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V.I.ROMANOVSKIY INSTITUTE OF MATHEMATICS

UZBEK MATHEMATICAL JOURNAL

Journal was founded in 1957. Until 1991 it was named by
"Izv. Akad. Nauk UzSSR, Ser. Fiz.-Mat. Nauk". Since 1991 it is known as
"Uzbek Mathematical Journal". It has 4 issues annually.

Volume 68 Issue 2 2024

Uzbek Mathematical Journal is abstracting and indexing by

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TASHKENT - 2024

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Address: University str., 9, Tashkent 100174, Uzbekistan

Uzbek Mathematical Journal
2024, Volume 68, Issue 2, pp.69-75
DOI: 10.29229/uzmj.2024-2-8

On dynamics of infinite dimensional Volterra QSOs

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Abstract. In this paper, we investigate the behavior of infinite-dimensional Volterra quadratic stochastic operators corresponding to the negative upper triangular skew-symmetric matrix. It is proved that the trajectory of such operators cannot consist of convergent subsequences. This property immediately implies non-ergodicity of the operator. Moreover, it is shown that the operator's dynamics with respect to pointwise convergence are regular.

Keywords: Volterra stochastic operator; trajectory, omega limiting sets, pointwise convergence.
MSC (2020): 37B25, 37A30, 46N60.

1. Introduction

Quadratic mappings are present in different areas of mathematics and have various applications: the theory of differential equations, probability theory, the theory of dynamical systems, mathematical economics, mathematical biology, statistical physics, etc (see [3, 4]).

The discrete dynamical system corresponding to quadratic stochastic operators (abbreviated QSO.) appeared in the works of Bernstein [2]. It is known [4] that QSOs is usually employed to describe the time evolution of species in biology. We notice that quadratic dynamical systems are crucial for analyzing dynamic properties and modeling in fields such as population dynamics [3], economics [10, 11] and mathematics [3, 4]. In the study of finite dimensional Volterra dynamical systems for a given biological population, one may pose the simple question: which genotypes will survive, and which ones will die out? Recall that a lot of papers are devoted to the investigations of finite dimensional Volterra operators (for more details see [7]).

In recent years, there has been an increased interest in the evolutionary and dynamical aspects of quadratic dynamical systems within game theory. Hofbauer and Sigmund's book [3] offers an excellent introduction to this theory. Akin and Losert have studied zero-sum games and their evolutionary dynamics within this context. In recent decades in the game theory, evolutionary and dynamical aspects of quadratic dynamical systems have dramatically increased in popularity [1].

In [9] Nagylaki examined the impact on the dynamics of the Volterra stochastic operators when the population size is large. This naturally leads our attention to the following problem: what is the dynamical behavior of Volterra operators on an infinite dimensional simplex? In [5, 6] a certain construction of infinite dimensional Volterra operators was studied, but the investigation of their dynamics were left out. In [8] the authors constructed the class of non-ergodic infinite-dimensional Volterra operators. We recall that the first finite dimensional non-ergodic QSO was constructed by Zakharevich [12]. The current paper continues the research from reference [8].

2. Preliminaries

In what follows, as usual, ℓ^1 denotes the space of all absolutely summable sequences with the norm $\|\mathbf{x}\|_1 = \sum\limits_{k=1}^{\infty} |x_k|$.

For a given $r > 0$ we denote

$$\mathbf{B}_r^+ = \{\mathbf{x} \in \ell^1 : \ x_k \geq 0 \text{ for all } k \in \mathbb{N}, \ \|\mathbf{x}\|_1 \leq r\}$$

and

$$S_r = \{\mathbf{x} \in \mathbf{B}_r^+ : \|\mathbf{x}\|_1 = r\}.$$

In the sequel, the unit sphere S_1 is called *an infinite dimensional simplex*. Furthermore, for the sake of simplicity, we write S instead of S_1.

It is known that $S = convh(ExtrS)$, where $Extr(S)$ is the extremal points of S and $convh(A)$ is the convex hall of a set A. Any extremal point of S has the following form:

$$\mathbf{e}_k = (\underbrace{0, \ldots, 0, 1}_{k}, , 0, \ldots), \quad k \in \mathbb{N}.$$

Here and henceforth we denote

$$int(S_r) = \{\mathbf{x} \in S_r : x_k > 0, \ k \in \mathbb{N}\}, \quad \partial S_r = S_r \setminus int(S_r).$$

Let $\{\mathbf{x}^{(n)}\}$ be a sequence in ℓ^1. In what follows we write $\mathbf{x}^{(n)} \xrightarrow{\|\cdot\|_1} \mathbf{a}$ instead of $\|\mathbf{x}^{(n)} - \mathbf{a}\|_1 \to 0$.

Note that for any positive real number r the sets S_r and $\mathbf{B}_r^+$ are not compact w.r.t. ℓ^1-norm. We notice that in the finite dimensional setting, analogues of these sets are compact, and hence, the investigation of the dynamics of nonlinear mappings over these kind of sets use well-known methods and techniques of dynamical systems. In our case, the non compactness (w.r.t. ℓ^1-norm) of the set $\mathbf{B}_r^+$ complicates our further investigation on dynamics of Volterra operators. Therefore, we need such a weak topology on ℓ^1 so that the set $\mathbf{B}_r^+$ would be compact with respect to that topology. It is obvious that one of weak topologies on ℓ^1 is the Tychonov topology which generates the pointwise convergence. We say that a sequence $\{\mathbf{x}^{(n)}\} \subset \ell^1$ converges *pointwise* to $\mathbf{x} = (x_1, x_2, \ldots) \in \ell^1$ if

$$\lim_{n\to\infty} x_k^{(n)} = x_k \quad \text{for every } k \geq 1.$$

and write $\mathbf{x}^{(n)} \xrightarrow{\text{p.w.}} \mathbf{x}$.

We notice that the set ℓ^1 is not closed w.r.t. pointwise topology, and its completion is s which is the space of all sequences. It is known that this topology is metrizable by the following metric:

$$\rho(\mathbf{a}, \mathbf{b}) = \sum_{k=1}^{\infty} 2^{-k} \frac{|a_k - b_k|}{1 + |a_k - b_k|}, \quad \mathbf{a}, \mathbf{b} \in s. \tag{2.1}$$

Hence, for a given sequence $\{\mathbf{x}^{(n)}\} \subset s$ the following statements are equivalent:

(i) $\mathbf{x}^{(n)} \xrightarrow{\text{p.w.}} \mathbf{x}$;

(ii) $\mathbf{x}^{(n)} \xrightarrow{\rho} \mathbf{x}$.

In the sequel, we will show that the unit ball of ℓ^1 is compact w.r.t. pointwise convergence, while whole ℓ^1 is not closed in s.

We recall that ℓ^∞ is defined to be the space of all bounded sequences endowed with the norm

$$\|\mathbf{x}\|_\infty = \sup_n \{|x_n|\}.$$

By c_0 we, as usual, denote the space of all null sequences, which is a closed subspace of ℓ^∞.

We notice that $\mathbf{B}_1^+$ is sequentially compact w.r.t. the pointwise convergence.

It is clear that $\mathbf{x}^{(n)} \xrightarrow{\|\cdot\|_1} \mathbf{a}$ implies $\mathbf{x}^{(n)} \xrightarrow{\text{p.w.}} \mathbf{a}$. A natural question arises: is there any equivalence criteria for these two types of convergence on some set? Next result gives a positive answer to this question.

Lemma 2.1. [8] *Let $\{\mathbf{x}^{(n)}\}$ be a sequence on S_r. Then the following statements are equivalent:*

(i) $\mathbf{x}^{(n)} \xrightarrow{\|\cdot\|_1} \mathbf{a}$*;*

(ii) $\mathbf{x}^{(n)} \xrightarrow{\text{p.w.}} \mathbf{a}$ *and* $\mathbf{a} \in S_r$*.*

Recall that a functional $\varphi : \ell^1 \to \mathbb{R}$ is called *pointwise continuous* if for any $\mathbf{a} \in \ell^1$ and any sequence $\{\mathbf{x}^{(n)}\} \subset \ell^1$ with $\mathbf{x}^{(n)} \xrightarrow{\text{p.w.}} \mathbf{a}$ one has $\varphi(\mathbf{x}^{(n)}) \to \varphi(\mathbf{a})$.

Now we provide a criteria for linear functionals to be pointwise continuous.

Given $\mathbf{b} \in \ell^\infty$, let us define

$$\varphi_{\mathbf{b}}(\mathbf{x}) = \sum_{k=1}^{\infty} b_k x_k, \qquad \mathbf{x} \in \ell^1. \tag{2.2}$$

Lemma 2.2. [8] *Let $\mathbf{b} \in \ell^\infty$, then the linear functional $\varphi_{\mathbf{b}}$ is pointwise continuous on $\mathbf{B}_1^+$ iff $\mathbf{b} \in c_0$.*

3. Infinite dimensional Volterra QSO

Let V be a mapping on the infinite dimensional simplex S defined by

$$(V(\mathbf{x}))_k = \sum_{i,j=1}^{\infty} p_{ij,k} x_i x_j, \quad k = 1, 2, 3, \dots \tag{3.1}$$

Here, $\{p_{ij,k}\}$ are the hereditary coefficients which satisfy

$$p_{ij,k} \geq 0, \quad p_{ij,k} = p_{ji,k}, \quad \sum_{k=1}^{\infty} p_{ij,k} = 1, \quad i, j, k = 1, 2, 3, \dots. \tag{3.2}$$

It is important to notice that the mapping V is well-defined i.e., $V(S) \subset S$. Such kind of mapping V is called *infinite dimensional quadratic stochastic operator* (in short QSO).

Likewise as a finite dimensional case a QSO $V : S \to S$ is called *Volterra QSO* if

$$p_{ij,k} = 0 \text{for any } (i, j, k) \in \mathbb{N}^3 \text{ such that } k \notin \{i, j\}. \tag{3.3}$$

It is known [7] that a quadratic stochastic operator V is *Volterra QSO* if and only if it can be represented as follows:

$$(V(\mathbf{x}))_k = x_k \left(1 + \sum_{i=1}^{\infty} a_{ki} x_i\right), \quad k \in \mathbb{N}, \tag{3.4}$$

where $\mathbf{x} = (x_1, x_2, \dots) \in S$ and

$$a_{ki} = -a_{ik}, \ |a_{ki}| \leq 1 \ \text{ for every } k, i \in \mathbb{N}. \tag{3.5}$$

By $\mathcal{V}$ we denote the set of all Volterra QSOs on infinite dimensional simplex S, and $\mathbb{A}$ denotes the set of all skew-symmetric matrices with (3.5). The representation (3.4) establishes a one-to-one correspondence $\mathfrak{f} : \mathcal{V} \to \mathbb{A}$ by $\mathfrak{f}(V) = (a_{ki})$. It is clear that $\mathfrak{f}$ is affine, hence $\mathcal{V}$ is convex, and moreover, this correspondence allows to investigate certain geometric properties of $\mathcal{V}$ by means of structure of the set $\mathbb{A}$ (see [6] for more details).

For a given operator V on S, by $\{V^n(\mathbf{x}_0)\}_{n=1}^{\infty}$ we denote the trajectory of a point $\mathbf{x}_0 \in S$ under V. By $\omega_V(\mathbf{x}_0)$ (respectively, $\omega_V^{(w)}(\mathbf{x}_0)$) we denote the set of limit points of $\{V^n(\mathbf{x}_0)\}_{n=1}^{\infty}$ with respect to ℓ^1-norm (respectively, pointwise convergence).

In what follows, by a *fixed point* of V we mean a vector $\mathbf{x} \in S$ such that $V(\mathbf{x}) = \mathbf{x}$. By $Fix(V)$ we denote the set of all fixed points of V.

Obviously, if $\omega_V(\mathbf{x}_0)$ consists of a single point, i.e. $\omega_V(\mathbf{x}_0) = \{\mathbf{x}^*\}$, then the trajectory $\{V^n(\mathbf{x}_0)\}_{n=1}^{\infty}$ converges to $\mathbf{x}^*$. Moreover, $\mathbf{x}^*$ is a fixed point of V. However, looking ahead, we remark that convergence of trajectories is not a typical case for the dynamical systems (3.4).

In [8] two subclasses of $\mathcal{V}$ have been defined:

$$\mathcal{V}^+ = \{V \in \mathcal{V} : \mathfrak{f}(V) \in \mathbb{A}^+\}, \quad \mathcal{V}^- = \{V \in \mathcal{V} : \ \mathfrak{f}(V) \in \mathbb{A}^-\},$$

where

$$\mathbb{A}^+ = \{(a_{ki}) \in \mathbb{A} : \ a_{ki} \geq 0 \text{ for all } k < i\},$$

$$\mathbb{A}^- = \{(a_{ki}) \in \mathbb{A} : \ a_{ki} \leq 0 \text{ for all } k < i\}.$$

We notice that $\mathcal{V}^+ \cap \mathcal{V}^- = \{Id\}$, here Id stands for the identity operator.

Definition 3.1. A ℓ^1-continuous function $\varphi : S \to \mathbb{R}$ is called a *Lyapunov function* for QSO V if the limit $\lim\limits_{n\to\infty} \varphi(V^n(\mathbf{x}))$ exists for any initial point $\mathbf{x} \in S$.

Obviously, if φ is Lyapunov function for QSO V and $\lim\limits_{n\to\infty} \varphi(V^n(\mathbf{x}_0)) = \mathbf{x}^*$, then $\omega_V(\mathbf{x}_0) \subset \varphi^{-1}(\mathbf{x}^*)$.

Let us denote

$$\mathbf{b}_\downarrow = (b_1, \dots, b_n, \dots), \quad \text{such that } b_1 \geq \dots \geq b_n \geq \cdots .$$

The following result is stated as Corollary 4.5 in [8].

Theorem 3.2. *Let $V \in \mathcal{V}^-$ and $\mathbf{b}_\downarrow \in \ell^\infty$. Then the functional $\varphi_{\mathbf{b}_\downarrow}$ given by (2.2) on S is a Lyapunov function for V.*

4. The dynamics of Volterra QSO

Let us consider a skew-symmetric matrix $A = (a_{ki})_{k,i\in\mathbb{N}}$ such that

$$-1 \leq a_{ki} < 0, \qquad \forall k < i. \tag{4.1}$$

Then corresponding Volterra QSO belongs to the class $\mathcal{V}^-$.

Lemma 4.1. *Let $A = (a_{ki})_{k,i\in\mathbb{N}}$ be a skew-symmetric matrix given by (4.1) and V be a Volterra QSO generated by A. Then $Fix(V) = \{\mathbf{e}_i : i \in \mathbb{N}\}$.*

Proof. It is obvious that $\{\mathbf{e}_i : i \in \mathbb{N}\} \subset Fix(V)$. Let us take an arbitrary $\mathbf{x} \in S \setminus \{\mathbf{e}_i : i \in \mathbb{N}\}$ and denote $i_0 = \min\{i \in \mathbb{N} : x_i \neq 0\}$. Then we have

$$(V(\mathbf{x}))_{i_0} = x_{i_0}\left(1 + \sum_{i\in supp(\mathbf{x})} a_{i_0 i} x_i\right).$$

Due to $a_{i_0 i_0} = 0$ and $a_{i_0 i} < 0$ for all $i > i_0$ one gets

$$(V(\mathbf{x}))_{i_0} < x_{i_0},$$

which implies that $\mathbf{x} \notin Fix(V)$. The lemma is proved. □

For a given positive integer m we consider the following functional on S

$$\varphi_m(\mathbf{x}) = \sum_{k=1}^m x_k, \qquad \forall \mathbf{x} = (x_1, x_2, \dots) \in S. \tag{4.2}$$

Due to Theorem 3.2 every φ_m is a Lyapunov function for V.

The following lemma plays a crucial role in our further investigations.

Lemma 4.2. *Let m be a positive integer and V be a Volterra QSO with (4.1). Then for any integer $n \geq 1$ one holds*

$$\varphi_m(V^n(\mathbf{x})) < \varphi_m(V^{n-1}(\mathbf{x})), \qquad \forall \mathbf{x} \in int(S). \tag{4.3}$$

Proof. Let us pick any $\mathbf{x} \in int(S)$ and calculate $\varphi_m(V^n(\mathbf{x}))$. By (3.4), (4.2) and keeping in mind $(a_{ki})_{k,i\in\mathbb{N}}$ is skew-symmetric we obtain

$$\begin{aligned}
\varphi_m(V^n(\mathbf{x})) &= \sum_{k=1}^m (V^n(\mathbf{x}))_k \\
&= \sum_{k=1}^m \left(V^{n-1}(\mathbf{x})\right)_k \left(1 + \sum_{i\neq k} a_{ki}\left(V^{n-1}(\mathbf{x})\right)_i\right) \\
&= \varphi_m\left(V^{n-1}(\mathbf{x})\right) + \sum_{k=1}^m \sum_{i=1}^m a_{ki}\left(V^{n-1}(\mathbf{x})\right)_k \left(V^{n-1}(\mathbf{x})\right)_i \\
&\quad + \sum_{k=1}^m \sum_{i>m} a_{ki}\left(V^{n-1}(\mathbf{x})\right)_k \left(V^{n-1}(\mathbf{x})\right)_i \\
&= \varphi_m\left(V^{n-1}(\mathbf{x})\right) + \sum_{k=1}^m \sum_{i>m} a_{ki}\left(V^{n-1}(\mathbf{x})\right)_k \left(V^{n-1}(\mathbf{x})\right)_i .
\end{aligned}$$

Then by $a_{ki} < 0$, $(k < i)$ we get

$$\sum_{k=1}^{m}\sum_{i>m} a_{ki}\left(V^{n-1}(\mathbf{x})\right)_k \left(V^{n-1}(\mathbf{x})\right)_i < 0.$$

Hence, (4.3) holds. The lemma is proved. □

Proposition 4.3. *Let $A = (a_{ki})_{k,i\in\mathbb{N}}$ be a skew-symmetric matrix given by (4.1) and V be a Volterra QSO generated by A. Then $|\omega_V^{(w)}(\mathbf{x})| = 1$ for any $\mathbf{x} \in int(S)$. Moreover, there exists $r \in [0;1)$ such that $\omega_V^{(w)}(\mathbf{x}) \subset \partial S_r$.*

Proof. Let us take an arbitrary $\mathbf{x} \in int(S)$. Thanks to lemma 4.2 for every $m \geq 1$ one can find $\alpha_m \in [0;1)$ such that

$$\alpha_m = \lim_{n\to\infty} \varphi_m(V^n(\mathbf{x})).$$

We notice that $\{\alpha_m\}$ is a decreasing sequence.

Now we show that for every $i \in \mathbb{N}$ there exists the following limit

$$\lim_{n\to\infty} (V^n(\mathbf{x}))_i.$$

Keeping in mind $(V^n(\mathbf{x}))_m = \varphi_m(V^n(\mathbf{x})) - \varphi_{m-1}(V^n(\mathbf{x}))$ for every $m \geq 2$ we get

$$\lim_{n\to\infty} (V^n(\mathbf{x}))_m = \alpha_m - \alpha_{m-1}.$$

And for $m = 1$ we use $(V^n(\mathbf{x}))_1 = \varphi_1(V^n(\mathbf{x}))$. This means that

$$\lim_{n\to\infty} (V^n(\mathbf{x}))_1 = \alpha_1.$$

Thus, we have shown that $\lim\limits_{n\to\infty} (V^n(\mathbf{x}))_m \geq 0$ for every $m \geq 1$. This means that $|\omega_V^{(w)}(\mathbf{x})| = 1$.

Let us suppose that there exists a positive integer i_0 such that

$$\lim_{n\to\infty} (V^n(\mathbf{x}))_{i_0} = r > 0.$$

Consequently, we have $\alpha_{i_0} > 0$. Moreover, since $\alpha_{i_0} < 1$ one gets $r < 1$. Then from

$$\varphi_{i_0}(V^n(\mathbf{x})) - \varphi_{i_0}(V^{n-1}(\mathbf{x})) = \sum_{k=1}^{i_0}\sum_{i>i_0} a_{ki}\left(V^{n-1}(\mathbf{x})\right)_k \left(V^{n-1}(\mathbf{x})\right)_i.$$

we immediately find

$$\lim_{n\to\infty}\sum_{k=1}^{i_0}\sum_{i>i_0} a_{ki}\left(V^{n-1}(\mathbf{x})\right)_k \left(V^{n-1}(\mathbf{x})\right)_i = 0.$$

Since $a_{ki} < 0$ we conclude that

$$\lim_{n\to\infty}\sum_{i>i_0} \left(V^{n-1}(\mathbf{x})\right)_k \left(V^{n-1}(\mathbf{x})\right)_i = 0, \quad \forall k \in \{1,\dots,i_0\},$$

which yields that

$$\lim_{n\to\infty}\sum_{i>i_0} \left(V^{n-1}(\mathbf{x})\right)_{i_0} \left(V^{n-1}(\mathbf{x})\right)_i = 0.$$

This together with assumption implies that

$$\lim_{n\to\infty} \left(V^{n-1}(\mathbf{x})\right)_i = 0, \quad \forall i > i_0.$$

On the other hand, if $i_0 \geq 2$ then

$$\lim_{n\to\infty} \left(V^{n-1}(\mathbf{x})\right)_i = 0, \quad \forall i < i_0. \tag{4.4}$$

Indeed, from the following equality

$$\begin{aligned}(V^n(\mathbf{x}))_{i_0}-(V^{n-1}(\mathbf{x}))_{i_0} &= (V^{n-1}(\mathbf{x}))_{i_0}\sum_{i=1}^{i_0}a_{i_0i}(V^{n-1}(\mathbf{x}))_i\\ &\quad+(V^{n-1}(\mathbf{x}))_{i_0}\sum_{i>i_0}a_{i_0i}(V^{n-1}(\mathbf{x}))_i\end{aligned}$$

by $\lim\limits_{n\to\infty}\sum\limits_{i>i_0}a_{i_0i}(V^{n-1}(\mathbf{x}))_i=0$ we get

$$\lim_{n\to\infty}\sum_{i=1}^{i_0}a_{i_0i}(V^{n-1}(\mathbf{x}))_i=0.$$

Henceforth, due to $a_{i_0i}>0$ $(i<i_0)$ we obtain (4.4). Thus, we have shown that

$$V^n(\mathbf{x})\xrightarrow{\text{p.w.}}(\underbrace{0,\dots,0,r}_{i_0},0,0,\dots),$$

where $0\le r<x_1+x_2+\dots+x_{i_0}$. □

By *support* of $\mathbf{x}=(x_1,\dots,x_n,\dots)$ we mean a set $supp(\mathbf{x})=\{i\in\mathbb{N}:\ x_i\neq 0\}$. Then as a corollary of proposition 4.3 we can formulate the following result.

Corollary 4.4. *Let $A=(a_{ki})_{k,i\in\mathbb{N}}$ be a skew-symmetric matrix given by (4.1) and V be a Volterra QSO generated by A. Then for any $\mathbf{x}\in S$ with $|supp(\mathbf{x})|=\infty$ there exist $r\in[0;1)$ and $i_0\in supp(\mathbf{x})$ such that*

$$\lim_{n\to\infty}(V^n(\mathbf{x}))_i=\begin{cases}0, & \text{if } i\neq i_0;\\ r, & \text{if } i=i_0.\end{cases}$$

Theorem 4.5. *Let $A=(a_{ki})_{k,i\in\mathbb{N}}$ be a skew-symmetric matrix given by (4.1) and V be a Volterra QSO generated by A. Then $\omega_V(\mathbf{x})\neq\emptyset$ for initial point $\mathbf{x}\in S$ iff $|supp(\mathbf{x})|<\infty$.*

Proof. Let $|supp(\mathbf{x})|<\infty$. Then one can find a positive integer i_0 such that $x_{i_0}>0$ and $x_i=0$ for every $i>i_0$. If $x_{i_0}=1$ then $\mathbf{x}=\mathbf{e}_{i_0}$. And due to $\mathbf{e}_{i_0}\in Fix(V)$ we infer that $\omega_V(\mathbf{x})=\{\mathbf{e}_{i_0}\}$. Now, we assume that $x_{i_0}\neq 1$. This case is only possible when $i_0\ge 2$. Then we have

$$\begin{aligned}\sum_{i<i_0}(V^{n+1}(\mathbf{x}))_i &= \sum_{i<i_0}(V^n(\mathbf{x}))_i+\sum_{k=i_0}^{\infty}\sum_{i<i_0}a_{ik}(V^n(\mathbf{x}))_i(V^n(\mathbf{x}))_k\\ &= \sum_{i<i_0}(V^n(\mathbf{x}))_i+\sum_{i<i_0}a_{ii_0}(V^n(\mathbf{x}))_i(V^n(\mathbf{x}))_{i_0}\\ &\le \sum_{i<i_0}(V^n(\mathbf{x}))_i\left(1+a_{i_0}-a_{i_0}\sum_{i<i_0}(V^n(\mathbf{x}))_i\right),\end{aligned}$$

where $a_{i_0}=\max\limits_{i<i_0}\{a_{ii_0}\}$. Hence,

$$\sum_{i<i_0}(V^{n+1}(\mathbf{x}))_i\le\sum_{i<i_0}\mathbf{x}_i\prod_{k=0}^{n}\left(1+a_{i_0}-a_{i_0}\sum_{i<i_0}\left(V^k(\mathbf{x})\right)_i\right)^k.$$

From the last inequality by

$$-a_{i_0}\left(V^k(\mathbf{x})\right)_i<-a_{i_0}\mathbf{x}_i,\qquad\forall k\ge 0,$$

we obtain

$$\sum_{i<i_0}(V^{n+1}(\mathbf{x}))_i\le\sum_{i<i_0}\mathbf{x}_i\left(1+a_{i_0}-a_{i_0}\sum_{i<i_0}\mathbf{x}_i\right)^{n+1}.$$

Finally, keeping in mind $0 < 1 + a_{i_0} - a_{i_0} \sum_{i<i_0} \mathbf{x}_i < 1$ one has

$$\lim_{n\to\infty} \sum_{i<i_0} (V^n(\mathbf{x}))_i = 0. \tag{4.5}$$

On the other hand, we get

$$\sum_{i=1}^{i_0} (V^n(\mathbf{x}))_i = 1, \qquad \forall n \geq 0.$$

This together with (4.5) yields that $\lim_{n\to\infty} (V^n(\mathbf{x}))_{i_0} = 1$. So, we conclude that $V^n(\mathbf{x}) \xrightarrow{\text{p.w.}} \mathbf{e}_{i_0}$. Since, $\mathbf{e}_{i_0} \in S$, thanks to lemma 2.1 one has

$$V^n(\mathbf{x}) \xrightarrow{\|\cdot\|_1} \mathbf{e}_{i_0}.$$

Let $|supp(\mathbf{x})| = \infty$. In this case according to Corollary 4.4 we get

$$V^n(\mathbf{x}) \xrightarrow{\text{p.w.}} \mathbf{x}^*,$$

where $\mathbf{x}^* \in S_r$ for some $r \in [0; 1)$. Then since $\mathbf{x}^* \notin S$ thanks to lemma 2.1 we infer that $\omega_V(\mathbf{x}) = \emptyset$.

The theorem is proved. □

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Uzbek Mathematical Journal
2024, Volume 68, Issue 2, pp.76-80
DOI: 10.29229/uzmj.2024-2-9

Geometric properties of geometric tripotents and split faces in neutral SFS-space

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Abstract. The present paper is devoted to the study of geometric properties of split faces of a unit ball of neutral SFS-space and to the study of properties of relations in the set of geometric tripotents. Namely, we give the condition under which the complete orthomodular lattice of geometric tripotents is a Boolean algebra and prove that the relations $\leq_r, \ \leq_c$ are pre-orders in the set of geometric tripotents.

Keywords: Neutral SFS-space; geometric tripotent; symmetric face; split face; strongly split face; Boolean algebra; pre-order.

MSC (2020): 46B20, 46E30

1. Introduction

The facially symmetric spaces first introduced and studied in [1, 2] by Y.Friedman and B.Russo provide an appropriate structure that allows us to study the problem of characterizing the unit ball of a predual space JBW*-triples, describing important properties of the convex set in geometric terms. In [1, 2, 3, 4], the face structure of the unit ball of a facially symmetric space and its dual space was deeply analyzed, and basic notions such as *orthogonality, projective unit, norm-exposed face, symmetric face, generalized (or geometric) tripotent,* and *generalized (or geometric) Pierce projections* were defined using purely geometric terms. In this paper, we will continue to study the geometric properties of these spaces and the aforementioned notions.

The structure of this paper is as follows. The second section introduces the necessary concepts and information from the theory of facially symmetric spaces, which are necessary to present the results of the study. Note that in this paper we use the terminology and notations used in [1, 2, 3, 4, 5].

In [2, Proposition 4.5] it was proved that for any fixed geometric tripotent ω in a neutral strongly facially symmetric space (SFS-space) Z the set $L_\omega := \{v \in G\mathfrak{U} : v \leq \omega\} \bigcup \{0\}$ is a complete orthomodular lattice with smallest element 0, largest element ω and orthomplement $v \mapsto v^\perp = \omega - v$, where $G\mathfrak{U}$ is the set of all geometric tripotents of the unit ball of the dual space Z^*. In [8, Definition 3.1] we defined a strongly split face of the unit ball of a neutral strongly facially symmetric space and proved that if for any $u \in L_\omega$ a symmetric face F_u is a strongly split face, and then L_ω is a Boolean algebra (see in [8, Theorem 3.2]). In the third section, we study geometric properties of split faces of the unit ball and show that in neutral SFS-space with condition (FE) the notions of split face and strongly split face coincide. Consequently, we give conditions (in Corollary 3.7) under which a complete orthomodular lattice $L_\omega := \{v \in G\mathfrak{U} : v \leq \omega\} \bigcup \{0\}$ is a Boolean algebra in neutral SFS-space with condition(FE).

In [5, Section 2.1] the relations, $\leq_r$, $\leq_c$ in the set of tripotents of JB^*-triples were defined and it is shown that they are pre-orders, i.e., they are reflexive, transitive and not antisymmetric. In Section 4, these relations are similarly defined on the set of geometric tripotents $G\mathfrak{U}$, in the dual space of SFS-space and it is shown that they are pre-orders. Moreover, we investigate geometric properties of these relations necessary for further study of the theory of facially symmetric spaces.

1.1. **Preliminaries.** Let Z be a real or complex normed space. Elements $f, g \in Z$ are called *mutually orthogonal* if $\|f+g\| = \|f-g\| = \|f\| + \|g\|$. Mutually orthogonal elements $f, g \in Z$ are denoted by $f \diamond g$. A *norm-exposed face* of a unit ball Z_1 of space Z is a non-empty set (not coincident with Z_1) of the form $F_x = \{f \in Z_1 : \ f(x) = 1\}$, where $x \in Z^*$, $\|x\| = 1$. For subsets $S, \ T$ of the space Z, $S \diamond T$ means that $f \diamond g$ for all $f \in S, \ g \in T$. For any subset $S \subset Z$, let $S^\diamond = \{f \in Z : \ f \diamond g, \ \ \forall g \in S\}$ and call $S^\diamond$ the *orthogonal complement* to S. An element $u \in Z^*$ is called a *projective unit* if $\|u\| = 1$ and $\langle u, \ F_u^\diamond \rangle = 0$. This means that a norm-exposed face F_u is "parallel" to $F_u^\diamond$. By $\mathfrak{F}$ and $\mathfrak{U}$ we denote the set of norm-exposed faces of Z_1 and projective units in Z^*, respectively. The mapping $\mathfrak{U} \ni u \mapsto F_u \in \mathfrak{F}$ is not a bijection (see [1, Example 4]).

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Computer imposition: *K.K. Abdurasulov*

The journal was registered by the Press and Information Agency of the Republic of Uzbekistan on December 22, 2006. Register. No 0044.

Handed over to the set on 11/03/2022. Signed for printing on 12/04/2023
Format 60×84 1/16. Literary typeface. Offset printing.

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Printed in a printing house "MERIT-PRINT"
Tashkent city, Yakkasaray district, Sh. Rustaveli street, 91