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A Variant of the Banach-Steinhaus Theorem for Weakly Additive, Order-Oreserving Operators

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Abstract. The Banach-Steinhaus theorem is one of the basic principles of Functional Analysis. We prove a weakly additive, order-preserving version of the Banach-Steinhaus theorem on spaces with order unit.

INTRODUCTION

Recently methods of Nonlinear Functional Analysis have more and more applications (see, for example [1, 2]). Many applied results support on principles of Functional Analysis. In [3] it was proved a variant of the Hahn-Banach theorem for the case of order-preserving functionals. Uniform boundedness principle for nonlinear operators on cones of functions was consdered in [4]. Before in [5] it was studied open mapping theorem for spaces of weakly additive homogeneous functionals on the space of continuous functions on a given compact Hausdorff space.

Recall the partially ordered vector space is [6] a couple $(E, \leq)$, where E is a vector space over the field of the real numbers $\mathbb{R}$, and $\leq$ is an order on E, satisfying the following conditions:

1) if $x \leq y$, then $x+u \leq y+u$ for every $x, y, u \in E$;

2) if $x \leq y$, then $\lambda x \leq \lambda y$ for all $x, y \in E$ and $\lambda \in \mathbb{R}_+$.

If 1) and 2) carry out then they say that $\leq$ is a vector order. Endowment of a vector space E over $\mathbb{R}$ with a vector order is equivalent to specifying the set $E_+ \subset E$ called a positive cone on E and having the following properties:

$$\begin{aligned} &E_+ + E_+ \subset E_+, \\ &\lambda E_+ \subset E_+, \qquad \lambda \geq 0, \\ &E_+ \cap E_- = 0, \end{aligned}$$

where $E_- = -E_+$. Here the order $\leq$ and the positive cone E_+ are related by

$$x \leq y \Leftrightarrow y - x \in E_+,$$

for all $x, y \in E_+$. Elements E_+ are called positive vectors.

Element $1 \in E_+$ is called (strong) order unit, if $E = \bigcup_{i=1}^{n} [-n1; n1]$. This is equivalent to the fact that for any $x \in E$ there exists $\lambda \in \mathbb{R}$, $\lambda > 0$, that $-\lambda 1 \leq x \leq \lambda 1$. In this case, the partially ordered vector space E is called a space with order unit.

Let $x \in E$. A partially ordered vector space E is called an Archimedean space, if the inequality $nx \leq 1$ executed for all positive integer n, implies that $x \leq 0$. In this case, a norm on E can be determined by the formula

$$\|x\| = \inf\{\lambda > 0 : -\lambda 1 \leq x \leq \lambda 1\}. \tag{1}$$

This norm is called the order norm. A partially ordered vector space E is called a space with ordinal unit, if on E there exists order unit and E is an Archimedean space. The topology on the E, generated by the norm (1), is called order (vector) topology. For a subset $X \subset E$ we denote by $Int\,X$ the interior of X in the order topology.

We accept [3] the following convention

$$x < y \Leftrightarrow y - x \in Int E_+.$$

Note, that every space E with an order unit has infinitely many order units. More precisely, every interior element of a positive cone E_+ is an order unit.

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Let $(E, \leq)$, $(F, \leq)$ be partially ordered sets.

Definition 1 [3]. A map $T: E \to F$ is called order-preserving, if for elements $x, y \in E$ the inequality $x \leq y$ on E implies the inequality $T(x) \leq T(y)$ on F.

Let E and F be a space with order unit, 1_E is the order unit of the space E.

Definition 2 [3]. An operator $T: E \to F$ we call weakly additive, if

$$T(x + \lambda 1_E) = T(x) + \lambda T(1_E)$$

holds for every $x \in E$, $\lambda \in \mathbb{R}$.

From this definition it immediately follows that for any weakly additive operator $T: E \to F$ we have

$$T(0_E) = T(1_E - 1_E) = T(1_E) - T(1_E) = 0_F,$$

i. e. $T(0_E) = 0_F$.

It is well known that the Banach-Steinhaus theorem is one of the basic principles of functional analysis. In this work we prove an option of the Banach-Steinhaus theorem for weakly additive, order-preserving operators on spaces with order unit.

MAIN PART

Monograph [7] is devoted to the automatic continuity of operators on Banach algebras. The following statement shows that every weakly additive, order-preserving operator on spaces with order unit is automatically continuous.

Theorem 1. If E and F are spaces with order unit, then every weakly additive, order-preserving operator $T: E \to F$ is continuous.

Proof. We show, that an operator T continuous at zero 0_E. First note the following. If $T(1_E) = 0_F$, then $T(x) = 0_F$ for all $x \in E$, since the operator T is weakly additive and order-preserving. On the other side for any $x \in E$ there exists $\lambda \in \mathbb{R}$, $\lambda > 0$, such that $-\lambda 1_E \leq x \leq \lambda 1_E$ since $1_E \in E_+$ is an order unit on E. Thus, $T(E) = 0_F$. We will not consider this case, and we will always assume that $T(1_E) \neq 0_F$. Under this assumption it is easy to see that $\|T(1_E)\| \neq 0$.

Let

$$V(0_F, \varepsilon) = \{y \in F : -\varepsilon 1_F < y < \varepsilon 1_F\}$$

is some neighborhood of zero 0_F on F, where $\varepsilon > 0$. Take a neighborhood $U = U\left(0_E, \frac{\varepsilon}{\|T(1_E)\|}\right)$ of zero 0_E on E. Then for every vector $x \in U$ we have

$$-\frac{\varepsilon}{\| T(1_E) \|} 1_E < x < \frac{\varepsilon}{\| T(1_E) \|} 1_E.$$

Since the operator T weakly additive and order-preserving, then

$$-\frac{\varepsilon}{\| T(1_E) \|} T(1_E) < T(x) < \frac{\varepsilon}{\| T(1_E) \|} T(1_E).$$

These inequalities imply that $\| T(x) \| < \varepsilon$, i. e. the operator T is continuous at zero. The following lemma completes the proof of the Theorem.

Lemma 1. If weakly additive, order-preserving operator $T: E \to F$ is continuous at zero 0_E, then it is continuous everywhere on E.

Remark 1. Obviously, that every linear non-negative operator on spaces with order unit is a weakly additive, order-preserving operator. The converse statement, generally speaking, incorrectly. But, nevertheless, such operators are linear on a one-dimensional subspace $\{\lambda 1_E : \lambda \in \mathbb{R}\} \subset E$. In this case, the image of the subspace $\{\lambda 1_E : \lambda \in \mathbb{R}\}$ on map T is as easy to see, one-dimensional subspace $\{\lambda T(1_E) : \lambda \in \mathbb{R}\} \subset F$. If 1_E is an order unit on E, then $T(1_E)$ is an order unit on $T(E) \subset F$ (but on F has not to be). Therefore, without loss of generality, can be considered, that $T(E) = F$ and

$$T(1_E) = 1_F. \tag{2}$$

Remark 2. From Theorem 1 and Remark 1 it follows that for any weakly additive, order-preserving operator $T: E \to F$ on spaces with order unit holds $T(1_E) < \infty$.

Recall the following concepts. Let X and Y be normed spaces. A subset A of the normed space X is called bounded, if there exists $R > 0$, such that A can be placed in a ball $\{x \in X : \| x \| \leq R\}$. A map $T\colon X \to Y$ called bounded, if it translates a bounded in X set onto a bounded in Y set. Obviously, that bounded the map $T\colon X \to Y$ it is equivalent the bounding of a set $\{\|T(x)\| : x \in X, \|x\| \leq K\}$ for every $K > 0$. In other words, for every bounded map T and for any $K > 0$ one has $\sup\{\|T(x)\| : x \in X, \|x\| \leq K\} < \infty$.

The following statement shows that every weakly additive, order-preserving operator on spaces with order unit is automatic bounded.

Theorem 2. Any weakly additive, order-preserving operator $T\colon E \to F$ the spaces E and F with order unit bounded, i. e. for every $K > 0$ we have

$$\sup\{\|T(x)\| : x \in E, \|x\| \leq K\} < \infty.$$

If Eq. (2) performed, then

$$\sup\{\|T(x)\| : x \in E, \|x\| \leq 1\} = 1.$$

The proof is trivial.

Remark 3. Since the operator $T\colon E \to F$ is weakly additive and preserves the order, then on Theorem 2 it is enough to consider 1 with any numbers $K > 0$. Really, let $sup\{\|T(x)\| : x \in E, \|x\| \leq 1\} < \infty$ and $K > 0$ be any number. If $\|x\| \leq K$, then we have $-K1_E \leq x \leq K1_E$. Therefore,

$$-KT(1_E) \leq T(x) \leq KT(1_E).$$

Here $\|T(x)\| \leq K\|T(1_E)\|$. But due to Remark 2, we have $T(1_E) < \infty$. Thus,

$$\sup\{\|T(x)\| : x \in E, \|x\| \leq K\} < \infty.$$

From Theorems 1, 2 and Remark 1 directly follows

Corollary 1. Any non-negative linear operator on the spaces with order unit is continuous (and bounded).

Let E and F be spaces with order unit 1_E and 1_F, correspondingly, and $\mathscr{H}$ some family of weakly additive,order-preserving operators $T\colon E \to F$. Taking account [8], we called a family $\mathscr{H}$ uniform continuous, if for every neighbourhood of zero V in F there exists neighbourhood U of zero in E such that $T(U) \subset V$ for any $T \in \mathscr{H}$. If the family $\mathscr{H}$ consists of only one weakly additive, order-preserving operator T, then the family $\mathscr{H}$ is uniform continuous, since is continuous T, and $\mathscr{H}$ uniform bounded from on bounded T. Next statement shows that every uniform continuous family weakly additive, order-preserving operators on spaces with order unit is uniformly bounded.

Theorem 3. Let E and F be spaces with order unit, $\mathscr{H}$ is a uniformly continuous family of weakly additive, order-preserving operators $T\colon E \to F$, and A be bounded subset on E. Then on F exits such a bounded set B, that $T(A) \subset B$ for every $T \in \mathscr{H}$.

Proof. We put $B = \bigcup\limits_{T \in \mathscr{H}} T(A)$. Since the family $\mathscr{H}$ is uniformly continuous, then for each neighbourhoods $V = V(0_F, \varepsilon)$ at zero F there exits a neighbourhood $U = U(0_E, \delta)$ at zero E , that $T(U) \subset V$ for every $T \in \mathscr{H}$. Due to the bounding of the set A on E we have $A \subset tU$ for enough large $t \in \mathbb{R}_+$. Clear, that $T(A) \subset T(tU)$.

Let $x \in tU$. Then

$$\|x\| < t\delta,$$

$$-t\delta 1_E < x < t\delta 1_E.$$

From order-proserving and weakly additivity of T we have

$$-t\delta T(1_E) < T(x) < t\delta T(1_E),$$

$$\|T(x)\| < t\delta\|T(1_E),$$

i. e.

$$\|\frac{1}{t}T(x)\| < \delta\|T(1_E)\| = \|T(\delta 1_E)\| \leq \varepsilon.$$

Therefore, $T(tU) \subset tV$. Thus, $T(A) \subset tV$ for all $T \in \mathscr{H}$. This means that $B \subset tV$, i. e., the set B bounded on F. Theorem 3 proved.

The following theorem is a variant of theorem Banach-Steinhaus, for weakly additive, order-preserving operators.

Theorem 4. Let E and F be spaces with order unit, $\mathscr{H}$ is some family of weakly additive, order-preserving operators $T: E \to F$, and A is the set all such points $x \in E$ the orbits

$$\mathscr{H}(x) = \{T(x) : T \in \mathscr{H}\}$$

of which are bounded in F. If A is a set of second category, then $A = E$ and the family $\mathscr{H}$ is uniformly continuous.

Proof. Let $V = V(0_F, \varepsilon)$ and $W = V(0_F, \varepsilon')$ be such neighbourhoods, that $\overline{V} + \overline{V} \subset W$, where $\overline{V}$ is closure of V. We put $B = \bigcap_{T \in H} T^{-1}(\overline{V})$.

Let $x \in A$. Then for some natural number n we have $\mathscr{H}(x) \subset nV$ from bounded $\mathscr{H}(x)$. Therefore, $T(x) \in nV$, or $x \in nT^{-1}(V)$ for all $T \in \mathscr{H}$. This means that $x \in nB$. Thus, $A \subset \bigcup_{n=1}^{\infty} nB$. Therefore, at least one of the sets nB is set of second categories, since A is set of second categories, according to the requirement of the theorem. The map $x \mapsto nx$ is a homeomorphism E on itself. Consequently, B is a set of second category on E. From continuously operators $T \in \mathscr{H}$ it follows that B closed on E. Since Because B is set of second category, then it contains interior points. From the construction of the set B, can see that the points of the form $\delta 1_E$ is interior points of B for enough small $\delta \in \mathbb{R}_+$. Let $\delta 1_E$ is such interior point B. Then the set $B - \delta 1_E = \{x - \delta 1_E : x \in B\}$ contains some neighborhood $U = U(0_E, \delta')$ of zero, and,

$$T(U) \subset T(B - \delta 1_E) = \{T(x - \delta 1_E) : x \in B\} = \{T(x) - \delta T(1_E) : x \in B\} = T(B) - \delta T(1_E) \subset \overline{V} - \overline{V} \subset W$$

for all $T \in \mathscr{H}$. This means that $\mathscr{H}$ is uniformly continuous. It means that $\mathscr{H}$ is uniform bounded by Theorem 3. Therefore the orbit $\mathscr{H}(x)$ is bounded in F for all $x \in E$. Consequently, $A = E$. Theorem 4 is proved.

Note that if a space with order unit is Banach space respect order norm, then it called complete space with order unit. Since all Banach space is set of second categories, then from Theorem 4 directly follows

Corollary 2. Let E is a complete space with order unit and F be a space with order unit, $\mathscr{H}$ is some family of weakly additive, order-preserving operators $T: E \to F$, and at every $x \in E$ the set

$$\mathscr{H}(x) = \{T(x) : T \in \mathscr{H}\}$$

bounded on F. Then the family $\mathscr{H}$ is uniformly continuous.

Since Theorem 3 holds, then the Corollary 2 means, that pointwise bounded any family of weakly additive, order-preserving operators from complete spaces with order unit on space with order unit attracts uniformly bounded this family. Let E and F be spaces with order unit, and $\{T_n\}$ a sequence of weakly additive, order-preserving operators $T_n\colon E \to F$. If there exists a limit $\lim_{n\to\infty} T_n(x)$, $x \in E$, then assuming

$$T(x) = \lim_{n\to\infty} T_n(x), x \in E. \tag{3}$$

we have

$$T(x + \lambda 1_E) = \lim_{n\to\infty} T_n(x + \lambda 1_E) = \lim_{n\to\infty} (T_n(x) + \lambda T_n(\lambda 1_E)) = T(x) + \lambda T(1_E)$$

and, if $x \le y$, then

$$T(x) = \lim_{n\to\infty} (T_n(x)) \le \lim_{n\to\infty} (T_n(y)) = T(y).$$

In other words, if there exists a limit of weakly additive, order-preserving operators, then this limit also is weakly additive, order-preserving. Besides, if to consider Theorems 2 and 3, then we get

Corollary 3. Let E and F be spaces with order unit, and $\{T_n\}$ a sequence of weakly additive, order-preserving operators $T_n\colon E \to F$. If there exists the limit $\lim_{n\to\infty} T_n(x)$, $x \in E$, then the operator $T: E \to F$, defined by the formula (3), also is weakly additive, order-preserving, and therefore, continuous.

APPLICATION TO THE ALGEBRA OF CONTINUOUS FUNCTIONS

In this part we state some applications of the obtained results to the algebra of all continuous (or bounded) functions on a compact Hausdorff space.

Note that the algebra $C(X)$ of continuous functions on a compact Hausdorff space X is a space with order unit with respect to the partial order introduced pointwise, i. e. with respect to such a partial order, according to it for continuous functions φ, $\psi \in C(X)$ the inequality $\varphi \leq \psi$ holds if and only if $\varphi(x) \leq \psi(x)$ is true for each $x \in X$.

Obviously, a set

$$C_+(X) = \{\varphi \in C(X) : \varphi \geq 0_X\}$$

is a positive cone in $C(X)$.

Since X is a compact Hausdorff space, any function $\varphi \in C_+(X)$ such that $\varphi(x) \geq a > 0$ for all $x \in X$, can be chosen as an order unit. But, for convenience, it is better to take a constant function 1_X which accepts the value 1 everywhere on X as an order unit in $C(X)$.

For a compact Hausdorff space X by $B(X)$ we denote the set of all bounded functions $\varphi\colon X \to \mathbb{R}$. It is clear that $C(X) \subset B(X)$. The sum $\varphi + \psi$ of elements from $B(X)$ is defined as

$$(\varphi + \psi)(x) = \varphi(x) + \psi(x), \qquad x \in X.$$

Multiplication of elements of $B(X)$ by a scalar is defined by the rule

$$(\lambda\varphi)(x) = \lambda\left(\varphi(x)\right), \qquad x \in X,$$

where $\lambda \in \mathbb{R}$ and $\varphi \in B(X)$. The norm of the function $\varphi \in B(X)$ is defined by the formula

$$\|\varphi\| = \sup\{|\varphi(x)| : x \in X\}.$$

It is easy to check that $(B(X), \|\cdot\|)$ is a Banach space. Moreover, $B(X)$ is a space with order unit 1_X. Similarly in the case of spaces with order unit, one can introduce the concepts of weakly additive, order-preserving and normalized functional on the spaces $C(X)$ of continuous functions and $B(X)$ of bounded functions on X with order unit 1_X.

The set of all weakly additive, order-preserving, of normalized functionals $T : B(X) \to \mathbb{R}$ we denote by $OB(X)$, and by $WB(X)$ is the set of all weakly additive order-preserving functionals. These sets are endowed with the pointwise convergence topology. The base of neighborhoods of a functional $T \in WB(X)$ is formed by sets of the form

$$\langle T; \varphi_1, \ldots, \varphi_n; \varepsilon\rangle = \{P \in OS(X) : |T(\varphi_i) - P(\varphi_i)| < \varepsilon,\ i = 1, \ldots, n\},$$

where $\varphi_i \in B(X)$, $i = 1, \ldots, n$, $n \in \mathbb{N}$, $\varepsilon > 0$.

Now we are ready to state results following from the results of the Main part.

Theorem 1 implies the following.

Theorem 5. Any weakly additive order-preserving the functional

$$T : B(X) \to \mathbb{R} \qquad (T : C(X) \to \mathbb{R})$$

is continuous.

From Theorem 2 we get the following

Theorem 6. Any weakly additive order-preserving functional

$$T : B(X) \to \mathbb{R} \qquad (T : C(X) \to \mathbb{R})$$

is bounded.

CONCLUSION

In the present paper we proved a weakly additive, order-preserving version of the Banach-Steinhaus theorem on spaces with order unit. This result improves results in [2, 3, 4, 5, 6, 7, 8]. To establish the main result at first we obtained that every weakly additive, order-preserving operator between spaces with order unit is automatically continuous and bounded. Then we showed that for an arbitrary uniformly continuous family $\mathscr{H}$ of weakly additive, order-preserving operators between spaces E and F with order unit, and for each bounded subset A in E there exits a bounded set B in F such that $T(A) \subset B$ for every $T \in \mathscr{H}$.

Finally we bring some applications of the obtained results to the algebra of all continuous (or bounded) functions on a compact Hausdorff space.

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