharoitda kamida 10 ta matorning ishlashi ehtimolini toping.

1937. Test 10 ta savoldan iborat bo'lib, har bir savolga yoki "ha" yoki "yo'q" deb javob berish kerak. Javobni tavakkaliga tanlash usuli orqali 80% dan kam bo'lmagan savollarga to'g'ri javob berish ehtimolini toping.

1938. Binoda 6 ta elektrolampochka bor. Har bir lampaning 1 yil davomida ishlashi ehtimoli 0,7 ga teng. 1 yil davomida 2 ta lampani almashtirishga to'g'ri kelishi ehtimolini toping.

1939. Radiosignalning har bir uzatishda qabul qilinishi ehtimoli 0,86 ga teng. 5 ta uzatishda radiosignal 4 martasida qabul bo'lishi ehtimolini toping.

1940. 18 ta avtobusdan har birining yo'lga chiqish ehtimoli 0,9 ga teng. Avtobaza normal ishlashi uchun 15 tadan kam bo'lmagan avtobus yo'lda bo'lishi kerak bo'lsa, avtobazaning normal ishlashi ehtimolini toping.

1941. 10 tup mevali daraxt ekildi. Bu daraxtlarning ko'karib ketish ehtimoli 0,7 ga teng. Ekilgan daraxtlarning 6 tasining ko'karib ketish ehtimolini toping.

1942. Televizor kineskopining kafolat muddatida nosoz ishlashi o'rtacha 12% ni tashkil etadi. 46 ta televizordan kamida 36 tasida kafolat muddatida nosozlik kuzatilmasligi ehtimolini hisoblang.

1943. Merganning har bir otishda nishonga tekkizishi ehtimoli 0,3 ga teng bo'lsa, 30 ta otishda 8 marta nishonga tekkizishi ehtimolini hisoblang.

1944. 800 ta mahsulotdan iborat partiyada oliy navli mahsulotlar soni k uchun $600 \le k \le 700$ o'rinli bo'lishi ehtimolini hisoblang. Ixtiyoriy mahsulotning oliy nav bo'lishi ehtimoli 0,62 ga teng.

1945. Bog'liqsiz 700 ta tajribada A hodisaning ro'y berishi chastotasi 460 va 600 orasida bo'lish ehtimolini hisoblang.

1946. 1000 ta chaqaloq orasida o'g'il bolalari soni 480 dan ko'p, 540 dan kam bo'lishi ehtimolini hisoblang (o'g'il bola tug'ilishi ehtimoli 0,515 ga teng deb olingan).

1947. Omborga 3 ta fabrikadan mahsulot olib kelinadi. Ombordagi mahsulotning 30% i 1-fabrikaga, 32% i – 2-fabrikaga, 38% i – 3-fabrikaga tegishli. 1-fabrikaning 60% i, 2-fabrikaning 25% i, 3-fabrikaning 50% – mahsuloti oliy navli bo'lsa, ombordan tavakkaliga olingan 300 ta mahsulotdan sifatlilari soni 130 va 130 ning orasida bo'lish ehtimolini hisoblang.

1948. Nishonga tekizish ehtimoli 0,75 bo'lsa, bog'liqsiz 300 ta otishda nishonga tegishlar soni k uchun $210 \le k \le 230$ o'rinli bo'lishi ehtimolini hisoblang.

1949. Ko'chani yorituvchi 2450 ta lampadan yilning oxiriga kelib 1500 dan 1600 tagachasi yonib turishi ehtimolini toping. Ihtiyoriy lampaning yil davomida yonib turishi ehtimolini 0,64 deb oling.

1950. Xaridorning 36-razmerli poyafzalga bo'lgan ehtiyojining ehtimoli 0,3 ga teng. 2000 ta xaridordan 575 tasi shu razmerli poyafzalga talabgor bo'lishi ehtimolini toping.

3-§. Tasodifiy miqdorlar

Ma'lum shart-sharoitlarda tasodifiy holatlarga bog'liq ravishda u yoki bu son qiymatlardan birini qabul qiladigan o'zgaruvchi miqdor **tasodifiy miqdor** deyiladi. Ular grek harflari bilan belgilanadi, masalan, ξ, η, ζ va h.k.

1⁰. Diskret tasodifiy miqdorlar va ularning taqsimot funksiyalari. Agar tasodifiy miqdorning qabul qilishi mumkin bo'lgan qiymatlari chekli yoki sanoqli (ya'ni bu qiymatlarni chekli yoki cheksiz ketma-ketlik shaklida yozish mumkin) bo'lsa, u **diskret tasodifiy miqdor** deyiladi.

Aytaylik, ξ tasodifiy miqdor, uning qabul qilishi mumkin bo'lgan qiymatlari

$$x_1, x_2, ..., x_n$$

bo'lsin. Bu tasodifiy miqdor yuqoridagi qiymatlarni mos ravishda $p_1, p_2, ..., p_n$ ehtimollar bilan qabul qilsin:

$$P\{\xi = x_1\} = p_1, \quad P\{\xi = x_2\} = p_2, ..., P\{\xi = x_n\} = p_n$$

Keltirilgan ma'lumotlardan ushbu

ξ	x_1	x_2	.....	x_n
$P\{\xi = x_i\}$	p_1	p_2	.....	p_n

jadvalni tuzamiz.

Ravshanki,

$$\{\xi = x_1\}, \ \{\xi = x_2\}, ..., \ \{\xi = x_n\}$$

hodisalar bir-biriga bog'liq bo'lmagan hodisalar bo'lib, tasodifiy miqdor, albatta, bitta qiymatni qabul qilishi kerakligidan

$$P\{\xi = x_1\} + P\{\xi = x_2\} + ... + P\{\xi = x_n\} = 1,$$

ya'ni

$$p_1 + p_2 + ... + p_n = 1$$

bo'ladi.

(1) jadval ξ tasodifiy miqdorni to'la xarakterlaydi. U ξ diskret tasodifiy miqdor ehtimollarining taqsimot qonuni deyiladi.

1-misol. Pul-buyum lotoreyasida 1 ta 1000000 so'm, 10 ta 100000 so'mdan, 100 ta 1000 so'mdan yutuq o'ynaladi. Lotoreya biletining umumiy soni 10000 ta. Bitta lotoreya biletiga ega bo'lgan kishining tasodifan yutishining taqsimot qonunini toping.

◀Ravshanki, tasodifiy miqdor ξ ning qabul qilishi mumkin bo'lgan qiymatlari

$$x_1 = 1000, \quad x_2 = 100000, \quad x_3 = 1000000, \quad x_4 = 0$$

bo'ladi. Ularning qabul qilish hodisalarining ehtimollarini topamiz:

$$p_1 = P\{\xi = x_1\} = \frac{100}{10000} = 0,01;$$

$$p_2 = P\{\xi = x_2\} = \frac{10}{10000} = 0,001;$$

$$p_3 = P\{\xi = x_3\} = \frac{1}{10000} = 0,0001;$$

$$p_4 = P\{\xi = x_4\} = 1 - (0,01 + 0,001 + 0,0001) = 0,9889.$$

Yutish taqsimot qonuni quyidagicha:

ξ	1000	100000	1000000	0
P	0,01	0,001	0,0001	0,9889

bo'ladi.▶

2⁰. Uzluksiz tasodifiy miqdorlar va ularning taqsimot funksiyalari. Agar tasodifiy miqdorniing qabul qilishi mumkin bo'lgan qiymatlari biror oraliqda joylashgan barcha qiymatlardan iborat bo'lsa, u uzluksiz tasodifiy miqdor deyiladi.

Uzluksiz tasodifiy miqdorlar ularning taqsimot funksiyalari orqali o'rganiladi. Faraz qilaylik, ξ ixtiyoriy tasodifiy miqdor, x esa biror haqiqiy son bo'lsin.

Ushbu

$$\{\xi < x\}$$

hodisaning, ya'ni tajriba natijasida sodir bo'lgan tasodifiy miqdorning x dan kichik bo'lishi hodisasi ehtimoli

$$P\{\xi < x\}$$

ξ tasodifiy miqdorning taqsimot funksiyasi deyiladi va $F(x)$ orqali belgilanadi:

$$F(x) = P\{\xi < x\}.$$

Taqsimot funksiyasi quyidagi xossalarga ega:

1) $0 \le F(x) \le 1$;

2) $F(x)$ kamaymaydigan funksiya;

3) $\lim\limits_{x\to+\infty} F(x) = 1, \quad \lim\limits_{x\to-\infty} F(x) = 0$.

Aytaylik, ξ tasodifiy miqdor, $F(x)$ esa uning taqsimot funksiyasi bo'lsin: $F(x) = P\{\xi < x\}$.

Agar $F(x)$ funksiya differensiallanuvchi bo'lsa, uning $F'(x)$ hosilasi ξ tasodifiy miqdorning ehtimol zichligi deyiladi va $p(x)$ kabi belgilanadi:

$$p(x) = F'(x).$$

$p(x)$ funksiya quyidagi xossalarga ega:

1) $p(x) \ge 0,$;

2) $\int\limits_{-\infty}^{-\infty} p(x)dx = 1;$

3) $F(x) = \int\limits_{-\infty}^{x} p(x)dx;$

4) $P\{x_1 < \xi < x_2\} = \int\limits_{x_1}^{x_2} p(x)dx$

5) agar ξ tasodifiy miqdorning taqsimot funksiyasi $F(x)$funksiya x_1 nuqtada uzluksiz bo'lsa, u holda

$$P\{\xi = x_1\} = 0$$

bo'lib,

$$P\{x_1 \le \xi < x_2\} = P\{x_1 < \xi < x_2\},$$
$$P\{x_1 < \xi < x_2\} = F(x_2) - F(x_1)$$

bo'ladi.

2-misol. Ushbu

ξ	-1	0	2
$P\{\xi = x_i\}$	0,2	0,3	0,5

qonuniyat bilan taqsimlangan ξ tasodifiy miqdorning taqsimot funksiyasini toping.

◀Ravshanki, ξ tasodifiy miqdorning qabul qiladigan qiymatlari

$$-1, 0, 2$$

bo'ladi.

Aytaylik, $x \le -1$ bo'lsin. Bu holda $\{\xi < x\}$ mumkin bo'lmagan hodisa bo'ladi, chunki tasodifiy miqdorning $\xi < x$ tengsizlikni qanoatlantiruvchi bitta ham qiymati yo'q. Demak,

$$F(x) = P\{\xi < x\} = 0.$$

Aytaylik, $-1 < x \le 0$ bo'lsin. Bu holda $\{\xi < x\} = \{\xi = -1\}$ bo'lib,

$$F(x) = P\{\xi < x\} = P\{\xi = -1\} = 0,2$$

bo'ladi.

Aytaylik, $0 < x \le 2$ bo'lsin. Bu holda

$$\{\xi < x\} = \{\xi = -1\} \cup \{\xi = 0\}$$

bo'lib,

$$F(x) = P\{\xi < x\} = P\{\xi = -1\} + P\{\xi = 0\} = 0,2 + 0,3 = 0,5$$

bo'ladi.

Aytaylik, $x > 2$ bo'lsin. Bu holda $\{\xi < x\}$ muqarrar hodisa,

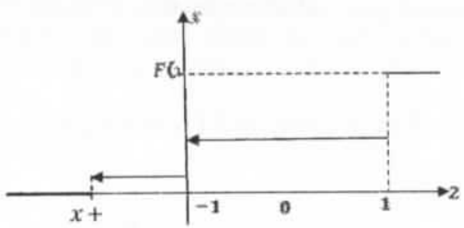
$$F(x) = P\{\xi < x\} = 1$$

bo'ladi.

Shunday qilib, ξ tasodifiy miqdorning taqsimot funksiyasi

$$F(x) = \begin{cases} 0, & агар \quad x \le -1, \\ 0,2, & агар \quad -1 < x \le 0, \\ 0,5, & агар \quad 0 < x \le 2, \\ 1, & агар \quad x > 2 \end{cases}$$

bo'ladi. Uning grafigi 1-chizmada tasvirlangan:



1-chizma

3-misol. Agar ξ uzluksiz tasodifiy miqdorning taqsimot funksiyasi

$$F(x) = \begin{cases} 0, & агар \quad x < 3 \\ C(x-3)^2, & агар \quad 3 \le x \le 5 \\ 1, & агар \quad x > 5 \end{cases}$$

bo'lsa, u holda

1) C – koeffitsiyentni toping;

2) zichlik funksiya $p(x)$ ni toping;

3) $P\{3 \le \xi < 4\}$ ehtimolni hisoblang.

◄1) Qaralayotgan tasodifiy miqdor uzluksiz tasodifiy miqdor bo'lganligi uchun, uning taqsimot funksiyasi $F(x)$ uzluksiz, jumladan, $x = 5$ nuqtada uzluksiz bo'ladi. Ayni paytda, $F(S) = 1$ bo'lgani uchun $C \cdot (5-3)^2 = 1$ bo'lib, undan $C = \frac{1}{4}$ bo'lishi kelib chiqadi. Demak,

$$F(x) = \begin{cases} 0, & агар \quad x < 3, \\ \frac{1}{4}(x-3)^2, & агар \quad 3 \le x \le 5, \\ 1, & агар \quad x > 5. \end{cases}$$

2) zichlik funksiyasi

$$p(x) = F'(x)$$

quyidagicha bo'ladi:

$$p(x) = \begin{cases} 0, & агар \quad x < 3, \\ \frac{1}{2}(x-3), & агар \quad 3 \le x \le 5, \\ 0, & агар \quad x > 5. \end{cases}$$

3) ushbu

$$P\{a \le \xi \le b\} = \int_a^b p(x)dx$$

formuladan foydalanib topamiz:

$$P\{3 \le \xi < 4\} = \int_3^4 \frac{1}{2}(x-3)dx = \frac{1}{4}. ►$$

Quyidagi masalalarni yeching

1951. 3 mergan nishonga qarata 1 tadan o'q uzishdi. Ularning nishonga tekkizishlari ehtimoli mos ravishda 0,5, 0,6, 0,8 ga teng. Nishonga tekkan o'qlar soni ξ tasodifiy miqdorning taqsimot qonunini tuzing.

1952. Nishonga tekkizish ehtimoli 0,7 bo'lsa, 2 ta bog'liqsiz otishda nishonga tekkan o'qlar soni ξ tasodifiy miqdorning taqsimot qonunini tuzing.

1953. ξ – diskret tasodifiy miqdorning taqsimot qonuni berilgan:

ξ_i:	−2	1	2	3
P_i:	0,08	0,4	0,32	0,2

ξ tasodifiy miqdorning taqsimot funksiyasi $F(x)$ ni toping.

1954. 16 ta sportchidan iborat komandada 6 tasi 1-razryadli. Tavakkaliga tanlangan 2 ta sportchidan 1 razryadlilari soni ξ tasodifiy miqdorning taqsimot qonunini tuzing, taqsimot funksiyasini toping.

1955. ATS 1500 abonentga xizmat ko‘rsatadi. 3 daqiqa mobaynida ATS ga chaqiruv kelishi ehtimoli 0,002 ga teng. 3 daqiqa davomida ATS ga kelgan chaqiruvlar soni ξ tasodifiy miqdorning taqsimot qonunini topish va $p\{\xi \ge 3\}$ -?

1956. ξ diskret tasodifiy miqdor butun qiymatlarni qabul qiladi:

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = 3, \ldots$$

$$p_n = \rho\{\xi = n\} = \frac{c}{n^2 + 3n + 2}$$

ma’lum bo‘lsa, o‘zgarmas son C ning qiymatini toping?

1957. ξ – uzluksiz tasodifiy miqdorning taqsimot funksiyasi berilgan:

$$F(x) = \begin{cases} 0, & x < -\pi \\ a(\cos x + c), & -\pi \le x \le 0 \\ 1, & x > 0 \end{cases}$$

O‘zgarmas sonlar – a va c ning qiymatini toping.

1958. ξ – uzluksiz tasodifiy miqdorning zichlik funksiyasi berilgan:

$$p(x) = \begin{cases} \frac{3}{x^4}, & x \ge 1 \\ 0, & x < 1 \end{cases}$$

Bu tasodifiy miqdor taqsimot funksiyasi $F(x)$ ni hisoblang.

1959. ξ uzluksiz tasodifiy miqdorning zichlik funksiyasi berilgan:

$$p(x) = \begin{cases} 0, & x \le 1 \\ 2x - 2, & 1 < x \le 2 \\ 0 & x > 2 \end{cases}$$

Quyidagi hodisalarning qaysi birining ehtimoli katta: ξ tasodifiy miqdorning $(1{,}6;1{,}8)$ intervalga tushishimi yoki $(1{,}9;2{,}6)$ intervalga tushishi.

1960. ξ tasodifiy miqdor taqsimot funksiyasi berilgan:

$$F(x) = \begin{cases} 3^x, & x \le 0 \\ 1, & x > 0 \end{cases}$$

Bu tasodifiy miqdorning zichlik funksiyasini toping.

1961. ξ tasodifiy miqdorning zichlik funksiyasi quyidagi ko‘rinishga ega:

$$p(x) = \begin{cases} 0, & x < -4 \\ -Ax, & -4 \le x < 0 \\ A\sqrt{x}, & 0 \le x < 4 \\ 0, & 4 \le x \end{cases}$$

A, $F(x)$, $p\{-1 < \xi < 5\}$ ni hisoblang.

4-§. Tasodifiy miqdorlarning sonli xarakteristikalari

1^0. Diskret tasodifiy miqdorning matematik kutilmasi va dispersiyasi

Faraz qilaylik, ξ diskret tasodifiy miqdor qabul qilishi mumkin bo‘lgan

$$x_1, x_2, \ldots, x_n$$

qiymatlarni mos ravishda $p_1, p_2, \ldots, p_n$ ehtimollar bilan qabul qilsin:

$$P\{\xi = x_1\} = p_1, \quad P\{\xi = x_2\} = p_2, \ldots, P\{\xi = x_n\} = p_n$$

Ushbu

$$x_1 \cdot p_1 + x_2 \cdot p_2 + \ldots + x_n p_n = \sum_{k=1}^{n} x_k p_k$$

yig‘indi ξ diskret tasodifiy miqdorning matematik kutilmasi deyiladi va $M\xi$ kabi belgilanadi:

$$M\xi = x_1 \cdot p_1 + x_2 \cdot p_2 + \ldots + x_n p_n = \sum_{k=1}^{n} x_k p_k \quad (1)$$

Diskret tasodifiy miqdorning matematik kutilmasi quyidagi xossalarga ega:

1) o‘zgarmas C sonning matematik kutilmasi shu songa teng:

$$M(C) = C, \qquad C = const.$$

2) ξ tasodifiy miqdor uchun

$$M(C \cdot \xi) = C \cdot M\xi \ (C = const)$$

bo‘ladi;

3) ξ va η tasodifiy miqdorlar uchun

$$M(\xi + \eta) = M\xi + M\eta$$

bo‘ladi;

4) o‘zaro bog‘liq bo‘lmagan ξ va η tasodifiy miqdorlar uchun

$$M(\xi \cdot \eta) = M\xi \cdot M\eta$$

bo‘ladi.

Ushbu

$$M(\xi - M\xi)^2$$

miqdor ξ tasodifiy miqdorning dispersiyasi deyiladi va $D\xi$ kabi belgilanadi:

$$D\xi = M(\xi - M\xi)^2 \quad (2)$$

Tasodifiy miqdorning dispersiyasini quyidagicha:

$$D\xi = M\xi^2 - (M\xi)^2 \quad (3)$$

ham ifodalash mumkin.

Diskret tasodifiy miqdorning dispersiyasi quyidagi xossalarga ega:

1) dispersiya har doim musbat

$$D\xi \geq 0;$$

2) o'zgarmas C sonning dispersiyasi nolga teng.

$$D(C)=0 \qquad (C=const) \ ;$$

3) ξ tasodifiy miqdor va C o'zgarmas uchun

$$D(C\cdot\xi)=C^2 D\xi$$

bo'ladi.

4) agar ξ va η bog'liq bo'lmagan tasodifiy miqdorlar bo'lsa, u holda

$$D(\xi+\eta)=D\xi+D\eta,$$
$$D(\xi-\eta)=D\xi+D\eta$$

bo'ladi.

Ushbu

$$\sqrt{D\xi}$$

miqdor ξ tasodifiy miqdorning o'rtacha kvadratik chetlanishi deyiladi va σ kabi belgilanadi:

$$\sigma=\sqrt{D\xi}\ .$$

1-misol. 100 ta lotereyada 1 ta 500000 so'mlik va 10 ta 10000 so'mlik yutuq o'ynalmoqda. Tavakkaliga sotib olingan 1 ta lotereya bileti ξ ning taqsimot qonunini toping va matematik kutilmasini hisoblang.

◀Tasodifiy miqdor ξ ning qabul qilishi mumkin bo'lgan qiymatlari

$$x_1=10000,\quad x_2=500000,\quad x_3=0$$

bo'ladi. Ularning qabul qilish hodisalarining ehtimollarini topamiz:

$$p_1=P\{\xi=10000\}=\frac{10}{100}=0,1$$

$$p_2=P\{\xi=500000\}=\frac{1}{100}=0,01$$

$$p_3=P\{\xi=0\}=1-(0,1+0,01)=1-0,11=0,89$$

Yutish taqsimot qonuni quyidagicha

ξ	10 000	500 000	0
p	0,1	0,01	0,89

Bu tasodifiy miqdorning matematik kutilmasini (1) formuladan foydalanib topamiz:

$$M\xi=10000\cdot 0,1+500000\cdot 0,01+0\cdot 0,89=$$
$$=1000+500=6000.$$ ▶

2-misol. Ushbu

$$\eta=\frac{\xi-a}{\sigma} \qquad (a,\sigma - \text{o'zgarmas})$$

tasodifiy miqdorning matematik kutilmasi va dispersiyasini toping.

◀Tasodifiy miqdorning matematik kutilmasi va dispersiyasining xossalaridan foydalanib topamiz:

$$M(\eta)=M\left(\frac{\xi-a}{\sigma}\right)=\frac{1}{\sigma}(M\xi-a),$$

$$D\eta=D\left(\frac{\xi-a}{\sigma}\right)=\frac{1}{\sigma^2}(D\xi+Da)=\frac{1}{\sigma^2}D\xi.$$ ▶

2⁰. Uzluksiz tasodifiy miqdorning matematik kutilmasi va dispersiyasi. Aytaylik, ξ uzluksiz tasodifiy miqdor bo'lib, $p(x)$ esa uning ehtimol zichligi bo'lsin.

Ushbu

$$\int_{-\infty}^{+\infty} xp(x)dx$$

xosmas integral ξ tasodifiy miqdorning matematik kutilmasi deyiladi. Demak,

$$M\xi=\int_{-\infty}^{+\infty} xp(x)dx \tag{4}$$

Ushbu

$$\int_{-\infty}^{+\infty}(x-M\xi)^2\, p(x)dx$$

xosmas integral ξ tasodifiy miqdorning dispersiyasi deyiladi. Demak,

$$D\xi=\int_{-\infty}^{+\infty}(x-M\xi)^2\cdot p(x)dx. \tag{5}$$

3-misol. Uzluksiz tasodifiy miqdor ξ ushbu

$$p(x)=\begin{cases}0,2 & -2\leq x\leq 3\\ 0 & x<-2,\quad x>3\end{cases}$$

ehtimol zichligiga ega bo'lsin. Shu tasodifiy miqdorning matematik kutilishi va dispersiyasini toping.

◀Tasodifiy miqdor ξ ning matematik kutilishi va dispersiyasini (4) va (5) formulalardan foydalanib topamiz:

$$M\xi=\int_{-\infty}^{+\infty} xp(x)dx=\int_{-\infty}^{-2} xp(x)dx+\int_{-2}^{3} xp(x)dx+\int_{3}^{+\infty} xp(x)dx=$$

$$=\int_{-2}^{3} x\cdot 0,2dx=0,2\cdot\frac{x^2}{2}\bigg|_{-2}^{3}=0,5;$$

$$D\xi=\int_{-\infty}^{+\infty}(x-M\xi)^2\cdot p(x)dx=\int_{-\infty}^{+\infty}(x-0,5)^2\cdot p(x)dx=\int_{-\infty}^{-2}(x-0,5)^2\cdot p(x)dx+$$
$$+\int_{-2}^{3}(x-0,5)^2\cdot p(x)dx+\int_{3}^{+\infty}(x-0,5)^2\cdot p(x)dx=$$
$$=\int_{-2}^{3}(x-0,5)^2\cdot 0,2dx=\frac{0,2}{3}(x-0,5)^3\Big|_{-2}^{3}=\frac{6,25}{3}\approx 2,1.$$ ▶

3⁰. Diskret tasodifiy miqdorning asosiy taqsimot qonunlari

a) Binomial taqsimot qonuni. n ta tajribada A hodisaning sodir bo'lishi soni tasodifiy miqdor bo'lib, bu ξ tasodifiy miqdorning qabul qilish mumkin bo'lgan qiymatlari

$$0, 1, 2, 3,...,n$$

ni quyidagi ehtimollar bilan qabul qilsin:

$$P(\xi=0)=q^n,\quad P(\xi=1)=C_n^1pq^{n-1},\quad P(\xi=2)=C_n^2p^2q^{n-2},...,P(\xi=k)=$$
$$=C_n^kp^kq^{n-k},...,P(\xi=n)=p^n$$

Tasodifiy miqdor ξ ning taqsimotini ifodalovchi bu qonun **binomial taqsimot qonuni** deyiladi.

Agar ξ tasodifiy miqdor deb, A hodisa i -tajribada sodir bo'lganda 1 ni, sodir bo'lmaganda 0 ni mos p va q ehtimollar bilan qabul qiladigan tasodifiy miqdor deyilsa, uning matematik kutilmasi

$$M\xi=np,$$

dispersiyasi

$$D\xi=npq,$$

o'rtacha kvadratik chetlanishi

$$\sigma=\sqrt{npq}$$

bo'ladi;

b) Puasson taqsimot qonuni. Bernulli sxemasida A hodisaning ehtimoli

$$P(A)=p_n\qquad (p_n>0)$$

bo'lib,

1) $n\to\infty$ da $p_n\to 0$,

2) $np_n=\lambda\quad (\lambda>0)\quad \lambda=const$

bo'lsa, $n\to\infty$ da

$$P_n(k)\to\frac{\lambda^k}{k!}e^{-\lambda}$$

bo'ladi.

Tajriba natijasida A hodisaning sodir bo'lishi soni ξ tasodifiy miqdor deyilsa, uning qabul qilishi mumkin bo'lgan qiymatlari

$$0, 1, 2, 3, \ldots, n, \ldots$$

bo'lib, ularni qabul qilish ehtimollari

$$P(\xi=0)=e^{-\lambda},\quad P(\xi=1)=\lambda e^{-\lambda},\quad P(\xi=2)=\frac{\lambda^2}{2!}e^{-\lambda},...,P(\xi=n)=\frac{\lambda^n}{n!}e^{-\lambda},...$$

bo'ladi.

Tasodifiy miqdor ξ ning taqsimotini ifodalovchi bu qonun **Puasson taqsimoti qonuni** deyiladi.

Puasson qonuni bo'yicha taqsimlangan tasodifiy miqdor ξ ning matematik kutilmasi

$$M\xi=\lambda,$$

dispersiyasi

$$D\xi=\lambda$$

bo'ladi.

4⁰. Uzluksiz tasodifiy miqdorning asosiy taqsimot qonunlari

a) Tekis taqsimot qonuni. Agar ξ tasodifiy miqdorning ehtimol zichligi biror oraliqda o'zgarmas funksiya bo'lib, oraliqdan tashqarida esa nolga teng bo'lsa, tasodifiy miqdor shu oraliqda tekis taqsimlangan deyiladi.

Aytayiik, ξ tasodifiy miqdorning ehtimol zichligi

$$p(x)=\begin{cases}\dfrac{1}{b-a}, & агар\ \ a\le x\le b\\ 0\ , & агар\ \ x<a,\ x>b\end{cases}$$

bo'lsin. Bu tasodifiy miqdorning taqsimot funksiyasi ushbu

$$F(x)=\begin{cases}0, & агар\quad x\le a,\\ \dfrac{x-a}{b-a}, & агар\quad a\le x\le b,\\ 1, & агар\quad x\ge b\end{cases}$$

bo'ladi.

4-misol. Tekis taqsimlangan tasodifiy miqdorning matematik kutilmasi va dispersiyasini toping.

◀(4) va (5) formulalardan foydalanib, tekis taqsimlangan tasodifiy miqdorning matematik kutilmasi va dispersiyasi topamiz:

$$M\xi=\int_{-\infty}^{+\infty}xp(x)dx=\int_{-\infty}^{a}xp(x)dx+\int_{a}^{b}xp(x)dx+\int_{b}^{+\infty}xp(x)dx=\int_{a}^{b}x\frac{1}{b-a}dx=$$
$$=\frac{1}{b-a}\cdot\frac{x^2}{2}\Big|_a^b=\frac{a+b}{2};$$

$$D\xi=\int_{-\infty}^{+\infty}(x-M\xi)^2\cdot p(x)dx=\int_a^b\left(x-\frac{b+a}{2}\right)^2\cdot\frac{1}{b-a}dx=\frac{1}{b-a}\left.\frac{\left(x-\frac{b+a}{2}\right)^3}{3}\right|_a^b=$$

$$=\frac{(b-a)^2}{12}.\blacktriangleright$$

b) Normal taqsimot qonuni. Agar uzluksiz tasodifiy miqdor ξ ning zichlik ehtimoli

$$p(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-a)^2}{2\sigma^2}}$$

bo'lsa, u normal qonun bo'yicha taqsimlangan tasodifiy miqdor deyiladi.

Bu tasodifiy miqdorning taqsimot funksiyasi

$$F(x)=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x}e^{-\frac{(t-a)^2}{2\sigma^2}}dt.$$

bo'ladi.

Ma'lumki,

$$P\{\alpha<\xi<\beta\}=F(\beta)-F(\alpha).$$

Demak,

$$P\{\alpha<\xi<\beta\}=\Phi\left(\frac{\beta-a}{\sigma}\right)-\Phi\left(\frac{\alpha-a}{\sigma}\right), \tag{6}$$

bunda

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt.$$

Eslatma. Ko'pincha, normal qonun bo'yicha taqsimlangan tasodifiy miqdor ξ va uning matematik kutilmasi (o'rta qiymat) $M\xi=a$ uchun ushbu

$$|\xi-a|<\varepsilon \quad (\varepsilon>0)$$

hodisaning ehtimolini, ya'ni

$$P\{|\xi-a|<\varepsilon\}$$

ni topishga to'g'ri keladi. Bu ehtimolni (6) formuladan foydalanib topish mumkin. (6) formulaga ko'ra

$$P\{|\xi-a|<\varepsilon\}=P\{a-\varepsilon<\xi<a+\varepsilon\}=\Phi\left(\frac{\varepsilon}{\sigma}\right)-\Phi\left(-\frac{\varepsilon}{\sigma}\right)=2\Phi\left(\frac{\varepsilon}{\sigma}\right)$$

bo'ladi. Xususan,

$$P\{|\xi-a|<3\sigma\}=2\Phi(3)=0,9973$$

bo'ladi.

5-misol. Normal qonun bo'yicha taqsimlangan tasodifiy miqdorning matematik kutilmasi va dispersiyasini toping.

◀(4) va (5) formulalardan foydalanib bu tasodifiy miqdorning matematik kutilmasi va dispersiyasini topamiz:

$$M\xi=\int_{-\infty}^{+\infty}xp(x)dx=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty}xe^{-\frac{(x-a)^2}{2\sigma^2}}dx=\left[\frac{x-a}{\sigma}=t,\quad dx=\sigma dt\right]=$$

$$=\frac{1}{\sigma\sqrt{2\pi}}\cdot\sigma\cdot\int_{-\infty}^{+\infty}(\sigma t+a)e^{-\frac{t^2}{2}}dt=\frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}te^{-\frac{t^2}{2}}dt+$$

$$+\frac{a}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-\frac{t^2}{2}}dt=\frac{\sigma}{\sqrt{2\pi}}\left[\int_{-\infty}^{0}te^{-\frac{t^2}{2}}dt+\int_{0}^{+\infty}te^{-\frac{t^2}{2}}dt\right]+\frac{a}{\sqrt{2\pi}}\cdot\sqrt{2\pi}=a.$$

Demak,

$$M\xi=a.$$

Shuningdek,

$$D\xi=\int_{-\infty}^{+\infty}(x-M\xi)^2\cdot p(x)dx=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty}(x-a)^2e^{-\frac{(x-a)^2}{2\sigma^2}}dx=\left[\frac{x-a}{\sigma}=t,\quad dx=\sigma dt\right]=$$

$$=\frac{1}{\sigma\sqrt{2\pi}}\cdot\sigma\cdot\int_{-\infty}^{+\infty}\sigma^2t^2e^{-\frac{t^2}{2}}dt=\frac{\sigma^2}{\sqrt{2\pi}}\cdot\int_{-\infty}^{+\infty}t^2e^{-\frac{t^2}{2}}dt=\left[\begin{matrix}u=t, & du=dt\\ te^{-\frac{t^2}{2}}dt=dv & v=-e^{-\frac{t^2}{2}}\end{matrix}\right]=$$

$$=\frac{\sigma^2}{\sqrt{2\pi}}\cdot\left[-te^{-\frac{t^2}{2}}\Big|_{-\infty}^{+\infty}+\int_{-\infty}^{-\infty}e^{-\frac{t^2}{2}}dt\right]=\frac{\sigma^2}{\sqrt{2\pi}}\sqrt{2\pi}=\sigma^2$$

bo'ladi. Demak,

$$D\xi=\sigma^2.\blacktriangleright$$

Quyidagi masalalarni yeching

1962. Agar har bir sinashda A hodisaning ro'y berish ehtimoli 0,6 ga teng bo'lsa, shu hodisaning 3 ta bog'liqsiz sinashda jami ro'y berishlar sonining taqsimot qonunini tuzing. Matematik kutilmasi va dispersiyani toping.

1963. Taqsimot funksiyasi $F(X)=\begin{cases}0, & агар\ x\le-2\\ 1/3, & агар\ -2<x\le0\\ 2/3, & агар\ 0<x\le2\\ 1, & агар\ x>2\end{cases}$ bo'lgan t.m. matematik kutilma va dispersiyasini toping

1964. Simmetrik tanga 7 marta tashlanadi. Tushgan gerb tomonlari soni ξ tasodifiy miqdorning matematik kutilmasi va dispersiyasini hisoblang.

1965. Nishonga toki 2 ta o'q tegmaguncha o'q uzilmoqda. Nishonga qarata o'q uzishlar sonining matematik kutilmasi va dispersiyasini hisoblang (Nishonga tegishi ehtimoli 0,2).

1966. ξ – diskret tasodifiy miqdorning taqsimot qonuni berilgan:

x_i	–2	–1	0	1	2	3
p_i	0,1	0,2	0,25	1,15	0,1	0,2

Bu tasodifiy miqdorning matematik kutilmasi va dispersiyasini hisoblang.

1967. Nomerlangan kub 10 marta tashlanganda kublar ustida tushgan ochkolar soni ξ tasodifiy miqdorni $M\xi$ va $D\xi$ hisoblang.

1968. ξ tasodifiy miqdorning matematik kutilmasi $\frac{7}{2}$ va dispersiyasi $35/12$ bo'lsa, $4\xi-1$ tasodifiy miqdorning matematik kutilmasi va dispersiyasini hisoblang.

1969. ξ tasodifiy miqdorning zichlik funksiyasi berilgan:

$$p(x)=\begin{cases}\frac{3}{26}(x-3)^2, & x\in[0,2]\\ 0, & x\notin[0,2]\end{cases}$$

$M\xi$ va $D\xi$ hisoblang.

1970. Korxona ishlab chiqaradigan mahsulotning 20% ini qo'shimcha qayta ishlash kerak. Tavakkaliga 150 ta mahsulot tanlanadi. Tanlangan mahsulotlar ichida qo'shimcha qayta ishlanishi kerak bo'lgan mahsulotlar soni ξ bo'lsa, uning matematik kutilmasi va dispersiyasini hisoblang.

1971. 1 ta otishda nishonga tekkizish ehtimoli 0,4 bo'lsa, o'rtacha nishonga 80 marta tekizish uchun necha marta o'q uzish kerak?

1972. ξ diskret tasodifiy miqdor $a=0{,}324$ parametrli Puasson qonuni bo'yicha taqsimlangan. Bu tasodifiy miqdorning matematik kutilmasi va o'rtacha kv tarqoqligini hisoblang.

1973. Do'konga 1000 dona meneral suv yuborilgan. Jo'natish vaqtida meneral suv idishining sinishi ehtimoli 0,002 bo'lsa, singan shishalar sonining o'rta qiymatini toping.

1974. ξ tasodifiy nuqta – $[a,b]$ oralig'ida tekis taqsimlangan. ξ tasodifiy miqdorning $[\alpha,\beta]$ oraliqqa tegishli bo'lishi ehtimolini hisoblang ($[\alpha,\beta]\subset[a,b]$).

1975. ξ tasodifiy miqdorning zichlik funksiyasi quyidagi ko'rinishga ega:

$$p(x)=\begin{cases}0,25\cdot A, & x\in[0,4]\\ 0, & x\notin[0,4]\end{cases}$$

A, $F(x)$, $M\xi$, $D\xi$ larni hisoblang.

1976. Kimdir soat 19^{00} va 20^{00} orasida telefon qo'ng'irog'ini kutmoqda. Qo'ng'iroqni kutish vaqti $\xi[19,20]$ oraliqdagi tekis taqsimotga ega. Qo'ng'iroq 19:22 dan 19:46 gacha bo'lishi ehtimolini toping.

1977. 1-kurs talabalarining o'zlashtirishi 80% ni tashkil qiladi. Tavakkaliga tanlangan 50 ta 1-kurs talabalari ichida o'zlashtirganlari soni ξ tasodifiy miqdorning matematik kutilmani va dispersiyasini hisoblang.

1978. Lampochkalarning 90% i 800 soat ishlagandan so'ng buziladi. ξ tasodifiy miqdor lampochkaning beto'xtov izlash vaqti bo'lsa, bu tasodifiy miqdorning 100 dan 200 gacha oraliqda o'zgarishini hisoblang.

5-§. Ehtimollar nazariyasining limit teoremalari

1°. Chebeshev tengsizligi. Agar ξ tasodifiy miqdorning matematik kutilmasi $M\xi$, dispersiyasi $D\xi$ bo'lsa, u holda ixtiyoriy $\varepsilon>0$ uchun

$$P\{|\xi-M\xi|\ge\varepsilon\}\le\frac{D\xi}{\varepsilon^2} \qquad (1)$$

bo'ladi.

(1) **Chebishev tengsizligi** deyiladi.

Chebishev tengsizligi tasodifiy miqdorning qabul qilishi mumkin bo'lgan qiymatlarini uning matematik kutilmasi (o'rta qiymati) atrofida joylashish darajasini ifodalaydi.

Chebishev tengsizligini taqsimoti noma'lum bo'lgan, tasodifiy miqdorga bog'liq hodisa ehtimolini baholash uchun ishlatish mumkun.

Chebishev tengsizligini quyidagicha ham yozish mumkin:

$$P\{|\xi-M\xi|<\varepsilon\}\ge 1-\frac{D\xi}{\varepsilon^2} \qquad (2)$$

1-misol. ξ diskret tasodifiy miqdor quyidagi taqsimot qonuni bilan berilgan:

x_i	0	2	6	10
p_i	0,2	0,3	0,4	0,1

Chebishev tengsizligi yordamida $|\xi-M\xi|<5$ hodisa ehtimolini baholang.

◀Avval, ξ tasodifiy miqdorning matematik kutilmasini va dispersiyasini hisoblaymiz:

$M\xi=0\cdot0{,}2+2\cdot0{,}3+6\cdot0{,}4+10\cdot0{,}1=4;$

$D\xi=0^2\cdot0{,}2+2^2\cdot0{,}3+6^2\cdot0{,}4+10^2\cdot0{,}1-4^2=25{,}6-16=9{,}6.$

(2) formulaga ko'ra quyidagiga ega bo'lamiz:

$$P\{|\xi-4|<5\}\ge1-\frac{9{,}6}{5^2}=1-0{,}384=0{,}616. ▶$$

2⁰. Limit teorema. (Chebishev teoremasi). Aytaylik,

$$\xi_1, \xi_2, ..., \xi_n, ... \qquad (3)$$

o'zaro bog'liq bo'lmagan tasodifiy miqdorlar ketma-ketligi bo'lsin.

Agar shunday $c > 0$ mavjud bo'lsaki,

$$D\xi_i \le c \qquad (i = 1, 2, 3, ...)$$

bo'lsa, u holda ixtiyoriy $\varepsilon > 0$ uchun

$$P\left\{\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i - \frac{1}{n}\sum_{i=1}^{n}M\xi_i\right| < \varepsilon\right\} \ge 1 - \frac{c}{n\varepsilon^2}$$

bo'ladi. Keyingi munosabatdan

$$\lim_{n\to\infty} P\left\{\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i - \frac{1}{n}\sum_{i=1}^{n}M\xi_i\right| < \varepsilon\right\} = 1$$

bo'lishi kelib chiqadi.

Odatda, ehtimoli birga yaqin bo'lgan hodisa deyarli muqarrar hodisa deb qaraladi. Bunda

$$\frac{1}{n}\sum_{i=1}^{n}\xi_i \approx \frac{1}{n}\sum_{i=1}^{n}M\xi_i$$

taqribiy formula hosil bo'ladi. Demak, n ning yetarlicha katta qiymatlarida tasodifiy miqdorlarning o'rta arifmetigi

$$\frac{\xi_1 + \xi_2 + ... + \xi_n}{n}$$

deyarli o'zgarmas miqdorga teng bo'lishini ifodalaydi.

2-misol. Elektrolampaning o'rtacha chidamliligi muddatini aniqlash uchun 200 ta bir xil yashikdan iborat partiyadan har bir yashikdan tavakkaliga 1 tadan lampa olinadi. Olingan lampalarning o'rtacha chidamlilik muddati partiyadagi barcha lampalarning o'rtacha chidamlilik muddati orasidagi farqining absolyut qiymati $5r$ dan kichik bo'lishi ehtimolini quyidan baholang (har bir yashikdagi lampalarning o'rtacha chidamlilik muddatining o'rtacha kvadratik tarqoqligi $7r$ dan kichik).

◀ ξ i – ichi yashikdan olingan lampaning chidamlilik muddati bo'lsin. $D\xi_i < 7^2 = 49$ berilgan. Tanlangan lampalarning o'rtacha chidamlilik muddati.

$$\frac{\xi_1 + \xi_2 + \cdots + \xi_{200}}{200}$$

ga teng. Butun partiyaniki esa

$$\frac{M\xi_1 + M\xi_2 + ... + M\xi_{200}}{200}$$

ga teng.

$$P\left\{\left|\frac{\xi_1 + \xi_2 + ... + \xi_{200}}{200} - \frac{M\xi_1 + M\xi_2 + ... + M\xi_{200}}{200}\right| < S\right\}$$

quyidan baholash kerak. $\xi_1, \xi_2, ..., \xi_{200}$ bog'liqsiz tan.nuq. lar bo'lgani uchun (2) tengsizlik o'ng qismidan foydalanib bu ehtimollikni quyidan baholash mumkin. Bu yerda $C = 49$, $\varepsilon = 5$, $n = 200$. Demak,

$$P \ge 1 - \frac{49}{200 \cdot 25} = 1 - \frac{49}{5000} = 1 - 0,0098 = 0,9902. ▶$$

Quyidagi masalalarni yeching

1979. Elektrostansiya 18000 ta lampa tarmog'iga xizmat ko'rsatadi. Har bir lampaning qish kechasi yonishi ehtimoli 0,9 ga teng. Qish kechasi tarmoqda yongan lampalar soni o'zining matematik kutilmasidan farqining absolyut qiymati 200 dan ko'p bo'lmasligi ehtimolini hisoblang.

1980. Detal uzunligining o'rta qiymati 50sm, dispersiya esa 0,1 ga teng. Chebishev tengsizligidan foydalanib, tavakkaliga olingan detal uzunligi 49,5 sm va 50,5 sm orasida bo'lish ehtimolini toping.

1981. ξ tasodifiy miqdor taqsimoti quyidagi jadvalda keltirilgan:

x	–1	0	2	4	6
$p(x)$	0,2	0,4	0,3	0,05	0,05

$|\xi - M\xi| < 5$ hodisa ehtimolini toping. Chebishev tengsizligidan foydalanib bu ehtimollikni baholang.

1982. Yilning yomg'irli kunlari $M\xi = 100$ bo'lgan tasodifiy miqdor bo'lsa, keyingi yilda yomg'irli kunlar 140 dan kam bo'lishi ehtimolini baholang.

1983. Avtoparkda 200 ta avtomobil bor. Ularning har biri biror t vaqt mobaynida bir-biriga bog'liq bo'lmagan holda 0,04 ehtimollik bilan ishdan chiqishi mumkin. Avtomobillar qismi, ixtiyoriy avtomobilning beto'xtov ishlashi ehtimolidan moduli bo'yicha farqi 0,1 dan ko'p bo'lmasligi ehtimolini baholang.

1984. Poyezd 49 ta vagondan iborat. Vagonning og'irligi ξ t.m. va $M\xi = 60$ t, $\delta(\xi) = 7$ t. Agar poyezdning og'irligi 3000 t. dan ortmasa, lokomativ poyezdni uni yurgaza oladi. Aks holda, qo'shimcha lokomativ ulanadi. Qo'shimcha lokomativ ulanadi. Qo'shimcha lokomativ ulanmasligi ehtimolini hisoblang.

1985. 5000 ga maydondagi o'rtacha hosildorlikni aniqlash uchun tavakkaliga har gektardan $1m^2$ maydon tanlanadi va bu maydonlardagi hosildorlik aniqlanadi. Tanlangan maydon o'rtacha hosildorligi umumiy maydon o'rtacha hosildorlikdan farqi 0,2 s dan oshmasligi ehtimolini baholash. Bu yerda har gektarning hosildorligi o'rtacha kvadratik tarqoqligi 5 s dan oshmaydi.

Nazorat savollari

1. Tasodifiy hodisalar deb nimaga aytiladi?
2. Hodisalar ustida amallarni izohlab bering.
3. Ehtimollarni qo'shish va ko'paytirish teoremalarini keltring.
4. To'la ehtimol formulasini izohlab bering.
5. Bayes formulasini izohlab bering.
6. Bernulli tajribalari sxemasini izohlab bering.
7. Bernulli formulasini izohlab bering.
8. Laplasning lokal teoremasini izohlab bering.
9. Laplasning integral teoremasini izohlab bering.
10. Diskret tasodifiy miqdorlar va ularning taqsimot funksiyalarini izohlab bering.
11. Uzluksiz tasodifiy miqdorlar va ularning taqsimot funksiyalarini izohlab bering.
12. Diskret tasodifiy miqdorning matematik kutilmasi va dispersiyasi deb nimaga aytiladi?
13. Uzluksiz tasodifiy miqdorning matematik kutilmasi va dispersiyasi deb nimaga aytiladi?
14. Diskret tasodifiy miqdorning asosiy taqsimot qonunlarini izohlab bering.
15. Uzluksiz tasodifiy miqdorning asosiy taqsimot qonunlarini izohlab bering.
16. Chebeshev tengsizligini izohlab bering.
17. Chebishev teoremasini izohlab bering.

Javoblar

11-bob

1346. $\sqrt{17}$. **1347. a)** I, **b)** VII. **1348. a)** $x<0, y<0, z>0$, **b)** $x<0, y>0, z<0$, **c)** $x>0, y<0, z<0$.. **1349.** $a) M(x,0,0)$, $b) M(0,y,0)$, $c) M(0,0,z)$. **1350.** $a) M(x,y,0)$, $b) M(0,y,z)$, $c) M(x,0,z)$. **1351.** $A-OX$ o'qida, $B-OY$ o'qida, $C-OZ$ o'qida, $D-YOZ$ tekisligida, $E-XOY$ tekisligida. **1352. a)** $5\sqrt{2}$, **b)** $4\sqrt{3}$.

1353. $A\left(\frac{14}{3},-8,12\right)$, $B\left(-\frac{11}{3},7,-13\right)$, $D\left(\frac{4}{3},-2,2\right)$, $E\left(-\frac{1}{3},1,-3\right)$.

1354. $M(-6,-4,3)$. **1355.** $M_1\left(4\sqrt{2},-4,4\right)$, $M_2\left(4\sqrt{2},4,4\right)$. **1356.** $\left(6,3,\frac{20}{3}\right)$.

1357. $(4,-1,3)$. **1360. a)** $(x-1)-2(y-2)+3(z+3)=0$, $4x+y-2z-3=0$ **b)** $3x+2y+z-8=0, 3x+3y+z-8=0$. **1361.** A, B va E. **1362.** a) tekislik OY o'qida parallel bo'ladi; b) tekislik, XOY tekisligiga parallel bo'ladi; c) tekislik, OZ o'qiga parallel bo'ladi; d) tekislik koordinata boshidan o'tadi; e) tekislik OZ o'qi orqali o'tadi; f) tekislik, OX o'qida parallel bo'ladi. **1363.** $z-3=0$. **1364. a)** masalan $2x+y-8z+5=0$;

b) $2(x-1)+4(y-1)+(z-1)=0$. **1365.** $3x+z=0$. **1366.** $9x-y+7z-40=0$

1367. $z=3$. **1368. a)** 0, **b)** 13,5, **c)** $\frac{5}{7}$, **d)** 4. **1369. a)** $\frac{x}{-2}+\frac{y}{-\frac{6}{5}}+\frac{z}{\frac{6}{7}}=1$,

b) $\frac{x}{3}+\frac{y}{6}+\frac{z}{-\frac{6}{5}}=1$, **c)** $\frac{x}{4}+\frac{y}{3}+\frac{z}{-4}=1$. **1370. a)** $\arccos 0{,}7$, **b)** $\frac{\pi}{2}$, **c)** 0,

d) $\arccos\frac{59}{63}$. **1371. a)** $\frac{5}{3}$, **b)** $\frac{3}{14}$. **1375. a)** $\frac{x-2}{1}=\frac{y}{2}=\frac{z+1}{3}$,

$\frac{x-2}{11}=\frac{y}{10}=\frac{z+1}{3}$) **b)** $\frac{x-2}{2}=\frac{y+1}{7}=\frac{z}{4}$, **c)** $\frac{x}{-5}=\frac{y+1}{12}=\frac{z-1}{13}$,

d) $\frac{x}{9}=\frac{y}{5}=\frac{z+3}{1}$. **1376. a)** $\cos\varphi=\frac{72}{77}$ $\left(\varphi=\arccos\frac{72}{77}\right)$, **b)** $\varphi=\frac{\pi}{2}$.

1377. $\cos\alpha=\frac{1}{\sqrt{6}}$, $\cos\beta=\frac{2}{\sqrt{6}}$, $\cos\gamma=\frac{1}{\sqrt{6}}$. **1378.** $\frac{x-2}{3}=\frac{y-5}{7}=\frac{z-4}{4}$.

1379. $\varphi=\frac{\pi}{3}$. **1380.** $m=3$. **1381.** $\frac{x+2}{0}=\frac{y+3}{1}=\frac{z-5}{0}$.

1382. $\frac{x-1}{1}=\frac{y+5}{\sqrt{2}}=\frac{z-3}{-1}$. **1383.** $\frac{1}{2}\sqrt{46}$. **1386.** $\arcsin\frac{18}{91}$.

1387. $\frac{x-3}{5}=\frac{y+2}{3}=\frac{z-4}{-7}$. **1388.** $8x-9y-22z-59=0$.

1389. $\frac{x}{33}=\frac{y}{-26}=\frac{z}{27}$. **1390.** $x^2+y^2+z^2-4x+4y-2z=0$.

1391. a) $(x+1)^2+(y+2)^2+z^2=9$, $r=3$ bo'lgan shar,

$x+1=\xi$, $y+2=\eta$, $z=\zeta$. $\xi^2+\eta^2+\zeta^2=9$.

b) $\frac{\xi^2}{4}+\frac{\eta^2}{2}+\frac{\zeta^2}{4}=1$, ellipsoid, $a=2$, $b=\sqrt{2}$, $c=2$,

c) $\frac{\xi^2}{4}+\frac{\eta^2}{2}-\frac{\zeta^2}{4}=1$ – bir pallali giperboloid, $a=2$, $b=\sqrt{2}$, $c=2$,

d) $\xi^2+2\eta^2=2\zeta$ – elliptik paraboloid, $p=1$, $q=\frac{1}{2}$,

e) $\frac{\xi^2}{4}-\eta^2-\frac{\zeta^2}{4}=1$ – ikki pallali giperboloid,

f) $\frac{\xi^2}{4}+\frac{\eta^2}{2}=1$ – elliptik silindr. **1392.** 1) yo'naltiruvchisi $x^2+z^2=9$ aylanadan iborat bo'lgan silindrik sirt; 2) yo'naltiruvchisi $\frac{y^2}{25}-\frac{z^2}{16}=1$ giperboladan iborat bo'lgan silindrik sirt; 3) yo'naltiruvchisi $y^2=6z$ paraboladan iborat bo'lgan silindrik sirt; 4) Ikkita $x-z=0$, $x+z=0$ tekisliklar; 5) $y=0$, $z=0$; OX o'qidan iborat to'g'ri chiziq; 6) fazoning biror nuqtasi berilgan tenglamani kanoatlantirmaydi; 7) YOZ koordinata tekisligi; 8) OZ o'qi; 9) $O(0,0,0)$ nuqta. **1393.** $x^2+\frac{y^2}{16}-\frac{z^2}{16}=1$. **1394.** $z=c$.

1395. a) $(3,4,-2)$, $(6,-2,2)$; b) to'g'ri chiziq hamda sirt umumiy nuqtaga ega emas. **1396.** $b=3$, $c=\sqrt{3}$.

12-bob

1397. a) 5,**b)** 3. **1398.** $\vec{a}+\vec{b}=\vec{i}-\vec{j}+6\vec{k}$, $\vec{a}-\vec{b}=5\vec{i}-3\vec{j}+6\vec{k}$, $3\vec{a}+2\vec{b}=5\vec{i}-4\vec{j}+18\vec{k}$. **1399.** $|\vec{a}|=2$, $\alpha=\frac{\pi}{3}$, $\beta=\frac{2\pi}{3}$, $\gamma=\frac{\pi}{4}$. **1400.** $x=y=z=\sqrt{3}$. **1401. a)** $\frac{3\pi}{4}$,**b)** $\frac{\pi}{3}$. **1402.** $\vec{a}^0=\frac{3}{7}\vec{i}-\frac{4}{7}\vec{j}+\frac{6}{7}\vec{k}$. **1403.** $\cos\alpha=\frac{1}{\sqrt{3}}$, $\cos\beta=\frac{1}{\sqrt{3}}$, $\cos\gamma=\frac{1}{\sqrt{3}}$.

1405. $|\vec{F}|=10$, $\alpha=\frac{\pi}{2}$, $\beta=0$, $\gamma=\frac{\pi}{2}$. **1406.** $\sqrt{129}$; 7. **1407.** $K_1(37,51,68)$, $K_2(-35,-45,-58)$. **1408.** 15. **1411.** $(\vec{a},\vec{b})=9$. **1412.** $(2\overrightarrow{BA}-\overrightarrow{CA},\ 2\overrightarrow{AC}-\overrightarrow{AB})=56$,

$\overrightarrow{BA}^2=29$, $\overrightarrow{CA}^2=108$. **1413.** $\sqrt{13}$. **1414.** $\varphi=\arccos\frac{2}{15}$. **1415.** $\varphi=\arccos\frac{18}{\sqrt{494}}$.

1416. $\arccos\frac{5}{\sqrt{133}}$. **1417.** $\lambda=-5$. **1418.** $|[\vec{a},\vec{b}]|=54$. **1419.** $(\vec{a},\vec{b})=\pm30$.

1420. $[\vec{a},\vec{b}]=4\vec{i}+7\vec{j}+13\vec{k}$. **1421.** $\sqrt{3}$; $5\sqrt{3}$. **1422.** $\frac{1}{2}\sqrt{3}$. **1423.** $S=8\sqrt{3}$.

1424. $\vec{M}=[\overrightarrow{OA},\vec{F}]=\{8,9,4\}$. **1425.** 20. **1426.** 2 ish br. **1427.** $-10\vec{i}+13\vec{j}+11\vec{k}$; $\alpha\approx120^0$, $\beta\approx49^0$, $\gamma\approx56^0$. **1428.** $|M|=15$, $\cos\alpha=\frac{1}{3}$, $\cos\beta=-\frac{2}{3}$, $\cos\gamma=-\frac{1}{3}$.

1431. $5x+y-3z+27=0$. **1432.** $3x+4y+12z+39=0$, $3x+4y+12z-39=0$.

1433. $9x+y+11z-7=0$. **1434.** $\cos\varphi=\pm\frac{98}{195}$. **1435.** $\frac{x+1}{1}=\frac{y}{\sqrt{2}}=\frac{z-5}{-1}$.

1436. $\frac{x-1}{2}=\frac{y+3}{-4}=\frac{z-5}{-5}$. **1437.** $2\sqrt{10}$. **1438.** $\cos\alpha=\frac{1}{\sqrt{6}}$, $\cos\beta=\frac{2}{\sqrt{6}}$, $\cos\gamma=\frac{1}{\sqrt{6}}$.

13-bob

1449. 1) tomonlari koordinata o'qlariga parallel bo'lgan to'g'ri to'rtburchak soha; 2) uchburchak ko'rinishidagi soha; 3) yarim tekislik soha; 4) uchburchak soha; 5) markazi koordinata boshida, radiusi 1 ga teng bo'lgan doira – doiraviy soha; 6) markazi koordinata boshida, radiuslari 1 va 2 bo'lgan konsentrik aylanalar orasidagi soha – halqa. **1450.** 1) tekislikning barcha nuqtalar to'plami: R^2; 2) yuqori yarim tekislik, bunda OX o'qining barcha nuqtalari ham, aniqlanish sohaga tegishli bo'ladi; 3) tekislikning $(0,0)$ nuqtadan boshqa barcha nuqtalari to'plami: $R^2\backslash\{(0,0)\}$; 4) I va III kvadratni nuqtalari to'plami; 5) $y=x$ o'qi to'g'ri chiziq nuqtalari va bu to'g'ri chiziqning o'ng tomonidan yarim tekislik nuqtalaridan iborat to'plam; 6) markazi koordinata boshida, radiusi 1 ga teng bo'lgan doira, bunda $x^2+y^2=1$ aylananing nuqtalari ham funksiyaning aniqlanish sohasiga kiradi; 7) $x+y=0$ to'g'ri chiziqning yuqori tomonida joylashgan yarim tekislik; 8) $x\geq0$, $y\geq0$, $x\geq\sqrt{y}$; 9) $y^2>4(x-2)$; 10) $|y|\leq|x|$, $(x\neq0)$. **1451.** $f(1,0)=1$, $f(1,1)=2$, $f(2,1)=\frac{9}{2}$.

1452. $f(2,-3)=-\frac{13}{12}$. **1453.** $f(x,y)=\frac{1}{8}(x^2-y^2)$. **1454.** a) mavjud emas, b) mavjud, v) mavjud emas, g) mavjud. **1455.** 1) $x+y=0$ to'g'ri chiziqqa parallel bo'lgan to'g'ri chiziqlar. 2) parallel to'g'ri chiziqlar. 3) koordinata boshidan o'tuvchi to'g'ri chiziqlar $(x\neq0, y\neq0)$. 4) teng yonli giperboialar. 5) markazi koordinata boshida bo'lgan konuentrik aylanalar. 6) o'qi OY bo'lgan parabolalar. 7) konuentrik aylanalar. 8) teng tomonli giperbolalar. 9) kvadratning konturlari.

10) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-c$ ko'rinishidagi ellipslar $(c<1)$. **1456.** 1) tekislik. 2) aylanma parabolid. 3) giperbolik parabolid. 4) konus. 5) sfera (markazi koordinata boshida, radiusi 1 ga teng sferaning XOY tekisligidan yuqori qismida joylashgan sfera. **1457.** a) -3, b) 1. **1458.** a) $+\infty$, b) 0. **1459.** a) mavjud emas, b) 2. **1460.** a) e^a, b) 2. **1461.** a) 0, b) 1). **1462.** a) tekislikning barcha nuqtalarida uzluksiz, b) $y=-x$ to'g'ri chiziq nuqtalarida uzilishga ega bo'lib, tekislikning qolgan barcha nuqtalarida uzluksiz. **1463.** a) markazi koordinata boshida, radiusi 2 ga teng aylananing nuqtalarida uzilishga ega bo'lib, tekislikning qolgan barcha nuqtalarida uzluksiz, b) $x+2y+1=0$ to'g'ri chiziq nuqtalarida uzilishga ega bo'lib, tekislikning qolgan barcha nuqtalarida uzluksiz. **1464.** a) $(0,0)$ nuqtada uzilishga ega bo'lib, qolgan barcha nuqtalarda uzluksiz, b) $y^2=-x$ da uzilishga ega bo'lib, qolgan barcha nuqtalarda uzluksiz. **1465.** a) $y=x$ to'g'ri chiziq nuqtalarida uzilishga ega bo'lib, qolgan nuqtalarda uzluksiz, b) $x^2+y^2=9$ aylanada uzilishga ega bo'lib, qolgan barcha funksiya aniqlanish sohasidagi nuqtalarda uzluksiz. **1466.** a) koordinata o'qlarida uzilishga ega, b) $x=0, y=0$ chiziqlarda uzilishga ega. **1467.** a) $y=0$ to'g'ri chiziqda uzilishga ega, b) $x^2+y^2=9$ aylanada uzilishga ega. **1468.** b) $x^2+y^2=4$ aylanada uzilishga ega. **1469.** $x=0, y=0$ da uzilishga ega.

1476. a) $\frac{\partial z}{\partial x}=3x^2-2y$, $\frac{\partial z}{\partial y}=2y^2-2x$; b) $\frac{\partial z}{\partial x}=10x+8y^2$, $\frac{\partial z}{\partial y}=16xy+3y^2$. **1477.** a) $\frac{\partial z}{\partial x}=3x^2-3ay$, $\frac{\partial z}{\partial y}=3y^2-3ax$;

b) $\frac{\partial z}{\partial x}=2x-2y$, $\frac{\partial z}{\partial y}=-2x+2y$. **1478.** a) $\frac{\partial z}{\partial x}=\frac{y^2}{(x+y)^2}$, $\frac{\partial z}{\partial y}=\frac{x^2}{(x+y)^2}$; b) $\frac{\partial z}{\partial x}=\frac{2y}{(x+y)^2}$, $\frac{\partial z}{\partial y}=-\frac{2x}{(x+y)^2}$.

1479. a) $\frac{\partial z}{\partial x}=-\frac{y}{x^2}$, $\frac{\partial z}{\partial y}=\frac{1}{x}$; b) $\frac{\partial z}{\partial x}=\frac{y}{2|y|\sqrt{xy}}$, $\frac{\partial z}{\partial y}=\frac{-x}{2|y|\sqrt{xy}}$. **1480.** a) $\frac{\partial z}{\partial x}=\frac{x}{\sqrt{x^2-y^2}}$, $\frac{\partial z}{\partial y}=-\frac{y}{\sqrt{x^2-y^2}}$;

b) $\frac{\partial z}{\partial x}=\frac{1}{2\sqrt{x+3y}}$, $\frac{\partial z}{\partial y}=\frac{3}{2\sqrt{x+3y}}$. **1481.** a) $\frac{\partial z}{\partial x}=-\frac{1}{2\sqrt{x}\left(\sqrt{x}-\sqrt{y}\right)^2}$, $\frac{\partial z}{\partial y}=\frac{1}{2\sqrt{y}\left(\sqrt{x}-\sqrt{y}\right)^2}$;

b) $\frac{\partial z}{\partial x}=\frac{y^2}{\left(x^2+y^2\right)^{3/2}}$, $\frac{\partial z}{\partial y}=-\frac{xy}{\left(x^2+y^2\right)^{3/2}}$.

1482. a) $\frac{\partial z}{\partial x}=\frac{y}{2\sqrt{x}}+\frac{1}{\sqrt{y}}$, $\frac{\partial z}{\partial y}=\sqrt{x}-\frac{x}{2y\sqrt{y}}$; **b)** $\frac{\partial z}{\partial x}=\frac{y^2}{\left(x^2+y^2\right)^{3/2}}$, $\frac{\partial z}{\partial y}=-\frac{xy}{\left(x^2+y^2\right)^{3/2}}$. **1483.** $f'_x(0,1)=2$,

$f'_y(0,1)=0$. **1484.** $f'_x(2,1)=\frac{1}{2}$, $f'_y(2,1)=0$. **1489.** $dz=\left(2xy-y^2\right)dx+\left(x^2-2xy\right)dy$. **1490.** $dz=6\left(x^2+y^2\right)^2xdx+6\left(x^2+y^2\right)^2ydy$.

1491. $dz=\sin 2xdx-\sin 2ydy$. **1492.** $dz=\frac{y}{1+x^2y^2}dx+\frac{x}{1+x^2y^2}dy$.

1493. $dz=e^{12x+5y}\left(12dx+5dy\right)$.

1494. $dz=(\sin x)^{\cos y}\left[\cos y\cdot ctgxdx-\sin y\cdot\ln\sin xdy\right]$. **1495.** $dz=\frac{1}{x+y}\left(dx-\frac{x}{y}dy\right)$.

1496. $dz=e^{xy}\left[\left(\frac{1}{y}+x\right)dx+\frac{x}{y}\left(x-\frac{1}{y}\right)dy\right]$. **1497.** $dz=\frac{1}{2\sqrt{xy}\,(1+xy)}(ydx+xdy)$.

1498. $df(1,1)=dx-2dy$. **1499.** $-0{,}1\cdot e^2$. **1500.** $\frac{4}{3}$. **1501.** $-0{,}008$. **1502.** **a)** $\approx -1{,}32$; **b)** $\approx 4{,}24$. **1503.** a) $\approx -0{,}05$; b) $\approx 1{,}05$. **1504.** $-\sin 2t\cdot e^{1+\cos^2 t}$. **1505.** $\frac{dz}{dt}=\frac{e^t(t\ln t-1)}{t\ln^2 t}$

1506. $\frac{\partial z}{\partial u}=\frac{2x}{y}\left(1-\frac{z}{y}\right)$, $\frac{\partial z}{\partial v}=-\frac{x}{y}\left(4+\frac{x}{y}\right)$. **1507.** $\frac{\partial z}{\partial t}=\frac{1}{\sqrt{y}}ctg\frac{x}{\sqrt{y}}\left(6-\frac{x}{2y^2}\right)$.

1508. $\frac{\partial z}{\partial u}=2u\cos 2v$, $\frac{\partial z}{\partial v}=-2u^2\sin 2v$. **1509.** $\frac{\partial z}{\partial x}=(\sin)^{\cos x}\left(\cos xctgx-\sin x\cdot\ln\sin x\right)$.

1510. $\frac{\partial z}{\partial x}=-\frac{aby}{a^2x^2-b^2y^2}$, $\frac{\partial z}{\partial y}=-\frac{abx}{a^2x^2-b^2y^2}$. **1511.** $\frac{\partial z}{\partial u}=\frac{y}{x}e^{\frac{y^2}{x}}\left(-\frac{y}{x}\frac{\partial x}{\partial u}+2\frac{\partial y}{\partial u}\right)$, $\frac{\partial z}{\partial v}=\frac{y}{x}e^{\frac{y^2}{x}}\left(-\frac{y}{x}\frac{\partial x}{\partial v}+2\frac{\partial y}{\partial v}\right)$.

1512. $\frac{d^2z}{dx^2}=12x^2-8y^2$, $\frac{d^2z}{dy^2}=12y^2-8x^2$, $\frac{d^2z}{dxdy}=-16xy$.

1513. $\frac{d^2z}{dx^2}=e^x\ln y$, $\frac{d^2z}{dy^2}=\frac{e^x}{y^2}$, $\frac{d^2z}{dxdy}=\frac{e^x}{y}$.

1514. $\frac{d^2z}{dx^2}=2\frac{y-x^2}{(x^2+y)^2}$, $\frac{d^2z}{dy^2}=\frac{-1}{(x^2+y)^2}$, $\frac{d^2z}{dxdy}=\frac{-2x}{(x^2+y)^2}$.

1515. $\frac{d^2z}{dx^2}=\frac{-y^2}{(2xy+y^2)^{3/2}}$, $\frac{d^2z}{dy^2}=\frac{-x}{(2xy+y^2)^{3/2}}$, $\frac{d^2z}{dxdy}=\frac{xy}{(2xy+y^2)^{3/2}}$.

1516. $\frac{d^2z}{dx^2}=\frac{2}{2y-3}$, $\frac{d^2z}{dy^2}=\frac{8x^2}{(2y-3)^3}$, $\frac{d^2z}{dxdy}=\frac{4x}{(2y-3)^2}$. **1517.** $\frac{d^2z}{dx^2}=e^x\ln y-\frac{\sin y}{x^2}$, $\frac{d^2z}{dy^2}=-\frac{e^x}{y^2}-\sin y\ln x$, $\frac{d^2z}{dxdy}=\frac{e^x}{y}+\frac{\cos y}{x}$.

1518. $\frac{d^2z}{dx^2}=\frac{y}{x^2}e^{\frac{z}{x}}\left(\frac{y}{x}-2\right)$, $\frac{d^2z}{dy^2}=-\frac{1}{x^2}e^{\frac{z}{x}}$, $\frac{d^2z}{dxdy}=\frac{1}{x^2}e^{\frac{z}{x}}\left(1-\frac{y}{x}\right)$.

1519. $\frac{d^2z}{dx^2}=\frac{1+\sin^2x}{4\sqrt{\sin^2x}}$, $\frac{d^2z}{dy^2}=-\frac{x}{y^2}-\sin y\ln x$, $\frac{d^2z}{dxdy}=\frac{1}{y}$.

1521. $\frac{d^2z}{dxdy^2}=-x^2y\cos(xy)-2x\sin(xy)$. **1522.** $\frac{d^2z}{dxdy^2}=-16x$.

1523. $\frac{d^2z}{dx^2dy}=-\sin(x-y)$. **1524.** $\frac{d^2z}{dx^2dy}=\frac{2}{(x+y)^2}$. **1525.** $\frac{d^2z}{dx^2dx^2}=\frac{15xy}{(1+x^2+y^2)^{7/2}}$.

1526. $d^2z=4dx^2-dxdy+2dy^2$. **1527.** $d^2z=e^{xy}\left[(ydx+xdy)^2+2dy^2)\right]$.

1528. $d^2z=-\frac{2}{y^3}dy(ydx-xdy)$. **1529.** $d^2z=\frac{(y^2-x^2)(dx^2-dy^2)-4xydxdy}{(x^2+y^2)^2}$.

1530. $d^2f(1,2)=6dx^2+2dxdy+4,5dy^2$.

1531. $d^2z=\sin x\cos ydx^2-2\cos x\sin ydxdy- -\sin x\cos ydy^2$.

1536. $2x+4y-z=0$. **1537.** $4(x-2)-2(y-1)-(z-3)=0$, . $\frac{x-2}{4}=\frac{y-1}{-2}=\frac{z-3}{-1}$. **1541.** $x+y=1\pm\sqrt{2}$. **1542.** $\left(0,\frac{115}{27},\frac{164}{27}\right)$.

1543. $y+xy+\frac{1}{3!}(3x^2y-y^3)$. **1544.** $y+\frac{1}{2!}(2xy-y^2)+\frac{1}{3!}(3x^2y-3xy^2+2y^3)$.

1545. $1-\frac{1}{2}(x^2+y^2)$. **1546.** $\left(\frac{1}{3},\frac{4}{3}\right)$. **1547.** $(0,0),\left(1,\frac{1}{2}\right)$. **1548.** $(0,0),(1,1)$.

1549. $(1,2),(2,1),(-1,-2),(-2,-1)$. **1550.** $\left(\frac{1}{3}(2m-n),\frac{1}{3}(2n-m)\right)$.

1551. $(1,0)$ nuqtada minimumga erishadi, $z_{min}=0$. **1552.** $(1,4)$ nuqtada minimumga erishadi, $z_{min}=-21$. **1553.** $(4,4)$ nuqtada maksimumga erishadi, $z_{max}=15$. **1554.** $\left(\frac{1}{3},\frac{1}{3}\right)$ nuqtada minimumga erishadi. **1555.** $(3,3)$ nuqtada minimumga erishadi, $z_{min}=0$.

1556. $(21,20)$ nuqtada maksimumga erishadi. **1557.** $(3,2)$ nuqtada maksimumga erishadi. **1558.** $(0,0)$ nuqtada minimumga $(0,1)$ va $(0,-1)$ nuqtada maksimumga erishadi. **1559.** $(-2,0)$ nuqtada minimumga erishadi, $z_{min}=-\frac{2}{e}$. **1560.** $(6,4)$ nuqtada maksimumga erishadi, $z_{max}=5ln2$. **1561.** Eng katta qiymati $z=-2$, eng kichik qiymati $z=-5$. **1562.** Eng katta qiymati $z=17$, eng kichik qiymati $z=-\frac{17}{4}$. **1563.** Eng katta qiymati $z=\frac{2}{3\sqrt{3}}$, eng kichik qiymati $z=-\frac{2}{3\sqrt{3}}$. **1564.** Eng katta qiymati $z=13$, eng kichik qiymati $z=-1$.

1565. Eng katta qiymati $z=3$, eng kichik qiymati $z=-3$. **1566.** Eng katta qiymati $z=128$, eng kichik qiymati $z=-4$. **1567.** $4\times4\times2$.

1568. $r=d+\sqrt[3]{\frac{v}{2\pi}}$, $h=2d+2\sqrt[3]{\frac{v}{2\pi}}$. **1569.** Qirrasining ulchovi $\frac{l}{12}$ bo'lgan kub.

1570. aniqlaydi. **1571.** aniqlamaydi.

1574. $y=-\cos x^2$, $\sqrt{\pi}\le|x|\le\sqrt{2\pi}$. **1575.** $y=\lg(10-10^x)$, $-\infty<x<1$.

1576. $y=\frac{x}{3}$, $-\infty<x<0$, $y=x$, $0\le x<+\infty$. **1577.** $y=\sqrt[3]{-x^2+\ln(x^6+5)}$. **1578.** $y'=-\frac{b^2x}{a^2y}$.

1579. $y'=\frac{y}{y-1}$. **1580.** $y'=\frac{y^x\ln y}{1-xy^{x-1}}$. **1581.** $y'=\frac{e^{2y}-\frac{y}{x}}{\ln x-2xe^{2y}}$. **1582.** $y'=\frac{x+y}{x-y}$

. **1583.** $y'=\frac{y^2(\ln x-1)}{x^2(\ln y-1)}$. **1584.** $y'=-\frac{y}{x}$.

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1585. 156. **1586.** $\frac{3}{20}$. **1587.** $(1-e^{-1})^2$. **1588.** $\ln\frac{25}{24}$. **1589.** $\frac{3}{20}$. **1590.** $\frac{\pi}{12}$.

1591. $-\frac{1}{20}$. **1592.** $\frac{4}{3}$. **1593.** 6. **1594.** $\frac{9}{4}$. **1595.** 6π. **1596.** $\frac{11}{120}$.

1597. $\int_2^4 dy\int_1^{-2} f(x,y)dx$. **1598.** $\int_0^1 dy\int_y^{-1} f(x,y)dx$. **1599.** $\int_0^1 dy\int_{\sqrt{y}}^1 f(x,y)dx$.

1600. $\int_0^1 dy\int_{y^3}^{\sqrt[3]{y}} f(x,y)dx$. **1601.** $\int_1^e dx\int_0^{\ln x} f(x,y)dy$.

1602. $\int\limits_0^{\ln 2} dx \int\limits_1^{e^x} f(x,y)dy + \int\limits_{\ln 2}^{1} f(x,y)dy + \int\limits_1^2 dx \int\limits_x^2 f(x,y)dy$.

1603. $\int\limits_{-1}^{0} dx \int\limits_{-\sqrt{4x+4}}^{\sqrt{4x+4}} f(x,y)dy + \int\limits_0^8 dx \int\limits_{-\sqrt{4x+4}}^{2-x} f(x,y)dy$.

1604. $\int\limits_{-1}^{0} dx \int_0^{\sqrt{1-x^2}} f(x,y)dy + \int_0^1 dx \int_0^{1-x} f(x,y)dy$.

1605. $\int\limits_0^{\frac{\sqrt{2}}{2}} dy \int_0^{\arcsin y} f(x,y)dx + \int\limits_{\frac{\sqrt{2}}{2}}^{1} dy \int_0^{\arccos y} f(x,y)dx$. **1606.** $5\frac{5}{6}$. **1607.** 4.

1608. $\frac{\pi^2}{16}$ **1609.** $3\frac{151}{210}$. **1610.** $\frac{56}{15}$. **1611.** $\frac{9}{4}$. **1612.** 50,4. **1613.** $\frac{1}{2}$. **1614.** $-\frac{1}{504}$.
1615. $\frac{1}{6}$. **1616.** $\frac{1}{2}$. **1617.** $\frac{2}{3}$. **1618.** $\frac{8\sqrt{2}}{3}$. **1619.** $\frac{21}{16}\pi$. **1620.** $\frac{21}{16}\pi$. **1621.** $\frac{\ln 2}{2}$.
1622. $\pi(e-1)$. **1623.** $\frac{\pi}{6}\left(2\sqrt{2}-1\right)$. **1624.** $\frac{\pi}{2}$. **1625.** $2\pi^2$. **1626.** $\frac{a^3}{12}$.
1627. $\frac{1}{3}\pi a^3$. **1628.** 2π. **1629.** 0. **1630.** $\frac{1}{8}\pi(\pi-2)$. **1631.** $\frac{\pi^2}{16}$. **1632.** $\frac{3}{2}\pi$.
1633. $\frac{\pi}{4}$. **1634.** 2. **1635.** 1. **1636.** $\frac{27}{64}$. **1637.** $\frac{1}{2}(15-16\ln 2)$.
1638. $\frac{1}{2}\left(3-\frac{\pi}{2}\right)$; $2+\frac{\pi}{2}$. **1639.** 4. **1640.** $a^2\ln 2$. **1641.** $\frac{1}{3}$. **1642.** $\frac{16}{3}\sqrt{15}$.
1643. π. **1644.** $\frac{(e-1)^2}{2}$. **1645.** πab. **1646.** $\frac{8}{3}$. **1647.** $\frac{1}{6}$. **1648.** $\frac{abc}{6}$.
1649. $\frac{ma^2b}{2}$. **1650.** $\frac{1}{3}$. **1651.** $\frac{68}{15}$. **1652.** $\frac{1}{12}$. **1653.** $\frac{3\pi-4}{6}$. **1654.** $\frac{32}{9}$.
1655. $\frac{\pi(b-a)}{3}\left(3r^2-a^2-ab-b^2\right)$. **1656.** $\frac{4}{3}\pi r^3$. **1657.** $\frac{abc}{3}$. **1658.** $\frac{88}{105}$.
1659. $\frac{1}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2}$. **1661.** $\frac{7}{128}\pi$. **1663.** 8π. **1664.** $\frac{1}{2}k\pi r^4$; k– proporsionallik koeffitsiyenti, r – doira aylanasining radiusi.
1665. $\frac{kab}{3}\left(a^2+b^2\right)$ **1666.** $\left(\frac{12}{5};0\right)$. **1667.** $\left(\frac{\pi}{2};\frac{\pi}{8}\right)$. **1668.** $(2,48;1,4)$.
1669. $\frac{ab\left(a^2+b^2\right)}{3}$.

1670. $a)\frac{ab}{2}\left(a^2+b^2\right), b)\frac{ab^3}{12}$. **1671.** $\frac{\pi ab}{4}\left(a^2+b^2\right)$. **1672.** $\frac{abc}{3}\left(a^2+b^2+c^2\right)$.
1673. $\frac{a^2h}{6}$. **1674.** $\frac{1}{8}$. **1675.** 6. **1676.** $\frac{a^{11}}{110}$. **1677.** $\frac{1}{12}$. **1678.** $\frac{1}{2}\ln 2-\frac{5}{16}$.
1679. $\frac{16}{3}\pi$. **1680.** $\frac{1}{720}$. **1681.** 11. **1682.** $\frac{a}{2}\pi$. **1683.** $\frac{8}{9}a^2$. **1684.** $\frac{8}{3}\left(\pi-\frac{4}{3}\right)$.
1685. $\frac{16}{9}a^2$. **1686.** $\frac{4}{3}\pi a^3$. **1687.** $\frac{3c^4}{8a}\pi$. **1688.** $\frac{19}{3}\pi$.
1689. $x_0=0,\quad y_0=0,\quad z_0=\frac{3}{4}$.
1690. $x_0=0,\quad y_0=0,\quad z_0=\frac{4}{3}$. **1691.** $\frac{17}{30}\pi$. **1692.** 104. **1693.** 45.
1694. $\frac{1}{12}\left(5\sqrt{5}-1\right)$. **1695.** $3\frac{\sqrt{5}}{2}$. **1696.** $\frac{2}{3}\left(5\sqrt{10}-\sqrt{2}\right)$. **1697.** $\frac{1}{6}\left(17\sqrt{17}-2\sqrt{2}\right)$.
1698. $\frac{1}{54}\left(10\sqrt{10}-1\right)$. **1699.** $3+2\sqrt{5}$. **1700.** $\frac{\pi}{2}$. **1701.** R^2. **1702.** $\frac{\pi}{2}R^3$.
1703. $16a$. **1704.** $\frac{k}{8}\left(2\sqrt{2}-1\right)$. **1705.** $x_0=0,\quad y_0=\frac{2}{\pi}R$. **1706.** $\approx 1,42$.
1707. $\frac{7}{4}$. **1708.** 190. **1709.** 2. **1710.** $\frac{16}{15}$. **1711.** 18. **1712.** $-R^2$. **1713.** $-\frac{ab^2}{3}$.
1714. $\frac{a^2}{8}$. **1715. 1).** 2π. **2).** $-\frac{9}{2}$. **3).** $\frac{3a^2\pi}{8}$. **4).** $\frac{3a^2}{2}$. **1716. 1)** $\frac{a^2\sqrt{3}}{2}$. **2)** 0.
3) $54\sqrt{14}$. **4).** $ah(4a+\pi h)$. **5).** $\frac{55+9\sqrt{3}}{65}$. **6).** $\frac{\pi\sqrt{2}}{2}$. **1717.** $\pi\ln 2$.
1718. 1). $\frac{4\pi}{5}R^2$. **2).** $-\frac{2\pi}{3}$. **3).** 0. **1719.** $\frac{\pi a^4}{2}$.

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1731. $y=e^{3x}$. **1732.** $y=3x^2$. **1733.** $y=Cx\ \left(x\neq 0,\ C\neq 0\right)$. **1734.** $y=\ln x+C$.
1735. $y=-2+Ce^x$. **1736.** $y=-\ln x+\left(c-e^x\right)$. **1737.** $x^2+y^2=C$.
1738. $y=C\left(x^2-1\right)$. **1739.** $y=C(x+1)e^{-x}$. **1740.** $\ln|x|=C+\sqrt{y^2+1}$.
1741. $\sqrt{2}y=c+\frac{1}{x}$. **1742.** $y=1+C(x+1)$. **1743.** $\arcsin y=C+\sqrt{1-y^2}$.
1744. $e^y=C+\ln\left|1-e^x\right|$. **1745.** $y=tg(\sin x+C)$. **1746.** $y=\ln\left(1\pm Ce^{-x}\right)$.
1747. $y=tg^2x+\sin^2 y=C$. **1748.** $|y-1|\left(x^2+1\right)=C$. **1749.** $\left(e^y+1\right)e^x=C$.

1750. $y=C\sin x$. **1751.** $y=\arcsin\frac{C}{x^2+3}$. **1752.** $2\sqrt{y}+\ln|y|-2\sqrt{x}=C$.

1753. $y=\frac{3}{2}x^2-3$. **1754.** $y=-2\cos x$. **1755.** $x+y=0$. **1756.** $\frac{\ln^2 y}{2}=ctgx-1$.

1757. $\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right|=arctgx-1$. **1758.** $2e^{y^2}=e^x+1$. **1759.** $y^2=2x^2\ln Cx$.

1760. $y=xe^{1+Cx}$. **1761.** $y=-x\ln\frac{C}{x}$. **1762.** $x^2+C(y+x)=0$. **1763.** $y=Ce^{\frac{y}{x}}$.

1764. $y=x\arcsin Cx$. **1765.** $y=xe^{c_1x}$. **1766.** $arctg\frac{y}{x}+\ln\sqrt{x^2+y^2}=0$.

1767. $\sqrt{x}+\sqrt{y}\ln Cy=0$. **1768.** $y=Cx+x^2$. **1769.** $y=Ce^{-2x}+e^{-x}$.

1770. $y=Ce^{-\frac{x^2}{2}}-1$. **1771.** $y=\frac{C}{x}+x\ln x-\frac{x}{2}$. **1772.** $y=(C+x)e^x$.

1773. $y=\frac{C}{\cos^2 x}$. **1774.** $y=\frac{1}{3}+Ce^{-x^3}$. **1775.** $y=-\frac{\cos x}{3}+\frac{C}{\cos^2 x}$.

1776. $y=\frac{1}{6}x^4+\frac{C}{x^2}$. **1777.** $y=(x+C)\sin x$. **1778.** $y(x^2+1)^2=x^3+3x+C$.

1779. $y=\left(x-2+Ce^{-\frac{x}{2}}\right)^2$. **1780.** $y=-e^{-x}\ln|1-x|$. **1781.** $y=-(x+1)$.

1782. $y=\frac{2}{x^2}-\frac{1}{x}$. **1783.** $y=\frac{\sin x+1}{x}$. **1784.** $y=x^2+1-2\sqrt{x^2+1}$.

1785. $y=x^2$. **1786.** $y=\frac{x}{\cos x}$. **1787.** $x=\frac{y^2-4}{9}-\ln y$. **1788.** $x^2y^2+7x=C$.

1789. $xe^y+ye^x+3x-2y=C$. **1790.** $\sin(x-y)=C$.

1791. $\frac{x^2}{y^2}+y^2+xy-e^y=C$. **1792.** $x^3+x^2y-xy^2-y^3=C$.

1793. $2xy-3x+y^2=C$. **1794.** $x\sin(x+y)=C$. **1795.** $x^2+y^2-2arctg\frac{y}{x}=C$.

1796. $x^2-y^2=Cy^3$. **1797.** $y=\frac{x^2}{2}-\frac{x^4}{12}+c_1x+c_2$. **1798.** $c_1y^2-1=(c_1x+c_2)^2$.

1799. $y=-\frac{1}{4}\cos 2x+c_1x+c_2$. **1800.** $y=\frac{1}{9}e^{-3x}+2x^2+c_1x+c_2$.

1801. $y=c_1+c_2\ln|x|$.

1802. $y=\ln\left|e^{2x}+c_1\right|-x+c_2$. **1803.** $y=c_1e^{c_2x}$.

1804. $y=\frac{x^2\ln|x|}{2}-\frac{3x^2}{4}+c_1x+c_2$. **1805.** $y=-\sin x+\frac{x^2}{2}+c_1x+c_2$.

1806. $y=c_1+\frac{(x+c_2)^2}{4c_1}$. **1807.** $y=c_1x+c_2-x\sin x-2\cos x$.

1808. $y=c_1\ln|x|-\frac{x^2}{4}+c_2$. **1809.** $y=c_2-\cos(x+c_1)$.

1810. $y=xe^x-e^x+c_1\frac{x^2}{2}+c_2$. **1811.** $y=\frac{1}{c_1x+c_2}$.

1812. $y=\frac{x^5}{5}-\frac{x^3}{3}+\frac{x^2}{2}+x-1$. **1813.** $2y^2-4x^2=1$. **1814.** $y^2=\frac{e}{e-1}+\frac{e^{-x}}{1-e}$.

1815. $y=\frac{x^2}{2}$. **1816.** $y=\frac{4}{(x-5)^2}$. **1817.** $y^3-y=3x$. **1818.** $y=C_1e^x+C_2x$.

1819. $y=C_1e^x+C_2e^{-\frac{1}{2}x}+e^{2x}\left(\frac{4}{5}x-\frac{28}{25}\right)$. **1820.** $y=(C_1+C_2x)e^x+\frac{1}{6}x^3e^x$.

1821. $y=C_1\cos x+C_2\sin x-\frac{x^2}{4}\cos x+\frac{x}{4}\sin x$.

1822. $y=C_1x\ln x+\frac{1}{2}x\ln^2 x+C_2x$. **1823.** $y=C_1\left(C_2x-1-x^2\right)$.

1824. $y=C_1e^{-3x}+C_2e^{-2x}$. **1825.** $y=C_1+C_2e^{-15x}$.

1826. $y=C_1\cos 7x+C_2\sin 7x$. **1827.** $y=C_1e^x+C_2e^{-2x}$.

1828. $y=e^{-x}(C_1\cos x+C_2\sin x)$. **1829.** $y=C_1e^{-\frac{x}{2}}+C_2e^{2x}$.

1830. $y=e^{2x}(C_1\cos 3x+C_2\sin 3x)$. **1831.** $y=e^{-\frac{x}{2}}\left(C_1e^{\frac{\sqrt{3}}{2}x}+C_2e^{-\frac{\sqrt{3}}{2}}\right)$.

1832. $y=e^{\frac{x}{6}}\left(C_1\cos\frac{\sqrt{11}}{6}x+C_2\sin\frac{\sqrt{\pi}}{6}x\right)$. **1833.** $y=e^{-3x}(C_1\cos 4x+C_2\sin 4x)$.

1834. $y=C_1+C_2e^{-x}+\frac{x}{2}$. **1835.** $y=C_1e^{2x}+C_2e^{3x}+\frac{1}{2}$.

1836. $y=C_1\cos 3x+C_2\sin 3x+\frac{2}{9}x$. **1837.** $y=Ce^{-2x}+e^{-x}$.

1838. $y=\frac{1}{3}e^{2x}+C_1e^x+C_2e^{-x}$. **1839.** $y=\frac{1}{6}x^2+\frac{5}{18}x+\frac{19}{108}+C_1e^{2x}+C_2e^{3x}$.

1840. $y=\frac{1}{3}\sin x+C_1\cos 2x+C_2\sin 2x$. **1841.** $y=C_1e^x+C_2e^{3x}+5xe^{3x}$.

1842. $y=C_1+C_2e^{5x}-3x^2+x$. **1843.** $y=C_1e^x+C_2e^{-2x}-\frac{2}{5}(3\sin 2x+\cos 2x)$.

1844. $y=C_1e^{2x}+C_2e^{-3x}+x\left(\frac{x}{10}-\frac{1}{25}\right)e^{3x}$. **1845.** $y=2-e^{-3x}$. **1846.** $y=\sin 2x$.

1847. $y=\frac{1}{3}e^{2x}-\frac{1}{3}e^{-x}$. **1848.** $y=3e^{-2x}\cdot\sin 5x$. **1849.** $y=e^x\left(\cos 3x-\frac{1}{3}\sin 3x\right)$.

1850. $y=-\cos 2x+\frac{1}{2}\sin 2x$. **1851.** $y=e^{3x}(x+2)$. **1852.** $y=e^x\sin x$.

1853. $y=e^x\left(\cos\sqrt{2}x+\sqrt{2}\sin\sqrt{2}x\right)$. **1854.** $y=1+x-e^x\cos x$.

1855. $y=\frac{7}{27}e^{3x}-\frac{1}{27}e^{-3x}+\frac{1}{9}x-\frac{2}{9}$. **1856.** $y=2\sin 2x+\frac{1}{2}x\sin x$.

1857. $y=e^x(-6\cos 3x+\sin 3x)+12\cos 3x+2\sin 3x$. **1858.** $y=3-2e^{6x}+3xe^{6x}$.

1859. $y=3\sin 2x-7\cos 3x-2\sin 3x$.

1860. $y=-3\cos x+\pi\sin x+x(4\cos x-3\sin x)$. **1861.** $y=ach\frac{x}{a}$.

1862. $y=2+x-\frac{x^2}{2}+\frac{5}{6}x^3+\frac{1}{8}x^4+\ldots$. **1863.** $y=1-\frac{x^3}{3!}+\frac{1\cdot4}{6!}x^6-\frac{1\cdot4\cdot7}{9!}x^9+\ldots$.

1864. $y=2+x-x^2-\frac{x^3}{2}-\frac{x^4}{12}+\ldots$. **1865.** $y=x+\frac{x^4}{3\cdot4}+\frac{x^7}{3\cdot4\cdot6\cdot7}+\ldots$.

1866. $y=1+2x+4x^2+8x^3+16x^4+\ldots$. **1867.** $y=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots=\sin x$.

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1870. $z^2=(x^2+y^2)\cdot\sin^2 C$. **1871** $x^2+y^2+z^2=R^2$.

1872. $2(\cos\beta-2\cos\alpha-2\cos\gamma)$; $-\frac{4}{3}$. **1873.** $2x, 2y, -2z$.

1874. $ye^{xy}, xe^{xy}-z^2, -2yz$. **1875.** $\frac{2x}{x^2+y^2+z^2}, \frac{2y}{x^2+y^2+z^2}, \frac{2z}{x^2+y^2+z^2}$.

1876. $(1,-12,-5)$. **1877.** $\frac{1}{2},-\frac{1}{2},\frac{\sqrt{3}}{2}$. **1878.** $1,1,-2$. **1879.** $\varphi=\arccos\frac{1}{\sqrt{5}}$.

1882. Ko'rsatma. Agar sim deb $0Z$ o'qi olinsa, u holda magnit maydonning $\vec{H}$ vektor kuchlanishi quydagi formula bilan topiladi.

$\vec{H}=\frac{2I}{\rho^2}(-y\cdot\vec{i}+x\cdot\vec{j})$. Bu holda $a_x=-\frac{2I}{\rho^2}y$, $a_y=\frac{2I}{\rho^2}x$, $a_z=0$. $x^2+y^2=2c$. **1883.** $W=2\pi a^2h$. **1884.** $W=\pi R^2H$. **1885.** $W=24a^3$.

1886. $W=\frac{3}{16}\pi$. **1887.** 29. **1889.** $div\vec{a}=0$. **1890.** $div\vec{a}=3$.

1891. $div\vec{a}=2(xy+yz+zk)$. **1892.** $div\frac{\vec{r}}{r}=\frac{2}{r}$. **1893.** $\mp\frac{\pi a^6}{8}$. **1894.** 3.

1895. $rot\vec{a}=-\vec{i}-\vec{j}-\vec{k}$. **1896.** $rot\vec{a}=-y\cdot\vec{i}-z\cdot\vec{j}-x\cdot\vec{k}$.

1897. $rot\vec{a}=\frac{x}{2y}(y-z)\cdot\vec{i}+\frac{y}{zx}(z-x)\cdot\vec{j}-\frac{z}{xy}(x-y)\cdot\vec{k}$. **1898.** $z=y$ tekislik nuqtalarida. **1899.** $u=2y^3+y\cdot\varphi(x)+\psi(x)$. **1900.** $u=xy+\varphi(x)+\psi(y)$.

1901. $\frac{\partial^2u}{\partial\xi\partial\eta}-\frac{1}{2\xi}\cdot\frac{\partial u}{\partial\eta}=0$. **1902.** $\frac{\partial^2u}{\partial\eta^2}=\frac{2\xi}{\xi^2+\eta^2}\cdot\frac{\partial u}{\partial\xi}$.

1903. $\frac{\partial^2u}{\partial\eta^2}=0$, $\xi=\frac{y}{x}$, $\eta=y$. **1904.** $\frac{\partial^2u}{\partial\xi\partial\eta}-\frac{\partial u}{\partial\xi}=0$, $\xi=x+y$, $\eta=3x+y$.

1905. $\frac{\partial^2u}{\partial\xi^2}+\frac{\partial^2u}{\partial\eta^2}+\frac{1}{2}\cdot\left(\frac{1}{\xi}\cdot\frac{\partial u}{\partial\xi}+\frac{1}{\eta}\cdot\frac{\partial u}{\partial n}\right)=0$, $\xi=y^2$, $\eta=x^2$. **1906.** $u=x^2+t^2$. **1907.** $u=xt$. **1908.** $u=\sin x\cdot\cos at+t$ ($t=\frac{\pi}{2a}$ da $u=\frac{\pi}{2a}$ abssissalar uqiga paralel). **1909.** $u=x(1-t)$. **1910.** $u=\frac{\cos x\cdot\sin at}{a}$. **1911.** $u=-\sin x$.

1912. $u(x,t)=\frac{8\ell^2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}\cos\frac{(2k-1)\pi at}{\ell}\sin\frac{(2k-1)\pi x}{\ell}$.

1913. $\frac{4\upsilon_0}{\pi a}\sum_{k=1}^{\infty}\frac{1}{k}\sin\frac{k\pi c}{\ell}\cdot\quad\cdot\sin\frac{k\pi x}{\ell}\sin\frac{k\pi at}{\ell}$ ga teng, chunki $\lim_{h\to0}\frac{\sin\frac{k\pi h}{\ell}}{h}=\frac{k\pi}{\ell}$.

914. $u(x,t)=A\sin\frac{\pi x}{\ell}\cos\frac{\pi at}{\ell}$. **1915.** $u(x,t)=\frac{9h}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\sin\frac{k\pi}{3}\cos\frac{k\pi a}{\ell}t\sin\frac{k\pi}{\ell}x$

1918. $u(x,t)=\frac{u_0}{2}\cdot\left[\Phi\left(\frac{x-x_1}{2a\sqrt{t}}\right)-,-\Phi\left(\frac{x-x_2}{2a\sqrt{t}}\right)\right]$ bu erda $\Phi(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-\mu^2}$.

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1919. $A+B$-detal 1-yoki 2-navli

$\overline{A+C}$ – detal 2-navli

$A\cdot B$ – mumkin bo'lmagan hodisa

$A\cdot B+C$ – detal 3-navli.

1920. 1) $A=B$, **2)** $A\cdot B=V$, **3)** $A=\cup$, **4)** $A=V$. **1922.** 1) gul sariq yoki oq rang, 2) gul qizil yoki sariq rang, 3) V (ϕ). 4) gul oq rang, 5) A,B va

C ixtiyoriy hodisalar bo'lsin. **1923.** 1) $A\bar{B}\bar{C}$, 2) $A\bar{B}\bar{C}+\bar{A}B\bar{C}+\bar{A}\bar{B}C$, 3) $AB\bar{C}+A\bar{B}C+\bar{A}BC$, 4) ABC, 5) $A+B+C$, 6) $\overline{ABC}$. **1924.** 0,89.

1925. 0,517. **1926.** $\frac{4}{7}$. **1927.** 0,087. **1928.** $\frac{1}{4}$. **1929.** 0,9187. **1930.** 0,56.

1931. 0,02. **1932.** 0,727. **1933.** 0,9865. **1934.** 0,06. **1935.** 0,217. **1936.** 0,559.

1937. $\approx 0,055$. **1938.** $\approx 0,324$. **1939.** 0,383. **1940.** $\approx 0,902$. **1941.** $\approx 0,2$.

1942. 0,972. **1943.** 0,147. **1944.** 0. **1945.** 0,993. **1946.** 0,929. **1947.** 0,719.

1948. 0,7258. **1949.** 0,91. **1950.** 0,0092.

1951.

x_i	0	1	2	3
p_i	0,04	0,26	0,46	0,24

1952.

x_i	0	1	2
p_i	0,09	0,42	0,49

1953.

x_i	-2	1	2	3
p_i	0,08	0,40	0,32	0,2

1954.

x_i	0	1	2
p_i	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{1}{8}$

1955.

x_i	0	1	2	....	1500
p_i	e^{-3}	$\frac{3\cdot e^{-3}}{1!}$	$\frac{3^2\cdot e^{-3}}{2!}$	....	$\frac{3^{1500}\cdot e^{-3}}{1500!}$

1956. $c=2$. **1957.** $a=\frac{1}{2}$, $c=1$. **1958.** $F(x)=\begin{cases}0, & x<1\\ 1-\frac{1}{x^3}, & x\geq 1\end{cases}$.

1959. (1,6;1,8). **1960.** $p(x)=\begin{cases}3^x\ln 3, & x\leq 0\\ 0, & x>1\end{cases}$.

1961. $A=\frac{3}{40}, F(x)=\begin{cases}0, & x<-4\\ \frac{3}{5}-\frac{3x^2}{80}, & -4\leq x<0\\ \frac{3}{5}+\frac{1}{20}\sqrt{x^3}, & 0\leq x<4\\ 1, & 4\leq x\end{cases}$, $p\{-1<\xi<5\}=\frac{7}{16}$.

1962. $M\xi=1,8$, $D\xi=0,72$. **1963.** $M\xi=0$, $D\xi=\frac{8}{3}$. **1964.** 3,5; 1,75.

1965. 10. **1966.** $M\xi=0,55$, $D\xi=2,6475$. **1967.** $M\xi=0,35$, $D\xi=29\frac{1}{6}$.

1968. $M(4\xi-1)=13$, $D(4\xi-1)=\frac{140}{3}$. **1969.** $M\xi\approx 0,692$, $D\xi\approx 0,259$.

1970. $M\xi=30$, $D\xi=24$. **1971.** 200. **1972.** 0,324; 0,569. **1973.** 2.

1974. $p\{\alpha\leq\xi\leq\beta\}=\frac{\beta-\alpha}{b-a}$. **1975.** $F(x)=\begin{cases}0, & x\leq 0\\ \frac{x}{4}, & 0<x\leq 4,\\ 1, & x>4\end{cases}$ $M\xi=2$, $D\xi=\frac{4}{3}$.

1976. 0,4. **1977.** 40; 8. **1978.** $\approx 0,19$ $(\lambda=0,0029)$. **1979.** 0,955. **1980.** 0,6.

1981. 0,95; $p\geq 0,872$. **1982.** $p\geq\frac{2}{7}$. **1983.** $p\geq 0,9808$. **1984.** $p\approx 0,89$.

1985. $p\geq\frac{7}{8}$

1-илова

$\varphi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ нинг жадали

x	0	1	2	3	4	5	6	7	8	9
0,0	0,3989	3989	3989	3988	3986	3984	3982	3980	3977	3973
0,1	3970	3965	3961	3956	3951	3945	3939	3932	3925	3918
0,2	3910	3902	3894	3885	3876	3867	3857	3847	3836	3825
0,3	3814	3802	3790	3778	3765	3752	3739	3726	3712	3697
0,4	3683	3668	3653	3637	3621	3605	3589	3572	3555	3538
0,5	3521	3503	3485	3467	3448	3429	3410	3391	3372	3352
0,6	3332	3312	3292	3271	3251	3230	3209	3187	3166	3144
0,7	3123	3101	3079	3056	3034	3011	2989	2966	2943	2920
0,8	2897	2874	2850	2827	2803	2780	2756	2732	2709	2685
0,9	2661	2637	2613	2589	2565	2541	2516	2492	2468	2444
1,0	0,2420	2396	2371	2347	2323	2299	2275	2251	2227	2203
1,1	2179	2155	2131	2107	2083	2059	2036	2012	1989	1965
1,2	1942	1919	1895	1872	1849	1826	1804	1781	1758	1736
1,3	1714	1691	1696	1647	1626	1604	1582	1561	1539	1518
1,4	1497	1476	1456	1435	1415	1394	1374	1354	1334	1315
1,5	1295	1276	1257	1238	1219	1200	1182	1163	1145	1127
1,6	1109	1092	1074	1057	1040	1023	1006	0989	0973	0957
1,7	0940	0925	0909	0893	0878	0863	0848	0833	0818	0804
1,8	0790	0775	0761	0748	0734	0721	0707	0694	0681	0669
1,9	0656	0644	0632	0620	0608	0596	0584	0573	0562	0551
2,0	0,0540	0529	0519	0508	0498	0488	0478	0468	0459	0449
2,1	0440	0431	0422	0413	0404	0396	0387	0379	0371	0363
2,2	0355	0347	0339	0332	0325	0317	0310	0303	0297	0290
2,3	0283	0277	0270	0264	0258	0252	0246	0241	0235	0229
2,4	0224	0219	0213	0208	0203	0198	0194	0189	0184	0180
2,5	0175	0171	0167	0163	0158	0154	0151	0147	0143	0139
2,6	0136	0132	0129	0126	0122	0119	0116	0113	0110	0107
2,7	0104	0101	0099	0096	0093	0091	0088	0086	0084	0081
2,8	0079	0077	0075	0073	0071	0069	0067	0065	0063	0061
2,9	0060	0058	0056	0055	0053	0051	0050	0048	0047	0046
3,0	0,0044	0043	0042	0040	0039	0038	0037	0036	0035	0034
3,1	0033	0032	0031	0030	0029	0028	0027	0026	0025	0025
3,2	0024	0023	0022	0022	0021	0020	0020	0019	0018	0018
3,3	0017	0017	0016	0016	0015	0015	0014	0014	0013	0013
3,4	0012	0012	0012	0011	0011	0010	0010	0010	0009	0009
3,5	0009	0008	0008	0008	0008	0007	0007	0007	0007	0006
3,6	0006	0006	0006	0005	0005	0005	0005	0005	0005	0004
3,7	0004	0004	0004	0004	0004	0004	0003	0003	0003	0003
3,8	0003	0003	0003	0003	0003	0002	0002	0002	0002	0002
3,9	0002	0002	0002	0002	0002	0002	0002	0002	0001	0001

2-илова

$\Phi(x)\approx\frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{x^2}{2}}dx$ нинг жадвали

x	Φ(x)	x	Φ(x)	x	Φ(x)	x	Φ(x)
0,00	0,0000	0,45	0,1736	0,90	0,3159	1,35	0,4115
0,01	0,0040	0,46	0,1772	0,91	0,3186	1,36	0,4131
0,02	0,0080	0,47	0,1808	0,92	0,3212	1,37	0,4147
0,03	0,0120	0,48	0,1844	0,93	0,3238	1,38	0,4162
0,04	0,0160	0,49	0,1879	0,94	0,3264	1,39	0,4177
0,05	0,0199	0,50	0,1915	0,95	0,3289	1,40	0,4192
0,06	0,0239	0,51	0,1950	0,96	0,3315	1,41	0,4207
0,07	0,0279	0,52	0,1985	0,97	0,3240	1,42	0,4222
0,08	0,0319	0,53	0,2019	0,98	0,3365	1,43	0,4236
0,09	0,0359	0,54	0,2054	0,99	0,3389	1,44	0,4251
0,10	0,0398	0,55	0,2088	1,00	0,3413	1,45	0,4265
0,11	0,0438	0,56	0,2123	1,01	0,3438	1,46	0,4279
0,12	0,0478	0,57	0,2157	1,02	0,3461	1,47	0,4292
0,13	0,0517	0,58	0,2190	1,03	0,3485	1,48	0,4306
0,14	0,0557	0,59	0,2224	1,04	0,3508	1,49	0,4319
0,15	0,0596	0,60	0,2257	1,05	0,3531	1,50	0,4332
0,16	0,0636	0,61	0,2291	1,06	0,3554	1,51	0,4345
0,17	0,0675	0,62	0,2324	1,07	0,3577	1,52	0,4357
0,18	0,0714	0,63	0,2357	1,08	0,3599	1,53	0,4370
0,19	0,0753	0,64	0,2389	1,09	0,3621	1,54	0,4382
0,20	0,0793	0,65	0,2422	1,10	0,3643	1,55	0,4394
0,21	0,0832	0,66	0,2454	1,11	0,3665	1,56	0,4406
0,22	0,0871	0,67	0,2486	1,12	0,3686	1,57	0,4418
0,23	0,0910	0,68	0,2517	1,13	0,3708	1,58	0,4429
0,24	0,0948	0,69	0,2549	1,14	0,3729	1,59	0,4441
0,25	0,0987	0,70	0,2580	1,15	0,3749	1,60	0,4452
0,26	0,1026	0,71	0,2611	1,16	0,3770	1,61	0,4463
0,27	0,1064	0,72	0,2642	1,17	0,3790	1,62	0,4474
0,28	0,1103	0,73	0,2673	1,18	0,3810	1,63	0,4484
0,29	0,1141	0,74	0,2703	1,19	0,3830	1,64	0,4495
0,30	0,1179	0,75	0,2734	1,20	0,3949	1,65	0,4505
0,31	0,1217	0,76	0,2764	1,21	0,3869	1,66	0,4515
0,32	0,1255	0,77	0,2794	1,22	0,3888	1,67	0,4525
0,33	0,1293	0,78	0,2823	1,23	0,3907	1,68	0,4535
0,34	0,1331	0,79	0,2852	1,24	0,3925	1,69	0,4545
0,35	0,1368	0,80	0,2881	1,25	0,3944	1,70	0,4554
0,36	0,1406	0,81	0,2910	1,26	0,3962	1,71	0,4564
0,37	0,1443	0,82	0,2939	1,27	0,3980	1,72	0,4573
0,38	0,1480	0,83	0,2967	1,28	0,3997	1,73	0,4582
0,39	0,1517	0,84	0,2995	1,29	0,4015	1,74	0,4591
0,40	0,1554	0,85	0,3023	1,30	0,4032	1,75	0,4599
0,41	0,1591	0,86	0,3051	1,31	0,4049	1,76	0,4608
0,42	0,1628	0,87	0,3078	1,32	0,4066	1,77	0,4616
0,43	0,1664	0,88	0,3106	1,33	0,4082	1,78	0,4625
0,44	0,1700	0,89	0,3133	1,34	0,4099	1,79	0,4633

x	Φ(x)	x	Φ(x)	x	Φ(x)	x	Φ(x)
1,80	0,4641	2,00	0,4772	2,40	0,4918	2,80	0,4974
1,81	0,4649	2,02	0,4783	2,42	0,4922	2,82	0,4976
1,82	0,4656	2,04	0,4793	2,44	0,4927	2,84	0,4977
1,83	0,4664	2,06	0,4803	2,46	0,4931	2,86	0,4979
1,84	0,4671	2,08	0,4812	2,48	0,4934	2,88	0,4980
1,85	0,4678	2,10	0,4821	2,50	0,4938	2,90	0,4981
1,86	0,4686	2,12	0,4830	2,52	0,4941	2,92	0,4982
1,87	0,4693	2,14	0,4838	2,54	0,4945	2,94	0,4984
1,88	0,4699	2,16	0,4846	2,56	0,4948	2,96	0,4985
1,89	0,4706	2,18	0,4854	2,58	0,4951	2,98	0,4986
1,90	0,4713	2,20	0,4861	2,60	0,4953	3,00	0,49865
1,91	0,4719	2,22	0,4868	2,62	0,4956	3,20	0,49931
1,92	0,4726	2,24	0,4875	2,64	0,4959	3,40	0,49966
1,93	0,4732	2,26	0,4881	2,66	0,4961	3,60	0,499841
1,94	0,4738	2,28	0,4887	2,68	0,4963	3,80	0,499928
1,95	0,4744	2,30	0,4893	2,70	0,4965	4,00	0,499968
1,96	0,4750	2,32	0,4898	2,72	0,4967	4,50	0,499997
1,97	0,4756	2,34	0,4904	2,74	0,4969	5,00	0,500000
1,98	0,4761	2,36	0,4909	2,76	0,4971		
1,99	0,4767	2,38	0,4913	2,78	0,4973		

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Nasridin Mirzoodilovich Jabborov

OLIY MATEMATIKA VA UNING TATBIQLARIGA DOIR MASALALAR TO'PLAMI

(Bakalavr ta'lim yo'nalishlari talabalari uchun o'quv qo'llanma)

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