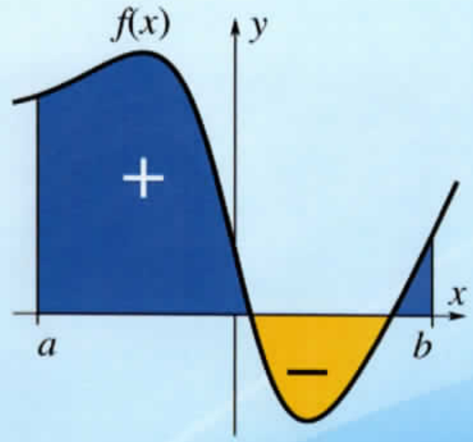
A.G.ABDURAXMANOV

ANIQMAS VA ANIQ INTEGRALLAR

(Matematik analiz)



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CHIRCHIQ DAVLAT PEDAGOGIKA UNIVERSITETI

« ALGEBRA VA MATEMATIK ANALIZ »
KAFEDRASI

A.G.ABDURAXMANOV

ANIQMAS VA ANIQ INTEGRALLAR

(Matematik analiz)

Kunduzgi, kechki va sirtqi ta’lim talabalari uchun

O‘quv qo‘llanma

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A.G.Abduraxmanov “Aniqmas va aniq integrallar” (Maple dasturi asosida). Kunduzgi, kechki va sirtqi ta’lim talabalari uchun o‘quv qo‘llanma. – T.: «BOOK TRADE 2022» nashriyoti, 2022. – 108 b.

O‘quv qo‘llanma 29.08.2020 da tasdiqlangan “Matematik analiz fan dasturi” asosida “Matematik analiz” fani bo‘yicha 60110600- Matematika va informatika yo‘nalishida tahsil olayotgan va 60110700- fizika va astranomiya ta’lim yo‘nalishida tahsil olayotgan talabalar uchun mo‘ljallangan. Unda kunduzgi,kechki va sirtqi ta’lim talabalari uchun oliy matematikaning aniqmas va aniq integrallar bo‘limiga tegishli masala va misollarni mustaqil ishlashlari uchun kerakli nazariy va amaliy mavzular berilgan.. Har bir mavzuga qisqacha nazariy ma’lumotlar keltirilib, ularning qo‘llanishi ko‘plab misollarda tushuntirilgan.

Taqrizchilar:

Raximov A.A. – Uzbekiston Milliy Universiteti “Algebra va funksional analiz” kafedrasi professori,

Gel'fanova D.D. – Fevzi Yaqubov nomli Qirim injener-pedagoglar Universiteti “Matematika” kafedrasi dotsenti (Rossiya),

Ichki taqrizchi:

Seytov A.J. – Chirchiq davlat pedagogika universiteti “Algebra va matematik analiz” kafedrasi dotsenti.

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ANIQMAS VA ANIQ INTEGRALLAR (MAPLE DASTURI ASOSIDA)

SO'Z BOSHI

Ushbu o‘quv qo‘llanmaning maqsadi talabaga integrallash texnikasini va ma’lum integrallarni qo‘llash uchun turli xil masalalarni echish qobiliyatini o‘rgatishdir. Bunda zamon talabidan kelib chiqqan xolda "Maple" dasturidan foydalanildi.

O‘quv qo‘llanma uchta bobdan iborat bo‘lib har bir bobi bir nechta paragraflardan tashkil topgan. Har bir bob asosiy ta’riflar, formulalar, teoremalarni isbotsiz o‘z ichiga olgan qisqa nazariy kirish bilan boshlanadi. Vazifalarni tanlashda , avvalambor, integrallash usullarini o‘zlashtirish yo‘lida talabalar duch kelishi mumkin bo‘lgan qiyinchiliklarni hisobga olishga harakat qilindi.

Ushbu o‘quv qo‘llanmada ko‘rsatilgan mavzular bo‘yicha batafsil echimlar bilan 53 ta misol keltirilgan. Yassi figuralarning yuzalari, egri chiziq uzunligini, fazoviy jismlarining hajmlarini hisoblashda echimlar aniqlik uchun “Maple” dasturida raqamlar va batafsil tushuntirishlar bilan tasvirlangan.

Mazkur kitob mavjud Davlat Ta’lim Standartida belgilangan va matematik ta’lim

mazmuniga kiritilgan ”Matematik ahaliz” kursining aniqmas va aniq integrallar mavzularini o‘zlashtirishda talabalarga o‘quv qo‘llanma sifatida tayyorlangan. Ushbu qo‘llanma "Funktsiyalar integrali" moduliga qo‘shimcha bo‘lib, unda "Aniqmas integrallar", "Aniq integrallar va ularning qo‘llanilishi" va "Hosmas integrallar" mavzularidagi individual topshiriqlar mavjud. Qo‘llanma pedagogika institutlarining aniq va tabiiy fanlari hamda oliy ta’lim muassasalarining texnika va iqtisodiy mutaxassisliklar bo‘yicha talim olayotgan birinchi bosqich talabalariga mo‘ljallangan.

Ushbu o‘quv qo‘llanma talabalarga modulni amalga oshirish va ushbu materialni o‘rganish bo‘yicha mustaqil ishlarda yordam beradi deb umid qilaman.

O‘quv qollanmani yozishda rus va o‘zbek tillarida chop etilgan mavjud adabiyotlardan foydalanildi.

O‘quv qollanma haqida fikr bildirgan barcha taklif va fikr-mulohazalarni minnatdorchilik bilan qabul qilaman.

Shartli belgilar:

$\forall$ -- ihtiyoriy, harqanday

$\exists$ -- shunday

1. ANIQMAS INTEGRALLAR

ASOSIY TUSHUNCHA VA TEOREMALAR.

1.1. Aniqmas integral va uni hisoblash usullari.

$f(x)$ va $F(x)$ funksiyalar biror (a,b) intervalda aniqlangan bo‘lib, $F(x)$ funksiya (a,b) intervalda differensiallanuvchi bo‘lsin.

Quyidagi masalani qaraymiz: $\exists F(x)$ funksiyani topish kerakki, $\forall x \in (a,b)$ uchun $F'(x) = f(x)$ bo‘lsin.

1-Ta’rif. *Agar* $\forall x \in (a,b)$ *uchun* $F'(x) = f(x)$ *bo‘lsa, u holda* $F(x)$ *funksiya* (a,b) *intervalda* $f(x)$ *funksiyaning* ***boshlang‘ich funksiyasi*** *deyiladi.*

Tabiiy savol tug‘iladi: *har qanday funksiya uchun boshlang‘ich funksiya mavjudmi?*

Masalan, $y = \mathrm{sgn}\, x$ funksiyaning $x=0$ nuqtani o‘z ichiga olmaydigan har qanday oraliqda boshlang‘ich funksiyasi mavjud bo‘lib, $x=0$ nuqtani o‘z ichiga oladigan har qanday oraliqda boshlang‘ich funksiyasi mavjud emas. Haqiqatdan ham, $x=0$ nuqtani o‘z ichiga olmaydigan har qanday oraliqda $y = \mathrm{sgn}\, x$ funksiya o‘zgarmas bo‘ladi. Masalan, [1;2] kesmada $\mathrm{sgn}\, x = 1$ bo‘lib, shu kesmada uning har qanday boshlang‘ich funksiyasi $F(x) = x + C$ ko‘rinishda bo‘ladi, bunda C - o‘zgarmas son. Endi, $x=0$ nuqtani o‘z ichiga oluvchi oraliqni qaraymiz. Teskarisini faraz qilaylik, masalan, $[-1;3]$ kesmada $y = \mathrm{sgn}\, x$ funksiya $F(x)$ boshlang‘ich funksiyaga ega bo‘lsin. U holda $F'(x) =$ s g ekanligidan, boshlang‘ich funksiya hosilasi $F'(x)$ funksiya $x=0$ nuqtada birinchi tur uzilishga ega ekanligi kelib chiqadi. Bu esa Lagranj teoremasining natijasiga zid.

Endi funksiya boshlangich funksiysi mavjudligining etarli shartini keltiramiz.

Teorema. *Agar* $f(x)$ *funksiya* (a,b) *intervalda uzluksiz bo‘lsa, u holda* $f(x)$ *funksiyaning* ***boshlang‘ich funksiyasi*** *mavjud bo‘ladi.*

Ma’lumki, $F(x)$ funksiya boshlang‘ich funksiya bo‘lsa, $F(x)+c$ ham boshlang‘ich funksiya bo‘ladi.

2-Ta’rif. (a,b) *intervalda berilgan* $f(x)$ *funksiya boshlang‘ich funksiyalarining umumiy ifodasi* $F(x)+c$ *shu* $f(x)$ *funksiyaning* ***aniqmas integrali*** *deb ataladi va*

$$\int f(x)dx$$

kabi belgilanadi.

Demak,

$$\int f(x)dx = F(x)+c \qquad \textbf{(1)}$$

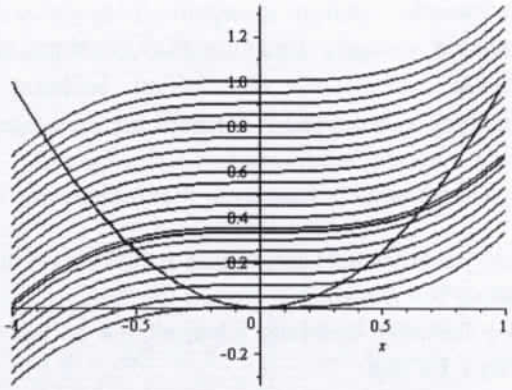
Odatda, $f(x)$ funksiya boshlang'ich funksiyasining grafigiga ***integral chiziq*** deb ataladi. Aniqmas integralning geometrik ma'nosi - integral chiziqlar oilasini anglatadi. Bunda $y=F(x)+C$ chiziqlar oilasi bitta integral chiziqni oy o'qi bo'ylab parallel ko'chirish natijasida hosil qilinadi.

Misol. $y=x^2$ funksiya uchun $(0;1)$ nuqtadan o'tuvchi integral chiziqni toping.

◁ $y=\int x^2dx=\frac{x^3}{3}+C$ - kubik parabolalar oilasini ifodalaydi. Boshlang'ich shartdan,

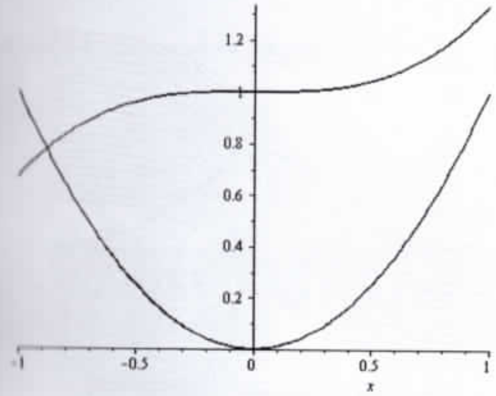
$1=0+C \Rightarrow y=\frac{x^3}{3}+1$ kelib chiqadi. ▷

```
> with(Student[Calculus1]) :
>
> AntiderivativePlot(x^2, x=-1..1, showclass)
```



A graph of $f'(x)=x^2$. The antiderivative $F(x)$ for which its value at the left end pointis 0. The members of the family of antiderivatives.


```
>
> AntiderivativePlot(x^2, x=-1..1, value=[0, 1])
```



A graph of $f'(x)=x^2$. The antiderivative $F(x)$ for which its value at0 is 1.

>

Izoh: Ma'lumki, elementar funksiyaning hosilasi yana elementar funksiya bo'lar edi, lekin integral olish uchun bu tasdiq o'rinli bo'lishi shart emas, ya'ni ba'zi bir elementar funksiyalarning integrallari elementar funksiya bo'lmay qolishi mumkin. Masalan, ushbu

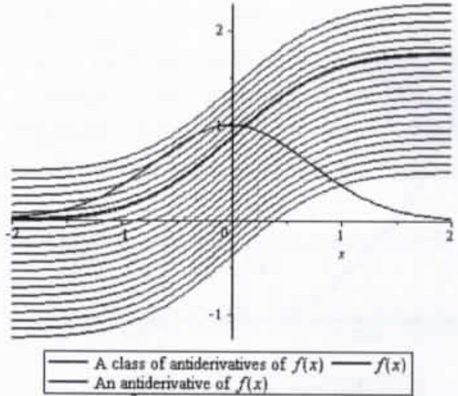
1. $\int e^{-x^2}dx$, 2. $\int \cos x^2 dx$,
3. $\int \sin x^2 dx$, 4. $\int \frac{dx}{\ln x}\ (x\ge 0, x\ne 1)$,
5. $\int \frac{\cos x}{x}dx\ (x\ne 0)$, 6. $\int \frac{\sin x}{x}dx$.

integrallarning har biri elementar funksiyalar yordamida ifodalanmaydi. Bu funksiyalar amaliyotda ko'p uchraganligi sababli ularning qiymatlarini hisoblash uchun alohida jadvallar tuzilgan va ularning grafiklari yasalgan. Shu yo'l bilan elementar funksiyalarda integrallanmaydigan funksiyalar ham to'liq o'rganilgan. Ushbu funksiyalarning grafigini keltiramiz.

```
> with(Student[Calculus1]) :
> [illegible]
> f := ∫ e^(-x^2) dx;
```

$$f := \frac{1}{2}\sqrt{\pi}\,\mathrm{erf}(x)$$

> *AntiderivativePlot*$\left(e^{-x^2}, -2..2, value = 0, showclass\right)$



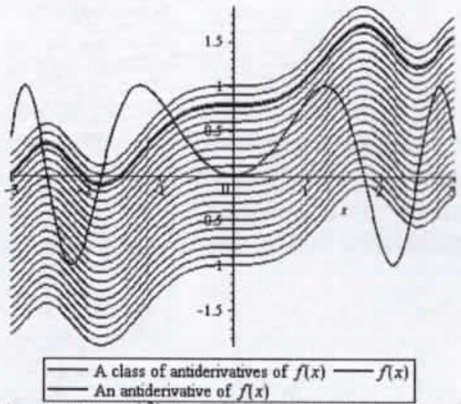
A graph of $f(x) = e^{-x^2}$. The antiderivative $F(x)$ for which its value at the left end pointis 0. The members of the family of antiderivatives.

> *Френель интеграллари* :

> $f := \int \sin(x^2)\,dx;$

$$f := \frac{1}{2}\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,x}{\sqrt{\pi}}\right)$$

> *AntiderivativePlot*$\left(\sin(x^2), -3..3, value = 0, showclass\right)$

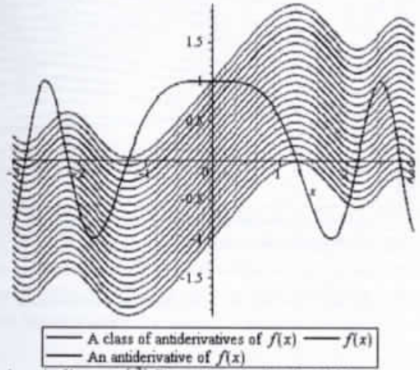


A graph of $f(x) = \sin(x^2)$. The antiderivative $F(x)$ for which its value at the left end pointis 0. The members of the family of antiderivatives.

> $f := \int \cos(x^2)\,dx;$

$$f := \frac{1}{2}\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,x}{\sqrt{\pi}}\right)$$

> *AntiderivativePlot*$\left(\cos(x^2), -3..3, value = 0, showclass\right)$



A graph of $f(x) = \cos(x^2)$. The antiderivative $F(x)$ for which its value at the left end pointis 0. The members of the family of antiderivatives.

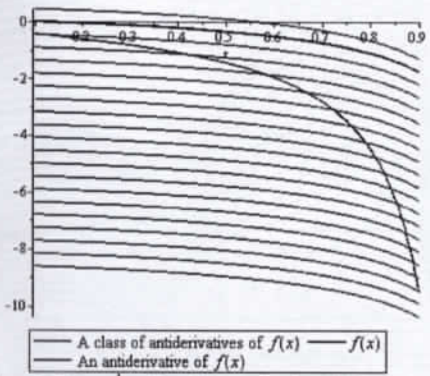
> *Интеграл логарифм* :

>

> $f := \int \frac{1}{\ln(x)}\,dx$

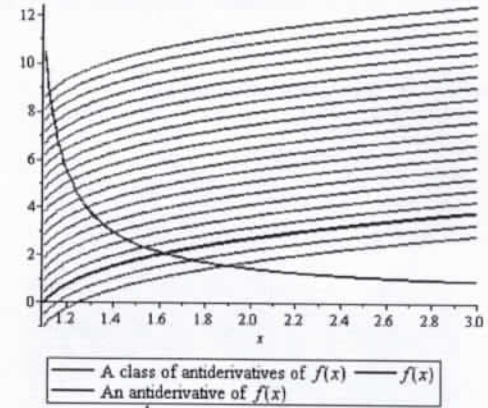
$$f := -\mathrm{Ei}(1, -\ln(x))$$

> *AntiderivativePlot*$\left(\frac{1}{\ln(x)}, 0.1..0.9, value = 0, showclass\right)$



A graph of $f(x) = \frac{1}{\ln(x)}$. The antiderivative $F(x)$ for which its value at the left end pointis 0. The members of the family of antiderivatives.

> *AntiderivativePlot*$\left(\frac{1}{\ln(x)}, 1.1..3, value=0, showclass\right)$



A graph of $f(x) = \frac{1}{\ln(x)}$. The antiderivative $F(x)$ for which its value at the left end pointis 0. The members of the family of antiderivatives.

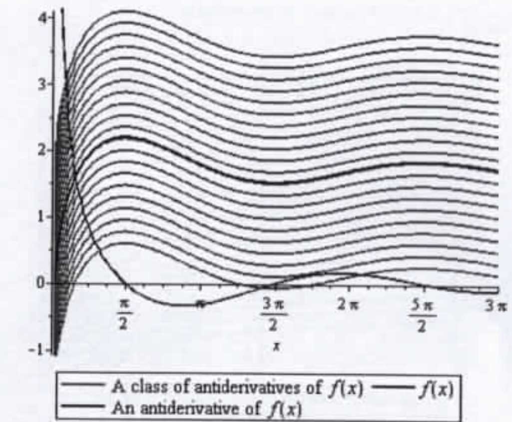
> *Интеграл косинус* :

>

> $f := \int \frac{\cos(x)}{x}\, dx$

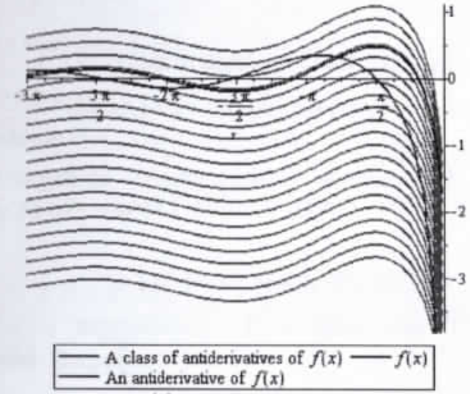
$$f := \mathrm{Ci}(x)$$

> *AntiderivativePlot*$\left(\frac{\cos(x)}{x}, 0.1..3\pi, value=0, showclass\right)$



A graph of $f(x) = \frac{\cos(x)}{x}$. The antiderivative $F(x)$ for which its value at the left end pointis 0. The members of the family of antiderivatives.

> *AntiderivativePlot*$\left(\frac{\cos(x)}{x}, -3\pi..-0.1, value=0, showclass\right)$



A graph of $f(x) = \frac{\cos(x)}{x}$. The antiderivative $F(x)$ for which its value at the left end pointis 0. The members of the family of antiderivatives.

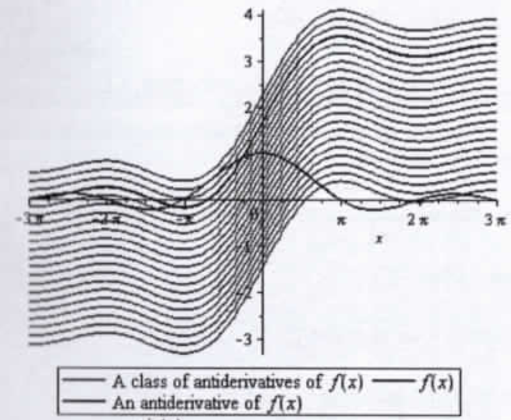
> *Интеграл синус* :

>

> $f := \int \frac{\sin(x)}{x}\, dx$

$$f := \mathrm{Si}(x)$$

> *AntiderivativePlot*$\left(\frac{\sin(x)}{x}, -3\pi..3\pi, value=0, showclass\right)$



A graph of $f(x) = \frac{\sin(x)}{x}$. The antiderivative $F(x)$ for which its value at the left end pointis 0. The members of the family of antiderivatives.

Aniqmas integralning ba’zi xossalarini ko‘raylik .

a) $\int dF(x) = F(x) + C;$

b) $(\int f(x)dx)' = f(x);$

c) $d(\int f(x)dx) = f(x)dx;$

d) $\int C \cdot f(x)dx = C \cdot \int f(x)dx$ bu erda C – o‘zgarmas;

e) $\int (f_1(x) \pm f_2(x) \pm ... \pm f_n(x))dx = \int f_1(x)dx \pm \int f_2(x)dx \pm ... \pm \int f_n(x)dx$

f) $\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C;$

j) Agar $\int f(x)dx = F(x) + C$ va $t = \varphi(x)$, u xolda $\int f(t)dt = F(t) + C.$ bo‘ladi.

Integral jadvali

1) $\int dx = x + C;$

2) $\int x^{\alpha}dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \quad \alpha \neq -1;$

3) $\int \frac{1}{x}dx = \ln|x| + C;$

4) $\int a^x dx = \frac{a^x}{\ln a} + C;$

5) $\int e^x dx = e^x + C;$

6) $\int \sin x dx = -\cos x + C;$

7) $\int \cos x dx = \sin x + C;$

8) $\int \frac{1}{\cos^2 x}dx = tgx + C;$

9) $\int \frac{1}{\sin^2 x}dx = -ctgx + C;$

10) $\int shx dx = chx + C;$

11) $\int chx dx = shx + C;$

12) $\int \frac{1}{sh^2 x}dx = -cthx + C;$

13) $\int \frac{1}{ch^2 x}dx = thx + C;$

14) $\int \frac{1}{1+x^2}dx = arctgx + C = -arcctgx + C;$

15) $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C = -\arccos x + C;$

16) $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}arctg\frac{x}{a} + C;$

17) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C;$

18) $\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\frac{x}{a} + C;$

19) $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}| + C;$

20) $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C;$

21) $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + C;$

22) $\int \frac{1}{\sin x}dx = \ln\left|tg\frac{x}{2}\right| + C;$

23) $\int \frac{1}{\cos x}dx = \ln\left|tg\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C;$

24) $\int tgx dx = -\ln|\cos x| + C;$

25) $\int ctgx dx = \ln|\sin x| + C.$

Aniqmas integralda o‘zgaruvchini almashtirish ikki xil usulda amalga oshiriladi:

1) $x = \varphi(t)$

$$\int f(x)dx = \int f(\varphi(t)) \cdot \varphi'(t)dt, \qquad \textbf{(2)}$$

Bu erda φ(t) – monoton, t o‘zgaruvchi bo‘yicha uzluksiz differensiallanuvchi funksiya ;

2) $\int f(g(x))g'(x)dx = \int f(u)du \qquad \textbf{(3)}$

u = g(x), u – yangi o‘zgaruvchi

Misol 1. Integralni xisoblang: $\int (2\sqrt{x} - \frac{7}{x^2} + 3x - 8)dx$.

Echish. Integrallash qoidasiga ko‘ra va integral jadvalidan foydalanib quydagilarni xosil qilamiz.

$$\int\left(2\sqrt{x} - \frac{7}{x^2} + 3x - 8\right)dx = 2\int x^{\frac{1}{2}}dx - 7\cdot\int x^{-2}dx + 3\int xdx - 8\int dx =$$

$$= 2\cdot\frac{x^{3/2}}{3/2} - 7\cdot\frac{x^{-1}}{-1} + 3\cdot\frac{x^2}{2} - 8\cdot x + C =$$

$$= \frac{4}{3}x\sqrt{x} + \frac{7}{x} + \frac{3}{2}x^2 - 8x + C.$$

Bu misolni Maple dasturida *with(Student[Calculus1])* : va IntTutor komandalari yordamida hisoblash mumkin.

```
> with(Student[Calculus1]) :
> IntTutor(2*sqrt(x) - 7/x^2 + 3*x - 8)
```

$$\int\left(2\sqrt{x}-\frac{7}{x^2}+3x-8\right)dx$$

$$=\int 2\sqrt{x}\,dx+\int -\frac{7}{x^2}\,dx+\int 3x\,dx+\int(-8)\,dx \quad [sum]$$

$$=2\left(\int\sqrt{x}\,dx\right)+\int -\frac{7}{x^2}\,dx+\int 3x\,dx+\int(-8)\,dx \quad [constantmultiple]$$

$$=2\left(\int 2u^2\,du\right)+\int -\frac{7}{x^2}\,dx+\int 3x\,dx+\int(-8)\,dx \quad [change, x=u^2, u]$$

$$=4\left(\int u^2\,du\right)+\int -\frac{7}{x^2}\,dx+\int 3x\,dx+\int(-8)\,dx \quad [constantmultiple]$$

$$=\frac{4u^3}{3}+\int -\frac{7}{x^2}\,dx+\int 3x\,dx+\int(-8)\,dx \quad [power]$$

$$=\frac{4x^{3/2}}{3}+\int -\frac{7}{x^2}\,dx+\int 3x\,dx+\int(-8)\,dx \quad [revert]$$

$$=\frac{4x^{3/2}}{3}-7\left(\int\frac{1}{x^2}\,dx\right)+\int 3x\,dx+\int(-8)\,dx \quad [constantmultiple]$$

$$=\frac{4x^{3/2}}{3}+\frac{7}{x}+\int 3x\,dx+\int(-8)\,dx \quad [power]$$

$$=\frac{4x^{3/2}}{3}+\frac{7}{x}+3\left(\int x\,dx\right)+\int(-8)\,dx \quad [constantmultiple]$$

$$=\frac{4x^{3/2}}{3}+\frac{7}{x}+\frac{3x^2}{2}+\int(-8)\,dx \quad [power]$$

$$=\frac{4x^{3/2}}{3}+\frac{7}{x}+\frac{3x^2}{2}-8x \quad [constant]$$

$$\int\left(2\sqrt{x}-\frac{7}{x^2}+3x-8\right)dx=\frac{4}{3}x^{3/2}+\frac{7}{x}+\frac{3}{2}x^2-8x$$

Misol 2. Integralni xisoblang: $\int\frac{\ln^3 x}{x}\,dx$.

Echish. Yuqoridagi j) xossadan foydalanib va integral jadvalidagi 2 formuladan foydalanib quyidagilarni xosil qilamiz

$\int\ln^3 x\cdot\frac{1}{x}\,dx=\int\ln^3 x\cdot d(\ln x)$, bu erda $d(\ln x)=(\ln x)'dx=\frac{1}{x}dx$

O'zgaruvchi sifatida $t=\ln x$ ni qabul qilamiz va darajali funksiya integralini xosil qilamiz

$$\int\ln^3 x\,d(\ln x)=\int t^3 dt=\frac{t^4}{4}+C=\frac{1}{4}\ln^4 x+C.$$

Buni maple dasturida quydagicha ko‘rsatamiz.

```
> with(Student[Calculus1]) :
> IntTutor(ln^3(x)/x)
```

$$\int\frac{\ln(x)^3}{x}\,dx$$

$$=\int u^3\,du \quad [change, u=\ln(x), u]$$

$$=\frac{u^4}{4} \quad [power]$$

$$=\frac{\ln(x)^4}{4} \quad [revert]$$

$$\int\frac{\ln(x)^3}{x}\,dx=\frac{1}{4}\ln(x)^4$$

Misol 3. Integralni xisoblang: $\int\sqrt{\sin x+8}\cdot\cos x dx$.

Echish. Yuqorida ko‘rilgan 2- misol kabi bunda xam xuddi shunday yo'l tutamiz va quyidagilarni xosil qilamiz.

$$\int\sqrt{\sin x+8}\cdot\cos x dx=\int(\sin x+8)^{1/2}d(\sin x)=$$

$$=\int(\sin x+8)^{1/2}d(\sin x+8)=\{t=\sin x+8\}=\int t^{1/2}dt=\frac{t^{3/2}}{3/2}+C=$$

$$=\frac{2}{3}(\sin x+8)^{3/2}+C=\frac{2}{3}(\sin x+8)\cdot\sqrt{\sin x+8}+C.$$

Bu maple dasturida quydagicha bo‘ladi.

```
> with(Student[Calculus1]) :
> IntTutor(sqrt(sin(x) + 8)cos(x))
```

$$\int\sqrt{\sin(x)+8}\,\cos(x)\,dx$$

$$=\int\sqrt{u+8}\,du \quad [change, u=\sin(x), u]$$

$$=\int 2ul^2\,dul \quad [change, u+8=ul^2, ul]$$

$$=2\left(\int ul^2\,dul\right) \quad [constantmultiple]$$

$$=\frac{2ul^3}{3} \quad [power]$$

$$=\frac{2(u+8)^{3/2}}{3} \quad [revert]$$

$$=\frac{2(\sin(x)+8)^{3/2}}{3} \quad [revert]$$

$$\int\sqrt{\sin(x)+8}\,\cos(x)\,dx=\frac{2}{3}(\sin(x)+8)^{3/2}$$

Misol 4. Integralni xisoblang: $\int (3x+10)^{15}dx$.

Echish. Yangi o'zgaruvchi kiritamiz $t=3x+10$, u xolda $x=\frac{1}{3}(t-10)$, $dx=\frac{1}{3}(t-10)'dt=\frac{1}{3}dt$.

Bundan quyidagilarni xosil qilamiz
(izox. f) xossadan foydalansa xam bo'ladi.)

Misol 5. Integralni xisoblang: $\int \frac{1}{x\cdot\sqrt{7x+1}}dx$.

Echish. Quyidagi almashtirish bajaramiz $t=\sqrt{7x+1}$, u xolda $7x+1=t^2$, $x=\frac{1}{7}(t^2-1)$, $dx=\frac{1}{7}\cdot(t^2-1)'dt=\frac{2}{7}\cdot tdt$.

17 formuladan foydalanib, quyidagilarni xosil qilamiz:

$$\int \frac{1}{x\sqrt{7x+1}}dx=\int \frac{1}{\frac{1}{7}\cdot(t^2-1)\cdot t}\cdot\frac{2}{7}\cdot t\cdot dt=2\int\frac{1}{t^2-1^2}dt=$$

$$=2\cdot\frac{1}{2}\cdot\ln\left|\frac{t-1}{t+1}\right|+C=\ln\left|\frac{\sqrt{7x+1}-1}{\sqrt{7x+1}+1}\right|+C.$$

Buni maple dasturida quydagicha ko'rsatamiz.

```
> with(Student[Calculus1]) :
> IntTutor(1/(x*sqrt(7x+1)))
```


$$\int \frac{1}{x\sqrt{7x+1}}\,dx$$

$$=\int \frac{2}{u^2-1}\,du \qquad [change, 7x+1=u^2, u]$$

$$=2\left(\int \frac{1}{u^2-1}\,du\right) \qquad [constantmultiple]$$

$$=2\left(\int\left(-\frac{1}{2(u+1)}+\frac{1}{2(u-1)}\right)du\right) \qquad [partialfractions]$$

$$=2\left(\int -\frac{1}{2(u+1)}\,du\right)+2\left(\int \frac{1}{2(u-1)}\,du\right) \qquad [sum]$$

$$=-\left(\int \frac{1}{u+1}\,du\right)+2\left(\int \frac{1}{2(u-1)}\,du\right) \qquad [constantmultiple]$$

$$=-\left(\int \frac{1}{uI}\,duI\right)+2\left(\int \frac{1}{2(u-1)}\,du\right) \qquad [change, uI=u+1, uI]$$

$$=-\ln(uI)+2\left(\int \frac{1}{2(u-1)}\,du\right) \qquad [power]$$

$$=-\ln(u+1)+2\left(\int \frac{1}{2(u-1)}\,du\right) \qquad [revert]$$

$$=-\ln(u+1)+\int \frac{1}{u-1}\,du \qquad [constantmultiple]$$

$$=-\ln(u+1)+\int \frac{1}{uI}\,duI \qquad [change, uI=u-1, uI]$$

$$=-\ln(u+1)+\ln(uI) \qquad [power]$$

$$=-\ln(u+1)+\ln(u-1) \qquad [revert]$$

$$=-\ln(\sqrt{7x+1}+1)+\ln(\sqrt{7x+1}-1) \qquad [revert]$$

$$\int \frac{1}{x\sqrt{7x+1}}\,dx=-\ln(\sqrt{7x+1}+1)+\ln(\sqrt{7x+1}-1)$$

1.2. Bo'laklab integrallash formulasi

Teorema. *Agar* $u=u(x)$ *va* $v=v(x)$ *funksiyalar* (a,b) *intervalda uzluksiz* $u'(x)$ *va* $v'(x)$ *hosilalariga ega bo'lsa, unda shu intervalda ushbu*

$$\int u(x)dv(x)=u(x)v(x)-\int v(x)du(x) \qquad (4)$$

bo'laklab integrallash formulasi *o'rinli bo'ladi.*

Amaliyot shuni ko'rsatadiki, bo'laklab integrallash usulini qo'llab hisoblanadigan integrallarni asosan uch guruhga ajratish mumkin.

Birinchi guruhga ko'paytuvchining biri ma'lum funksiyaning hosilasi bo'lgan, ikkinchisi esa ushbu

$$\ln(x), \arcsin x, \arccos x, arctgx, (arctgx)^2, (\arccos x)^2, \ln\varphi(x), \ldots$$

funksiyalardan biriga teng bo'lgan funksiyalarning integrallari kiritiladi. Bu holda $u(x)$ deb shu funksiyalar belgilanadi.

Ikkinchi guruhga $\int(ax+b)^n\cos(cx)dx, \int(ax+b)^n\cdot\sin(cx)dx$ va $\int(ax+b)^n e^{cx}dx$ kurinishidagi integrallar kiritiladi. Bu holda $u(x)=(ax+b)^n$ deb olinib, bo'laklab integrallash formulasi n marta qo'llaniladi.

Uchinchi guruhga $\int e^{ax}\cos bxdx, \int e^{ax}\sin bxdx, \int\sin(\ln x)dx, \int\cos(\ln x)dx, \ldots$ ko'rinishidagi integrallar kiritiladi. Bunda integralni I deb belgilab, bo'laklab integrallash formulasini ikki martta qo'llasak, I ga nisbatan chiziqli tenglamaga kelamiz.

Bu uchta guruhga kirmagan ba'zi bir integrallarni ham bo'laklab integrallash usuli bilan hisoblash mumkin.Masalan,

$$I_n=\int\frac{dx}{(x^2+a^2)^n}, (n\in N)$$

integral yuqoridagi uchta guruhga kirmaydi, lekin bu integralni ham bo'laklab integrallash usuli bilan **rekkurent formulaga** keltirish yordamida hisoblash mumkin:

$$I_{n+1}=\frac{1}{2na^2}\cdot\frac{x}{(x^2+a^2)^n}+\frac{2n-1}{2n}\cdot\frac{1}{a^2}I_n \qquad \textbf{(5)}$$

$I_1=\int\frac{dx}{x^2+a^2}=\frac{1}{a}arctg\frac{x}{a}+c$ Agar (5)-tenglikda $n=1$ desak,

$I_2=\int\frac{dx}{(x^2+a^2)^2}=\frac{1}{2a^2}\cdot\frac{x}{x^2+a^2}+\frac{1}{2a^2}arctg\frac{x}{a}+c$ ekanini topamiz.

Misol 6. Integralni xisoblang: $\int(2x+3)\cdot\sin xdx$.

Echish. Agar $u=2x+3$ desak , u xolda $du=2\cdot dx$; $dv=\sin xdx$, bundan $v=-\cos x$ ekanligini topamiz.

Bo'laklab integrallash formulasiga ko'ra quyidagini xosil qilamiz

$$\int(2x+3)\cdot\sin xdx=-(2x+3)\cdot\cos x-\int(-\cos x)\cdot 2\cdot dx=$$

$$=-(2x+3)\cdot\cos x+2\sin x+C.$$

Buni maple dasturida quydagicha ko'rsatamiz.

```
> with(Student[Calculus1]) :
> IntTutor((2x + 3)sin(x))
```


$\int(2x+3)\sin(x)\,dx$

$=\int(2\sin(x)\,x+3\sin(x))\,dx$	[*rewrite*, $(2x+3)\sin(x)=2\sin(x)\,x+3\sin(x)$]
$=\int 2\sin(x)\,x\,dx+\int 3\sin(x)\,dx$	[*sum*]
$=2\left(\int\sin(x)\,x\,dx\right)+\int 3\sin(x)\,dx$	[*constantmultiple*]
$=-2x\cos(x)-2\left(\int-\cos(x)\,dx\right)+\int 3\sin(x)\,dx$	[*parts*, x, $-\cos(x)$]
$=-2x\cos(x)+2\left(\int\cos(x)\,dx\right)+\int 3\sin(x)\,dx$	[*constantmultiple*]
$=-2x\cos(x)+2\sin(x)+\int 3\sin(x)\,dx$	[cos]
$=-2x\cos(x)+2\sin(x)+3\left(\int\sin(x)\,dx\right)$	[*constantmultiple*]
$=-2x\cos(x)+2\sin(x)-3\cos(x)$	[sin]

$$\int(2x+3)\sin(x)\,dx=-2x\cos(x)+2\sin(x)-3\cos(x)$$

Misol 7. Integralni xisoblang: $\int x^3\cdot\ln xdx$.

Echish. Yuqoridagi 6 misol kabi yo'l tutib quyidagini xosil qilamiz.

$$\int x^3\cdot\ln xdx=\begin{Bmatrix}u=\ln x, & du=\frac{1}{x}\cdot dx\\ dv=x^3\cdot dx, & v=\frac{x^4}{4}\end{Bmatrix}=\frac{x^4}{4}\cdot\ln x-\int\frac{x^4}{4}\cdot\frac{1}{x}dx=$$

$$=\frac{x^4}{4}\cdot\ln x-\frac{1}{4}\int x^3dx=\frac{x^4}{4}\cdot\ln x-\frac{1}{4}\cdot\frac{x^4}{4}+C=\frac{x^4}{4}\cdot\ln x-\frac{x^4}{16}+C.$$

Ba'zi integrallarni xisoblashda bo'laklab integrallash formulasini bir necha martta qo'llashga to'gri kelishi mumkin.

Misol 8. Integralni xisoblang: $J=\int 3^x\cdot\cos 5x\cdot dx$.

Echish. Bo'laklab integrallash formulasini ikki martta qo'llab quyidagini xosil qilamiz

$$J=\int 3^x\cdot\cos 5x\cdot dx=\begin{Bmatrix}u=3^x, & du=3^x\cdot\ln 3\cdot dx\\ dv=\cos 5xdx, & v=\frac{\sin 5x}{5}\end{Bmatrix}=$$

$$=3^x\cdot\frac{1}{5}\cdot\sin 5x-\int\frac{1}{5}\cdot\sin 5x\cdot 3^x\cdot\ln 3dx=\begin{Bmatrix}u=3^x, & du=3^x\cdot\ln 3dx\\ dv=\sin 5xdx, & v=-\frac{\cos 5x}{5}\end{Bmatrix}=$$

$$=\frac{1}{5}\cdot 3^x\cdot\sin 5x-\frac{1}{5}\cdot\ln 3\cdot\left(3^x\cdot\left(-\frac{\cos 5x}{5}\right)+\int\frac{\cos 5x}{5}\cdot 3^x\cdot\ln 3dx\right)=$$

$$=\frac{1}{5}\cdot 3^x\cdot\sin 5x+\frac{\ln 3}{25}\cdot 3^x\cdot\cos 5x-\frac{\ln^2 3}{25}\cdot\int 3^x\cdot\cos 5x dx.$$

J noma'lum integralli tenglama xosil qilamiz:

$$J=\frac{1}{5}\cdot 3^x\cdot\sin 5x+\frac{\ln 3}{25}\cdot 3^x\cdot\cos 5x-\frac{\ln^2 3}{25}\cdot J \quad \text{yoki}$$

$$J+\frac{\ln^2 3}{25}\cdot J=\frac{3^x}{5}\cdot\sin 5x+\frac{\ln 3}{25}\cdot 3^x\cdot\cos 5x,$$

$$\frac{25+\ln^2 3}{25}\cdot J=\frac{3^x}{25}(5\sin 5x+\ln 3\cdot\cos 5x), \text{ bu erdan J ni topamiz}$$

$$J=\frac{3^x}{25+\ln^2 3}\cdot(5\sin 5x+\ln 3\cdot\cos 5x)+C.$$

Bu misol Maple dasturida quydagicha ko'rinishga ega bo'ladi.

```
> with(Student[Calculus1]):
> IntTutor(3^x cos(5x))
```

$$\int 3^x\cos(5x)\,dx$$

$$=\int e^{x\ln(3)}\cos(5x)\,dx \qquad \left[rewrite,\ 3^x\cos(5x)=e^{x\ln(3)}\cos(5x)\right]$$

$$=\int\frac{e^{\frac{\ln(3)u}{5}}\cos(u)}{5}\,du \qquad [change,\ u=5x,\ u]$$

$$=\frac{\int e^{\frac{\ln(3)u}{5}}\cos(u)\,du}{5} \qquad [constantmultiple]$$

$$=\frac{\cos(u)e^{\frac{\ln(3)u}{5}}}{\ln(3)}-\frac{\int-\frac{5e^{\frac{\ln(3)u}{5}}\sin(u)}{\ln(3)}\,du}{5} \qquad \left[parts,\ \cos(u),\ \frac{5e^{\frac{\ln(3)u}{5}}}{\ln(3)}\right]$$

$$=\frac{\cos(u)e^{\frac{\ln(3)u}{5}}}{\ln(3)}+\frac{\int e^{\frac{\ln(3)u}{5}}\sin(u)\,du}{\ln(3)} \qquad [constantmultiple]$$

$$=\frac{\cos(u)e^{\frac{\ln(3)u}{5}}}{\ln(3)}+\frac{\frac{5e^{\frac{\ln(3)u}{5}}\sin(u)}{\ln(3)}-\int\frac{5\cos(u)e^{\frac{\ln(3)u}{5}}}{\ln(3)}\,du}{\ln(3)} \qquad \left[parts,\ \sin(u),\ \frac{5e^{\frac{\ln(3)u}{5}}}{\ln(3)}\right]$$

$$=\frac{\cos(u)e^{\frac{\ln(3)u}{5}}}{\ln(3)}+\frac{\frac{5e^{\frac{\ln(3)u}{5}}\sin(u)}{\ln(3)}-\frac{5\int e^{\frac{\ln(3)u}{5}}\cos(u)\,du}{\ln(3)}}{\ln(3)} \qquad [constantmultiple]$$

$$=\frac{e^{\frac{\ln(3)u}{5}}(\ln(3)\cos(u)+5\sin(u))}{\ln(3)^2+25} \qquad [solve]$$

$$=\frac{e^{x\ln(3)}(\ln(3)\cos(5x)+5\sin(5x))}{\ln(3)^2+25} \qquad [revert]$$

$$\int 3^x\cos(5x)\,dx=\frac{e^{x\ln(3)}(\ln(3)\cos(5x)+5\sin(5x))}{\ln(3)^2+25}$$

1.3. Ratsional funksiyalarni integrallash

Sodda kasirlarning integrallarini ko'rib chiqamiz:

I. $\int\frac{A}{x-a}dx=A\ln|x-a|+C;$

II. $\int\frac{A}{(x-a)^k}dx=\frac{A}{1-k}\cdot\frac{1}{(x-a)^{k-1}}+C,\quad k\neq 1;$

III. $\int\frac{Ax+B}{x^2+px+q}dx;$

IV. $\int\frac{Ax+B}{(x^2+px+q)^k}dx,$

Bu erda A, B, p, q, a – xaqiyqiy sonlar.

III va IV tip integrallarni xisoblashni misollar orqali ko'rib chiqamiz.

Misol 9. Integralni xisoblang: $\int\frac{dx}{x^2-6x+18}$.

Yechish. Integral ostida turgan kvadrat uch xaddan to'la kvadrat ajratamiz va quyidagini xosil qilamiz:

$x^2-6x+18=(x^2-2\cdot 3\cdot x+9)+9=(x-3)^2+3^2.$

U xolda

$$\int\frac{dx}{x^2-6x+18}=\int\frac{d(x-3)}{(x-3)^2+3^2}=\{t=x-3\}=$$

$$=\int\frac{dt}{t^2+3^2}=\frac{1}{3}\cdot arctg\frac{t}{3}+C=\frac{1}{3}arctg\frac{x-3}{3}+C.$$

Integral jadvalidagi 16 formuladan foydalandik.

Misol 10. Integralni xisoblang: $\int\frac{5x+1}{x^2+4x-1}dx$.

Yechish. Integral ostidagi kasirning suratidagi chiziqli funksiyani maxrajdagi funksiyaning xosilasiga moslaymiz ya'ni:

$(x^2+4x-1)'=2x+4,$

$5x+1=\frac{5}{2}(2x+4)-10+1=\frac{5}{2}(2x+4)-9.$

U xolda quyidagiga ega bo'lamiz:

$$\int\frac{5x+1}{x^2+4x-1}dx=\int\frac{\frac{5}{2}(2x+4)-9}{x^2+4x-1}dx=$$

$$=\frac{5}{2}\int\frac{(2x+4)dx}{x^2+4x-1}-9\cdot\int\frac{dx}{x^2+4x-1}=J.$$

Endi birinchi integralda $(2x+4)dx=d(x^2+4x-1)$ deb olamiz. Yangi o'zgaruvchi kiritamiz $t=x^2+4x-1$ va integral jadvalidan foydalanib 3 formulani qo'llaymiz. Ikkinchi integralda kvadrat uchxaddan to'la kvadrat ajratamiz: $x^2+4x-1=(x+2)^2-5$, natijada integral jadvalidagi 17 formuladan foydalanib quyidagini xosil qilamiz .

$$J=\frac{5}{2}\int\frac{d(x^2+4x-1)}{x^2+4x-1}-9\cdot\int\frac{d(x+2)}{(x+2)^2-(\sqrt{5})^2}=$$

$$=\{t=x^2+4x-1, z=x+2\}=$$

$$=\frac{5}{2}\int\frac{dt}{t}-9\int\frac{dz}{z^2-(\sqrt{5})^2}=\frac{5}{2}\ln|t|-9\cdot\frac{1}{2\sqrt{5}}\ln\left|\frac{z-\sqrt{5}}{z+\sqrt{5}}\right|+C=$$

$$=\frac{5}{2}\cdot\ln|x^2+4x-1|-\frac{9}{2\sqrt{5}}\ln\left|\frac{x+2-\sqrt{5}}{x+2+\sqrt{5}}\right|+C.$$

Maple dasturida bu misol quydagi ko‘rinishga ega bo‘ladi.

> *with*(*Student*[*Calculus1*]) :

> *IntTutor*$\left(\frac{(5x+1)}{(x^2+4x-1)}\right)$

$$\int\frac{5x+1}{x^2+4x-1}dx$$

$$=\int\left(-\frac{(-9+5\sqrt{5})\sqrt{5}}{10(-x-2+\sqrt{5})}+\frac{(9+5\sqrt{5})\sqrt{5}}{10(x+2+\sqrt{5})}\right)dx \quad [partialfractions]$$

$$=\int-\frac{(-9+5\sqrt{5})\sqrt{5}}{10(-x-2+\sqrt{5})}dx+\int\frac{(9+5\sqrt{5})\sqrt{5}}{10(x+2+\sqrt{5})}dx \quad [sum]$$

$$=-\frac{(-9+5\sqrt{5})\sqrt{5}\left(\int\frac{1}{-x-2+\sqrt{5}}dx\right)}{10}+\int\frac{(9+5\sqrt{5})\sqrt{5}}{10(x+2+\sqrt{5})}dx \quad [constantmultiple]$$

$$=-\frac{(-9+5\sqrt{5})\sqrt{5}\left(\int-\frac{1}{u}du\right)}{10}+\int\frac{(9+5\sqrt{5})\sqrt{5}}{10(x+2+\sqrt{5})}dx \quad [change, u=-x-2+\sqrt{5}, u]$$

$$=\frac{(-9+5\sqrt{5})\sqrt{5}\left(\int\frac{1}{u}du\right)}{10}+\int\frac{(9+5\sqrt{5})\sqrt{5}}{10(x+2+\sqrt{5})}dx \quad [constantmultiple]$$

$$=\frac{(-9+5\sqrt{5})\sqrt{5}\ln(u)}{10}+\int\frac{(9+5\sqrt{5})\sqrt{5}}{10(x+2+\sqrt{5})}dx \quad [power]$$

$$=\frac{(-9+5\sqrt{5})\sqrt{5}\ln(-x-2+\sqrt{5})}{10}+\int\frac{(9+5\sqrt{5})\sqrt{5}}{10(x+2+\sqrt{5})}dx \quad [revert]$$

$$=\frac{(-9+5\sqrt{5})\sqrt{5}\ln(-x-2+\sqrt{5})}{10}+\frac{(9+5\sqrt{5})\sqrt{5}\left(\int\frac{1}{x+2+\sqrt{5}}dx\right)}{10} \quad [constantmultiple]$$

$$=\frac{(-9+5\sqrt{5})\sqrt{5}\ln(-x-2+\sqrt{5})}{10}+\frac{(9+5\sqrt{5})\sqrt{5}\left(\int\frac{1}{u}du\right)}{10} \quad [change, u=x+2+\sqrt{5}, u]$$

$$=\frac{(-9+5\sqrt{5})\sqrt{5}\ln(-x-2+\sqrt{5})}{10}+\frac{(9+5\sqrt{5})\sqrt{5}\ln(u)}{10}$$

$$=\frac{(-9+5\sqrt{5})\sqrt{5}\ln(-x-2+\sqrt{5})}{10}+\frac{(9+5\sqrt{5})\sqrt{5}\ln(x+2+\sqrt{5})}{10} \quad [revert]$$

$$\int\frac{5x+1}{x^2+4x-1}dx=\frac{1}{10}(-9+5\sqrt{5})\sqrt{5}\ln(-x-2+\sqrt{5})+\frac{1}{10}(9+5\sqrt{5})\sqrt{5}\ln(x+2+\sqrt{5})$$

IV tip ratsional kasirlarni integrallashda rekurrent formuladan foydalanamiz:

$$\int\frac{dt}{(t^2+a^2)^n}=J_n;$$

$$J_n=\frac{t}{2a^2(n-1)(t^2+a^2)^{n-1}}+\frac{1}{a^2}\cdot\frac{2n-3}{2n-2}\cdot J_{n-1}.$$

Misol 11. Integralni xisoblang: $\int\frac{dx}{(x^2+4)^2}=J_2$

Yechish. Bu erda $n=2;\ a^2=4$. Rekurrent formulani qo‘llaganimizdan so‘ng quyidagini xosil qilamiz:

$$J_2=\int\frac{dx}{(x^2+4)^2}=\frac{x}{2\cdot4\cdot(2-1)\cdot(x^2+4)^{2-1}}+\frac{1}{4}\cdot\frac{2\cdot2-3}{2\cdot2-2}\cdot J_1=$$

$$=\frac{x}{8(x^2+4)}+\frac{1}{8}\int\frac{dx}{x^2+4}=\frac{x}{8(x^2+4)}+\frac{1}{16}arctg\frac{x}{2}+C.$$

Agar $n>2$, u xolda rekurrent formuladan bir necha martta foydalanib integral jadvaliga keltiramiz .

Misol 12. Integralni xisoblang: $\int\frac{x+5}{(x^2+4x+5)^2}dx.$

Yechish. Integral ostidagi funksiyada quyidagicha almashtirish bajaramiz. Kasrning suratida chiziqli funksiyani maxrajdagi kvadrat uchhadning differensialiga moslab olamiz. So‘ngra integralni ikkiga ajratamiz birinchi integralni to‘gridan to‘gri jadvaldan foydalanib topamiz, ikkinchi integralni rekurrent formula yordamida topamiz:

$d(x^2+4x+5)=(2x+4)dx.$

$$\frac{x+5}{(x^2+4x+5)^2}=\frac{\frac{1}{2}(2x+4)+3}{(x^2+4x+5)^2}=\frac{1}{2}\cdot\frac{2x+4}{(x^2+4x+5)^2}+\frac{3}{(x^2+4x+5)^2}.$$

U xolda quyidagiga ega bo‘lamiz

$$\int\frac{x+5}{(x^2+4x+5)^2}dx=\frac{1}{2}\int\frac{2x+4}{(x^2+4x+5)^2}dx+\int\frac{dx}{(x^2+4x+5)^2}=$$

$$=\frac{1}{2}\int\frac{d(x^2+4x+5)}{(x^2+4x+5)^2}+3\int\frac{d(x+2)}{((x+2)^2+1)^2}=$$

$$=\{x^2+4x+5=t;\ \ x+2=z\}=\frac{1}{2}\int\frac{dt}{t^2}+3\int\frac{dz}{(z^2+1)^2}=$$

$$=\frac{1}{2}\cdot\frac{t^{-1}}{-1}+3\cdot\left(\frac{z}{2\cdot1\cdot(z^2+1)}+\frac{2\cdot2-3}{2\cdot2-2}\cdot\int\frac{dz}{z^2+1}\right)=$$

$$=\frac{-1}{2t}+\frac{3z}{2(z^2+1)}+\frac{3}{2}arctgz+C=$$

$$=-\frac{1}{2(x^2+4x+5)}+\frac{3(x+2)}{2(x^2+4x+5)}+\frac{3}{2}arctg(x+2)+C.$$

Agar integral ostida murakkab ratsional funksiya bo'lsa u holda quyidagi almashtirishlarni bajaramiz:

1) Agar noto'gri ratsional kasr bo'lsa u xolda noto'gri kasrning butun qismini ajratib olamiz. So'ngra noto'gri kasrni butun qismi xamda $\frac{Q(x)}{P(x)}$; to'gri ratsional kasr qismi yigindisi ko'rinishida yozib olamiz.

2) Kasirning maxrajida to'rgan ratsional ko'pxadning ildizlarini xisobga olgan xolda chiziqli va kvadratik funksiyalar ko'paytmalari ko'rinishida yozib olamiz. $P(x)=(x-a)^m\cdot\ldots\cdot(x^2+px+q)^n\cdot\ldots$

Bu erda x^2+px+q kvadrat uchxad xaqiyqiy ildizga ega emas bunda p va q – xaqiyqiy sonlar;

3) $\frac{Q(x)}{P(x)}$ to'gri ratsional kasrni sodda kasrlaga ajratamiz (bunda R(x) ko'pxadning darajasi Q(x) ko'pxadning darajasidan katta)

$$\frac{Q(x)}{P(x)}=\frac{A_1}{(x-a)^m}+\frac{A_2}{(x-a)^{m-1}}+\ldots+\frac{A_m}{x-a}+\ldots+$$
$$+\frac{B_1x+C_1}{(x^2+px+q)^n}+\frac{B_2x+C_2}{(x^2+px+q)^{n-1}}+\ldots+\frac{B_nx+C_n}{x^2+px+q};$$

4) $A_1, A_2, \ldots, B_1, C_1, \ldots, B_n, C_n$. aniqmas koeffitsientlar topiladi.

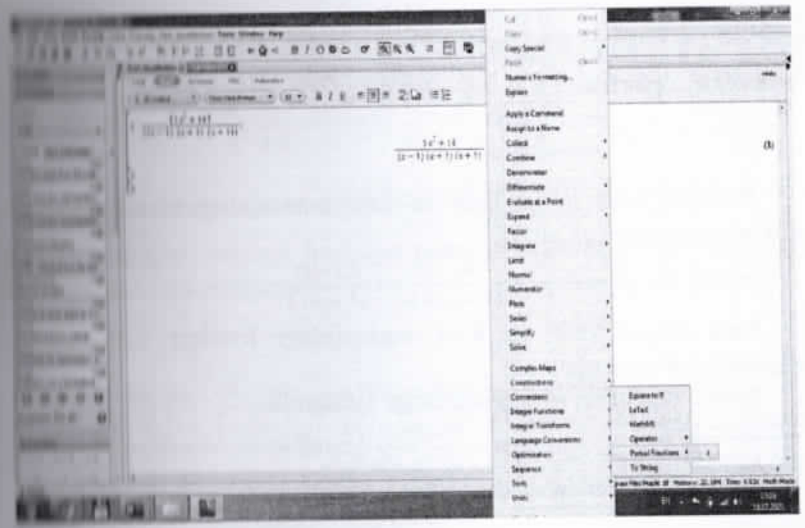
Natijada ratsional funksiyalarni integrallash ko'pxad xamda sodda kasrlani yigindisini integrallashdan iborat bo'ladi.

Ixtiyoriy to'gri ratsional kasrni sodda kasrlar yigindisi ko'rinishida yozish mumkin. Buni quyidagi misollar orqali ko'rsatamiz.

Misol 13. $\frac{5x^2+14}{(x-1)(x+3)(x+5)}=\frac{A}{x-1}+\frac{B}{x+3}+\frac{C}{x+5}.$

Bu erda to'gri kasr va kasrning maxraji sodda ko'paytuvchilarga ajratilgan va ular turli xil. Ko'paytuvchilarning xar biriga I tip sodda kasr mos keladi.

Maple dasturida quydagicha bo'ladi. Sichqoncha o'ng tomonini bosib conversions—>Partial Fractions-->x Camandasi yordamida kasr funksiya sodda kasrlarga ajratiladi.



> $\frac{(5x^2+14)}{((x-1)(x+3)(x+5))}$;

$$\frac{5x^2+14}{(x-1)(x+3)(x+5)}$$

> **convert(, 'parfrac', x);**

$$-\frac{59}{8(x+3)}+\frac{19}{24(x-1)}+\frac{139}{12(x+5)}$$

Misol 14. $\frac{7x^2+8x-1}{(x+3)^4}=\frac{A}{(x+3)^4}+\frac{B}{(x+3)^3}+\frac{C}{(x+3)^2}+\frac{D}{x+3}.$

Kasr to'gri kasr va kasr maxraji karrali echimga ega (izox. echim bitta va karrasi 4 ga teng).

Maple dasturida bu misol quydagi sodda kasrlarga ajratiladi.

> $\frac{(7x^2+8x-1)}{(x+3)^4}$;

$$\frac{7x^2+8x-1}{(x+3)^4}$$

> **convert(, 'parfrac', x);**

$$\frac{38}{(x+3)^4}+\frac{7}{(x+3)^2}-\frac{34}{(x+3)^3}$$

>

Misol 15. $\frac{5x^2+2x+4}{(x^2+x+1)(x^2+x+5)}=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{x^2+x+5}.$

Maple dasturida soda kasrlarga ajratamiz.

> $\frac{(5x^2+2x+4)}{((x^2+x+1)(x^2+x+5))}$;

$$\frac{5x^2+2x+4}{(x^2+x+1)(x^2+x+5)}$$

> convert(, 'parfrac', x);

$$\frac{1}{4}\frac{-3x-1}{x^2+x+1}+\frac{1}{4}\frac{3x+21}{x^2+x+5}$$

Bu erda kasr to‘gri kasr va kasr maxrajidagi kvadrat uch xadlar xaqiyqiy echimga ega emas.

Misol 16. $\frac{3x^2+x-1}{(x^2+x+1)^2}=\frac{Ax+B}{(x^2+x+1)^2}+\frac{Cx+D}{x^2+x+1}$.

Kasr to‘gri kasr va kasr maxrajidagi kvadrat uch xad karrali kompleks echimga ega.

Maple dasturida sodda kasrlarga ajratamiz.

> $\frac{(3\cdot x^2+x-1)}{(x^2+x+1)^2}$;

$$\frac{3x^2+x-1}{(x^2+x+1)^2}$$

> convert(, 'parfrac', x);

$$\frac{3}{x^2+x+1}+\frac{-2x-4}{(x^2+x+1)^2}$$

Misol 17.

$$\frac{3x^4+7x-1}{(x+2)x^2(x^2+x+5)^2(x^2-x+2)}=$$

$$=\frac{A}{x+2}+\frac{B}{x^2}+\frac{C}{x}+\frac{Dx+F}{(x^2+x+5)^2}+\frac{Cx+E}{x^2+x+5}+\frac{Lx+M}{x^2-x+2}.$$

To‘gri ratsional kasrning bu ko‘rinishi 13–16.misollarning taxlilidan kelib chiqadi

Sodda kasrlardagi A, B, C, D, … koeffitsientlar aniqmas koeffitsenlar metodi orqali topiladi. Bunda quyidagicha yo‘l tutiladi. Kasrlani umumiy maxrajga keltiramiz. Tenglikning chap va o‘ng tomonini suratlarini x oldidagi koeffitsentlarni mos ravishda tenglaymiz. Xosil bo‘lgan sistemani echib noma’lum koeffitsentlarni topamiz va sodda kasrlani xisoblash metodlaridan foydalanib integral xisoblaymiz.

Maple dasturida bu misol quydagi sodda kasrlarga ajratiladi.

> $\frac{(3x^4+7x-1)}{(x+2)\cdot x^2\cdot(x^2-x+2)\cdot(x^2+x+5)^2}$;

$$\frac{3x^4+7x-1}{(x+2)\,x^2\,(x^2-x+2)\,(x^2+x+5)^2}$$

> convert(, 'parfrac', x);

$$\frac{1}{[illegible]}\frac{-2474x-879}{(x^2+x+5)^2}+\frac{1}{16928}\frac{471x-269}{x^2-x+2}+\frac{37}{500x}+\frac{33}{1568\,(x+2)}-\frac{1}{100x^2}$$
$$+\frac{1}{3340125}\frac{-398113x-267993}{x^2+x+5}$$

Misol 18. Integralni xisoblang: $\int\frac{x^2+2x+5}{x+2}dx$.

Yechish. Integral ostidagi funksiya noto‘gri ratsional kasr. Quydagi almashtirishni amalga oshiramiz:

$$\frac{x^2+2x+5}{x+2}=\frac{x(x+2)+5}{x+2}=x+\frac{5}{x+2}.$$

$$\int\frac{x^2+2x+5}{x+2}dx=\int(x+\frac{5}{x+2})dx=\frac{x^2}{2}+5\ln|x+2|+C.$$

Misol 19 Integralni xisoblang: $\int\frac{7x-3}{x^3-x^2+x-1}dx$.

Yechish. Integral ostidagi funksiya to‘gri ratsional kasr. Sodda kasrlarga ajratamiz.

$$\frac{7x-3}{x^3-x^2+x-1}=\frac{7x-3}{(x^3-x^2)+(x-1)}=\frac{7x-3}{x^2(x-1)+(x-1)}=$$

$$=\frac{7x-3}{(x^2+1)(x-1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x-1}=\frac{(Ax+B)(x-1)+C(x^2+1)}{(x^2+1)(x-1)}.$$

Birinchi va oxirgi kasrni taqqoslaylik. Agar koeffitsientlar x ning bir xil darajalarida teng bo‘lsa, ikkita ko‘pxad teng deb hisoblanadi:

$7x-3=(Ax+B)\cdot(x-1)+C(x^2+1)$

$7x-3=(A+C)x^2+(-A+B)x+C-B.$

$$\left.\begin{array}{ll} x^2 & A+C=0, \\ x & -A+B=7, \\ x^0 & C-B=-3 \end{array}\right\}$$

Uchta tenglikni qo‘shib quyidagilarni xosil qilamiz

$2C=4$ yoki $C=2$.

Sistemadagi birinchi tenglikdan $A=-C$ yoki $A=-2$.

Sistemadagi ikkinchi tenglikdan $B=7+A$ yoki $B=7-2=5$.

U holda,

$$\frac{7x-3}{(x^2+1)(x-1)}=\frac{-2x+5}{x^2+1}+\frac{2}{x-1}.$$

Natijada quydagilarni xosil qilamiz.

$$\int\frac{7x-3}{x^3-x^2+x-1}dx=\int\left(\frac{-2x+5}{x^2+1}+\frac{2}{x-1}\right)dx=$$

$$=-\int\frac{2x}{x^2+1}dx+5\int\frac{dx}{x^2+1}+2\ln|x-1|=$$

$$= -\int \frac{d(x^2+1)}{x^2+1} + 5arctgx + 2\ln|x-1| =$$

$$= -\ln(x^2+1) + 5\mathrm{arctg}x + 2\ln|x-1| + C =$$

$$= 5arctgx + \ln\frac{(x-1)^2}{x^2+1} + C.$$

Maple dasturi yordamida bu misol quydagicha ishlanadi. Bunda Windows oynasining o'ng tomonida misolni echish bo'yicha izoh berilgan.

```
> with(Student[Calculus1]) :
> IntTutor((7x-3)/(x^3-x^2+x-1))
```

$$\int \frac{7x-3}{x^3-x^2+x-1}\,dx$$

$$= \int \left(\frac{2}{x-1} + \frac{-2x+5}{x^2+1}\right) dx \quad [partialfractions]$$

$$= \int \frac{2}{x-1}\,dx + \int \frac{-2x+5}{x^2+1}\,dx \quad [sum]$$

$$= 2\left(\int \frac{1}{x-1}\,dx\right) + \int \frac{-2x+5}{x^2+1}\,dx \quad [constantmultiple]$$

$$= 2\left(\int \frac{1}{u}\,du\right) + \int \frac{-2x+5}{x^2+1}\,dx \quad [change,\ u=x-1,\ u]$$

$$= 2\ln(u) + \int \frac{-2x+5}{x^2+1}\,dx \quad [power]$$

$$= 2\ln(x-1) + \int \frac{-2x+5}{x^2+1}\,dx \quad [revert]$$

$$= 2\ln(x-1) + \int \left(-\frac{2x}{x^2+1} + \frac{5}{x^2+1}\right) dx \quad \left[rewrite,\ \frac{-2x+5}{x^2+1} = -\frac{2x}{x^2+1} + \frac{5}{x^2+1}\right]$$

$$= 2\ln(x-1) + \int -\frac{2x}{x^2+1}\,dx + \int \frac{5}{x^2+1}\,dx \quad [sum]$$

$$= 2\ln(x-1) - 2\left(\int \frac{x}{x^2+1}\,dx\right) + \int \frac{5}{x^2+1}\,dx \quad [constantmultiple]$$

$$= 2\ln(x-1) - 2\left(\int \frac{1}{2u}\,du\right) + \int \frac{5}{x^2+1}\,dx \quad [change,\ u=x^2+1,\ u]$$

$$= 2\ln(x-1) - \left(\int \frac{1}{u}\,du\right) + \int \frac{5}{x^2+1}\,dx \quad [constantmultiple]$$

$$= 2\ln(x-1) - \ln(u) + \int \frac{5}{x^2+1}\,dx \quad [power]$$

$$= 2\ln(x-1) - \ln(x^2+1) + \int \frac{5}{x^2+1}\,dx \quad [revert]$$

$$= 2\ln(x-1) - \ln(x^2+1) + 5\left(\int \frac{1}{x^2+1}\,dx\right) \quad [constantmultiple]$$

$$= 2\ln(x-1) - \ln(x^2+1) + 5\left(\int 1\,du\right) \quad [change,\ x=\tan(u),\ u]$$

$$= 2\ln(x-1) - \ln(x^2+1) + 5u \quad [constant]$$

$$= 2\ln(x-1) - \ln(x^2+1) + 5\arctan(x) \quad [revert]$$

$$\int \frac{7x-3}{x^3-x^2+x-1}\,dx = 2\ln(x-1) - \ln(x^2+1) + 5\arctan(x)$$

Misol 20. Integralni xisoblang: $\int \frac{x^5+1}{x^4-8x^2+16}dx$.

Yechish. Integral ostida noto'gri ratsional kasr. Uni butun qism va ratsional kasr yig'indisi sifatida ifodalaymiz. Avval ushbu kasrning butun qismini ajratib olamiz.

$$\begin{array}{r|l} x^5+1 & x^4-8x^2+16 \\ \underline{x^5-8x^3+16x} & x \\ 8x^3-16x+1 & \end{array}$$

Quyidagiga ega bo'lamiz.

$$\frac{x^5+1}{x^4-8x^2+16} = x + \frac{8x^3-16x+1}{x^4-8x^2+16} = x + \frac{8x^3-16x+1}{(x^2-4)^2} =$$

$$= x + \frac{8x^3-16x+1}{(x-2)^2(x+2)^2} = x + \frac{A}{(x-2)^2} + \frac{B}{x-2} + \frac{C}{(x+2)^2} + \frac{D}{x+2}.$$

$$\frac{8x^3-16x+1}{(x-2)^2(x+2)^2} = \frac{A}{(x-2)^2} + \frac{B}{x-2} + \frac{C}{(x+2)^2} + \frac{D}{x+2} =$$

$$= \frac{A(x+2)^2 + B(x-2)(x+2)^2 + C(x-2)^2 + D(x+2)(x-2)^2}{(x-2)^2(x+2)^2}.$$

Teng maxrajli kasrlar, agar ularning suratlari ham teng bo'lsa, u holda bunday kasrlar teng bo'ladi.

$8x^3-16x+1 = A(x+2)^2 + B(x-2)(x+2)^2 + C(x-2)^2 + D(x+2)(x-2)^2$. A, B, C, D Koeffitsientlarni quydagicha topamiz.

A va C – tanlash usuli bilan, B va D – aniqmas koeffitsientlar usuli orqali topamiz

$x=-2$ bo'lsin u xolda $8\cdot(-2)^3+16\cdot2+1 = A\cdot0+B\cdot0+C\cdot16+D\cdot0$ yoki

$16C=-31$; $\quad C=-\frac{31}{16}$.

Agar $x=2$ bo'lsa, u xolda

$8\cdot2^3-16\cdot2+1 = 16A+B\cdot0+C\cdot16+D\cdot0$ Yoki

$16A=33$; $\quad A=\frac{33}{16}$.

Tenglikni quyidagicha yozib olamiz.

$$A(x+2)^2 + B(x-2)(x+2)^2 + C(x-2)^2 + D(x-2)^2(x+2) =$$

$$= A(x^2+4x+4) + B(x^3+2x^2-4x+8) + C(x^2-4x+4) +$$

$$+ D(x^3-2x^2-4x+8) = (B+D)x^3 + (A+2B+C-2D)x^2 +$$

$$+ (4A-4B-4C-4D)x + (4A+8A+4C+8D)$$

Yoki

$$8x^3-16x+1=(B+D)x^3+(A+2B+C-2D)x^2+$$
$$+(-4A-4B-4C-4D)x+4A+8B+4C+8D.$$

Oxirgi tenglikda x ning bir xil darajalari oldidagi koeffitsientlarni tenglashtirsak, A, B, C va D noma'lumlar uchun chiziqli tenglamalar sistemasini hosil qilamiz.

$x^3 \quad B+D=8,$

$x^2 \quad A+2B+C-2D=0,$

$x \quad -4A-4B-4C-4D=-16,$

$x^0 \quad 4A+8B+4C+8D=1$

Agar $A=\frac{33}{16},\ C=-\frac{31}{16}$ ekanligini xisobga olsak, Tenglamalar sistemasining birinchi va ikkinchi tenglamasidan foydalansak

$$\begin{cases} D=8-B, \\ \frac{33}{16}+2B-\frac{31}{16}-2(8-B)=0 \end{cases} \quad \text{yoki} \quad \begin{cases} D=\frac{129}{32}, \\ B=\frac{127}{32}. \end{cases}$$

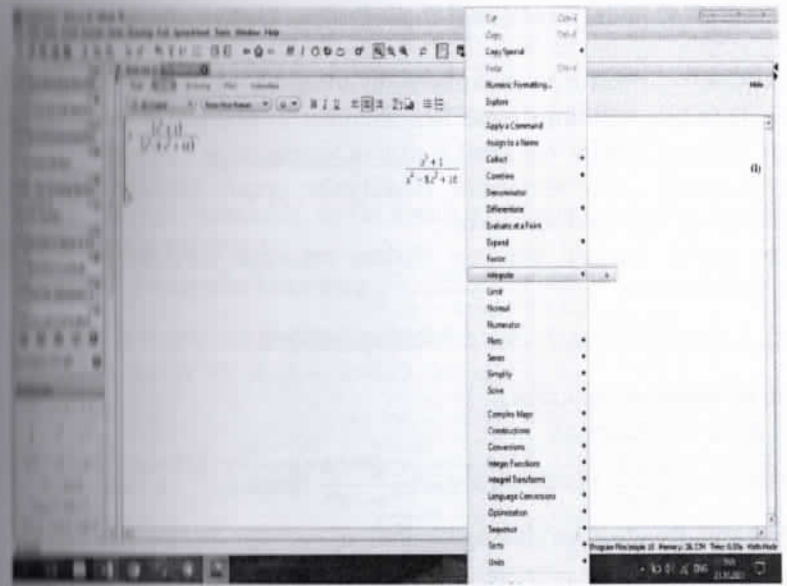
Bularni integralga keltirib qo'ysak quyidagiga ega bo'lamiz.

$$\int\frac{x^5+1}{x^4-8x^2+16}dx=$$
$$=\int(x+\frac{33}{16}\cdot\frac{1}{(x-2)^2}+\frac{127}{32}\cdot\frac{1}{x-2}-\frac{31}{16}\cdot\frac{1}{(x+2)^2}+\frac{129}{32}\cdot\frac{1}{x+2})\,dx=$$
$$=\frac{x^2}{2}-\frac{33}{16(x-2)}+\frac{127}{32}\ln|x-2|+\frac{31}{16(x+2)}+\frac{129}{32}\cdot\frac{1}{x+2}+C.$$

Maple dasturida to'gridan -to'gri hisoblash mumkun.

Sichqoncha o'ng tomonini bosib integrate—>x Camandasi yordamida hisoblaymiz.

Windows oynasida quydagi ko'rinishga ega.



$$\frac{x^5+1}{x^4-8x^2+16}$$

$$\frac{1}{2}x^2+\frac{127}{32}\ln(x-2)+\frac{129}{32}\ln(x+2)+\frac{31}{16(x+2)}-\frac{33}{16(x-2)}$$

Misol 21. Integralni xisoblang: $\int\frac{x^3}{(x+5)^5}dx$.

Yechish. Integral belgisi ostida oddiy ratsional kasr joylashgan bo'lib, bu kasrni oddiy kasrlarning yig'indisi sifatida ifodalash orqali integralni topish mumkin. Shu bilan birga, o'zgaruvchini almashtirish orqali integralni sodda ko'rinishga keltirish mumkin: $x+5=t;\ x=t-5;\ dx=dt$.

U holda

$$\int\frac{x^3}{(x+5)^5}dx=\int\frac{(t-5)^3}{t^6}dx=\int\frac{t^3-15t^2+75t-125}{t^6}dt=$$
$$=\int\left(\frac{1}{t^3}-15\cdot\frac{1}{t^4}+75\cdot\frac{1}{t^5}-125\cdot\frac{1}{t^6}\right)dt=$$
$$=-\frac{1}{2}\cdot\frac{1}{t^2}+\frac{15}{-3t^3}+\frac{75}{-4t^4}-\frac{125}{-5t^5}+C=$$
$$=-\frac{1}{2(x+5)^2}+\frac{5}{(x+5)^3}-\frac{75}{4(x+5)^4}+\frac{25}{(x+5)^5}+C.$$

1.4. Ba'zi irratsional funksiyalarni integrallash

Ixtiyoriy irratsional funktsiyaning integrali elementar funktsiyalar orqali ifodalash mumkun emas.Biz shunday $t=\omega(x)$, almashtirish bajarib integral ostidagi ifodani ratsional funksiya ko'rinishiga keltirishga xarakat qilamiz . Agar $\omega(x)$ elementar funksiyalar orqali ifodalansa u xolda integralni oson xisoblash mumkun .

Bu usulni integral ostidagi ifodani ratsional ko'rinishga keltirish metodi deb ataymiz.

1) $\int R\left(x,\sqrt[m]{\frac{\alpha x+\beta}{\gamma x+\delta}}\right)dx$, ko'rinishidagi integral

Bu erda R ratsional funksiya,

$m\in N,\alpha,\beta,\gamma,\delta$ – o'zgarmas.

Agar, $t=\omega(x)=\sqrt[m]{\frac{\alpha x+\beta}{\gamma x+\delta}},\ t^m=\frac{\alpha x+\beta}{\gamma x+\delta},\ x=\varphi(x)=\frac{\delta t^m-\beta}{\alpha-\gamma t^m}$ .desak

Integral quyidagi ko'rinishga keladi.

$$\int R(\varphi(t),t)\cdot\varphi'(t)dt,$$

Bu erda $R,\varphi(t),\varphi'(t)$ – ratsional funksiyalar.

Ushbu integralni ratsional funktsiyalarni integrallash qoidalari bo'yicha hisoblab , biz eski o'zgaruvchi $t=\omega(x)$. ga qaytamiz

(1) Ko'rinishdagi integralga $\int R\left(x,\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)^r,\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)^s,...\right)dx$, 9

ko'rinishdagi umiymiroq integrallani xam keltirish mumkin

Bu erda r, s,… – ratsional sonlar.

Bu erda r, s, larni umumiy m maxrajga keltirib integral ostidagi ifodani x ning xamda $\sqrt[m]{\frac{\alpha x+\beta}{\gamma x+\delta}}$. radikalning funksiyasiga aylantiramiz.

Misol 22. Integralni xisoblang: $\int\frac{dx}{\sqrt[3]{(x-1)(x+1)^2}}$.

Echish. $\int\frac{dx}{\sqrt[3]{(x-1)(x+1)^2}}=\int\sqrt[3]{\frac{x+1}{x-1}}\frac{dx}{x+1}=\left\{\begin{array}{l}t=\sqrt[3]{\frac{x+1}{x-1}},\\ x=\frac{t^3+1}{t^3-1},\ dx=\frac{-6t^2dt}{(t^3-1)^2}\end{array}\right\}=$

$$=\int\frac{t\cdot(-6t^2)dt}{\left(\frac{t^3+1}{t^3-1}+1\right)(t^3-1)^2}=\int\frac{-3dt}{t^3-1}=\int\frac{-3dt}{(t-1)(t^2+t+1)}=$$

$$=\int\left(-\frac{1}{t-1}+\frac{t+2}{t^2+t+1}\right)dt=\frac{1}{2}\ln\left|\frac{t^2+t+1}{(t-1)^2}\right|+\sqrt{3}arctg\frac{2t-1}{\sqrt{3}}+C,$$

Bu erda $t=\sqrt[3]{\frac{x+1}{x-1}}$.

2) $\int\frac{dx}{\sqrt{ax^2+bx+c}}$. ko'rinishidagi integral

Agar kvadrat uchxadda to'liq kvadrat ajratilsa, bunday integrallar integral jadvaliga keltiriladi.

Misol 23. Integralni xisoblang: $\int\frac{dx}{\sqrt{x^2-6x+8}}$.

Echish. Kvadrat uch xadda quydagicha almashtish bajaramiz.

$x^2-6x+8=(x-3)^2-1;\quad x-3=t,\ x=t+3,\ dx=dt$.

U xolda

$$\int\frac{dx}{\sqrt{x^2-6x+8}}=\int\frac{d(x-3)}{\sqrt{(x-3)^2-1}}=\int\frac{dt}{\sqrt{t^2-1}}=\left\{\text{int}\,egral\ 19\right\}$$

$$=\ln|t+\sqrt{t^2-1}|+C=\ln|x-3+\sqrt{x^2-6x+8}|+C.$$

3) $\int\frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$.

Ko'rinishdagi integral

Bu ko'rinishdagi integralni xisoblash uchun integral ostidagi ifodani suratida maxrajidagi kvadrat uchxadning differensialiga moslab olamiz. So'ngra integralni ikkiga ajratib olamiz. Birinchi integral to'gridan to'gri integral jadvali yordamida topiladi ikkinchisi esa 23 misol kabi topiladi.

Misol 24. Integralni xisoblang: $\int\frac{7x+2}{\sqrt{-x^2+4x+5}}dx$.

Echish. Integral ostidagi ifodaning suratida maxrajdagi kvadrat uchxadning differensialiga moslab olamiz.

$(-x^2+4x+5)'=-2x+4$.

$7x+2=-\frac{7}{2}(-2x+4-4)+2=-\frac{7}{2}(-2x+4)+16.$

U xolda quyidagilarga ega bo'lamiz.

$$\int\frac{7x+2}{\sqrt{-x^2+4x+5}}dx=\int\frac{-7/2(-2x+4)+16}{\sqrt{-x^2+4x+5}}dx=$$

$$=-\frac{7}{2}\int\frac{(-2x+4)dx}{\sqrt{-x^2+4x+5}}+16\cdot\int\frac{dx}{-x^2+4x+5}=$$

$$=\left\{\begin{array}{l}(-2x+4)dx=d(-x^2+4x+5)\\ -x^2+4x+5=-(x^2-4x+4-4)+5=-(x-2)^2+9\end{array}\right\}=$$

$$=-\frac{7}{2}\int\frac{d(-x^2+4x+5)}{\sqrt{-x^2+4x+5}}+16\cdot\int\frac{d(x-2)}{\sqrt{9-(x-2)^2}}=$$

$$=\{-x^2+4x+5=t;\ x-2=z\}=$$
$$=-\frac{7}{2}\int\frac{dt}{\sqrt{t}}+16\int\frac{dz}{\sqrt{9-z^2}}=-7\sqrt{t}+16\arcsin\frac{z}{3}+C=$$
$$=-7\sqrt{-x^2+4x+5}+16\arcsin\frac{x-2}{3}+C.$$

Maple dasturida quydagicha bo'ladi.

```
> with(Student[Calculus1]):
> IntTutor((7x+2)/(sqrt(-x^2+4x+5)))
```

$$\int\frac{7x+2}{\sqrt{-x^2+4x+5}}\,dx$$

$=\int\frac{16+7u}{\sqrt{-u^2+9}}\,du$ [change, $u=x-2$, u]

$=\int\left(\frac{7u}{\sqrt{-u^2+9}}+\frac{16}{\sqrt{-u^2+9}}\right)du$ [rewrite, $\frac{16+7u}{\sqrt{-u^2+9}}=\frac{7u}{\sqrt{-u^2+9}}+\frac{16}{\sqrt{-u^2+9}}$]

$=\int\frac{7u}{\sqrt{-u^2+9}}\,du+\int\frac{16}{\sqrt{-u^2+9}}\,du$ [sum]

$=7\left(\int\frac{u}{\sqrt{-u^2+9}}\,du\right)+\int\frac{16}{\sqrt{-u^2+9}}\,du$ [constantmultiple]

$=7\left(\int(-1)\,du1\right)+\int\frac{16}{\sqrt{-u^2+9}}\,du$ [change, $-u^2+9=u1^2$, $u1$]

$=-7\,u1+\int\frac{16}{\sqrt{-u^2+9}}\,du$ [constant]

$=-7\sqrt{-u^2+9}+\int\frac{16}{\sqrt{-u^2+9}}\,du$ [revert]

$=-7\sqrt{-u^2+9}+16\left(\int\frac{1}{\sqrt{-u^2+9}}\,du\right)$ [constantmultiple]

$=-7\sqrt{-u^2+9}+16\left(\int 1\,du1\right)$ [change, $u=3\sin(u1)$, $u1$]

$=-7\sqrt{-u^2+9}+16\,u1$ [constant]

$=-7\sqrt{-u^2+9}+16\arcsin\left(\frac{u}{3}\right)$ [revert]

$=-7\sqrt{-x^2+4x+5}+16\arcsin\left(\frac{x}{3}-\frac{2}{3}\right)$ [revert]

$$\int\frac{7x+2}{\sqrt{-x^2+4x+5}}\,dx=-7\sqrt{-x^2+4x+5}+16\arcsin\left(\frac{1}{3}x-\frac{2}{3}\right)$$

4) $\int R(x,\sqrt{ax^2+bx+c})dx,\ (a\neq 0)$ ko'rinishidagi integral

Bunday integrallar Eylerning quyidagi almashtirishlari orqali ratsional funksiyalarning integaliga keltiriladi.

Eylerning 1-almashtirishi. Agar a > 0 bo'lsa, u xolda

$\sqrt{ax^2+bx+c}=\pm\sqrt{a}\cdot x+t$.

Quyidagi xolni ko'ramiz.

$\sqrt{ax^2+bx+c}=\sqrt{a}\cdot x+t$. U xolda

$ax^2+bx+c=ax^2+2\sqrt{a}\cdot x\cdot t+t^2$, $x=\frac{t^2-c}{b-2\sqrt{a}\cdot t}$, va

$\sqrt{ax^2+bx+c}=\sqrt{a}\cdot x+t=\sqrt{a}\cdot\frac{t^2-c}{b-2t\sqrt{a}}+t$ –bu yangi o'zgaruvchi t ning ratsional funksiyasi dx xam t o'zgaruvchi orqali ratsional ko'rinishda ifodalanadi.

Eylerning 2-almashtirishi. Agar c > 0 bo'lsa, u xolda

$\sqrt{ax^2+bx+c}=xt\pm\sqrt{c}$.

Aniqlik uchun c oldidagi ishorani «+» deb olamiz. U xolda

$ax^2+bx+c=x^2t^2+2xt\sqrt{c}+c$, $x=\frac{2\sqrt{c}t-b}{a-t^2}$.

Bunda dx va $\sqrt{ax^2+bx+c}$ ifoda yangi o'zgaruchi t orqali ratsional ko'rinishda ifodalanadi. U xolda $\int R(x,\sqrt{ax^2+bx+c})dx$ integral yangi o'zgaruvchili t-ratsional funksiyaning integraliga keladi.

Misol 25. Integralni xisoblang: $\int\frac{(1-\sqrt{1+x+x^2})^2}{x^2\sqrt{1+x+x^2}}dx$.

Yechish. Eylerning 2 almashtirishini qo'llaymiz.

$\sqrt{1+x+x^2}=xt+1$; $1+x+x^2=x^2t^2+2xt+1$,

$x=\frac{2t-1}{1-t^2}$, $dx=\frac{2t^2-2t+2}{(1-t^2)^2}dt$;

$\sqrt{1+x+x^2}=xt+1=\frac{t^2-t+1}{1-t^2}$, $1-\sqrt{1+x+x^2}=\frac{-2t^2+1}{1-t^2}$.

$$\int\frac{(1-\sqrt{1+x+x^2})^2}{x^2\sqrt{1+x+x^2}}dx=\int\frac{(-2t^2+t)^2(1-t^2)^2\cdot(1-t^2)(2t^2-2t+2)}{(1-t^2)^2(2t-1)^2(t^2-t+1)(1-t^2)^2}dt=$$

$$=2\int\frac{t^2}{1-t^2}dt=-2\int\frac{-t^2+1-1}{1-t^2}dt=-2\int\left(1+\frac{2}{1-t^2}\right)dt=$$

$$=-2t+\ln\left|\frac{1+t}{1-t}\right|+C,$$

Bunda $t=\frac{\sqrt{1+x+x^2}-1}{x}$.

$$\int \frac{\left(1-\sqrt{x^2+x+1}\right)^2}{x^2\sqrt{x^2+x+1}}\,dx$$

$$= \int \frac{2\left(-2+\sqrt{4u^2+3}\right)^2}{\sqrt{4u^2+3}\,(2u-1)^2}\,du \qquad [change,\ u = -x+\frac{1}{2},\ u]$$

$$= 2\left(\int \frac{\left(-2+\sqrt{4u^2+3}\right)^2}{\sqrt{4u^2+3}\,(2u-1)^2}\,du\right) \qquad [constantmultiple]$$

$$= 2\left(\int \frac{-ul^2+6ul-9}{2ul^3+12ul^2+18ul}\,dul\right) \qquad [change,\ ul = (4u^2+3)^{1/2}-2u,\ ul]$$

$$= 2\left(\int\left(-\frac{1}{2ul}+\frac{6}{(ul+3)^2}\right)dul\right) \qquad [partialfractions]$$

$$= 2\left(\int -\frac{1}{2ul}\,dul\right)+2\left(\int \frac{6}{(ul+3)^2}\,dul\right) \qquad [sum]$$

$$= -\left(\int \frac{1}{ul}\,dul\right)+2\left(\int \frac{6}{(ul+3)^2}\,dul\right) \qquad [constantmultiple]$$

$$= -\ln(ul)+2\left(\int \frac{6}{(ul+3)^2}\,dul\right) \qquad [power]$$

$$= -\ln(ul)+12\left(\int \frac{1}{(ul+3)^2}\,dul\right) \qquad [constantmultiple]$$

$$= -\ln(ul)+12\left(\int \frac{1}{u2^2}\,du2\right) \qquad [change,\ u2 = ul+3,\ u2]$$

$$= -\ln(ul)-\frac{12}{u2} \qquad [power]$$

$$= -\ln(ul)-\frac{12}{ul+3} \qquad [revert]$$

$$= -\frac{12}{\sqrt{4u^2+3}-2u+3}-\ln\left(\sqrt{4u^2+3}-2u\right) \qquad [revert]$$

$$= -\frac{12}{\sqrt{4x^2+4x+4}-2x+2}-\ln\left(\sqrt{4x^2+4x+4}-2x-1\right) \qquad [revert]$$

$$\int \frac{\left(1-\sqrt{x^2+x+1}\right)^2}{x^2\sqrt{x^2+x+1}}\,dx = -\frac{12}{\sqrt{4x^2+4x+4}-2x+2}-\ln\left(\sqrt{4x^2+4x+4}-2x-1\right)$$

Eylerning 3 almashtirishi. ax^2+bx+c kvadrat uch xad xaqiyqiy α va β ildizlarga ega . U xolda $\int R(x,\sqrt{ax^2+bx+c})\,dx$ integral yangi o‘zgaruvchi t ning ratsional funksiyasini integrallashga keladi.

a $\neq 0$ $\sqrt{ax^2+bx+c}=(x-\alpha)t$ yoki

$\sqrt{a(x-\alpha)(x-\beta)}=(x-\alpha)t, \quad x=\dfrac{\alpha\beta-a t^2}{a-t^2}.$

1.5. Binamial ifodalarni integrallash.

Binomial differensial deb quyidagi ifodaga aytiladi.

$$x^m(a+bx^n)^p dx,$$

Bu erda m,n,p – ratsional sonlar, a,b –esa o‘zgarmas sonlar. Quydagi integralni ko‘ramiz. $\int x^m(a+bx^n)^p dx$ **(6)**

1) n –butun son . Bu integral t ning ratsional funksiyasini integrallashga keltiriladi. Bunda $t=\sqrt[\lambda]{x}$, almashtirish bajaramiz. λ – m va n sonlarining eng kichik umumiy karralisi.

2) $\frac{m+1}{n}$ – butun son, u xolda integral ostidagi funksiyani quydagicha almashtirish bajarib ratsional kasr ko‘rinishiga keltirish mumkin.

$t=\sqrt[\nu]{a+bx^n}$, ν – r-kasrning maxraji.

3) $\frac{m+1}{n}+p$ – butun son.

$t=\sqrt[\nu]{ax^{-n}+b}$ almashtirish orqali berilgan integralni ratsional kasrlarni integrallashga keltiramiz bu erda ν – r-kasrning maxraji

Bu almashtirishlar ingliz matematigi Nyutonga ma’lum bo‘lgan. Biroq bu almashtirishlarning isbotini utgan asrning urtalarida rus matematigi P.L.Chebishev keltirgan. Shuning uchun bu 1-3 almashtirishlar *Chebishev* almashtirishlari deyiladi.

Misol 26. Integralni xisoblang: $\int \frac{dx}{x\sqrt[3]{1+x^5}}$.

Echish. $\int \frac{dx}{x\sqrt[3]{1+x^5}}=\int x^{-1}(1+x^5)^{-\frac{1}{3}}dx.$

$m=-1, n=5, p=-\frac{1}{3}; \quad \nu=3.$

Chebishevning 2 almashtirishini qo‘llaymiz, u xolda. $\frac{m+1}{n}=0$ – butun son.

$t=\sqrt[3]{1+x^5}, \quad 1+x^5=t^3, \quad x=(t^3-1)^{1/5}; \quad dx=\frac{3}{5}t^2(t^3-1)^{-\frac{4}{5}}dt.$

$\int \frac{dx}{x\sqrt[3]{1+x^5}}=\frac{3}{5}\int \frac{t\,dt}{t^3-1}=\frac{1}{5}\int\left(\frac{1}{t-1}-\frac{t-1}{t^2+t+1}\right)dt=$

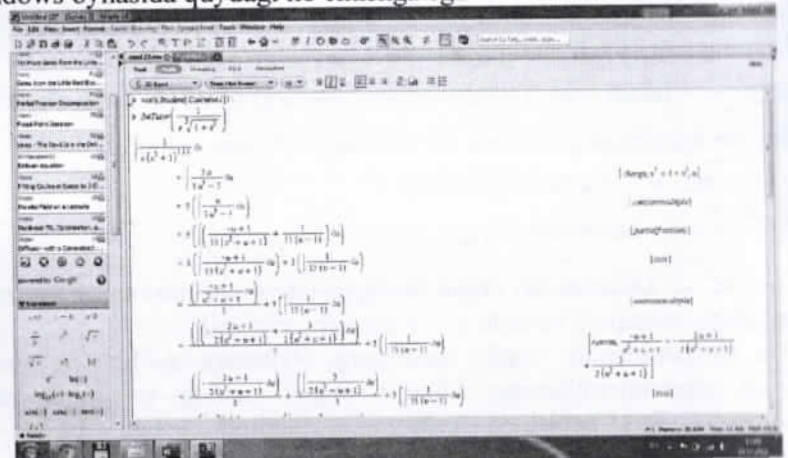
$=\left\{(t^2+t+1)'=2t+1, \quad t-1=\frac{1}{2}(2t+1)-\frac{3}{2}\right\}=\frac{1}{5}\ln|t-1|-\frac{1}{10}\int\frac{dt}{t^2+t+1}=$

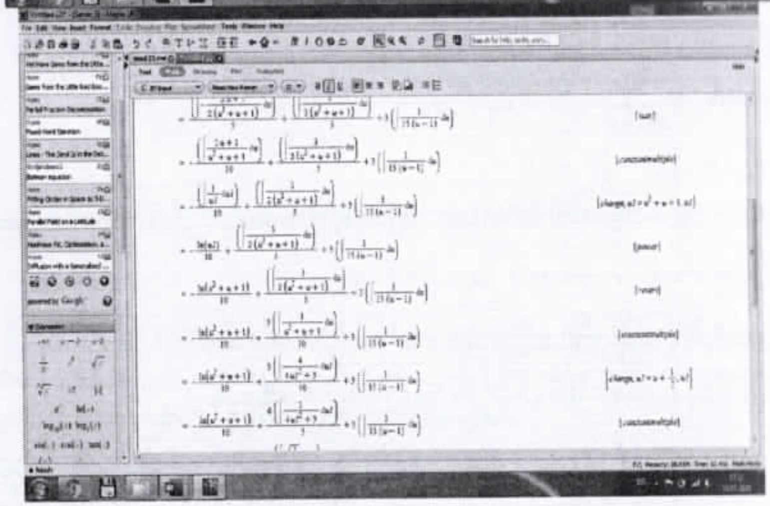
$$=\frac{1}{5}\ln|t-1|-\frac{1}{10}\int\frac{d(t^2+t+1)}{t^2+t+1}+\frac{3}{10}\int\frac{d\left(t+\frac{1}{2}\right)}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}=$$

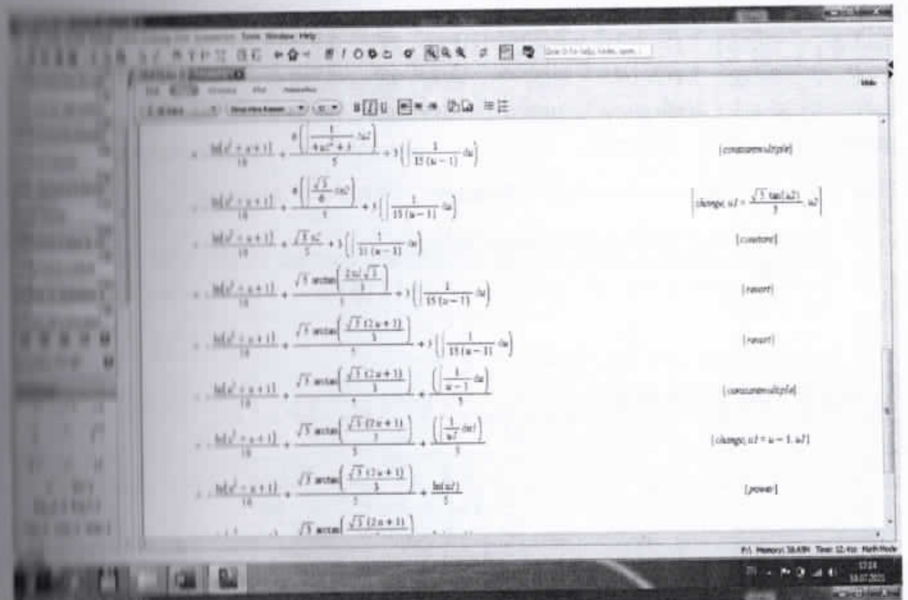
$$=\frac{1}{5}\ln|t-1|-\frac{1}{10}\ln(t^2+t+1)+\frac{3}{10}\cdot\frac{1\cdot 2}{\sqrt{3}}arctg\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}+C=$$

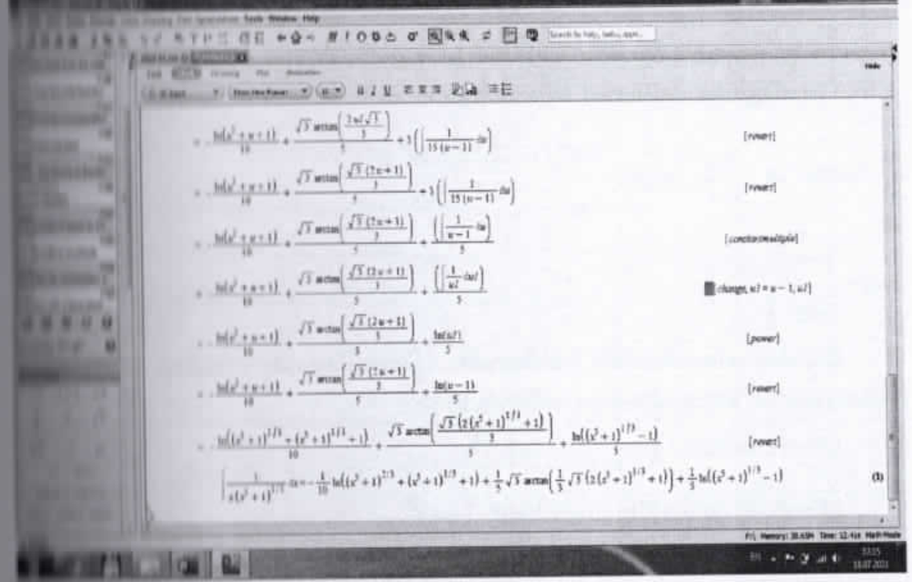
$$=\frac{1}{10}\ln\frac{(t-1)^2}{t^2+t+1}+\frac{\sqrt{3}}{5}arctg\frac{2t+1}{\sqrt{3}}+C.$$

Albatta bu misolni maple dasturi yordamida ishlash mumkun. Bu Windows oynasida quydagi ko'rinishga ega









Misol 27. Integralni xisoblang: $\int\frac{dx}{x^3\sqrt[3]{2-x^3}}$.

Echish. Integral ostidagi funksiyani quyidagicha yozish mumkin $\frac{(m+1)}{n}+p=\frac{(-3+1)}{3}-\frac{1}{3}=-1$ – butun son. Shuning uchun binamial funksiyalarni integrallashda *Chebishev* almashtirishlaridan foydalansak (3- almashtirish) quydagini xosil qilamiz: $2x^{-3}-1=t^3$ almashtirish bajaramiz u xolda $d(2x^{-3}-1)=dt^3$ yoki $-6x^{-4}dx=3t^2dt$ bundan $x^{-4}dx=-\frac{1}{2}t^2dt$ desak.

Integral quyidagi ko'rinishga keladi.

$$\int\frac{dx}{x^3\cdot\sqrt[3]{2-x^3}}=\int x^{-3}\cdot(2-x^3)^{-\frac{1}{3}}dx=\int x^{-3}\cdot\left(x^3(2x^{-3}-1)\right)^{-\frac{1}{3}}dx=$$

$$=\int x^{-3}\cdot x^{-1}\cdot(2x^{-3}-1)^{-\frac{1}{3}}dx=\int(2x^{-3}-1)^{-\frac{1}{3}}x^{-4}dx=$$

$$=\int(t^3)^{-\frac{1}{3}}\cdot\left(-\frac{1}{2}t^2\right)dt=-\frac{1}{2}\int t\,dt=-\frac{1}{2}\cdot\frac{t^2}{2}+C=-\frac{1}{4}t^2+C=$$

$$=-\frac{1}{4}\cdot\sqrt[3]{(2x^{-3}-1)^2}+C=-\frac{1}{4}\cdot\sqrt[3]{\left(\frac{2}{x^3}-1\right)^2}+C=-\frac{\sqrt[3]{(2-x^3)^2}}{4x^2}+C.$$

1.6. Ba'zi trigonometric funksiyalarni integrallash

1) $\int R(\sin x,\cos x)dx$ ko'rinishdagi integrallar.

Quydagicha universal trigonometric almashtirish bajaramiz. $tg\frac{x}{2}=t$,

$$x=2arctgt,\quad dx=\frac{2dt}{1+t^2},\quad \sin x=\frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{2t}{1+t^2},$$

$$\cos x=\frac{1-tg\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}.$$

Bunday almashtirish yordamida $\int R(\sin x,\cos x)dx$ integral ratsional funksiyalarni integrallashga keltiriladi.

$$\int R(\sin x,\cos x)dx=\int R\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2}.$$

Misol 28. Integralni xisoblang: $\int\frac{dx}{3\sin x+2}$.

Echish:

$$\int\frac{dx}{3\sin x+2}=\int\frac{\frac{2dt}{1+t^2}}{\frac{6t}{1+t^2}+2}=\int\frac{dt}{t^2+3t+1}=\int\frac{d\left(t+\frac{3}{2}\right)}{\left(t+\frac{3}{2}\right)^2-\frac{5}{4}}=$$

$$=\frac{1}{2\cdot\frac{\sqrt{5}}{2}}\ln\left|\frac{t+\frac{3}{2}-\frac{\sqrt{5}}{2}}{t+\frac{3}{2}+\frac{\sqrt{5}}{2}}\right|+C=\frac{\sqrt{5}}{5}\ln\left|\frac{2t+3-\sqrt{5}}{2t+3+\sqrt{5}}\right|+C=$$

$$=\frac{\sqrt{5}}{5}\ln\left|\frac{2tg\frac{x}{2}+3-\sqrt{5}}{2tg\frac{x}{2}+3+\sqrt{5}}\right|+C.$$

Bu misolni maple dasturi yordamida quydagicha ishlash mumkun.

```
> with(Student[Calculus1]):
> IntTutor(1/(3sin(x)+2))
```

$$\left[change, u=\tan\left(\frac{x}{2}\right), u\right]$$

$$\int\frac{1}{3\sin(x)+2}\,dx=$$

$$=\int\frac{1}{u^2+3u+1}\,du$$

$$=\int\left(-\frac{2\sqrt{5}}{5(-2u-3+\sqrt{5})}-\frac{2\sqrt{5}}{5(2u+3+\sqrt{5})}\right)du$$

$$=\int-\frac{2\sqrt{5}}{5(-2u-3+\sqrt{5})}\,du+\int-\frac{2\sqrt{5}}{5(2u+3+\sqrt{5})}\,du$$

$$=-\frac{2\sqrt{5}\left(\int\frac{1}{-2u-3+\sqrt{5}}\,du\right)}{5}+\int-\frac{2\sqrt{5}}{5(2u+3+\sqrt{5})}\,du$$

$$=-\frac{2\sqrt{5}\left(\int-\frac{1}{2u1}\,du1\right)}{5}+\int-\frac{2\sqrt{5}}{5(2u+3+\sqrt{5})}\,du$$

$$=\frac{\sqrt{5}\left(\int\frac{1}{u1}\,du1\right)}{5}+\int-\frac{2\sqrt{5}}{5(2u+3+\sqrt{5})}\,du$$

$$=\frac{\sqrt{5}\ln(u1)}{5}+\int-\frac{2\sqrt{5}}{5(2u+3+\sqrt{5})}\,du$$

$$=\frac{\sqrt{5}\ln(-2u-3+\sqrt{5})}{5}+\int-\frac{2\sqrt{5}}{5(2u+3+\sqrt{5})}\,du$$

$$=\frac{\sqrt{5}\ln(-2u-3+\sqrt{5})}{5}-\frac{2\sqrt{5}\left(\int\frac{1}{2u+3+\sqrt{5}}\,du\right)}{5}$$

$$=\frac{\sqrt{5}\ln(-2u-3+\sqrt{5})}{5}-\frac{2\sqrt{5}\left(\int\frac{1}{2u1}\,du1\right)}{5}$$

$$=\frac{\sqrt{5}\ln(-2u-3+\sqrt{5})}{5}-\frac{\sqrt{5}\left(\int\frac{1}{u1}\,du1\right)}{5}$$

$$=\frac{\sqrt{5}\ln(-2u-3+\sqrt{5})}{5}-\frac{\sqrt{5}\ln(u1)}{5}$$

$$=-\frac{\sqrt{5}\ln(2u+3+\sqrt{5})}{5}+\frac{\sqrt{5}\ln(-2u-3+\sqrt{5})}{5}$$

$$=-\frac{\sqrt{5}\left(\ln\left(2\tan\left(\frac{x}{2}\right)+3+\sqrt{5}\right)-\ln\left(-2\tan\left(\frac{x}{2}\right)-3+\sqrt{5}\right)\right)}{5}$$

$$\left[change, u1=-2u-3+\sqrt{5}, u1\right]$$

$$\int \frac{1}{3\sin(x)+2}\,dx = -\frac{1}{5}\sqrt{5}\left(\ln\left(2\tan\left(\frac{1}{2}x\right)+3+\sqrt{5}\right)-\ln\left(-2\tan\left(\frac{1}{2}x\right)-3+\sqrt{5}\right)\right)$$

2) $\int R(\sin x)\cdot\cos x dx$ yoki $\int R(\cos x)\cdot \sin x dx$ ko‘rinishidagi integrallar

a) $\int R(\sin x)\cdot\cos x dx$ integral $\sin x = t, \cos x dx = dt.$ almashtirish yordamida $\int R(t)dt$ ko‘rinishidagi integralga keltiriladi.

b) $\int R(\cos x)\cdot\sin x dx$ integral $\cos x = t, \sin x dx = -dt.$ almashtirish yordamida $\int(-R(t)dt)$, ko‘rinishidagi integralga keltiriladi.

3) $\int R(tgx)dx,\ \int R(\sin^{2k}x, \cos^{2m}x)dx$ ko‘rinishidagi integrallar.

Agar interal ostidagi funksiya faqat tg(x) boglik bo‘lsa yoki faqat sin(x) va cos(x) larning juft darajalariga boglik bo‘lsa u xolda quyidagi almashtirishlarni bajaramiz.

$$tgx = t,\ x = arctgx,\ \ dx = \frac{dt}{1+t^2}$$

$$\cos^2 x = \frac{1}{1+tg^2x} = \frac{1}{1+t^2};\quad \sin^2 x = \frac{tg^2x}{1+tg^2x} = \frac{t^2}{1+t^2}.$$

Natijada ratsional funksiylarning integraliga kelamiz:

Misol 29. Integralni xisoblang: $\int \frac{dx}{3+\sin^2 x}$

Echish:

$$\int\frac{dx}{3+\sin^2 x} = \{tgx = t\} = \int \frac{dt}{\left(3+\frac{t^2}{1+t^2}\right)(1+t^2)} = \int\frac{dt}{4t^2+3} =$$

$$= \frac{1}{4}\int\frac{dt}{t^2+\frac{3}{4}} = \frac{1}{4}\cdot\frac{1}{\sqrt{\frac{3}{4}}}arctg\frac{t}{\sqrt{\frac{3}{4}}}+C = \frac{1}{2\sqrt{3}}arctg\left(\frac{2tgx}{\sqrt{3}}\right)+C.$$

4) $\int \sin^m x\cdot\cos^n x dx$ ko‘rinishidagi integral.

a) m va n larning kamida bittasi toq son. Aniqlik uchun n toq son bo‘lsin. U xolda $n=2p+1$ almashtirish bajaramiz va quyidagini xosil qilamiz.

$$\int \sin^m x\cdot \cos^n x dx = \int sim^m x\cdot\cos^{2p}x\cdot\cos x dx =$$

$$= \int\sin^m x\cdot(1-\sin^2 x)^p d\sin x = \{\sin x = t\} = \int t^m\cdot(1-t^2)^p dt = \int R(t)dt.$$

b) m va n – nomanfiy juft sonlar . U xolda $m=2p,\ n=2q$ almashtirishlar bajaramiz va quyidagini xosil qilamiz.

$$\sin^2 x = \frac{1}{2}(1-\cos 2x);\quad \cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$\int\sin^m x\cdot\cos^n x dx = \int(\sin^2 x)^p\cdot(\cos^2 x)^q dx =$$

$$= \int\left(\frac{1}{2}-\frac{1}{2}\cos 2x\right)^p\cdot\left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)^q dx.$$

Qavslarni ochib cos2x ning juft va toq darajalariga bogliq integrallarni xosil qilamiz. cos2x ning toq darajadagi Integrallar a). xolidagi kabi integrallanadi. cos2x ning juft darajadagilarini yuqoridagi kabi darajasi pasaytiriladi. Shu tarzda davom etib $\int\cos kx dx$, ko‘rinishidagi integralga kelamiz.

Misol 30. Integralni xisoblang: $\int\sin^2 x\cdot\cos^3 x dx.$

Echish:

$$\int\sin^2 x\cdot\cos^3 x dx = \int\sin^2 x\cdot(1-\sin^2 x)d\sin x = \frac{\sin^3 x}{3}-\frac{\sin^5 x}{5}+C.$$

Misol 31. Integralni xisoblang: $\int\sin^2 3x\cdot\cos^2 3x dx.$

Echish:

$$\int\sin^2 3x\cdot\cos^2 3x dx = \frac{1}{4}\int(1-\cos 6x)(1+\cos 6x)dx =$$

$$= \frac{1}{4}\int(1-\cos^2 6x)dx = \frac{1}{4}x-\frac{1}{4}\cdot\int\frac{1+\cos 12x}{2}dx =$$

$$= \frac{1}{4}x-\frac{1}{8}x-\frac{1}{8}\cdot\frac{\sin(12x)}{12}+C = \frac{1}{8}x-\frac{1}{96}\sin(12x)+C.$$

v) m va n – juft sonlar. Biroq ularning birontasi manfiy qiymatga ega.

Bu xolda quydagicha almashtirish bajaramiz. $tgx = t$ yoki $ctgx = t$).

Misol 32. Integralni xisoblang: $\int\frac{\sin^2 x}{\cos^6 x}dx.$

Echish:

$$\int\frac{\sin^2 x}{\cos^6 x}dx = \int\frac{\sin^2 x(\sin^2 x+\cos^2 x)^2}{\cos^2 x\cdot\cos^4 x}dx =$$

$$= \int tg^2 x\cdot(tg^2 x+1)^2 dx = \left\{\begin{matrix} tgx = t; x = arctg\,t \\ dx = \frac{dt}{1+t^2}\end{matrix}\right\} = \int t^2(1+t^2)^2\frac{dt}{1+t^2} =$$

$$= \int t^2(1+t^2)dt = \int(t^2+t^4)dt = \frac{t^3}{3}+\frac{t^5}{5}+C = \frac{tg^3x}{3}+\frac{tg^5x}{5}+C.$$

5) $\int\cos mx\cdot\sin nx dx;\ \int\cos mx\cdot\cos nx dx;\ \int\sin mx\cdot\sin nx dx\quad (m\neq n)$. Ko‘rinishidagi integrallar

Bunday ko‘rinishda funksiyalarni integrallash uchun quydagi trigonometric almashtirishlarni bajarish etarli:

$$\cos mx\cdot\cos nx = \frac{1}{2}[\cos(m+n)x+\cos(m-n)x]$$

$$\sin mx\cdot\cos nx=\frac{1}{2}[\sin(m+n)x+\sin(m-n)x]$$

$$\sin mx\cdot\sin nx=\frac{1}{2}[\cos(m-n)x-\cos(m+n)x]$$

U xolda quyidagilarni xosil qilamiz.

$$\int\sin mx\cdot\cos nx dx=\frac{1}{2}\int[\sin(m+n)x+\sin(m-n)]dx=$$

$$=-\frac{\cos(m+n)x}{2(m+n)}-\frac{\cos(m-n)x}{2(m-n)}+C.$$

Qolgan ikkita integral xam xuddi shunday xisoblanadi.

Misol 33. Integralni xisoblang: $\int\sin 4x\cdot\cos 6x dx$.

Echish:

$$\int\sin 4x\cdot\cos 6x dx=\frac{1}{2}\int(\sin 10x-\sin 2x)dx=$$

$$=\frac{1}{2}\cdot\left(-\frac{\cos 10x}{10}+\frac{\cos 2x}{2}\right)+C=-\frac{\cos 10x}{20}+\frac{\cos 2x}{4}+C.$$

2. Aniq integrallar

Aniq integral tushunchasi va uni hisoblash usullari maktab ko'rsida qisman va ma'ruzalarda batafsil o'tilishini hisobga olib, biz aniq integralni geometrik ma'nosi, hisoblash usullariga qisman to'xtalamiz hamda asosiy etiborimizni uning tatbiqlariga qaratamiz.

$f(x)$ funksiya $[a,b]$ kesmada chegeralangan bo'lsin. Quyidagi belgilashlarni kiritamiz:

$$m_k=\inf_{[x_k,x_{k+1}]}\{f(x)\},\qquad M_k=\sup_{[x_k,x_{k+1}]}\{f(x)\},$$

$$\underline{S}=\sum_{k=0}^{n-1}m_k\Delta x_k,\qquad \overline{S}=\sum_{k=0}^{n-1}M_k\Delta x_k.$$

2-ta'rif. *$\underline{S}$ va $\overline{S}$ yig'indilar mos ravishda **Darbuning quyi va yuqori yig'indilari** deb ataladi.*

Darbu yig'indilari quyidagi xossalarga ega.

1^0. *Agar [a,b] kesmaning bo'linish nuqtalariga yangilari ko'shilsa, unda $\underline{S}$ faqat ortishi, $\overline{S}$ esa kamayishi mumkin.*

Demak, $\{\underline{S}\}\uparrow$ va $\{\overline{S}\}\downarrow$.

2^0. *Darbuning ixtiyoriy quyi yig'indisi uning ixtiyoriy yuqori yig'indisidan katta bo'la olmaydi (agar u boshqa bo'linishga mos kelsa ham).*

Agar ushbu:

$$I_*=Sup\{\underline{S}\}\ \text{va}\ I^*=\inf\{\overline{S}\}$$

tengliklar yordamida Darbuning **quyi va yuqori integrallarini** aniqlasak, unda:

$$\underline{S}\le I_*\le I^*\le\overline{S}$$

tengsizliklar o'rinli bo'ladi.

1-teorema. *Aniq integralining mavjud bo'lishi uchun ushbu:*

$$\lim_{\lambda\to 0}(\overline{S}-\underline{S})=0$$

yoki

$$\lim_{\lambda\to 0}\sum_{k=0}^{n-1}\omega_k\Delta x_k=0$$

tenglikning bajarilishi zarur va etarli $(\omega_k=M_k-m_k)$.

Aniq integral uchun mavjudlik teoremasi [a, b] oralig'ida uzluksiz bo'ladigan har bir f (x) funktsiya unga integrallanishi mumkinligini ko'rsatadi.

Bundan buyon integral integral uzluksiz deb qabul qilinadi.

Shunday qilib quyidagiga ega bo'lamiz.

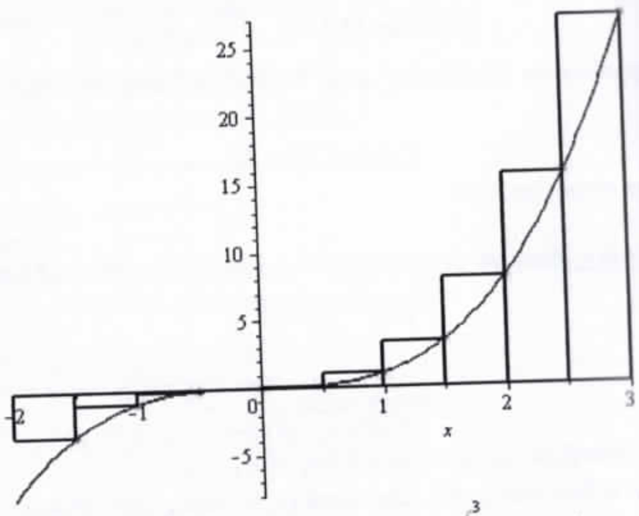
$$\int_a^b f(x)dx=\lim_{\max\Delta x_k\to 0}\sum_{k=1}^{n}f(\xi_k)\cdot\Delta x_k.$$

a va b sonlar integralning kuyi va yukori chegaralari deyiladi, f(x) – integral osti funksiyasi, [a,b] –integrallash soxasi.

$\int f(x)dx$ chekli integrali mavjud bo'lgan f(x) Funksiya [a,b],oraliqda integrallanuvchi deyiladi va yuqoridagi limit [a,b] segmentning bo'linishiga va ξ_k nuqtaning tanlanishiga bogliq emas.

Misol 34. $f(x)=x^3$ funksiya uchun $[-2;3]$ kesmani teng n ta bo'laklarga ajratgan holda, Darbuning yuqori va quyi yig'indilarini toping.

```
> with(Student[Calculus1]):
>
> f(x) := x^3
f := x -> x^3
> RiemannSum(x^3, -2..3, method = upper, output = animation)
```



An animated upper Riemann sum approximation of $\int_{-2}^{3} f(x)\,dx$, where $f(x) = x^3$ and the partition is uniform. The approximate value of the integral is 25.31250000 . Number of subintervals used:10.

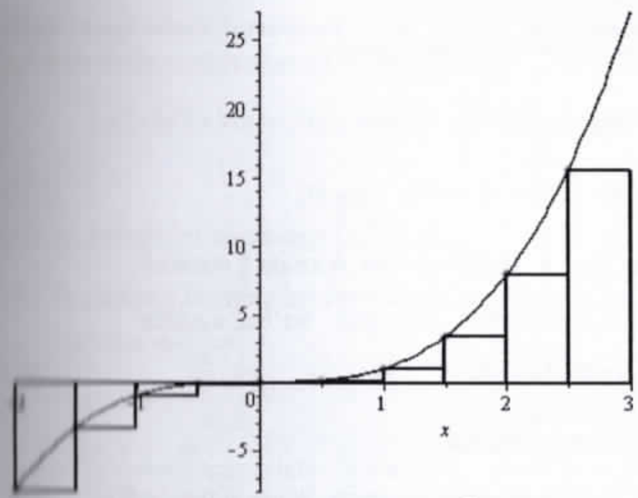
$$> S = \sum_{i=1}^{n} \frac{5}{n} \cdot \left(-2 + \frac{5}{n} \cdot i\right)^3$$

$$S = -\frac{40\,(n+1)}{n} + \frac{150\,(n+1)^2}{n^2} - \frac{150\,(n+1)}{n^2} - \frac{250\,(n+1)^3}{n^3} + \frac{375\,(n+1)^2}{n^3} - \frac{125\,(n+1)}{n^3} + \frac{625}{4}\,\frac{(n+1)^4}{n^4} - \frac{625}{2}\,\frac{(n+1)^3}{n^4} + \frac{625}{4}\,\frac{(n+1)^2}{n^4} + \frac{40}{n}$$

> simplify( );

$$S = \frac{5}{4}\,\frac{13\,n^2 + 70\,n + 25}{n^2}$$

$> RiemannSum\left(x^3, -2..3, method = lower, output = animation\right)$



An animated lower Riemann sum approximation of $\int_{-2}^{3} f(x)\,dx$, where $f(x) = x^3$ and the partition is uniform. The approximate value of the integral is 7.812500000 . Number of subintervals used:10.

$$> s := \sum_{i=1}^{n} \frac{5}{n} \cdot \left(-2 + \frac{5}{n} \cdot (i-1)\right)^3$$

$$s = -\frac{40\,(n+1)}{n} - \frac{450\,(n+1)}{n^2} - \frac{1625\,(n+1)}{n^3} - \frac{1875\,(n+1)}{n^4} + \frac{150\,(n+1)^2}{n^2} + \frac{1125\,(n+1)^2}{n^3} + \frac{8125}{4}\,\frac{(n+1)^2}{n^4} - \frac{250\,(n+1)^3}{n^3} - \frac{1875}{2}\,\frac{(n+1)^3}{n^4} + \frac{625}{4}\,\frac{(n+1)^4}{n^4} + \frac{40}{n} + \frac{300}{n^2} + \frac{750}{n^3} + \frac{625}{n^4}$$

> simplify( );

$$s = \frac{5}{4}\,\frac{13\,n^2 - 70\,n + 25}{n^2}$$

Aniq integrallarning ba'zi xossalarini batafsil tushuntirishlarsiz taqdim etamiz.

Aniq integralning xossalari

1. $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$.

2. $\int_a^b c \cdot f(x)dx = c \cdot \int_a^b f(x)dx, \quad c = const.$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \quad c \in (a,b)$.

4. Agar $f(x) \geq 0$ [a,b]da bo'lsa, u xolda $\int_a^b f(x)dx \geq 0$.

5. Agar $\forall x \in [a,b]$uchun $f(x) \leq g(x)$ bo'lsa, u xolda

a) $\int_a^b f(x)dx \leq \int_a^b g(x)dx$;

b) $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|\, dx$.

6. O'rta qiymat xaqidagi teorema: $\exists \xi \in [a,b]$,mavjudki

$\int_a^b f(x)dx = f(\xi)(b-a)$, bunda $f(x)$ – [a,b] da uzluksiz.

7. $\int_a^a f(x)dx = 0$.

8. $\int_a^b f(x)dx = -\int_b^a f(x)dx.$

2.1. Nyuton-Leybnits formulsi. Agar $f(x)$ funksiya $[a,b]$ kesmada uzluksiz bo'lsa va $F'(x) = f(x)$ tenglik bajarilsa, u xolda

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b \qquad (7)$$

formula o'rinli bo'ladi.
Formulaning isbotida uzluksiz $f(x)$ funksiya uchun ham bajariladigan

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

tenglikdan foydalaniladi.

Misol 35. Integralni xisoblang: $\int_0^2 \frac{1}{x^2+4}dx$.

Echish: Nyutona-Leybnits formulasidan foydalanib, integral jadvalidagi 16 formulaga ko'ra quyidagiga ega bo'lamiz.

$$\int_0^2 \frac{1}{x^2+4}dx = \frac{1}{2}arctg\frac{x}{2}\Big|_0^2 = \frac{1}{2}(arctg\,1 - arctg\,0) = \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}.$$

2.2. Bo'laklab integrallash formulasi. Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ kesmada uzluksiz differensiallanuvchi bo'lsa, u xolda

$$\int_a^b f(x)g'(x)dx = f(x)\cdot g(x)\Big|_a^b - \int_a^b f'(x)g(x)dx \qquad \textbf{(8)}$$

bo'ladi.

Misol 36. Integralni xisoblang: $\int_1^2 x^2 \ln x dx$.

Echish: Yuqoridagi formuladan foydalanib quydagini xosil qilamiz.

$$\int_1^2 x^2 \ln x dx = \begin{Bmatrix} u = \ln x, & du = \frac{1}{x}dx \\ dv = x^2dx, & v = \frac{x^3}{3} \end{Bmatrix} =$$

$$= \frac{x^3}{3}\ln x\Big|_1^2 - \int_1^2 \frac{x^3}{3}\cdot\frac{1}{x}dx = \left(\frac{8}{3}\ln 2 - \frac{1}{3}\cdot\ln 1\right) - \frac{1}{3}\int_1^2 x^2dx =$$

$$= \frac{8}{3}\ln 2 - \frac{1}{3}\cdot\frac{x^3}{3}\Big|_1^2 = \frac{8}{3}\ln 2 - \left(\frac{1}{9}\cdot 8 - \frac{1}{9}\cdot 1\right) = \frac{8}{3}\ln 2 - \frac{7}{9}.$$

Maple dasturida quydagicha bo'ladi.

```
> with(Student[Calculus1]):
> [illegible](x^2 ln(x))
```

$$\int_1^2 x^2 \ln(x)\, dx$$

$$= \frac{8\ln(2)}{3} - \left(\int_1^2 \frac{x^2}{3}\,dx\right) \quad \left[parts, \ln(x), \frac{x^3}{3}\right]$$

$$= \frac{8\ln(2)}{3} - \frac{\left(\int_1^2 x^2\,dx\right)}{3} \quad [constantmultiple]$$

$$= \frac{8\ln(2)}{3} - \frac{7}{9} \quad [power]$$

$$\int_1^2 x^2 \ln(x)\,dx = \frac{8}{3}\ln(2) - \frac{7}{9}$$

2.3. O‘zgaruvchini almashtirish. Agar $\varphi(t)$ funksiya $[\alpha,\beta]$ kesmada uzluksiz differensiallanuvchi va $\varphi(t)\in[\alpha,\beta]$, $a=\varphi(\alpha)$, $b=\varphi(\beta)$ bo‘lib, $f(x)$ funksiya $[a,b]$ kesmada uzluksiz bo‘lsa, unda

$$\int_a^b f(x)dx=\int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt \qquad (9)$$

bo‘ladi.

Misol 37. Integralni xisoblang: $\int_2^3\sqrt{4x-x^2-3}\,dx$.

Echish: Ildiz ostidagi ifoda uchun quydagi almashtirishni bajaramiz.

$4x-x^2-3=1-(x^2-4x+4)=1-(x-2)^2$.

Yangi o‘zgaruvchi kiritamiz. $x-2=\sin t$, u xolda $x=2+\sin t$,

$dx=d(2+\sin t)$ yoki $dx=(2+\sin t)'dt=\cos t\,dt$.

Yangi t o‘zgaruvchi uchun integral chegarasini topib olamiz:

Agar $x_1=2$, bo‘lsa u xolda to $0=\sin t \Rightarrow t_1=0$

Agar $x_2=3$, bo‘lsa u xolda $1=\sin t \Rightarrow t_2=\frac{\pi}{2}$.

Aniq integralda o‘zgaruvchini almashtirish formulasidan foydalanib quydagini xosil qilamiz.

$$\int_2^3\sqrt{4x-x^2-3}dx=\int_2^3\sqrt{1-(x-2)^2}dx=\int_0^{\pi/2}\sqrt{1-\sin^2 t}\cos t dt=$$

$$=\int_0^{\pi/2}\cos^2 t dt=\frac{1}{2}\int_0^{\pi/2}(1+\cos 2t)dt=\frac{1}{2}\left(t+\frac{\sin 2t}{2}\right)\Big|_0^{\pi/2}=$$

$$=\frac{1}{2}\left(\frac{\pi}{2}+\frac{\sin\pi}{2}\right)-\frac{1}{2}\left(0+\frac{\sin 0}{2}\right)=\frac{\pi}{4}.$$

Maple dasturida quydagicha bo‘ladi.

```
> with(Student[Calculus1]) :
> IntTutor(sqrt(4x - x^2 - 3))
```


$$\int_2^3\sqrt{-x^2+4x-3}\,dx$$

$$=\int_0^1\sqrt{-u^2+1}\,du \qquad [change,\ u=x-2,\ u]$$

$$=\int_0^{\frac{\pi}{2}}(-\sin(u1)^2+1)\,du1 \qquad [change,\ u=\sin(u1),\ u1]$$

$$=\int_0^{\frac{\pi}{2}}-\sin(u1)^2\,du1+\int_0^{\frac{\pi}{2}}1\,du1 \qquad [sum]$$

$$=-\left(\int_0^{\frac{\pi}{2}}\sin(u1)^2\,du1\right)+\int_0^{\frac{\pi}{2}}1\,du1 \qquad [constantmultiple]$$

$$=-\left(\int_0^{\frac{\pi}{2}}\left(\frac{1}{2}-\frac{\cos(2u1)}{2}\right)du1\right)+\int_0^{\frac{\pi}{2}}1\,du1 \qquad \left[rewrite,\ \sin(u1)^2=\frac{1}{2}-\frac{\cos(2u1)}{2}\right]$$

$$=-\left(\int_0^{\frac{\pi}{2}}\frac{1}{2}\,du1\right)-\left(\int_0^{\frac{\pi}{2}}-\frac{\cos(2u1)}{2}\,du1\right)+\int_0^{\frac{\pi}{2}}1\,du1 \qquad [sum]$$

$$=-\frac{\pi}{4}-\left(\int_0^{\frac{\pi}{2}}-\frac{\cos(2u1)}{2}\,du1\right)+\int_0^{\frac{\pi}{2}}1\,du1 \qquad [constant]$$

$$=-\frac{\pi}{4}+\frac{\int_0^{\frac{\pi}{2}}\cos(2u1)\,du1}{2}+\int_0^{\frac{\pi}{2}}1\,du1 \qquad [constantmultiple]$$

$$=-\frac{\pi}{4}+\frac{\int_0^{\pi}\frac{\cos(u2)}{2}\,du2}{2}+\int_0^{\frac{\pi}{2}}1\,du1 \qquad [change,\ u2=2u1,\ u2]$$

$$=-\frac{\pi}{4}+\frac{\int_0^{\pi}\cos(u2)\,du2}{4}+\int_0^{\frac{\pi}{2}}1\,du1 \qquad [constantmultiple]$$

$$=-\frac{\pi}{4}+\int_0^{\frac{\pi}{2}}1\,du1 \qquad [cos]$$

$$=\frac{\pi}{4} \qquad [constant]$$

$$\int_2^3\sqrt{-x^2+4x-3}\,dx=\frac{1}{4}\pi$$

Izox. aniq integralda o‘zgaruvchini almashtirish bajarilganda avvalgi x o‘zgaruvchiga qaytish extiyoji yo‘qoladi. Sababi aniq integral bu –aniq bir son. Aniqmas integral esa bu –m’lum bir funksiya. Shuning uchun aniqmas integralda avvalgi x o‘zgaruvchiga qaytish kerak.

b) Ko‘p xollarda $x=\varphi(t)$ almashtirish o‘rniga $t=g(x)$ teskari almashtirish bajarish mumkun.

Quyidagi misolni ko'ramiz.

Misol 38. Integralni xisoblang:. $\int_1^e \frac{dx}{x(5+\ln x)}$

Echish: $t = \ln x$ bo'lsin, u xolda $\frac{1}{x}dx = d\ln x = dt$. bo'ladi.

Agar $x_1 = 1$, bo'lsa $t_1 = \ln 1 = 0$, bo'ladi agar $x_2 = e$,bo'lsa

U xolda $t_2 = \ln e = 1$. bo'ladi.

Quyidagini xosil qilamiz,

$$\int_1^e \frac{dx}{x(5+\ln x)} = \int_1^e \frac{d(\ln x)}{5+\ln x} = \int_0^1 \frac{dt}{5+t} = \ln|t+5|\Big|_0^1 = \ln 6 - \ln 5 =$$

$$= \ln\frac{6}{5} = \ln 1{,}2.$$

2.4. O'rta qiymat haqidagi birinchi teorema.

Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ kesmada chegaralangan va integrallanuvchi bo'lib, $g(x)$ funksiya (a,b) da ishorasini o'zgartirmasa, shunday $\mu \in [m, M]$ $\left(m = \inf_{[a,b]}\{f(x)\},\ M = \sup_{[a,b]}\{f(x)\} \right)$ nuqta topiladiki,

$$\int_a^b f(x)g(x)dx = \mu \cdot \int_a^b g(x)dx \qquad \textbf{(10)}$$

tenglik bajariladi.

2.5 Aniq integral yordamida tekis shaklning yuzasini hisoblash.

a) Dekart koordinatalar sistemasida berilgan shaklning yuzasini hisoblash.

$f(x) \in C[a,b]$ bo'lib, $\forall x \in [a,b]$ uchun $f(x) \geq 0$ tengsizlik bajarilsin va D soha quyidagicha aniqlansin:

$$D = \begin{cases} a \leq x \leq b \\ 0 \leq y \leq f(x) \end{cases} \text{-egri chiziqli trapetsiya.}$$

Unda

$$S = \int_a^b f(x)dx \qquad \textbf{(11)}$$

tenglik o'rinli.

Agar $f_1(x) \in C[a,b]$, $f_2(x) \in C[a,b]$ bo'lib,

$$D = \begin{cases} a \leq x \leq b \\ f_1(x) \leq y \leq f_2(x) \end{cases}$$

bo'lsa, u holda

$$S = \int_a^b [f_2(x) - f_1(x)]dx \qquad \textbf{(12)}$$

bo'ladi.

Misol 39. $y = 2x - x^2$ va $y = -x$. Chiziqlar bilan chegaralangan figuraning yuzasini toping.

Echish: $y = 2x - x^2$ – parabola . Uning uchini va koordinata o'qlari bilan kesishish nuqtalarini topamiz.

$y' = 2 - 2x$; $y' = 0$ yoki $2 - 2x = 0$, $x = 1$

Agar $x_0 = 1$, bo'lsa $y_0 = 2 - 1 = 1$. bo'ladi $M_0(1;1)$ – parabolaning uchi .

$y = 0$ yoki $2x - x^2 = 0$ yoki $x(2-x) = 0$ $x = 0$; $x = 2$.

$y = -x$ -to'g'ri chiziq.

To'g'ri chiziq va parabolaning kesishish nuqtalarining absissasini topamiz.

$2x - x^2 = -x$ yoki $x^2 - 3x = 0$ $x_1 = 0$; $x_2 = 3$. Yuza xisoblash uchun

$$S = \int_0^3 (2x - x^2 - (-x))dx =$$

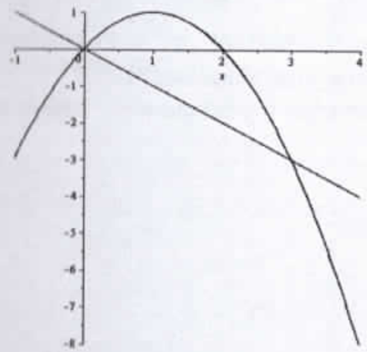
(11) formuladan foydalanamiz. $= \int_0^3 (3x - x^2)dx = \left(3 \cdot \frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^3 =$

$$= \frac{27}{2} - \frac{27}{3} = \frac{9}{2} = 4{,}5 \text{ (kv.birlik).}$$

Maple dasturida quydagicha bo'ladi.

Hosil bo'lgan figurani chizib olamiz. Buning uchun "plot" komandasidan foydalanamiz.

```
> plot([2x-x^2,-x],x=-1..4)
```



To'g'ri chiziq va parabolaning kesishish nuqtalarining absissasini topamiz. Buning uchun "solve" komandasidan foydalanamiz.

$$f := -x^2 + 2x$$

> $g := -x;$

$$g := -x$$

> $solve(\{f = g\}, x);$

$$\{x = 0\}, \{x = 3\}$$

Hosil bo'lgan figuraning yuzasini hisoblaymiz.

> $with(Student[Calculus1]):$

> $IntTutor(2x - x^2 - (-x))$

$$\int_0^3 (-x^2 + 3x)\,dx$$

$$= \int_0^3 -x^2\,dx + \int_0^3 3x\,dx \qquad [sum]$$

$$= -\left(\int_0^3 x^2\,dx\right) + \int_0^3 3x\,dx \qquad [constantmultiple]$$

$$= -9 + \int_0^3 3x\,dx \qquad [power]$$

$$= -9 + 3\left(\int_0^3 x\,dx\right) \qquad [constantmultiple]$$

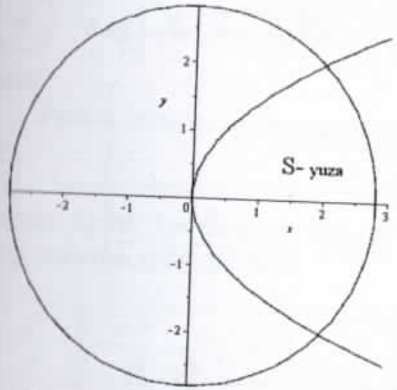
$$= \frac{9}{2} \qquad [power]$$

$$\int_0^3 (-x^2 + 3x)\,dx = \frac{9}{2}$$

Misol 40. $x^2 + y^2 = 8$ aylanani $y^2 = 2x$ parabola bilan kesishdan xosil bo'lgan S soxaning yuzi xisoblansin .
Echish: Quydagi chizmadan foydalanamiz . Maple dasturi yordamida chizib olamiz.

> $with(plots):$

> $implicitplot(\{x^2 + y^2 - 8 = 0, y^2 - 2x = 0\}, x = -3..3, y = -3..3);$



$x^2 + y^2 = 8$ – markazi koordinata boshida bo'lgan va radiusi $R = \sqrt{8}$ bo'lgan aylana

$y^2 = 2x$ – uchi O(0,0) nuqtada bo'lgan parabola.
Aylana va parabolaning kesishish nuqtalarini topamiz.

$$\begin{cases} x^2 + y^2 = 8 \\ y^2 = 2x \end{cases} \Rightarrow 8 - x^2 = 2x \Rightarrow \begin{matrix} x^2 + 2x - 8 = 0 \\ x_1 = -4; x_2 = 2 \end{matrix}$$

$x = -4$ da $y^2 = 2x$ tenglamani qanoatlantirmaydi.
Agar $x = 2$ bo'lsa u xolda $y^2 = 4$ yoki $y_1 = -2$, $y_2 = 2$ bo'ladi.
Buni maple dasturida quydagicha topamiz.

> $s := solve(\{x^2 + y^2 - 8 = 0, y^2 - 2x = 0\}, \{x, y\});$

$$s := \{x = 2, y = 2\}, \{x = 2, y = -2\}, \{x = -4, y = 2\,RootOf(\_Z^2 + 2)\}$$

S sohaning yuzasini hisoblaymiz.

$$S = \int (g_2(y) - g_1(y))\,dy.$$

$$\begin{cases} x^2 + y^2 = 8 \\ x \ge 0 \end{cases} \Rightarrow x = \sqrt{8 - y^2};$$

$$\begin{cases} y^2 = 2x \\ x \ge 0 \end{cases} \Rightarrow x = \frac{y^2}{2}.$$

$$s_1=\int_{-2}^{2}\left(\sqrt{8-y^2}-\frac{y^2}{2}\right)dy=2\int_{0}^{2}\left(\sqrt{8-y^2}-\frac{y^2}{2}\right)dy \quad \textcircled{=}$$

$$\left\{\begin{array}{l} y=\sqrt{8}\sin t;\quad dy=\sqrt{8}\cos t\,dt \\ Agar\ y=0,\to t=0 \\ Agar\ y=2,\to 2=\sqrt{8}\,\text{sint},\ \text{sint}=\frac{\sqrt{2}}{2};\ t=\frac{\pi}{4} \end{array}\right\}$$

$$\textcircled{=}\,2\int_0^2\sqrt{8-y^2}\,dy-\left.\frac{y^3}{3}\right|_0^2=2\int_0^2\sqrt{8-y^2}\,dy-\frac{8}{3}=$$

$$=2\int_0^{\pi/4}\sqrt{8-8\sin^2 t}\cdot\sqrt{8}\cos t\,dt-\frac{8}{3}=16\int_0^{\pi/4}\cos^2 t\,dt-\frac{8}{3}=$$

$$=8\int_0^{\pi/4}(1+\cos 2t)\,dt-\frac{8}{3}=8\left(t+\frac{\sin 2t}{2}\right)\Big|_0^{\pi/4}-\frac{8}{3}=8\left(\frac{\pi}{4}+\frac{1}{2}\right)-\frac{8}{3}=2\pi+\frac{4}{3}.$$

Quydagi soxalarning yuzasini xisoblaymiz.

$$S_{ayl.}=\pi R^2;\quad S_{ayl.}=\pi\cdot(\sqrt{8})^2=8\pi$$

$$S_2=S_{ayl.}-S_1=8\pi-\left(2\pi+\frac{4}{3}\right)=6\pi-\frac{4}{3}.$$

Maple dasturida to'gridan -to'g'ri xisoblasak quydagicha bo'ladi. Bunda " int " komandasidan foydalanamiz.

$> S1 := int\left(\sqrt{8-y^2}-\frac{y^2}{2}, y=-2..2\right)$

$$S1 := \frac{4}{3}+2\pi$$

$> R=\sqrt{8};$

$$2\sqrt{2}=2\sqrt{2}$$

$> S2 := \text{Pi}\cdot R^2;$

$$S2 := 8\pi$$

$> S := S2 - S1;$

$$S := 6\pi-\frac{4}{3}$$

b) Qutb koordinatalar sistemasida berilgan shaklning yuzasini hisoblash.

Agar D soha qutb koordinatalar sistemasida

$$D=\begin{cases}\alpha\le\varphi\le\beta\\ 0\le r\le r(\varphi)\end{cases}$$

ko'rinishida berilgan bo'lib, $r(\varphi)\in C[\alpha,\beta]$ bo'lsa,

$$S=\frac{1}{2}\int_\alpha^\beta r^2(\varphi)\,d\varphi \qquad \textbf{(13)}$$

formula o'rinli bo'ladi.

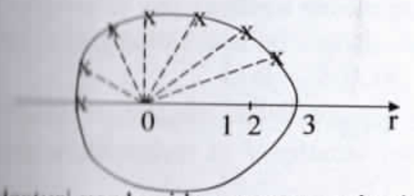
Misol 41. Paskal ulitkasi bilan chegaralangan soxaning yuzi hisoblansin $r=2+\cos\varphi$.

Yechish: (13) formuladan foydalanamiz.Integral chegarasini quyidagicha topib olamiz. Qutib koordinatalarida $r=2+\cos\varphi$ egri chiziqni chizib olamiz. Quydagi jadvalni xosil qilamiz.

Jadval 1

φ	0^o	30^o	45^o	60^o	90^o	120^o	135^o	150^o	180^o
$\cos\varphi$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	−1
$r=2+\cos\varphi$	3	$2+\frac{\sqrt{3}}{2}$	$2+\frac{\sqrt{2}}{2}$	2,5	2	1,5	$2-\frac{\sqrt{3}}{2}$	$2-\frac{\sqrt{2}}{2}$	1

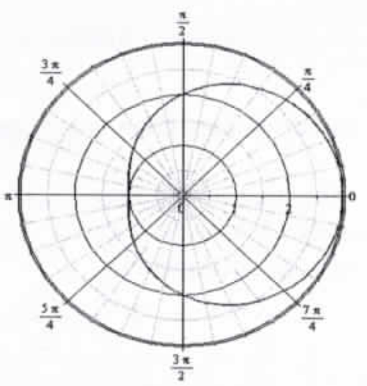
$\cos\varphi$ funksiya – juft funksiya bo'lgani uchun, $r=2+\cos\varphi$ funksiya grafigi gorizontal o'qqa nisbatan $\varphi\in(180^o,360^o]$ qiymatlar uchun simmetrik bo'ladi.Funksiya grafigini $\varphi\in[0;180^o)$da chizish uchun r qutib chizig'ini chizamiz; r o'qlarda jadval 1 da berilgan qiymatlarni belgilaymiz va grafigini chizamiz. Paskal ulitkasi nomli yopiq egri chiziqni xosil qilamiz. (2.1 rasm).



2.1 rasm

Buni Maple dasturi yordamida onson va aniq chizish mumkun.

```
> with(plots):
> polarplot(2+cos(theta), theta = 0..2·Pi, scaling = constrained);
```



Xosil bo‘lgan figuraning yuzasi quydagiga teng.

$$S=\frac{1}{2}\int_0^{2\pi}(2+\cos\phi)^2d\phi=\frac{1}{2}\int_0^{2\pi}\left(4+4\cos\phi+\frac{1+\cos 2\phi}{2}\right)d\phi=$$

$$=\frac{1}{2}\left(4,5\phi+4\sin\phi+\frac{1}{4}\sin 2\phi\right)\Big|_0^{2\pi}=4,5\pi \quad (\kappa v.\text{birlik.})$$

2.6 Aniq integral yordamida yoy uzunligini hisoblash.

a) Dekart koordinatalar sistemasida berilgan yoy uzunligini xisoblash. $f(x)$ funksiya $[a,b]$ kesmada aniqlangan bo‘lsin. Uning grafigi quyidagi

$$\{(x,f(x)):x\in[a,b]\}$$

nuqtalar to‘plamidagi iborat. Shu grafikdagi $A(a,f(a))$ va $B(b,f(b))$ nuqtalar orasidagi $\breve{AB}$ egri chiziq yoyi uzunligi l ni topish talab qilinsin. Agar $f'(x)\in C[a,b]$ bo‘lsa, unda

$$l=\int_a^b\sqrt{1+[f'(x)]^2}\,dx \qquad \textbf{(14)}$$

bo‘ladi.

Agar (14) da $b=x$ desak, $l(x)=\int_a^x\sqrt{1+[f'(x)]^2}\,dx$ bo‘lib,

$$\frac{dl}{dx}=\sqrt{1+[f'(x)]^2}\Rightarrow dl=\sqrt{1+[f'(x)]^2}\,dx.$$

Bu ifodaga **yoy differensiali** deb ataladi.

b) Parametrik ko‘rinishda berilgan egri chiziq yoyining uzunligi hisoblash.

Agar

$$\breve{AB}:\begin{cases}x=\varphi(t)\\ y=\psi(t),\end{cases} \quad \alpha\le t\le\beta$$

bo‘lib, $\varphi'(t)\in C[\alpha,\beta]$ va $\psi'(t)\in C[\alpha,\beta]$ bo‘lsa,

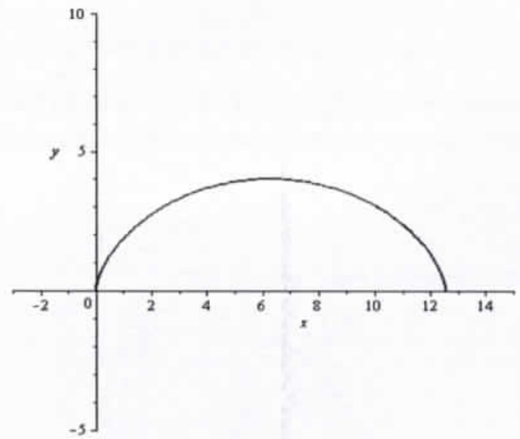
$$l=\int_\alpha^\beta\sqrt{\varphi'(t)^2+[\psi'(t)]^2}\,dt \qquad \textbf{(15)}$$

bo‘ladi.

Misol 42. Sikloidaning bitta arkasi yoyining uzunligini hisoblang.

$\begin{cases}x=a(t-\sin t)\\ y=a(1-\cos t),\end{cases}$ $0\le t\le 2\pi$. Maple dasturi yordamida chizib olamiz

```
> plot([2·(t − sin(t)), 2(1 − cos(t)), t = 0..2·Pi], x = −3..15, y = −5..10);
```



Maple dasturi yordamida a=2 bo‘lgan xol uchun ko‘ramiz. (15) formuladan foydalanamiz.

```
> with(Student[Calculus1]) :
> x := a·(t−sin(t));
                                x := a (t − sin(t))
> d/dt x;
                                a (1 − cos(t))
> y := a·(1−cos(t));
                                y := a (1 − cos(t))
> d/dt y;
                                a sin(t)
> IntTutor(2·√((1−cos(t))^2 + (sin(t))^2))
```

$$\int_0^{\pi} 2\sqrt{(1-\cos(t))^2+\sin(t)^2}\,dt$$

$$= \int_0^{\pi} 2\sqrt{2-2\cos(t)}\,dt \qquad \left[rewrite, 2\sqrt{(1-\cos(t))^2+\sin(t)^2} = 2\sqrt{2-2\cos(t)}\right]$$

$$= 2\left(\int_0^{\pi}\sqrt{2-2\cos(t)}\,dt\right) \qquad [constantmultiple]$$

$$= 2\left(\int_0^{\infty}\frac{4u}{(u^2+1)^{3/2}}\,du\right) \qquad \left[change, u=\tan\left(\frac{t}{2}\right), u\right]$$

$$= 8\left(\int_0^{\infty}\frac{u}{(u^2+1)^{3/2}}\,du\right) \qquad [constantmultiple]$$

$$= 8\left(\int_1^{\infty}\frac{1}{u^{3/2}}\,du\right) \qquad [change, u^2+1=u1^2, u1]$$

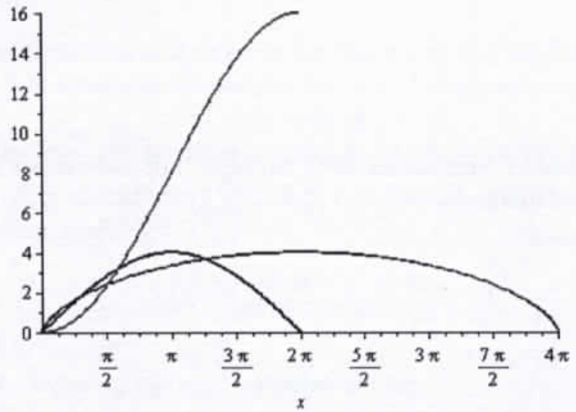
$$= 8 \qquad [power]$$

$$I= \int_0^{\pi} 2\sqrt{(1-\cos(t))^2+\sin(t)^2}\,dt = 8$$

Bu erdan yoy uzunligi L=2*I =16 ga teng bo‘ladi

Bu misolni “Maple” dasturida quydagicha usulda xam echish mumkun.

```
> with(Student[Calculus1]) :
> ArcLength([2·(x − sin(x)), 2·(1−cos(x))], output = plot, x = 0 ..2·Pi);
```



The arc length of $f(x) = [2x - 2\sin(x),\ 2 - 2\cos(x)]$ on the interval $[0, 2\pi]$. The coordinate system is Cartesian.

```
> ArcLength([2·(x − sin(x)), 2·(1−cos(x))], x = 0 ..2·Pi, output = integral);
```


$$\int_0^{2\pi} 2\sqrt{\cos(x)^2+\sin(x)^2-2\cos(x)+1}\,dx$$

```
> ArcLength([2·(x − sin(x)), 2·(1−cos(x))], x = 0 ..2·Pi);
```

c) Qutb koordinatalar sistemasida berilgan egri chiziq yoyining uzunligi hisoblash.

Agar

$$\overset{\smile}{AB}:\begin{cases}\alpha\le\varphi\le\beta,\\ r=r(\varphi)\end{cases}$$

bo‘lib, $r'(\varphi)\in C[\alpha,\beta]$ bo‘lsa, unda

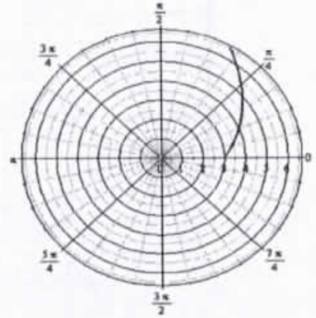
$$l=\int_\alpha^\beta\sqrt{r^2(\varphi)+[r'(\varphi)]^2}\,d\varphi$$ **(16)** formula o‘rinli bo‘ladi.

Misol 43. Egri chiziqning uzunligi topilsin.

$r=3\cdot e^{\frac{3\varphi}{4}},\ 0\le\varphi\le\frac{\pi}{3}.$

Maple dasturi yordamida chizib olamiz.

```
> with(plots) :
> polarplot(3·e^(3·(theta)/4), theta = 0 .. Pi/3, scaling = constrained);
```



Echish: $r=3\cdot e^{\frac{3\varphi}{4}}$ egri chiziq qutb koordinatalar sistemasida berilgan. (16) formuladan foydalanamiz.

$r'(\varphi)$ ni topib olamiz.

$$r'=\left(3\cdot e^{\frac{3\varphi}{4}}\right)'=3\cdot e^{\frac{3\varphi}{4}}\cdot\frac{3}{4}=\frac{9}{4}\cdot e^{\frac{3\varphi}{4}}.$$

$$r^2+(r')^2=9\cdot\left(r^{\frac{3\varphi}{4}}\right)^2+\frac{81}{16}\cdot\left(r^{\frac{3\varphi}{4}}\right)^2=\frac{225}{16}\cdot\left(r^{\frac{3\varphi}{4}}\right)^2$$

$$L=\int_0^{\pi/3}\sqrt{\frac{225}{16}\cdot\left(r^{\frac{3\varphi}{4}}\right)^2}\,d\varphi=\frac{15}{4}\int_0^{\pi/3}e^{\frac{3}{4}\varphi}d\varphi=\frac{15}{4}\cdot\frac{4}{3}e^{\frac{3}{4}\varphi}\Big|_0^{\pi/3}=$$

$= 5\cdot e^{\frac{3}{4}\cdot\frac{\pi}{3}} - 5e^{0} = 5\cdot(e^{\pi/4}-1)$ (birlik).

2.7. Aylanma sirtning yuzasi.

Aytaylik, $f(x)\in C[a,b]$ bo'lib, $f(x)\geq 0$ bo'lsin. $\breve{AB}$ yoyni OX o'qi atrofida aylantiramiz va **aylanma sirtni** hosil qilamiz. Agar $f'(x)\in C[a,b]$ bo'lsa, unda shu aylanma sirtning yuzasi ushbu

$$S = 2\pi\int_a^b f(x)\cdot\sqrt{1+[f'(x)]^2}\,dx \qquad \textbf{(17)}$$

formula yordamida hisoblanadi.

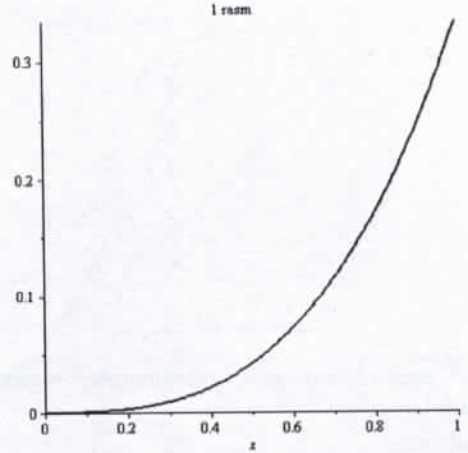
Misol 44. OX o'qi atrofida $3y-x^3=0,\ 0\leq x\leq 1$.egri chiziq aylanishidan xosil bo'lgan figura sirtining yuzasi topilsin.
Maple dasturi yordamida chizib olamiz.

> $restart : with(plots) : with(plottools) : y := x \to \frac{1}{3}x\char`^3 :$

> $F := plot(y(x), x = 0..1, thickness = 2) :$

> $plots[display]([F], scaling = unconstrained, title = \text{" 1 rasm"});$

1 rasm



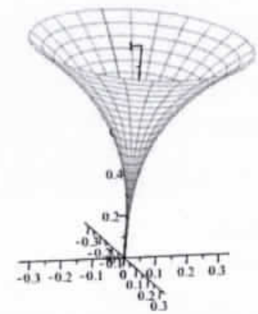
> print(`aylanma figurani chizamiz:`);

aylanma figurani chizamiz:

> $F1 := plot3d\left(\frac{1}{3}h\char`^3, a=-Pi..Pi, h=0..1, coords=cylindrical, axes=normal\right) :$

> $plots[display]([F1], scaling = unconstrained, style = hidden, title = \text{"2 rasm"});$

2 rasm



Echish: $3y-x^3=0$ yoki $y=\frac{1}{3}x^3$

$$y'=\left(\frac{1}{3}x^3\right)'=x^2$$

(17) formuladan foydalanamiz

$$Q_X = 2\pi\int_0^1\frac{1}{3}x^3\cdot\sqrt{1+(x^2)^2}\,dx = \frac{2\pi}{3\cdot 4}\int_0^1(1+x^4)^{\frac{1}{2}}d(1+x^4) =$$

$$= \frac{\pi}{6}\cdot\frac{2}{3}(1+x^4)^{\frac{3}{2}}\Big|_0^1 = \frac{2}{9}\pi(2^{\frac{3}{2}}-1) = \frac{2}{9}\pi(2\sqrt{2}-1).$$

Buni maple dasturida quydagicha topamiz

> $with(Student[Calculus1]) :$

> $IntTutor\left(2\cdot Pi\cdot\frac{x^3}{3}\cdot\sqrt{1+(x^2)^2}\right)$

$$\int_0^1 \frac{2\pi x^3\sqrt{x^4+1}}{3}\,dx$$

$$= \frac{2\pi\left(\int_0^1 x^3\sqrt{x^4+1}\,dx\right)}{3} \qquad [constantmultiple]$$

$$= \frac{2\pi\left(\int_1^{\sqrt{2}}\frac{u^2}{2}\,du\right)}{3} \qquad [change, x^4+1=u^2, u]$$

$$= \frac{\pi\left(\int_1^{\sqrt{2}}u^2\,du\right)}{3} \qquad [constantmultiple]$$

$$= \frac{2\pi\left(\frac{\sqrt{2}}{3}-\frac{1}{6}\right)}{3} \qquad [power]$$

$$\int_0^1\frac{2}{3}\pi x^3\sqrt{x^4+1}\,dx = \frac{2}{3}\pi\left(\frac{1}{3}\sqrt{2}-\frac{1}{6}\right)$$

2.8 Aniq integral yordamida hajm hisoblash.

Faraz qilaylik, bizga biror T jism berilgan bo‘lib, uning OY uqiga parallel bo‘lgan kesimlarining yuzasi ma’lum bo‘lsin. Bu yuza x o‘zgaruvchining funksiyasi bo‘ladi, uni $S=S(x)$ deb belgilaylik. Agar $S(x)\in C[a,b]$ bo‘lsa, unda T jismning xajmi V ushbu

$$V=\int_a^b S(x)dx \qquad \textbf{(18)}$$

formula yordamida hisoblanadi.

Natija. (Aylanma jismning hajmi). *Ushbu*

$$D=\begin{cases} a\le x\le b \\ 0\le y\le f(x)\end{cases}$$

egri chiziqli trapetsiyani OX o‘*qi atrofida aylantirishdan hosil bo‘lgan aylanma jismning hajmi*

$$V=\pi\int_a^b [f(x)]^2dx \qquad \textbf{\textit{(19)}}$$

formula yordamida hisoblanadi.

Misol.45. $y=chx$, $0\le x\le 1$ egri chiziqni OX o‘qi atrofida aylanishdan xosil bo‘lgan figuraning xajmini xisoblang.

Echish: $y=chx=\frac{e^x+e^{-x}}{2}$ Egri chiziq zanjir chizig‘i deyiladi. Buning grafigi

1 rasmda tasvirlangan. OX o‘qi atrofida aylanishdan xosil bo‘lgan figuraning xajmi 2 rasmda tasvirlangan. 19. formula yordamida xisoblaymiz.

Maple dasturi yordamida chizib olamiz.

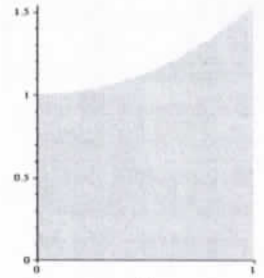
```
> restart : with(plots) : with(plottools) :
> y(x) := (e^x + e^(-x))/2;
```

$$y:=x\to \frac{1}{2}e^x+\frac{1}{2}e^{-x}$$

```
> print( `OXY uqida chizamiz: `);
```

OXY uqida chizamiz:

```
> Y := plot(y(x), x = 0..1, color = RED, thickness = 2) :
> YF := plot(y(x), x = 0..1, filled = true, color = GREEN, thickness = 2) :
> plots[display]([Y, YF], xtickmarks = 2, scaling = constrained);
```

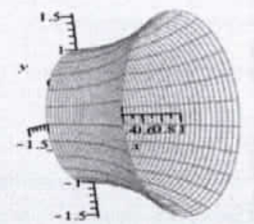



(1. rasm)

```
> Theta := Pi :
> print( `x uqi atrofida aylantiramiz: `);
```

x uqi atrofida aylantiramiz:

```
> a1 := plot3d([e^h + e^(-h)]/2, a = -Pi..Theta, h = 0..1, coords = cylindrical, style = HIDDEN) :
> plots[display]([a1], style = hidden, scaling = constrained, orientation = [-45, 50], labels = [z, y, x], axes = normal);
```



(2. rasm)

$$V_X=\pi\int_0^1 ch^2xdx=\pi\int_0^1\left(\frac{e^x+e^{-x}}{2}\right)^2dx=\frac{\pi}{4}\int_0^1(e^{2x}+2\cdot e^x\cdot e^{-x}+e^{-2x})dx=$$

$$=\frac{\pi}{4}\int_0^1(e^{2x}+2+e^{-2x})dx=\frac{\pi}{4}\cdot\left(\frac{e^{2x}}{2}+2x-\frac{e^{-2x}}{2}\right)\Big|_0^1=$$

$$=\frac{\pi}{4}\left(\frac{e^2}{2}+2-\frac{e^{-2}}{2}\right)-\frac{\pi}{4}\left(\frac{e^0}{2}+0-\frac{e^0}{2}\right)=\frac{\pi}{4}\left(2+\frac{e^2-e^{-2}}{2}\right)=\frac{\pi}{4}(2-sh2)\cdot$$

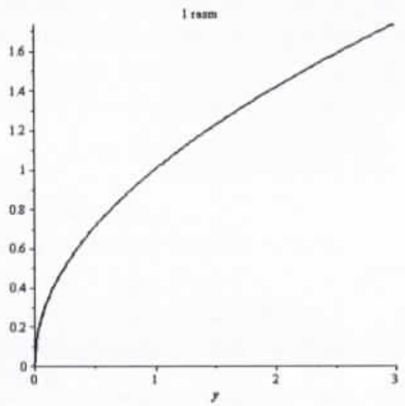
Misol.46. Radiusi R, va balandligi – H ga.teng bo‘lgan paraboloidning xajmi topilsin.

Maple dasturi yordamida chizib olamiz.

```
> restart : with(plots) : with(plottools) : x := y->y^(1/2) :
```

> $F := plot(x(y), y = 0..3, thickness = 2)$:

> $plots[display]([F], scaling = unconstrained, title = \text{" 1 rasm"})$;

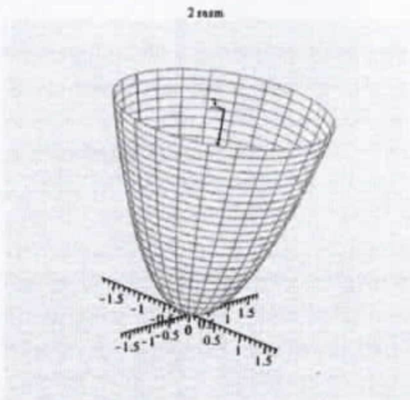


> $print(\text{`aylanma figurani chizamiz:`})$;

aylanma figurani chizamiz:

> $F1 := plot3d\left(h\hat{}\frac{1}{2}, a = -Pi..Pi, h = 0..3, coords = cylindrical, axes = normal\right)$:

> $plots[display]([F1], scaling = unconstrained, style = hidden, title = \text{"2 rasm"})$;



Echish: Bu paroboloid OY o'qi atrofida $y = kx^2$ parabolani aylantirishdan xosil qilinadi, $0 \le y \le H$ (1 rasm,2 rasm), bunda k quydagicha topiladi.

Agar $x = R$, bo'lsa u xolda $y = H$, quydagini xosil qilamiz.

$$H = kR^2 \Rightarrow k = \frac{H}{R^2} \Rightarrow y = \frac{H}{R^2} \cdot x^2.$$

Natijada xajm quydagicha topiladi.

$$V_y = \pi \cdot \int_0^H x^2(y)dy.$$

Agar $y = \frac{H}{R^2} \cdot x^2$, desak u xolda $x^2 = \frac{R^2}{H} \cdot y$ bundan

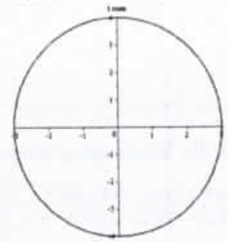
$$V_y = \pi \cdot \int_0^H \frac{R^2}{H} \cdot y dy = \pi \cdot \frac{R^2}{H} \cdot \frac{y^2}{2}\bigg|_0^H = \frac{\pi \cdot R^2}{H} \cdot \frac{H^2}{2} = \frac{1}{2}\pi R^2 H \text{ (kub birlik).}$$

Misol.47. $x = 3\cos t$, $y = 4\sin t$ egri chiziqni OX o'qi atrofida aylanishidan xosil bo'lgan figuraning xajmi topilsin .
Maple dasturi yordamida chizib olamiz.

> $restart : with(plots) : with(plottools)$:

> $F := plot([3 \cdot \cos(t), 4 \cdot \sin(t), t = 0..2 \cdot Pi], thickness = 2)$:

> $plots[display]([F], scaling = unconstrained, title = \text{" 1 rasm"})$;

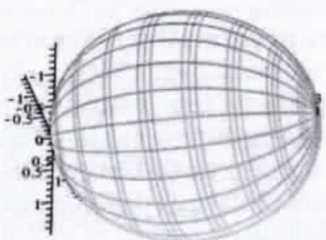


> $print(\text{`aylanma figurani chizamiz:`})$;

aylanma figurani chizamiz:

> $F1 := plot3d(3 \cdot \cos(h), a = -Pi...Pi, h = 0..4, coords = spherical, axes = normal)$:

> $plots[display]([F1], scaling = unconstrained, style = hidden, title = \text{"2 rasm"})$;



Echish: Bu egri chiziq parametrik ko'rinishda berilgan bo'lib ellips (1 rasm) xosil qiladi. Agar OX o'qi atrofida aylantirsak ellipsoid xosil bo'ladi (2 rasm). V_X xajmni topamiz

19. Formula.

$$V_X = \pi\int_a^b y^2(x)dx.$$

Agar $x=-3$ bo'lsa, u xolda $3\cos t=-3,\ \cos t=-1,\ t_1=\pi$.
Agar $x=3$ bo'lsa, u xolda $3\cos t=3,\ \cos t=1,\ t_2=0$.

$$V_X=\pi\int_{-3}^{3} y^2dx=\pi\int_{\pi}^{0}(4\sin t)^2 d(3\cos t)=\pi\cdot 16\cdot 3\cdot\int_{\pi}^{0}(\sin t)^2 d(\cos t)=$$
$$=48\pi\int_{\pi}^{0}(1-\cos^2 t)d(\cos t)=48\pi\left(\cos t-\frac{\cos^3 t}{3}\right)\Big|_{\pi}^{0}=$$
$$=48\pi\left(-1+\frac{1}{3}\right)-48\pi\left(\cos\pi-\frac{\cos^3\pi}{3}\right)=48\pi\cdot\frac{2}{3}-48\pi\left(-1+\frac{1}{3}\right)=$$
$$=\frac{4}{3}\cdot 48\pi=64\pi \text{ (kub.birlik.).}$$

2.9. O'zgaruvchi kuchning bajargan ishi.

OX o'qida shu o'q bo'ylab biror jism $F=F(x)$ kuch ta'sirida harakat qilayotgan bo'lsin. Agar $F(x)\in C[a,b]$ bo'lsa, $F=F(x)$ kuch ta'sirida jismni a nuqtadan b nuqtaga o'tkazishda bajarilgan ish ushbu

$$A=\int_a^b F(x)dx \qquad \textbf{(20)}$$

formula yordamida hisoblanadi.

2.10. Statik moment. Og'irlik markazi.

Aytaylik ,m massaga ega bo'lgan M(x,y)-material nuqta berilgan bo'lsin. my va mx ko'paytmalarga mos ravishda berilgan nuqtaning **OX va OY o'qlarga nisbatan statik momentlari** deb ataladi.

Egri chiziqning OX va OY o'qlarga nisbatan **statik momentlari** M_X va M_y lar ham shu kabi aniqlanadi hamda

$$M_x=\int_0^l ydl,\ M_y=\int_0^l xdl \qquad \textbf{(21)}$$

formulalar yordamida hisoblanadi. Bu erda $dl=\sqrt{(dx)^2+(dy)^2}$ -yoy differensiali, l esa berilgan egri chiziq uzunligi.

Berilgan egri chiziq og'irlik markazining koordinatalari esa ushbu

$$\bar{x}=\frac{M_y}{l},\ \bar{y}=\frac{M_x}{l} \qquad \textbf{(22)}$$

formulalar yordmida hisoblanadi.

2.11. Geometrik figuralarning statik momentlari va og'irlik markazi.

Agar geometrik figura

$$D=\begin{cases} a\le x\le b\\ 0\le y\le f(x)\end{cases}$$

egri chiziqli trapetsiyadan iborat bo'lsa, unda

$$M_x=\frac{1}{2}\int_a^b y^2dx,\ M_y=\frac{1}{2}\int_a^b xydx \qquad \textbf{(23)}$$

va

$$\left(\bar{x},\bar{y}\right)=\left(\frac{M_y}{S},\frac{M_x}{S}\right) \qquad \textbf{(24)}$$

bo'ladi. Bu erda $S=\int_a^b y(x)dx$ -trapetsiyaning yuzi.

2.12.Elliptik integrallar.

5-Ta'rif. *Ushbu*

$$F(k,\varphi)=\int_0^{\varphi}\frac{dx}{\sqrt{1-k^2\sin^2 x}} \qquad \textbf{(25)}$$

$$E(k,\varphi)=\int_0^{\varphi}\sqrt{1-k^2\sin^2 x}dx \qquad \textbf{(26)}$$

ko'rinishdagi integrallar I va II-tipdagi ***elliptik integrallarning Lejandr formasi*** *deb ataladi.*

(25) va (26)-integral ostidagi funksiyalarning boshlang'ich funksiyalari elementar funksiyalar yordamida ifodalanmaydi. Shuning uchun ham ularning qiymatlarini hisoblash uchun maxsus jadvallar yaratilgan.

Agar (25) va (26)-integrallarda $\varphi=\frac{\pi}{2}$ bo'lsa, u holda bunday integrallar **to'liq elliptik integrallar** deb ataladi va ular $F(k), E(k)$ kabi belgilanadi.

Demak,

$$F(k)=\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{1-k^2\sin^2 x}} \qquad \textbf{(27)}$$

$$E(k)=\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2 x}\,dx \qquad \textbf{(28)}$$

To'liq elliptik integrallarning qiymatlari ham maxsus jadvallar yordamida hisoblanadi.

Misol.48. Ushbu $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellips yoyining uzunligi hisoblansin.

◁Ellipsni parametrik ko'rinishida $\begin{cases} x=a\sin t \\ y=b\sin t, 0\le t\le 2\pi \end{cases}$ kabi ifodalab olamiz.

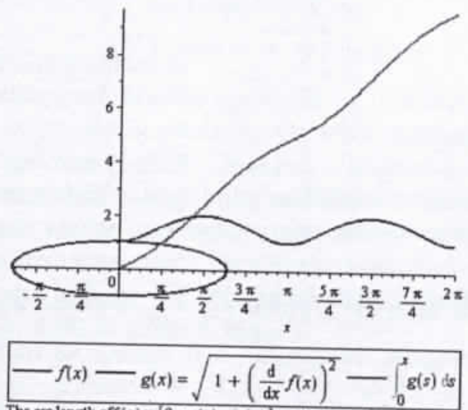
Unda

$$l=4l_1=4\cdot\int_0^{\frac{\pi}{2}}\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt=4\int_0^{\frac{\pi}{2}}\sqrt{a^2\cos^2 t+b^2\sin^2 t}\,dt=4\int_0^{\frac{\pi}{2}}\sqrt{a^2\left(1-\sin^2 t\right)+b^2\sin^2 t}\,dt=$$

$$=4a\int_0^{\frac{\pi}{2}}\sqrt{1-\frac{a^2-b^2}{a^2}\sin^2 t}\,dt=4aE\left(\frac{\sqrt{a^2-b^2}}{a}\right)=4aE(\varepsilon)$$ bu erda $\varepsilon=\frac{\sqrt{a^2-b^2}}{a}$-ellipsning ekssentrisiteti.▷

> $with(Student[Calculus1])$:

> $ArcLength([\,2\cos(x), \sin(x)\,], output=plot, x=0..2\pi)$



The arc length of $f(x)=[2\cos(x), \sin(x)]$ on the interval $[0, 2\pi]$. The coordinate system is Cartesian.

> $ArcLength([\,2\cos(x), \sin(x)\,], x=0..2\pi, output=integral)$

$$\int_0^{2\pi}\sqrt{4\sin(x)^2+\cos(x)^2}\,dx$$

> $ArcLength([\,2\cos(x), \sin(x)\,], x=0..2\pi)$

$$8\,\mathrm{EllipticE}\left(\frac{1}{2}\sqrt{3}\right)$$

> $evalf\left(8\,\mathrm{EllipticE}\left(\frac{1}{2}\sqrt{3}\right)\right)$

$$9.688448224$$

3. Xosmas integrallar

Aniq integralning tarifini kiritganda va uning xossalari integrallash metodlarini ko'rganimizda f(x) funksiya [a,b] oraliqda uzluksiz va chekli deb faraz qildik.

Umuman olganda f(x) funksiya [a,b] oraliqda uzluksiz va chekli bo'lishi shart emas. Bu xolda biz xosmas integral tushunchasiga kelamiz.

3.1. Birinchi tur xosmas integrallar(integrallash chegarasi cheksiz).

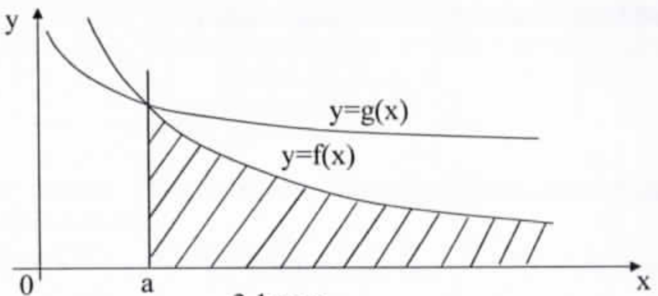
$y=f(x)$ funksiya $[a;+\infty)$.oraliqda uzluksiz bo'lsin.

f(x) funksiyaning $[a;+\infty)$ oraliqdagi xosmas integrali deb quyidagi limitga aytiladi $\lim_{A\to\infty}\int_a^A f(x)dx$:

$$\int_a^{+\infty}f(x)dx=\lim_{A\to+\infty}\int_a^A f(x)dx. \qquad \textbf{(29)}$$

Agar yuqoridagi limit mavjud va chekli bo'lsa u xolda xosmas integral yaqinlashuvchi bo'ladi aks xolda o'zoqlashuvchi deyiladi.

Agar $[a,+\infty)$ oraliqda $f(x)>0$ va $\int_a^{+\infty}f(x)dx<\infty$ bo'lsa, u xolda $[a;+\infty)$ cheksiz intervalda $y=f(x)$egri chiziq bilan va $x=a$ to'gri chiziq bilan chegaralangan cheksiz egri chiziqli trapetsiyaning yuzini xosil qilamiz.



3.1rasm

$(-\infty; b]$ intervaldagi xosmas integral xam xuddi shunday aniqlanadi.

$$\int_{-\infty}^{b} f(x)dx = \lim_{B\to-\infty}\int_{B}^{b} f(x)dx, \qquad \textbf{(30)}$$

$(-\infty; +\infty)$ intervalda esa quyidagi formula orqali topiladi.

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx, \qquad \textbf{(31)}$$

Bu erda s – ixtiyoriy xaqiyqi son.

Agarda 2 ta egri chiziqli trapetsiyalarni ko'radigan bo'lsak 3.1 rasm, u xolda bu funksiyalarning mos ravishda xosmas integralini chekli yoki cheksizligi $y=f(x)$ va $y=g(x)$ funksiyalarning $x\to+\infty$ xususiyatiga bogliq.

Masalan. $\int_{1}^{+\infty}\frac{dx}{x^{\alpha}}$ integral $\alpha>1$da yaqinlashadi va $\alpha\le 1$ da uzoqlashadi.

$\int_{1}^{A}\frac{1}{x^{\alpha}}dx$, integralni $A\to+\infty$ da xisoblaymiz.

Agar $f(x)=\frac{1}{x}$, bo'lsa u xolda $A\to\infty$da $\int_{1}^{A}\frac{1}{x}dx = \ln|x|\Big|_{1}^{A} = \ln A - \ln 1 = \ln A \to +\infty$ bo'ladi demak $\int_{1}^{+\infty}\frac{1}{x}dx$ – uzoqlashuvchi, Bundan mos ravishda egri chiziqli tropetsiyaning yuzasi chegaralanmagan deb xulosa qilishimiz mumkun.

Quyidagi integralni ko'ramiz. $\int_{1}^{A}\frac{1}{x^2}dx = -\frac{1}{x}\Big|_{1}^{A} = -\frac{1}{A}+1$

$\int_{1}^{+\infty}\frac{1}{x^2}dx = \lim_{A\to+\infty}\left(1-\frac{1}{A}\right)=1$ – xosmas integral yaqinlashuvchi demak $y=\frac{1}{x^2}$, $x=1$ chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuzasi $[1;+\infty)$ intervalda chekli va uning yuzasi 1 ga teng.

Buni maple dasturida quydagicha topamiz

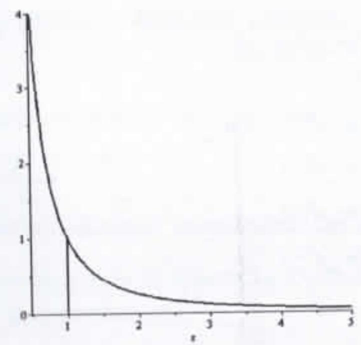
```
> Int(1/x^2, x = 1 ..infinity) = int(1/x^2, x = 1 ..infinity);
```

$$\int_{1}^{\infty}\frac{1}{x^2}\,dx = 1$$

```
> f := x -> 1/x^2;
```

$$f := x \to \frac{1}{x^2}$$

```
> p1 := plot(f(x), x = .5 ..5, thickness = 2) :
> p2 := plottools[line]([1, 0], [1, f(1)], color = blue, thickness = 2) :
> plots[display](p1, p2);
```




```
> ad := int(f(x), x);
```

$$ad := -\frac{1}{x}$$

```
> inttob := subs(x = b, ad) - subs(x = 1, ad);
```

$$inttob := -\frac{1}{b} + 1$$

```
> Int(1/x^2, x = 1 ..infinity) = limit(inttob, b = infinity);
```

$$\int_{1}^{\infty}\frac{1}{x^2}\,dx = 1$$

Misol.49. $\int_{-\infty}^{0} x\cdot e^{x}dx$. Xosmas integralni yaqinlashuvchilikka tekshiring.

Echish: Quyi chegarasi cheksiz bulgan xosmas integral tarifidan va bo'laklab integrallash formulasidan foydalanib quyidagini topamiz.

$$\int_{-\infty}^{0} x\cdot e^{x}dx = \lim_{\beta\to-\infty}\int_{\beta}^{0} x\cdot e^{x}dx = \begin{Bmatrix} u=x, & du=dx \\ dv=e^{x}dx, & v=e^{x}\end{Bmatrix} =$$

$$= \lim_{\beta\to-\infty}\left(x\cdot e^{x}\Big|_{\beta}^{0} - \int_{\beta}^{0} e^{x}dx\right) = \lim_{\beta\to-\infty}(x\cdot e^{x}-e^{x})\Big|_{\beta}^{0} =$$

$$=\lim_{\beta\to-\infty}(0-\beta\cdot e^{\beta}-e^{0}+e^{\beta})=\lim_{\beta\to-\infty}\left(-\frac{\beta}{e^{-\beta}}-1+\frac{1}{e^{-\beta}}\right)=-1.$$

Maple dasturida quydagisha bo‘ladi

```
> Int(x·exp(x), x=-infinity..0) = int(x·exp(x), x=-infinity..0);
```

$$\int_{-\infty}^{0} x\,e^{x}\,dx=-1$$

```
> inttoa := int(x·exp(x), x = a..0);
```

$$inttoa := -e^{a}\,a+e^{a}-1$$

```
> limit(inttoa, a=-infinity);
```

$$-1$$

Demak xosmas integral yaqinlashuvchi.

Bu erda limit-- Maple dasturida “Limits Tutor” komandasi yordamida quydagicha hisoblanadi.

$$\lim_{a\to-\infty}(-e^{a}a+e^{a}-1)$$
$$=\lim_{a\to-\infty}-e^{a}a+\lim_{a\to-\infty}e^{a}+\lim_{a\to-\infty}-1$$
$$=\lim_{a\to-\infty}-e^{a}a+\lim_{a\to-\infty}e^{a}-1$$
$$=-\lim_{a\to-\infty}e^{a}a+\lim_{a\to-\infty}e^{a}-1$$
$$=-\lim_{a\to-\infty}-e^{a}+\lim_{a\to-\infty}e^{a}-1$$
$$=2\lim_{a\to-\infty}e^{a}-1$$
$$=-1$$

Misol.50. Xosmas integralni xisoblang.

$$\int_{-\infty}^{+\infty}\frac{dx}{x^2+4x+9}.$$

Echish: Chegaralari cheksiz bo‘lgan xosmas integral ta’rifidan foydalanamiz. $c=-2$ deb olamiz va quyidagiga ega bo‘lamiz.

$$\int_{-\infty}^{+\infty}\frac{dx}{x^2+4x+9}=\int_{-\infty}^{+\infty}\frac{d(x+2)}{(x+2)^2+5}=\lim_{B\to-\infty}\int_{B}^{-2}\frac{d(x+2)}{(x+2)^2+5}+\lim_{A\to+\infty}\int_{-2}^{A}\frac{d(x+2)}{(x+2)^2+5}=$$

$$=\lim_{B\to-\infty}\frac{1}{\sqrt5}arctg\frac{(x+2)}{\sqrt5}\Big|_{B}^{-2}+\lim_{A\to+\infty}\frac{1}{\sqrt5}arctg\frac{(x+2)}{\sqrt5}\Big|_{-2}^{A}=$$

$$=\frac{1}{\sqrt5}\lim_{B\to-\infty}\left(arctg0-arctg\frac{B+2}{\sqrt5}\right)+\frac{1}{\sqrt5}\lim_{A\to+\infty}\left(arctg\frac{A+2}{\sqrt5}-arctg0\right)=$$

$$=\frac{1}{\sqrt5}\left(0+\frac{\pi}{2}\right)+\frac{1}{\sqrt5}\left(\frac{\pi}{2}-0\right)=\frac{\pi}{\sqrt5}=\frac{\sqrt5}{5}\pi.$$

Demak xosmas integral yaqinlashuvchi.

Taqqoslash alomati. $[a;+\infty)$ oraliqda f(x) va g(x) funksiyalar uzluksiz va $0\le f(x)\le g(x)$ bo‘lsin. Agar $\int_a^{+\infty}g(x)dx$ integral yaqinlashsa, u xolda $\int_a^{+\infty}f(x)dx$ integral xam yaqinlashadi. Agar $\int_a^{+\infty}f(x)dx$ integral uzoqlashsa, u xolda $\int_a^{+\infty}g(x)dx$ integral xam uzoqlashadi.

Izox. Taqqoslash alomati $\int_{-\infty}^{b}f(x)dx=\lim_{B\to-\infty}\int_{B}^{b}f(x)dx,$ va $\int_{-\infty}^{+\infty}f(x)dx=\int_{-\infty}^{c}f(x)dx+\int_{c}^{+\infty}f(x)dx,$ xosmas integallar uchun xam o‘rinli.

Misol. 51. $\int_1^{+\infty}\frac{xdx}{\sqrt{(x^2+3)^4}}$ xosmas integralni yaqinlashuvchilikka tekshiring.

Echish: Taqqoslash alomatidan foydalanamiz.

$$\frac{x}{\sqrt{(x^2+3)^4}}<\frac{x}{\sqrt{x^8}}=\frac{x}{x^4}=\frac{1}{x^3}\quad(1\le x<+\infty).$$

Bizga $\int_1^{+\infty}\frac{dx}{x^3}$ xosmas integral yaqinlashuvchi ekanligi ma’lum. $\alpha=3$ (Buning isbotini o‘quvchiga xovola qilamiz). Demak taqqoslash alomatiga ko‘ra xosmas integral yaqinlashuvchi.

3.2 Ikkinchi tur xosmas integrallar (chegaralanmagan funksiyaning aniq integrali).

$y=f(x)$ funksiya [a,b] oraliqda a va b, yoki $c\in(a,b)$,nuqtada II tur uzilishga ega bo‘lsin. U xolda uzilishga ega bo‘lgan $y=f(x)$ funksiyaning xosmas integrali quyidagicha aniqlanadi:

1) $x=a$ – uzulish nuqtasi u xolda

$$\int_a^b f(x)dx=\lim_{\varepsilon\to0}\int_{a+\varepsilon}^{b}f(x)dx; \qquad \textbf{(32)}$$

2) x = b – uzulish nuqtasi u xolda

$$\int_a^b f(x)dx=\lim_{\varepsilon\to0}\int_{a}^{b-\varepsilon}f(x)dx, \qquad \textbf{(33)}$$

3) $x=c,\ c\in(a,b)$, s – uzulish nuqtasi u xolda

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx. \qquad \textbf{(34)}$$

Agar yuqoridagi limitlar mavjud va chekli bo'lsa u xolda xosmas integral yaqinlashuvchi deyiladi aks xolda uzoqlashuvchi deyiladi.

Taqqoslash alomati. f(x) va g(x) funksiyalar [a,b) oraliqda uzluksiz va $x = b$ nuqtada II tur uzulishga ega. $0 \le f(x) \le g(x)$ bo'lsin. Agar $\int_a^b g(x)$ integral yaqinlashsa, u xolda $\int_a^b f(x)dx$ integral xam yaqinlashadi. Agar $\int_a^b f(x)$ integral uzoqlashsa, u xolda $\int_a^b g(x)dx$ integral xam uzoqlashadi.

Misol: 52. $\int_0^3 \frac{dx}{(x-1)^2}$ xosmas integralni yaqinlashuvchilikka tekshiring.

Echish: $y = \frac{1}{(x-1)^2}$ Funksiya $x = 1$ nuqtada II tur uzulishga ega u xolda quyidagiga ega bo'lamiz.

$$\int_0^3 \frac{dx}{(x-1)^2} = \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon}\frac{d(x-1)}{(x-1)^2} + \lim_{\varepsilon\to 0}\int_{1+\varepsilon}^{3}\frac{d(x-1)}{(x-1)^2} = \lim_{\varepsilon\to 0}\left(-\frac{1}{x-1}\right)\Big|_0^{1-\varepsilon} +$$

$$+\lim_{\varepsilon\to 0}\left(-\frac{1}{x-1}\right)\Big|_{1+\varepsilon}^{3} = \lim_{\varepsilon\to 0}\left(-\frac{1}{1-\varepsilon-1}+\frac{1}{-1}\right) + \lim_{\varepsilon\to 0}\left(-\frac{1}{3-1}+\frac{1}{1+\varepsilon-1}\right) =$$

$$= \lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}-1\right) + \lim_{\varepsilon\to 0}\left(-\frac{1}{2}+\frac{1}{\varepsilon}\right) = \infty + \infty = \infty$$

Integral uzoqlashuvchi .

Maple dasturida quydagisha bo'ladi

```
> Int(1/(x-1)^2,x=0..3)=int(1/(x-1)^2,x=0..3);
```

$$\int_0^3 \frac{1}{(x-1)^2}\,dx = \infty$$

```
> ad := int(1/(x-1)^2,x);
```

$$ad := -\frac{1}{x-1}$$

```
> properint := subs(x=1-epsilon,ad) - subs(x=0,ad);
```

$$properint := \frac{1}{\varepsilon} - 1$$

```
> Int(1/(x-1)^2,x=0..1)=limit(properint,epsilon=0,right);
```

$$\int_0^1 \frac{1}{(x-1)^2}\,dx = \infty$$


```
> properint := subs(x=3,ad) - subs(x=1+epsilon,ad);
```

$$properint := -\frac{1}{2} + \frac{1}{\varepsilon}$$

```
> Int(1/(x-1)^2,x=1..3)=limit(properint,epsilon=0,right);
```

$$\int_1^3 \frac{1}{(x-1)^2}\,dx = \infty$$

```
> Int(1/(x-1)^2,x=0..3)=int(1/(x-1)^2,x=0..3);
```

$$\int_0^3 \frac{1}{(x-1)^2}\,dx = \infty$$

Misol 53. xosmas integralni yaqinlashuvchilikka tekshiring.

$$\int_0^1 \frac{2x+chx}{\sqrt[4]{x^2+shx}}dx.$$

Echish: $x = 0$ nuqtada funksiya maxraji 0 ga teng, su'rati 1 ga teng, demak, $x = 0$ – nuqtada funksiya II tur uzilishga ega . (0;1] oraliqning barcha nuqtalarida integral osti funksiya uzluksiz .

Agar $(2x+chx)dx = d(x^2+shx)$, ekanligini e'tiborga olsak

$$\int \frac{2x+chx}{\sqrt[4]{x^2+shx}}dx = \int (x^2+shx)^{-\frac{1}{4}} d(x^2+shx) = \{x^2+shx = t\} =$$

$$= \int t^{-\frac{1}{4}}dt = \frac{4t^{3/4}}{3} + C = \frac{4}{3}\sqrt[4]{x^2+shx} + C.$$

Chegaralanmagan funksiyaning xosmas integrali ta'rifidan xamda Nyutona-Leybnits formulasidan foydalanib quyidagini xosil qilamiz.

$$\int_0^1 \frac{2x+chx}{\sqrt[4]{x^2+shx}}dx = \lim_{\varepsilon\to 0}\int_{0+\varepsilon}^{1}\frac{2x+chx}{\sqrt[4]{x^2+shx}}dx = \lim_{\varepsilon\to 0}\frac{4}{3}\sqrt[4]{x^2+shx}\Big|_{\varepsilon}^{1} =$$

$$= \frac{4}{3}\lim_{\varepsilon\to 0}\left(\sqrt[4]{1+sh1} - \sqrt[4]{\varepsilon^2+sh\varepsilon}\right) = \frac{4}{3}\cdot\sqrt[4]{1+sh1}.$$

Integral yaqinlashuvchi.

4. Nazorat savollari.

1. Boshlang'ich funksiya tushunchasi.
2. Aniqmas integral va uning xossalari.
3. Aniqmas integralda o'zgaruvchini almashtirish.
4. Aniqmas integralda bo'laklab integrallash formulasi.
5. Ratsional funksiyalarni integrallash.
6. Ba'zi irratsional ko'rinishdagi funksiyalarni integrallash.
7. Eyler almashtirishlari.
8. Binomial differensiallarni integrallash.
9. Trigonometrik funksiyalarni integrallash.
10. Aniq integral tushunchasi.
11. Nyuton-Leybnits formulasi.
12. Aniq integralda bo'laklab integrallash formulasi.
13. Aniq integralda o'zgaruvchini almashtirish.
14. O'rta qiymat haqidagi birinchi teorema.
15. Dekart koordinatalar sistemasida berilgan shaklning yuzasini hisoblash.
16. Qutb koordinatalar sistemasida berilgan shaklning yuzasini hisoblash.
17. Dekart koordinatalar sistemasida berilgan yoy uzunligini hisoblash.
18. Parametrik ko'rinishda berilgan egri chiziq yoyining uzunligini hisoblash.
19. Qutb koordinatalar sistemasida berilgan yoy uzunligini hisoblash.
20. Aylanma sirtning yuzasini hisoblash.
21. Aniq integral yordamida hajm hisoblash.
22. O'zgaruvchi kuchning bajargan ishi.
23. Egri chiziqning koordinata o'qlariga nisbatan statik momentlarini topish.
24. Egri chiziq og'irlik markazining koordinatalarini topish.
25. Geometrik figuralarning statik momentlari.
26. Geometrik figura og'irlik markazining koordinatalarini topish.
27. Elliptik integrallar.
28. Xosmas integrallar.

5. Mustaqil echish uchun misol va masalalar.

1-masala. Aniqmas integral topilsin.

1.1 $\int \frac{dx}{\sqrt{1-x}}$;

1.2 $\int \frac{\cos x}{1-2\sin x}dx$;

1.3 $\int\left(\cos\frac{x}{3}+1\right)dx$;

1.4 $\int \frac{xdx}{\cos^2 x}$.

1.5 $\int (x^2+3)\cdot\cos 2x\,dx$.

1.6 $\int xe^{-2x}\,dx$;

1.7 $\int (x^2+1)\cdot\sin 2x\,dx$.

1.8 $\int x^2 \ln x\,dx$;

1.9 $\int x\cdot arctgx\,dx$;.

1.10 $\int (2x^2-5x)e^{-3x}dx$.

1.11 $\int (x\sqrt{2}-3)\cos 2xdx$.

1.12 $\int (x+1)\cdot 3^x\,dx$.

1.13 $\int (x^2-2)\cos xdx$.

1.14 $\int \sin 7x \sin 3xdx$.

1.15 $\int arctg\sqrt{3x-1}dx$.

1.16 $\int arctg\sqrt{5x-1}dx$.

1.17 $\int 2^x(4x+6)\,dx$.

1.18 $\int e^x\cdot x^2\cdot dx$.

1.19 $\int 3x^2\ln(x+2)dx$;.

1.20 $\int (2-4x)\sin 2xdx$.

1.21 $\int (3x-7)\cos 5xdx$.

2-masala. Aniqmas integral hisoblansin.

2.1 $\int\left(x^2-2x+\frac{3}{\sqrt{x}}\right)dx$;

2.2 $\int \frac{1+\ln x}{x}dx$.

2.3 $\int \frac{xdx}{(5x^2+1)^2}$;

2.4 $\int 7^x\sqrt{3\cdot 7^x+4}dx$;

2.5 $\int \frac{xdx}{\sqrt{x^4+x^2+1}}$.

2.6 $\int \frac{e^{-x}}{e^{-x}+4}dx$;

2.7 $\int tgx\cdot\ln(\cos x)dx$.

2.8. $\int tg3x\,dx$;

2.9. $\int \frac{x^3}{(x^2+1)^2}dx$.

2.10 $\int \frac{1-\cos x}{(x-\sin x)^2}dx$.

2.11. $\int \frac{\cos x\,dx}{\sin^2 x-3}$;

2.12 $\int \frac{x\cos x+\sin x}{(x\sin x)^2}dx$.

2.13 $\int \frac{x^3+x}{x^4+1}dx$.

2.14 $\int \cos x\cos 4xdx$

2.15. $\int \frac{xdx}{\sqrt[3]{x-1}}$.

2.16 $\int \frac{1+\ln(x-1)}{x-1}dx$.

2.17 $\int \frac{(x^2+1)dx}{(x^3+3x+1)^5}$.

2.18 $\int \frac{4arctgx-x}{1+x^2}dx$.

2.19 $\int \frac{x^3dx}{x^2+4}$.

2.20 $\int \frac{x+\cos x}{x^2+2\sin x}dx$.

2.21 $\int \frac{2x-\sin x}{(x^2+\cos x)^2}dx$.

3-masala. Aniqmas integral hisoblansin.

3.1 $\int \frac{x^2+2x-2}{x^3-9x}dx$;

3.2 $\int \frac{2x-1}{x^2-3x+2}dx$;

3.3 $\int \frac{x^3+1}{x^3-2x}dx$;

3.4 $\int \frac{x^4+2x^3+9x^2+5x+2}{x^2(x+1)}dx$.

3.5 $\int \frac{3x^2-1}{x^3-x}dx$;

3.6 $\int \frac{x^3-3x^2+7x-1}{2x^3-x^2+3}dx$.

3.7 $\int \frac{2x^3+5}{x^2-x-2}dx$.

3.8 $\int \frac{x^3+1}{x^2-4}dx$;

3.9 $\int \frac{2x^3-1}{x^2+x-6}dx$.

3.10 $\int \frac{x^3-3x^2-12}{(x-4)(x-2)x}dx$.

3.11 $\int \frac{3x^3+25}{x^2+3x+2}dx$.

3.12 $\int \frac{x^4+2}{x^3+3x}dx$;

3.13 $\int \frac{x^3+2x^2+3}{(x-1)(x-2)(x-3)}dx$.

3.14 $\int \frac{3x^2+x+3}{(x-1)^3(x^2+1)}dx$.

3.15 $\int \frac{3x^3+2x^2+1}{(x+2)(x-2)(x-1)}dx$.

3.16 $\int \frac{2x^4+8x^3+x^2+x-2}{x^3(x+5)}dx$.

3.17 $\int \frac{x^3+6x^2+11x+7}{(x^3+3x+2)(x+2)}dx$.

3.18 $\int \frac{x^2-3x+2}{x(x^2+2x+1)}dx$.

3.19 $\int \frac{-x^5+25x^3+1}{x^2+5x}dx$.

3.20 $\int \frac{x^3-x-2}{x^3(x-2)}dx$.

3.21 $\int \frac{x^6}{x^2-x+1}dx$.

4-masala. Aniqmas integral hisoblansin.

4.1 $\int \frac{x-3}{x^3+8}dx$;.

4.2 $\int \frac{x+3}{x^3+10x^2+25x}dx$;

4.3 $\int \frac{x^3-2}{x^3+2x^2+x}dx$;

4.4 $\int \frac{x^3+2x^2+10x}{(x+1)^2\cdot(x^2-x+1)}dx$.

4.5 $\int \frac{x^5+3x^2}{x^2+x}dx$;

4.6 $\int \frac{4x^3+24x^2+20x-28}{(x+3)^2\cdot(x^2+2x+2)}dx$.

4.7 $\int \frac{3x^2+x+3}{(x-1)^3(x^2+1)}dx$

4.8 $\int \frac{x^3+x+1}{(x^2+x+1)(x^2+1)}dx$.

4.9 $\int \frac{2x^3+11x^2+16x+10}{(x+2)^2\cdot(x^2+2x+3)}dx$.

4.10 $\int \frac{x^2-1}{x^2+3x+2}dx$.

4.11 $\int \frac{3x^3+6x^2+5x-1}{(x+1)^2\cdot(x^2+2)}dx$.

4.12 $\int \frac{4x^2+3x+4}{(x^2+1)(x^2+x+1)}dx$.

4.14 $\int \frac{2x^2-x+1}{(x^2-x+1)(x^2+1)}dx$.

4.13 $\int \frac{x^3+9x^2+21x+21}{(x+3)^2\cdot(x^2+3)}dx$.

4.15 $\int \frac{x^3+6x^2+8x+8}{(x+2)^2\cdot(x^2+4)}dx$.

4.16 $\int \frac{x^3+x+1}{(x^2-x+1)\cdot(x^2+1)}dx$.

4.17 $\int \frac{x^3+5x^2+12x+4}{(x+2)^2\cdot(x^2+4)}dx$.

4.18 $\int \frac{x^3+2x^2+x+1}{(x^2+x+1)\cdot(x^2+1)}dx$.

4.19 $\int \frac{2x^3-4x^2-16x-12}{(x-1)^2\cdot(x^2+4x+5)}dx$.

4.20 $\int \frac{2x^3+2x^2+2x+1}{(x^2+x+1)(x^2+1)}dx$.

4.21 $\int \frac{x^2+3x+6}{x^3-5x^2+6x}dx$.

5-masala. Aniqmas integral hisoblansin.

5.1 $\int \frac{\sqrt[3]{2x-7}\cdot dx}{1-\sqrt{2x-7}}$.

5.2 $\int \frac{\sqrt[3]{x}\cdot dx}{\sqrt{x}+\sqrt[3]{x}}$.

5.3 $\int \frac{\sqrt{x-7}+7}{\sqrt[3]{x-7}}dx$.

5.4 $\int \frac{\sqrt[4]{x}}{1+5\sqrt{x}}dx$.

5.5 $\int \frac{2+\sqrt{x}}{1-\sqrt{x}}dx$.

5.6 $\int \frac{dx}{\sqrt{3x+1}+\sqrt[4]{3x+1}}$;

5.7 $\int \frac{\sqrt[4]{x}}{1+5\sqrt{x}}dx$.

5.8 $\int \frac{\sqrt[4]{x}}{1-\sqrt{x}}dx$;

5.9 $\int \frac{dx}{\sqrt{2x-1}+\sqrt[4]{2x-1}}$.

5.10 $\int \frac{\sqrt[3]{x}\cdot dx}{\sqrt{x}+\sqrt[3]{x}}$

5.11 $\int \frac{\sqrt{x}}{1-\sqrt[3]{x}}dx$;.

5.12 $\int \frac{\sqrt[3]{\left(1+\sqrt{x}\right)^4}}{x\cdot\sqrt[10]{x^9}}dx$.

5.13 $\int \frac{x+\sqrt{2+x}}{\sqrt[3]{2+x}}dx$.

5.14 $\int \frac{\sqrt{x-7}+7}{\sqrt[3]{x-7}}dx$.

5.15 $\int \frac{\sqrt[3]{2x-7}\cdot dx}{1-\sqrt{2x-7}}$.

5.16 $\int \frac{\sqrt{1+\sqrt[5]{x^4}}}{x^2\cdot\sqrt[5]{x}}dx$.

5.17 $\int \frac{x+\sqrt{1+x}}{\sqrt[3]{x+1}}$.

5.18 $\int \frac{4x dx}{\sqrt{2x^2-1}+\sqrt[4]{2x^2-1}}$.

5.19 $\int \frac{\sqrt{1+x}}{x^2\cdot\sqrt{x}}dx$.

5.20 $\int \frac{dx}{\sqrt{5x^2-4}+\sqrt[4]{5x^2-4}}$;

5.21 $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$.

6-masala. Aniq integral hisoblansin.

6.1 $\int_1^{\sqrt{e}} \frac{dx}{x\sqrt{1-\ln^2 x}}$.

6.2 $\int_0^8 \left(\sqrt{2x}+\sqrt[3]{x}\right)dx$.

6.4 $\int_1^2 \left(\frac{1}{x^4}+x^2\right)dx$.

6.3 $\int_1^e \frac{\sin \ln x}{x} dx$.

6.5 $\int_{\pi^2/9}^{\pi^2} \frac{\cos\sqrt{x}}{\sqrt{x}} dx$.

6.7 $\int_0^{\ln 4} \frac{e^x}{\sqrt{e^x+1}} dx$.

6.9 $\int_1^{\sqrt{e}} \frac{dx}{x\sqrt{1-\ln^2 x}}$.

6.11 $\int_0^{\sqrt{3}} \frac{x-(arctgx)^4}{1+x^2} dx$.

6.13 $\int_0^{\pi/2} \frac{\cos x dx}{\sin^2 x+1}$.

6.15 $\int_0^9 \frac{x^3 dx}{\sqrt{x^4+4}}$.

6.17 $\int_1^e \frac{x^2+\ln x^2}{x} dx$.

6.19 $\int_0^1 \frac{x^3 \cdot dx}{(x^2+1)^2}$.

6.21 $\int_0^{15} \frac{xdx}{\sqrt{1+x}}$.

6.6 $\int_0^{\pi/2} \frac{\cos(x)dx}{\sin^5(x)}$.

6.8 $\int_0^2 \frac{x^2 dx}{(x^3+1)^4}$.

6.10 $\int_0^{\sqrt{3}} \frac{arctgx+x}{1+x^2} dx$.

6.12 $\int_0^1 \frac{x^4 dx}{x^5+1}$.

6.14 $\int_4^9 \frac{\sqrt{x}}{\sqrt{x}-1} dx$.

6.16 $\int_1^e \frac{1+\ln x}{x} dx$.

6.18 $\int_0^{\pi/2} \frac{dx}{2+\cos x}$.

6.20 $\int_\pi^{2\pi} \frac{1-\cos x}{(x-\sin x)^2} dx$.

7-masala Aniq integral hisoblansin.

7.1 $\int_0^1 xe^{-x} dx$.

7.3 $\int_0^\pi e^x \cos(x) dx$.

7.5 $\int_{\pi/4}^{\pi/3} x \sin^{-2}(x) dx$.

7.7 $\int_1^e \frac{\ln^2(x)}{x} dx$.

7.9 $\int_0^{2\pi} (3x^2+5)\cos 2x dx$.

7.11 $\int_0^{2\pi} (3-7x^2)\cos 2x dx$.

7.13 $\int_{-1}^0 (x^2+2x+1)\sin 3x dx$.

7.2 $\int_0^{\pi/2} x \cos(x) dx$.

7.4 $\int_0^\pi e^x \sin(x) dx$.

7.6 $\int_{\pi/4}^{\pi/3} x \cos^{-2}(x) dx$.

7.8 $\int_0^1 \frac{e^x}{x} dx$.

7.10 $\int_0^{2\pi} (2x^2-15)\cos 3x dx$.

7.12 $\int_0^{2\pi} (1-8x^2)\cos 4x dx$.

7.14 $\int_1^e \ln(x) dx$.

7.15 $\int (x^2-3x+2)\sin x dx$.

7.17 $\int (x^2+6x+9)\sin 2x dx$.

7.19 $\int (3x-x^2)\sin 2x dx$.

7.21 $\int x^2 \cos x dx$.

7.16 $\int_0^{\pi/2} (x^2-5x+6)\sin 3x dx$.

7.18 $\int_0^{\pi/2} (1-5x^2)\sin x dx$.

7.20 $\int_1^2 x \ln^2 x dx$.

8-masala Aniq integral hisoblansin.

8.1 $\int \frac{\cos x dx}{1+\sin x-\cos x}$.

8.3 $\int \frac{\sin x}{5+3\sin x} dx$.

8.5 $\int \frac{dx}{\sin^2 2x(2+\cos 2x)}$.

8.7 $\int \frac{\cos x dx}{(1+\cos x+\sin x)^2}$.

8.9 $\int \frac{\cos x}{(1++\cos x+\sin x)^2} dx$.

8.11 $\int \frac{dx}{\sin^2 x(1+\cos x)}$.

8.13 $\int \frac{dx}{\sin x\cdot(1-\sin x)}$.

8.15 $\int \frac{\cos x-\sin x}{(1+\sin x)^2} dx$.

8.17 $\int \frac{dx}{\sin^2 x\cdot(1+\cos x)}$.

8.19 $\int \frac{1}{3\cos^2 x-5\sin^2 x} dx$.

8.21 $\int \frac{\cos x-\sin x}{(1+\sin x)^2} dx$.

8.2 $\int_{-\frac{\pi}{2}}^0 \frac{\sin x}{(1+\cos x-\sin x)^2} dx$.

8.4 $\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\sin x)^2} dx$.

8.6 $\int_0^{\frac{\pi}{2}} \frac{1}{1-2\cos x+3\sin x} dx$.

8.8 $\int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x+2\sin^2 x} dx$.

8.10 $\int_0^{\pi/2} \frac{(1+\cos x)dx}{1+\cos x+\sin x}$.

8.12 $\int_0^{2\pi/3} \frac{1+\sin x}{1+\cos x+\sin x} dx$.

8.14 $\int_{2arctg\frac{1}{2}}^{\pi/2} \frac{dx}{(1+\sin x-\cos x)^2}$.

8.16 $\int_{2arctg2}^{2arctg3} \frac{dx}{\cos x\cdot(1-\cos x)}$.

8.18 $\int_0^{\frac{\pi}{4}} \frac{tg^4 x}{\cos^4 x} dx$.

8.20 $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x dx}{(1+tg^2 x)\sin 2x}$.

9-masala. Aniq integral hisoblansin.

9.1 $\int_0^{2\pi} \sin^8 \frac{x}{4} dx$.

9.2 $\int_0^{2\pi} \sin^6 \frac{x}{4} \cdot \cos^2 \frac{x}{4} dx$.

9.3 $\int_{\frac{\pi}{2}}^{\pi} 2^8 \sin^8 x dx$.

9.4 $\int_0^{\pi} 2^4 \sin^4 x \cos^4 x dx$.

9.5 $\int_0^{\pi} 2^4 \sin^6 \frac{x}{2} \cdot \cos^2 \frac{x}{2} dx$.

9.6 $\int_0^{\pi} 2^4 \sin^4 \frac{x}{2} \cdot \cos^4 \frac{x}{2} dx$.

9.7 $\int_0^{2\pi} \sin^4 x \cdot \cos^4 x dx$.

9.8 $\int_0^{2\pi} \sin^2 x \cos^6 x dx$.

9.9 $\int_{-\frac{\pi}{2}}^{0} 2^8 \sin^4 x \cdot \cos^4 x dx$.

9.10 $\int_{-\frac{\pi}{2}}^{0} 2^8 \sin^2 x \cdot \cos^6 x dx$.

9.11 $\int_0^{2\pi} \sin^2 \frac{x}{4} \cdot \cos^6 \frac{x}{4} dx$.

9.12 $\int_0^{2\pi} \cos^8 \frac{x}{4} dx$.

9.13 $\int_0^{\pi} 2^4 \cdot \cos^8 \frac{x}{2} dx$.

9.14 $\int_{-\pi}^{0} 2^8 \sin^6 x \cos^2 x dx$.

9.15 $\int_{-\frac{\pi}{2}}^{0} 2^8 \sin^8 x dx$.

9.16 $\int_0^{2\pi} \sin^6 x \cos^2 x dx$.

9.17 $\int_0^{2\pi} \sin^8 x dx$.

9.18 $\int_0^{\pi} 2^4 \sin^2 x \cos^6 x dx$.

9.19 $\int_{\pi/2}^{\pi} 2^8 \sin^6 x \cos^2 x dx$.

9.20 $\int_0^{2\pi} \sin^4 3x \cos^4 3x dx$.

9.21 $\int_0^{\pi} 2^4 \cos^8 \frac{x}{2} dx$.

10-masala Aniq integral hisoblansin.

10.1 $\int_0^{2\sqrt{4}} x^5 \sqrt{x^2+4}\, dx$.

10.2 $\int_0^{3} \frac{dx}{(9+x^2)^{3/2}}$.

10.3 $\int_2^{4} \frac{\sqrt{x^2-4}}{x} dx$.

10.4 $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{(4-x^2)^3}}$.

10.5 $\int_1^{\sqrt{3}} \frac{dx}{\sqrt{(1+x^2)^3}}$.

10.6 $\int_0^{1/2} x^2 \sqrt{1-x^2} dx$.

10.7 $\int_0^{9} \frac{dx}{(81+x^2)\sqrt{81+x^2}}$.

10.8 $\int_0^{3} \frac{dx}{(9+x^2)^{3/2}}$.

10.9 $\int_0^{\sqrt{3}/2} \frac{-3dx}{(1-x^2)\sqrt{1-x^2}}$.

10.10 $\int_0^{2\sqrt{3}} \frac{\sqrt{4+x^2}}{x^4} dx$.

10.11 $\int \frac{x^4}{\sqrt{(1-x^2)^3}} dx$.

10.12 $\int_0^{3/2} x^2 \sqrt{9-x^2} dx$.

10.13 $\int \frac{dx}{(16+x^2)^{3/2}}$.

10.14 $\int_{\sqrt{2}}^{\sqrt{6}} \frac{dx}{x\sqrt{2+x^2}}$.

10.15 $\int \frac{x^4 dx}{\sqrt{(8-x^2)^3}}$.

10.16 $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{(4-x^2)^3}}$.

10.17 $\int \frac{x^4 dx}{\sqrt{(1-x^2)^3}}$.

10.18 $\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{(4-x^2)^3}}$.

10.19 $\int \frac{dx}{(9+x^2)^{3/2}}$.

10.20 $\int_1^{\sqrt{2}} \frac{dx}{(4-x^2)\sqrt{4-x^2}}$.

10.21 $\int \sqrt{\frac{9-2x}{2x-21}} dx$.

11-masala. Quyidagi chiziqlar bilan chegaralangan shaklning yuzasi hisoblansin.

11.1 $y=\sin\frac{x}{2},\ y=\cos\frac{x}{2},\ x=0$

11.2 $y=\frac{1}{x\sqrt{\ln x}},\ y=0$.

11.3 $y=x^2;\ x+y=2;\ x=0$

11.4 $y=\sqrt{e^x-1},\ y=0,\ x=\ln 5$

11.5 $y=x^2 tgx;\ y=0;\ x=\frac{\pi}{4}$.

11.6 $y=2^x-1;\ y=\frac{3}{4}x(4-x)$.

11.7 $y=\frac{1}{x\sqrt{1+\ln x}};\ y=0, x=1; x=e^3$.

11.8 $y=x^2 tgx;\ y=0;\ x=\frac{\pi}{4}$

11.9 $y=\arccos x;\ y=0;\ x=0$.

11.10 $y=e^x,\ y=e^{-x},\ x=1$.

11.11 $y=x\sqrt{36-x^2};\ y=0;(0\le x\le 6)$.

11.12 $y=\sin^2 x\cdot\cos x,\ y=0,\ x=\frac{\pi}{2}$

11.13 $y=x\,arctgx, y=0, x=\sqrt{3}$.

11.14 $y=x^2\sqrt{8-x^2};\ y=0;(0\le x\le 2\sqrt{2})$.

11.15 $x=\sqrt{e^y-1};\ x=0;\ y=\ln 2$.

11.16 $y=x\sqrt{4-x^2};\ y=0;(0\le x\le 2)$.

11.17 $y=e^{1-x},\ y=0,\ x=0,\ x=1$

11.18 $y=x^2,\ x=y^2$.

11.19 $y=\frac{x}{(x^2+1)^2}\ y=0; x=1$.

11.20 $x=4-y^2; x=y^2-2y$.

11.21 $y=(x-2)^3, y=4x-8$.

12-masala. Tenglamalari qutb koordinatalar sistemasida berilgan chiziqlar bilan chegarlangan shaklning yuzasi hisoblansin.

12.1 $\rho=4\cos 2\varphi,\ \rho=2\ \rho\ge 2$

12.2 $r=2\sin\varphi;\ r=4\sin\varphi$.

12.3 $\rho=4\sin\frac{3\varphi}{2},\ \rho=2,\ \rho\ge 2$.

12.4 $\rho=1+\sin 2\varphi$.

12.5 $r=\cos\varphi+\sin\varphi$.

12.6 $\rho^2=3\cos\left(\varphi-\frac{\pi}{3}\right)$.

12.7 $\rho^2 = 3\cos 3\varphi$. 12.8 $\rho = 4\cos^2\left(2\varphi - \frac{\pi}{4}\right)$.

12.9 $r = \frac{3}{2}\cos\varphi; r = \frac{5}{2}\cos\varphi$.

12.10 $\rho = \sin\varphi . \rho = \cos\varphi, \quad \varphi \in \left[0, \frac{\pi}{2}\right]$

12.11 $\rho = 3 - \sin\varphi$.

12.13 $r = \frac{1}{2} + \cos\varphi$. 12.12 $r = \frac{5}{2}\sin\varphi; r = \frac{3}{2}\sin\varphi$.

12.15 $r = \sin\varphi; r = 2\sin\varphi$. 12.14 $r = \cos\varphi; r = 2\cos\varphi$.

12.17 $\rho = 2\sin^2\varphi$. 12.16 $r = 6\cos 3\varphi; r = 3(r \geq 3)$.

12.19 $\rho = 2\sin 3\varphi$. 12.18 $r = 6\sin 3\varphi; r = 3(r \geq 3)$.

12.21 $r = 4\cos 3\varphi; r = 2.(r \geq 2)$. 12.20. $\rho = 3\sin 4\varphi$.

13-masala. Parametrik ko‘rinishda berilgan egri chiziq yoyining uzunligi hisoblansin.

13.1 $x = 2\cos^2 t, \quad y = 2\sin^2 t \; 0 \leq t \leq \frac{\pi}{4}$.

13.2 $x = e^{2t}\sin t, \quad y = e^{2t}\cos t, \quad 0 \leq t \leq \frac{\pi}{4}$.

13.3 $x = 6\cos^3 t, \quad y = 6\sin^3 t, \quad \frac{\pi}{4} \leq t \leq \frac{\pi}{2}$.

13.4 $x = 3\cos^3 t, \quad y = 3\sin^3 t, \quad 0 \leq t \leq \frac{\pi}{2}$.

13.5 $x = e^t\cos t, \quad y = e^t\sin t, \quad 0 \leq t \leq \frac{\pi}{2}$.

13.6 $x = 2(\sin t + \cos t), \quad y = 2(\sin t - t\cos t), \; 0 \leq t \leq \frac{\pi}{2}$.

13.7 $x = 3(t - \sin t); y = 3(1 - \cos t); \pi \leq t \leq 2\pi$.

13.8 $x = \frac{1}{2}\cos t - \frac{1}{4}\cos 2t; y = \frac{1}{2}\sin t - \frac{1}{4}\sin 2t; \frac{\pi}{2} \leq t \leq \frac{2\pi}{3}$.

13.9 $x = 3(\cos t + t\sin t); y = 3(\sin t - t\cos t); 0 \leq t \leq \frac{\pi}{3}$.

13.10 $x = (t^2 - 2)\sin t + 2t\cos t; y = (2 - t^2)\cos t + 2t\sin t; 0 \leq t \leq \frac{\pi}{3}$.

13.11 $x = 6\cos^3 t; y = 6\sin^3 t; 0 \leq t \leq \frac{\pi}{3}$.

13.12 $x = e^t(\cos t + \sin t); y = e^t(\cos t - \sin t); \frac{\pi}{2} \leq t \leq \pi$.

13.13 $x = 2{,}5(t - \sin t); y = 2{,}5(1 - \cos t); \frac{\pi}{2} \leq t \leq \pi$.

13.14 $x = 3{,}5(2\cos t - \cos 2t); y = 3{,}5(2\sin t - \sin 2t); 0 \leq t \leq \frac{\pi}{2}$.

13.15 $x = 6(\cos t + t\sin t); y = 6(\sin t - t\sin t); 0 \leq t \leq \pi$.

13.16 $x = (t^2 - 2)\sin t + 2t\cos t; y = (2 - t^2)\cos t + 2t\sin t; 0 \leq t \leq \frac{\pi}{2}$.

13.17 $x = 8\cos^3 t; y = 8\sin^3 t; 0 \leq t \leq \frac{\pi}{6}$.

13.18 $x = e^t(\cos t + \sin t); y = e^t(\cos t - \sin t); 0 \leq t \leq 2\pi$.

13.19 $x = 4(t - \sin t); y = 4(1 - \cos t); \frac{\pi}{2} \leq t \leq \frac{2\pi}{3}$.

13.20 $x = 2(2\cos t - \cos 2t); y = 2(2\sin t - \sin 2t); 0 \leq t \leq \frac{\pi}{3}$.

13.21 $x = 3(t - \sin t), y = 3(1 - \cos t); \; 0 \leq x \leq \pi$.

14-masala. Quyidagi sirtlar bilan chegaralangan jismning hajmi topilsin.

14.1 $x^2/3 + y^2/4 = 1$, $z = y\sqrt{3}$ $z = 0$ $(y \geq 0)$. 14.2 $z = x^2 + 8y^2$, $z = 4$.

14.3 $\frac{x^2}{9} + \frac{y^2}{4} - z^2 = 1; z = 0; z = 3$. 14.4 $x^2/3 + y^2/16 = 1$, $z = 0$, $z = y\sqrt{3}$ $(y \geq 0)$.

14.5 $\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} = 1; z = 1; z = 0$. 14.6 $x^2 + y^2 = 9; z = y; z = 0.(y \geq 0)$.

14.7 $x^2/81 + y^2/25 - z^2 = 1$, $z = 0$, $z = 2$. 14.8 $\frac{x^2}{4} + y^2 - z^2 = 1; z = 0; z = 3$.

14.9 . $x^2/9 + y^2/25 - z^2/36 = -1$, $z = 12$.. 14.10 $\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{16} = 1; z = 2; z = 0$.

14.11 $z = 2x^2 + 8y^2; z = 4$. 14.12 . $x^2/16 + y^2/9 + z^2/36 = 1$, $z = 0$, $z = 3$.

14.13 $x^2/9 + y^2/4 + z^2 = 1$, $z = 0$, $z = 4$.. 14.14 $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{36} = 1; z = 0; z = 3$.

14.15 $z = x^2 + 5y^2, z = 5$. 14.16 $\frac{x^2}{9} + \frac{y^2}{4} - z^2 = 1; z = 0; z = 4$.

14.17 $\frac{x^2}{9} + \frac{y^2}{25} - \frac{z^2}{100} = -1; z = 20$. 14.18 $\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{64} = 1; z = 4; z = 0$.

14.19 $z = 4x^2 + 9y^2, z = 6$. 14.20 $z = 2x^2 + 18y^2, z = 6$.

14.21 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$;

15-masala. Quyidagi chiziqlar bilan chegaralangan shaklni 1-5 variantlarda OY o‘qi atrofida, 6-21 variantlarda esa OX o‘qi atrofida aylantirishdan hosil bo‘lgan jismning hajmi topilsin.

15.1 $\{y = x^2, y = 4$. 15.2 $\left\{\frac{x^2}{4} - \frac{y^2}{9} = 1, \; y = \pm 6,\right.$.

15.3 $\left\{y = 1 - \frac{x^2}{2}, x + y = 1,\right.$. 15.4 $\left\{y = 2^x, y = \frac{3x+5}{4}\right.$

15.5 $y = x^2 + 1, y = x, x = 0, x = 1$. 15.6 $\left\{x = \sin x, x = 0, y = \frac{2}{\pi}x\right.$,.

15.7 $\{y = x^2, x = y^2$. 15.8 $\{x = 3(t - \sin t), y = 3(1 - \cos t), \; y = 0, \; 0 \leq x \leq 6\pi$,

15.9 $\{y = x^2 + 1, \; x \pm 2, \; y = 0,$.

15.11 $\{(y-1)^2 = x, x = 1.$.

15.13 $\{y = x^3, x = 0, y = 8,$.

15.15 $\{2y^2 = x^3, x = 4.$

15.17 $\{x = 2\cos^3 t, y = 2\sin^3 t.$

15.19 $y = x^2, \ y = 1, \ x = 2.$

15.21 $y = 2x - x^2, y = -x + 2.$.

15.10 $\left\{y = 1 - \cos 2x, \quad y = 0, x = \frac{\pi}{2}, \right.$.

15.12 $\{x = 3\cos t, y = 5\sin t.$

15.14 $\{y = \sin x, \quad y = 0, \ 0 \le x \le \pi,$.

15.16 $\{y^3 = 4x^2, \ y = 2.$

15.18 $\{y = e^x, \quad y = 1, x = 1,$.

15.20 $\{x^2 - y^2 = 9, x = \pm 6.$

16-masala.

Quyidagi chiziqlarni aylantirishdan hosil bo'lgan aylanma sirtlarning yuzalari hisoblansin.

16.1 . $y = x^2, \ x = 0, x = 2\ y = 0.$ OY o'qi atrofida.

16.2 $y = \frac{x^3}{3}, \ -\frac{1}{2} \le x \le \frac{1}{2},$ OX o'qi atrofida.

16.3 $3x^2 + 4y^2 = 12$ ellipisni OY o'qi atrofida.

16.4 $x = \frac{1}{4}y^2 - \frac{1}{2}\ln y \quad (1 \le y \le e)$ OX o'qi atrofida.

16.5 . $y = x^2 + 1, \ y = x, x = 1, x = 0.$ OY o'qi atrofida.

16.6 $3y = x^2, \ \ 0 \le x \le 2,$ OX o'qi atrofida.

16.7 $x = e^t \sin t; y = e^t \cos t \ \left(0 \le t \le \frac{\pi}{2}\right)$ OX o'qi atrofida.

16.8 $x = 2\cos t - \cos 2t, y = 2\sin t - \sin 2t$ ni OX o'qi atrofida.

16.9 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipsning OX o'qidan yuqorida joylashgan bo'lagigining koordinata o'qlariga nisbatan statik momentlari topilsin.

16.10 $x + y = 1, x = 0, y = 0$ chiziqlar bilan chegaralangan uchburchakning OX va OY o'qlarga nisbatan statik momentlari topilsin.

16.11 $y^2 = 2x, (y > 0, 0 \le x \le 2)$ parabola yoyining OX va OY o'qlarga nisbatan statik momentlari topilsin.

16.12 $y = \cos x\left(-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right)$ egri chiziq yoyining OX o'qiga nisbatan statik momenti topilsin.

16.13 $\frac{x}{a} + \frac{y}{b} = 1$ to'g'ri chiziqning koordinata o'qlari orasida joylashgan kesmasining koordinata o'qlariga nisbatan statik momentlari topilsin.

16.14 $y = \frac{2}{1+x^2}$ va $y = x^2$ chiziqlari bilan chegaralangan shaklning OX o'qiga nisbatan statik momenti topilsin.

16.15 $x^2 + y^2 = a^2, y \ge 0$-yarim aylananing og'irlik markazi topilsin.

16.16 $x^{2/3} + y^{2/3} = a^{2/3}, x \ge 0, y \ge 0$ -astroida yoyining og'irlik markazi topilsin.

16.17 $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 (x \ge 0, y \ge 0)$ ning og'irlik markazi topilsin.

16.18 $x = \frac{1}{4}y^2 - \frac{1}{2}\ln y \quad (1 \le y \le 2)$ chiziq yoyining og'irlik markazi topilsin.

Quyidagi chiziqlar bilan chegaralangan tekis shaklning og'irlik markazi topilsin.

16.19 $ax = y^2, \ ay = x^2 \ (x \succ 0).$

16.20 $y = \frac{2}{\pi}x, \ y = \sin x \ (x \ge 0).$

16.21 $x^2 + 4y - 16 = 0; y = 0.$

17 masala.

Xosmas integrallar xisoblansin.

17.1 a) $\int_e^{\infty} \frac{dx}{x\ln^3 x};$ b) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}.$

17.2 a) $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^3}};$ b) $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^3}};$

17.3 a) $\int_0^{\infty} e^{-x} dx;$ b) $\int_0^1 \ln x dx.$

17.4 a) $\int_0^1 \frac{dx}{\sqrt[3]{x}};$ b) $\int_{-\infty}^{+\infty} \frac{dx}{x^2 + 6x + 11}.$

17.5 a) $\int_2^{\infty} \frac{xdx}{x^2+4};$ b) $\int_1^2 \frac{dx}{(x-1)^2}.$

17.6 a) $\int_e^{\infty} \frac{dx}{x\sqrt{\ln x}};$ b) $\int_{-1}^1 \frac{dx}{(x+1)^4}.$

17.7 a) $\int_{-\infty}^{+\infty} \frac{dx}{x^2+4x+8};$ b) $\int_1^e \frac{dx}{x\ln^3 x}.$

17.8 a) $\int_2^4 \frac{dx}{\sqrt{6x-x^2-8}}.$ b) $\int_e^{\infty} \frac{dx}{x\sqrt[3]{\ln x}}.$

17.9 a) $\int_0^{\infty} xe^{-x^2}dx$; b) $\int_0^1 \frac{dx}{x+x^2}$.

17.10 a) $\int_0^2 \frac{xdx}{\sqrt{4-x^2}}$; b) $\int_2^{\infty} \frac{xdx}{\sqrt{(x^2+5)^3}}$.

17.11 a) $\int_{-\infty}^{\infty} e^x dx$ b) $\int_{-1}^1 \frac{dx}{\sqrt[3]{x^2}}$

17.12 a) $\int_0^{\infty} \cos x \cdot dx$ b) $\int_0^2 \frac{dx}{\sqrt[3]{1-x}}$

17.13 a) $\int_0^{\infty} \frac{1}{1+x^2}dx$ b) $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

17.14 a) $\int_0^{\infty} \frac{x+1}{x^2+2x+2}dx$ b) $\int_1^e \frac{dx}{x \cdot \ln^2 x}$

17.15 a) $\int_0^{\infty} \frac{x}{\sqrt{x^2+4}}dx$ b) $\int_{-3}^0 \frac{dx}{\sqrt{x+3}}$

17.16 a) $\int_0^{\infty} \frac{x}{x^2+1}dx$ b) $\int_0^1 \frac{dx}{\sqrt[3]{(x-1)^2}}$

17.17 a) $\int_0^{\infty} \frac{dx}{x^2+2x+2}$ b) $\int_1^e \frac{dx}{x \cdot \sqrt{\ln x}}$

17.18 a) $\int_1^{\infty} \frac{1+\ln x}{x}dx$ b) $\int_0^2 \frac{dx}{\sqrt{4-x^2}}$

17.19 a) $\int_2^{\infty} \frac{dx}{x\sqrt{2}}$ b) $\int_0^1 \frac{dx}{(x-1)^2}$

17.20 a) $\int_1^{\infty} \frac{dx}{x^2+x}$ b) $\int_0^{\pi/4} tg2x\ dx$

17.21 a) $\int_0^{\infty} \frac{xdx}{\sqrt{x^2+4}}$; b) $\int_0^4 \frac{dx}{\sqrt{16-x^2}}$

6. Namunaviy variant yechimi.

1.21-masala. $\int (3x-7)\cos 5xdx$ **aniqmas integral hisoblansin.**

◂ Bu integralni bo‘laklab integrallash usulidan foydalanib hisoblaymiz:

$$u = 3x-7,\ dv = \cos 5xdx,$$

$$du = 3dx, v = \int \cos 5dx =$$

$= \frac{1}{5}\int \cos 5xd(5x) = \frac{1}{5}\sin 5x.$ U holda

$$J = \frac{1}{5}(3x-7)\sin 5x$$

$-\frac{3}{5}\int \sin 5xdx = \frac{1}{5}(3x-7)\sin 5x + \frac{3}{25}\cos 5x + C.$

2.21-masala. $\int \frac{2x-\sin x}{(x^2+\cos x)^2}dx$ **aniqmas integral hisoblansin.**

◂ Bu integralni o‘zgaruvchilarni almashtirish usulidan foydalanib hisoblaymiz:

$x^2+\cos x = t,$ $(2x-\sin x)dx = dt.$ U holda $\int \frac{dt}{t^2} = \int t^{-2}dt$

$= \frac{t^{-1}}{-1} + C = -\frac{1}{t} + C = -\frac{1}{x^2+\cos x} + C.$ yoki

$$\int \frac{2x-\sin x}{(x^2+\cos x)^2}dx = \int (x^2+\cos x)^{-2}d(x^2+\cos x)$$

$= \frac{(x^2+\cos x)^{-1}}{-1} + C =$

$$= -\frac{1}{x^2+\cos x} + C.$$

3.21-masala. $\int \frac{x^6}{x^2-x+1}dx$ **aniqmas integral hisoblansin.**

◂ Biz bu integralni ratsional funksiyani integrallash usulidan foydalanib hisoblaymiz. Avval noto‘g‘ri kasrni to‘g‘ri kasrga keltiramiz, so‘ngra uni sodda kasrlarga yoyamiz:

$\frac{x^6}{x^2-x+1} = x^4+x^3-x-1+\frac{1}{x^2-x+1}$ kvadrat uch hadda tula kvadrat ajratamiz

$x^2-x+1 = \left(x-\frac{1}{2}\right)^2 - \frac{1}{4} + 1 = \left(x-\frac{1}{2}\right)^2 + \frac{3}{4}.$ U holda

$$\int\left(x^4+x^3-x-1+\frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\right)dx=\int x^4dx+\int x^3dx-\int xdx-\int 1dx+\int\frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=$$

$$=\frac{x^5}{5}+\frac{x^4}{4}-\frac{x^2}{2}-x+\frac{1}{\frac{\sqrt{3}}{2}}\text{arctg}\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}+C=\frac{x^5}{5}+\frac{x^4}{4}-\frac{x^2}{2}-x+\frac{2}{\sqrt{3}}\text{arctg}\frac{2x-1}{\sqrt{3}}+C;$$

4.21-masala. $\int\frac{x^2+3x+6}{x^3-5x^2+6x}dx$ **aniqmas integral hisoblansin.**

◃ Bu integral ostida ham ratsional funksiya turibdi. Bu funksiyani sodda kasrlarga yoyaamiz.

$$\frac{x^2+3x+6}{x^3-5x^2+6x}=\frac{x^2+3x+6}{x(x-2)(x-3)}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x-3}=$$

Bu tenglikning o‘ng tomonidagi noma’lum A,B,C larni noma’lum koeffitsientlar usulidan foydalanib topamiz. Buning uchun tenglikning o‘ng tomonini umumiy maxrajga keltiramiz va berilgan kasr hamda hosil bo‘lgan kasrlarning suratlarini bir-biriga tenglaymiz:

$$=\frac{A(x-2)(x-3)+Bx(x-3)+Cx(x-2)}{x(x-2)(x-3)}=$$

$$=\frac{Ax^2-5Ax+6A+Bx^2-3Bx+Cx^2-2Cx}{x(x-2)(x-3)};$$

$$x^2+3x+6=Ax^2-5Ax+6A+Bx^2-3Bx+Cx^2-2Cx$$

$\Downarrow$

$$\left.\begin{array}{l}x^2:\ 1=A+B+C\\ x^1:\ 3=-5A-3B-2C\\ x^0:\ 6=6A\end{array}\right\}\Leftrightarrow\begin{cases}1=1+B+C\\ 8=-3B-2C\\ A=1\end{cases}\Leftrightarrow\begin{cases}A=1,\\ B=-8,\\ C=8.\end{cases}$$

U holda

$$\frac{x^2+3x+6}{x^3-5x^2+6x}=\frac{1}{x}-\frac{8}{x-2}+\frac{8}{x-3}.$$

$$\int\left(\frac{1}{x}-\frac{8}{x-2}+\frac{8}{x-3}\right)dx=\int\frac{dx}{x}-8\int\frac{dx}{x-2}+8\int\frac{dx}{x-3}=:$$

hadma had integrallab quydagini hosil qilamiz

$$=\ln|x|-8\ln|x-2|+8\ln|x-3|+C;$$

5.21-masala. $\int\frac{dx}{\sqrt{x}+\sqrt[3]{x}}$ aniqmas integral hisoblansin.

◃ Integralni quydagi almashtirishlarni bajarib hisoblaymiz:

$t=\sqrt[6]{x}\quad x=t^6,\ dx=6t^5dt$

$$\int\frac{dx}{\sqrt{x}+\sqrt[3]{x}}=\int\frac{6t^5dt}{t^3+t^2}=6\int\frac{t^3dt}{t+1}=6\int\frac{t^3+1-1}{t+1}dt=$$

$$=6\int\frac{(t+1)(t^2-t+1)}{t+1}dt=6\int\frac{dt}{t+1}=6\int(t^2-t+1)dt-6\int\frac{dt}{t+1}=$$

$$=2t^3-3t^2+6t-6\ln|t+1|+c=2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\ln|\sqrt[6]{x}+1|+c.$$

6.21-masala. $\int_0^{15}\frac{xdx}{\sqrt{1+x}}$. **aniq integral hisoblansin.**

$\sqrt{1+x}=t$, Almashtirish bajarsak $t^2=1+x,\ x=t^2-1,\ dx=2tdt.\ \ x=0\ \ t^2-1=0,\ \ t=1;$ Quydagiga ega bulamiz $x=15,\ \ 15=t^2-1\ \ t^2=16,\ t=4.$ U holda

$$\int_1^4\frac{(t^2-1)\cdot 2tdt}{t}=\int_1^4(2t^2-2)dt=\left(\frac{2t^3}{3}-2t\right)\Big|_1^4=\frac{2}{3}\cdot 4^3-2\cdot 4-\left(\frac{2}{3}-2\right)=36.$$

7.21-masala. $\int_0^{2\pi}x^2\cos xdx$ **aniq integral hisoblansin.**

◃ Bu integralni bo‘laklab integrallash usulidan foydalanib hisoblaymiz:

$$\int_0^{2\pi}x^2\cos xdx=\left|\begin{array}{l}u=x^2\\ du=2xdx\\ dv=\cos xdx\\ v=\sin x\end{array}\right|=x^2\sin x\Big|_0^{2\pi}-\int 2x\sin xdx=$$

yana bir martta bo‘laklab $=\left|\begin{array}{l}u_1=x\\ du_1=dx\\ dv_1=\sin xdx\\ v_1=-\cos x\end{array}\right|=4\pi^2\sin 2\pi-0-2x\cdot(-\cos x)\Big|_0^{2\pi}-2\int_0^{2\pi}\cos xdx=$

$$=2\cdot 2\pi\cos 2\pi-0-2\sin x\Big|_0^{2\pi}=4\pi.$$

8.21-masala. $\int_0^{\pi/2}\frac{\cos x-\sin x}{(1+\sin x)^2}dx$ **aniq integral hisoblansin.**

◃ Bu integralni hisoblash uchun $tg\frac{x}{2}=t$ universal almashtirish bajaramiz.

◃ $$\int_0^{\pi/2}\frac{\cos x-\sin x}{(1+\sin x)^2}dx=\left|\begin{array}{ll}tg\frac{x}{2}=t & \cos x=\frac{1-t^2}{1+t^2}\\ dx=\frac{2dt}{1+t^2} & \sin x=\frac{2t}{1+t^2}\end{array}\right|=\int_0^1\frac{\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}}{\left(1+\frac{2t}{1+t^2}\right)^2}\cdot\frac{2dt}{1+t^2}=$$

$$=\int_0^1\frac{2(1-2t-t^2)}{(1+t)^4}dt.$$

◃ $\frac{2(1-2t-t^2)}{(1+t^4)}$ sodda kasrlarga ajratamiz

$$\triangleleft \quad \frac{2-4t-2t^2)}{(1+t)^4}=\frac{A}{1+t}+\frac{B}{(1+t)^2}+\frac{C}{(1+t^3)}+\frac{D}{(1+t)^4}=$$
$$=\frac{A(1+t)^3+B(1+t)^2+C(1+t)+D}{(1+t)^4}.$$

$\triangleleft$ $A(1+t)^3+B(1+t)^2+C(1+t)+D=2-4t-2t^2$.

$t=-1$,da $D=4$;

$\triangleleft$ Mos koefisentlarni tenglaymiz t^3, $A=0$;

$\triangleleft$ t^2, $3A+B=-2\Rightarrow B=-2$;

$\triangleleft$ t, $3A+2B+C=-4\Rightarrow C=0$;

$\triangleleft$ Bu erdan $\int_0^1\left(\frac{4}{(1+t)^4}-\frac{2}{(1+t)^2}\right)dt=\left(-\frac{4}{3(1+t)^3}+\frac{2}{1+t}\right)\Big|_0^1=-\frac{4}{3\cdot 8}+1+\frac{4}{3}-2=\frac{1}{6}$.

$\triangleleft$

9.21-masala. $\int_0^{\pi}2^4\cos^8\frac{x}{2}dx$ **aniq integral hisoblansin.**

$$\int_0^{\pi}2^4\cos^8\frac{x}{2}dx=\int_0^{\pi}(1+\cos x)^4dx=\int_0^{\pi}(1+2\cos x+\cos^2x)^2dx=$$
$$=\int_0^{\pi}(1+3\cos x+6\cos^2x+4\cos^3x+\cos^4x)dx=$$
$$=\int_0^{\pi}\left(\frac{35}{8}+3\cos x+\frac{7}{2}\cos 2x+\frac{1}{8}\cos 4x\right)dx+4\int_0^{\pi}(1-\sin^2x)\cos x dx=$$
$$=\left(\frac{35}{8}x+3\sin x+\frac{7}{4}\sin 2x+\frac{1}{32}\sin 4x\right)\Big|_0^{\pi}+4\int_0^{\pi}(1-\sin^2x)d(\sin x)=$$
$$=\frac{35}{8}\pi+4(\sin x-\frac{1}{3}\sin^3x)\Big|_0^{\pi}=\frac{35}{8}\pi.$$

10.21-masala. $\int_6^9\sqrt{\frac{9-2x}{2x-21}}dx$ **aniq integral hisoblansin.**

$$\int\sqrt{\frac{9-2x}{2x-21}}dx=\left|\begin{matrix}\frac{9-2x}{2x-21}=t^2\\ dx=\frac{12t}{(t^2+1)}dt\end{matrix}\right|=aniqmas\quad int\ egra\ lni\ hisoblaymi\quad z=$$
$$=12\int t\frac{t}{(t^2+1)^2}dt=12\int\frac{t^2}{(t^2+1)^2}dt=$$
$$=\left|\begin{matrix}t=tga\\ dt=\frac{da}{\cos^2a}\end{matrix}\right|=12\int tg^2\cos^2ada\ =12\int\sin^2ada=6\int(1-\cos\ 2a)\ da=$$
$$=6arctg\ T-3\sin(2arctg\ \ t)=\left(6arctg\ \sqrt{\frac{9-2x}{2x-21}}-3\sin\left(2arctg\ \sqrt{\frac{9-2x}{2x-21}}\right)\right)\Big|_6^9=$$
$$=6arctg\sqrt{3}-3\sin(2arctg\sqrt{3})-6arctg\frac{1}{3}+3\sin(2arctg\frac{1}{\sqrt{3}})=2\pi-3\sin\frac{2\pi}{3}-$$
$$-\pi+3\sin\frac{\pi}{3}=\pi-3\frac{\sqrt{3}}{2}+3\frac{\sqrt{3}}{2}=\pi.$$

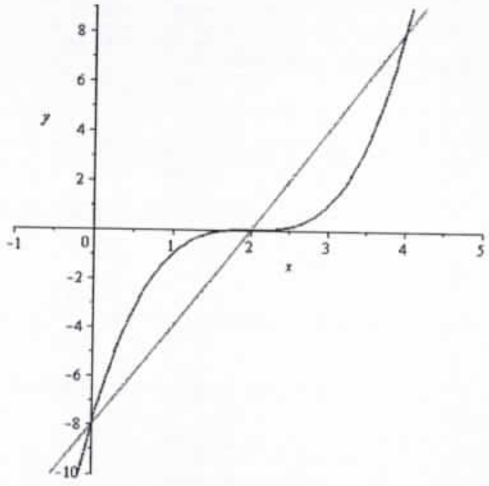
11.21-masala. Quyidagi $y=(x-2)^3, y=4x-8.$ **chiziqlar bilan chegaralangan shaklning yuzasi hisoblansin.**

$\begin{cases}y=4x-8.\\ y=(x-2)^3\end{cases}$ Sistemani echib, bu chiziqlarning kesishish nuqtalarini topamiz:

$M_1(0,-8)$, $M_2(2,0)$ va $M_3(4,8)$ U holda

$$S=2\int_0^2((x-2)^3-4x+8)dx=2\int_0^2(x^3-6x^2+12x-8-4x+8)dx=$$
$$=2\int_0^2(x^3-6x^2+8x)dx=2(\frac{1}{4}x^4-2x^3+4x^2)\Big|_0^2=\frac{1}{2}\cdot 2^4-4\cdot 2^3+8\cdot 2^2=8.$$

```
> with(plots) :
> solve({y = 4·x − 8, y = (x − 2)^3}, {x, y})
                {x = 0, y = −8}, {x = 2, y = 0}, {x = 4, y = 8}
> plot([4·x − 8, (x − 2)^3], x = −1..5, y = 9..−10, color = [red, blue]);
```



> with(Student[Calculus1]) :

> IntTutor(2·((x − 2)^3 − 4·x + 8))

$$\int_0^2 (2(x-2)^3 - 8x + 16)\,dx$$

$$= \int_0^2 2(x-2)^3\,dx + \int_0^2 -8x\,dx + \int_0^2 16\,dx \qquad [sum]$$

$$= 2\left(\int_0^2 (x-2)^3\,dx\right) + \int_0^2 -8x\,dx + \int_0^2 16\,dx \qquad [constantmultiple]$$

$$= 2\left(\int_{-2}^0 u^3\,du\right) + \int_0^2 -8x\,dx + \int_0^2 16\,dx \qquad [change, u = x-2, u]$$

$$= -8 + \int_0^2 -8x\,dx + \int_0^2 16\,dx \qquad [power]$$

$$= -8 - 8\left(\int_0^2 x\,dx\right) + \int_0^2 16\,dx \qquad [constantmultiple]$$

$$= -24 + \int_0^2 16\,dx \qquad [power]$$

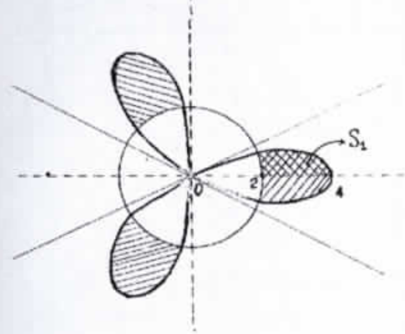
$$= 8 \qquad [constant]$$

$$\int_0^2 (2(x-2)^3 - 8x + 16)\,dx = 8$$

12.21-masala. Tenglamlari qutb koordinatalari sistemasida berilgan chiziqlar bilan chegaralangan shakilning yuzasi hisoblansin.

$$r = 4\cos 3\varphi;\ \ r = 2, (r \ge 2)$$

◁ Birinchi navbatda bu funksiyaning aniqlanish sohasini topamiz va uning chizmasini chizamiz.



6-chizma.

$$D(r) = \left[0, \frac{\pi}{6}\right] \cup \left[\frac{\pi}{2}, \frac{5\pi}{6}\right] \cup \left[\frac{7\pi}{6}, \frac{3\pi}{2}\right] \cup \left[\frac{11\pi}{6}, 2\pi\right]$$

Yuzasini hisoblashimiz kerak bo‘lgan soha 6-chizmada shtrixlab ko‘rsatilgan.Integrallash chegarasini topishimiz uchun $\begin{cases} r = 4\cos 3\varphi \\ r = 2 \end{cases}$ sistemasidan φ ni topamiz

$$\Rightarrow \cos 3\varphi = \frac{1}{2} \Rightarrow 3\varphi = \frac{\pi}{3} \Rightarrow \varphi = \frac{\pi}{9} \Rightarrow$$

$$S = 6S_1 = 6\cdot\frac{1}{2}\int_0^{\frac{\pi}{9}}\left[(4\cos 3\varphi)^2 - 2^2\right]d\varphi = 3\cdot\left[16\int_0^{\pi/9}\cos^2 3\varphi d\varphi - 4\int_0^{\pi/9} d\varphi\right] =$$

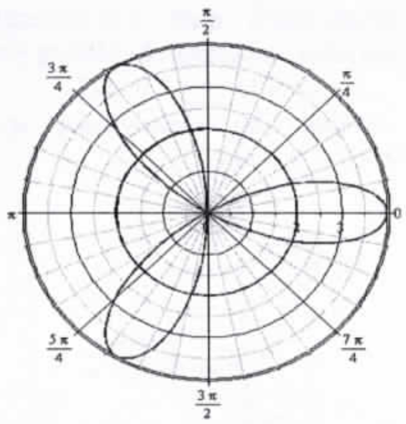
$$= 3\left[8\cdot\int_0^{\pi/9}(1+\cos 6\varphi)d\varphi - 4\varphi\ \Big|_0^{\frac{\pi}{9}}\right] = 3\left[8\left(\varphi + \frac{1}{6}\sin 6\varphi\right)\Big|_0^{\frac{\pi}{9}} - \frac{4\pi}{9}\right] = 3\left[\frac{8\pi}{9} + \frac{4}{3}\sin\frac{2\pi}{3} - \frac{4\pi}{9}\right] =$$

$$3\left[\frac{4\pi}{9} + \frac{2\sqrt{3}}{3}\right] = \frac{4\pi}{3} + 2\sqrt{3}. \triangleright$$

Maple dasturida quydagicha ko‘rinishga ega bo‘ladi.

>

> polarplot([[4·cos(3·t), t, t = 0..2·π], [2, t, t = 0..2·π]], numpoints = 50)



> $S=\frac{6\cdot 1}{2}\int_0^{\frac{\pi}{9}}\left((4\cdot\cos(3\cdot x))^2-4\right)\mathrm{d}x$

$$S=2\sqrt{3}+\frac{4}{3}\pi$$

13.21-masala. Parametrik ko'rinishda berilgan egri chiziq yoyining uzunligi hisoblansin.

$$x=3(t-\sin t),\, y=3(1-\cos t);\ \ 0\le x\le\pi$$

$$\begin{cases}x=3(t-\sin t),\\ y=3(1-\cos t);\end{cases}\ \begin{cases}x'(t)=3(1-\cos t),\\ y'(t)=3\cdot\sin t.\end{cases}$$

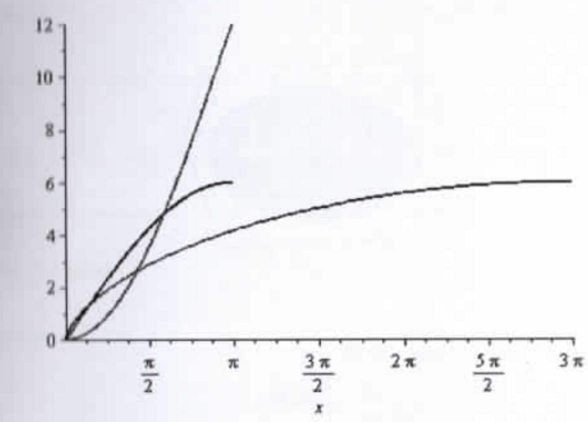
$$l=\int_0^{\pi}\sqrt{(3(1-\cos t))^2+(3\sin t)^2}\,dt=3\int_0^{\pi}\sqrt{1-2\cos t+\cos^2 t+\sin^2 t}\,dt=\{\cos^2 t+\sin^2 t=1\}=$$

$$=3\int_0^{\pi}\sqrt{2-2\cos t}\,dt=3\int_0^{\pi}\sqrt{2(1-\cos t)}\,dt=\left\{1-\cos t=2\sin^2\frac{t}{2}\right\}=3\int_0^{\pi}\sqrt{2\cdot 2\sin^2\left(\frac{t}{2}\right)}\,dt=$$

$$=6\int_0^{\pi}\sin\frac{t}{2}\,dt=-6\cdot 2\cos\frac{t}{2}\Big|_0^{\pi}=-12\left(\cos\frac{\pi}{2}-\cos 0\right)=-12\cdot(0-1)=12;$$

Maple dasturida quydagicha bo'ladi.

```
> with(Student[Calculus1]) :
> ArcLength([3·(x − sin(x)), 3·(1 − cos(x))], output = plot, x = 0..π)
```

—— $f(x)$ —— $g(x)=\sqrt{1+\left(\frac{d}{dx}f(x)\right)^2}$ —— $\int_0^x g(s)\,ds$

The arc length of $f(x)=[3x-3\sin(x),\ 3-3\cos(x)]$ on the interval $[0,\pi]$. The coordinate system is Cartesian.

```
> ArcLength([3·(x − sin(x)), 3·(1 − cos(x))], x = 0..π, output = integral)
```

$$\int_0^{\pi}3\sqrt{\cos(x)^2+\sin(x)^2-2\cos(x)+1}\,dx$$

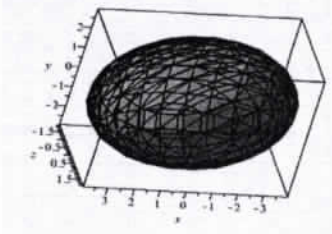
```
> ArcLength([3·(x − sin(x)), 3·(1 − cos(x))], x = 0..π)
```

$$12$$

14.21-masala. Quyidagi $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ sirt bilan chegaralangan jismning hajmi topilsin.

```
> with(plots) :
> implicitplot3d( x^2/16 + y^2/9 + z^2/4 = 1, x = −4..4, y = −3..3, z = −2..2, scaling = CONSTRAINED,
);
```

Maple dasturi yordamida chizib olamiz.



◃Hajimni (18)-formulaga ko‘ra

$$V=\int_{x_1}^{x_2} S(x)dx$$

formula yordamida hisoblaymiz. Buning uchun $S(x)$ ni topish lozim.

O‘zgaruvchi x ni fiksirlasak, ellipsoid kesimida.

$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1-\frac{x^2}{a^2}$ yoki $\frac{y^2}{\left(b\sqrt{1-\frac{x^2}{a^2}}\right)^2}+\frac{z^2}{\left(c\sqrt{1-\frac{x^2}{a^2}}\right)^2}=1$ ellips hosil bo‘ladi. Bizga ma’lumki, $\frac{y^2}{m^2}+\frac{z^2}{n^2}=1$ ellipsning yuzasi πmn ga teng edi.$\Rightarrow$

$$S(x)=\pi\cdot b\sqrt{1-\frac{x^2}{a^2}}\cdot c\sqrt{1-\frac{x^2}{a^2}}=\pi bc\cdot\left(1-\frac{x^2}{a^2}\right)\Rightarrow V=\int_{-a}^{a}S(x)dx=\pi bc\int_{-a}^{a}\left(1-\frac{x^2}{a^2}\right)dx=$$

$$=\pi bc\left(x-\frac{x^3}{3a^2}\right)\Big|_{-a}^{a}=\frac{4}{3}\pi abc.\triangleright$$

Natija. Agar $a=b=c=R$ bo‘lsa, ellipsoid sharga aylanadi va shar xajmini hisoblash usuli

$$V=\frac{4}{3}\pi R^3$$

formulani hosil qilamiz.

15.21-masala. Quyidagi

$$y=2x-x^2, y=-x+2.$$

chiziqlar bilan chegaralangan shaklni OX o‘qi atrofida aylantirishdan hosil bo‘lgan jismning hajmi topilsin.

◃Avval OX o‘qi atrofida aylantirish kerak bo‘lgan D sohani chizib olamiz **(7-chizma).**

$$V=\pi\int y^2dx$$

$$V=\pi\int_1^2(2x-x^2+x-2)^2dx=\pi\int_1^2(-x^2+3x-2)^2dx=$$

$$=\pi\int_1^2(x^4-6x^3+13x^2-12x+4)dx=\pi\left(\frac{1}{5}x^5-\frac{3}{2}x^4+\frac{13}{3}x^3-6x^2+4x\right)\Big|_1^2=$$

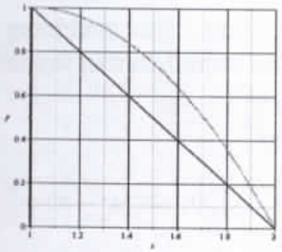
$$=\pi\left(\frac{32}{5}-24+\frac{104}{3}-24+8-\frac{1}{5}+\frac{3}{2}-\frac{13}{3}+6-4\right)=\frac{\pi}{30}.$$

> with(Student[Calculus1]) :

> with(Student[Calculus1]) :

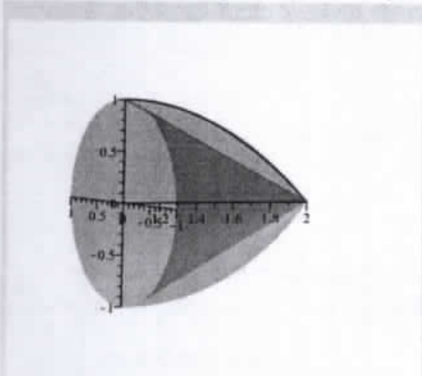
> with(plots) :

> plot([2·x − x², −x + 2], x = 1 ..2, y = 0 ..1, color = [red, blue]);



7-chizma.

> VolumeOfRevolution(2·x − x², −x + 2, x = 1 ..2, output = plot, axis = horizontal)



The solid of revolution created on $1\le x\le 2$ by rotation of $f(x)=-x^2+2x$ and $g(x)=-x+2$ about the axis $y=0$

> $VolumeOfRevolution(2\cdot x - x^2, -x + 2, x = 1..2, output = integral, axis = horizontal)$

$$\int_1^2 \pi\left|x^4 - 4x^3 + 3x^2 + 4x - 4\right| dx$$

> $VolumeOfRevolution(2\cdot x - x^2, -x + 2, x = 1..2, axis = horizontal)$

$$\frac{1}{5}\pi$$

16.21-masala. Quyidagi

$$x^2 + 4y - 16 = 0,\ y = 0$$

chiziqlar bilan chegaralangan shakilning og'irlik markazi topilsin.

◅Masala shartidan ko'rinadiki berilgan chiziqlar bilan chegaralangan D sohani ushbu

$$D = \begin{cases} -4 \le x \le 4 \\ 0 \le y \le 4 - \frac{x^2}{4} \end{cases}$$

ko'rinishda ifodalash mumkin. Bu shaklning og'irlik markazining koordinatalarini (23) va (24)-formulalardan foydalanib topamiz.

$$S = \int_{-4}^{4}\left(4 - \frac{x^2}{4}\right)dx = \left(4x - \frac{x^3}{12}\right)_{-4}^{4} = \frac{64}{3}.$$

$$M_x = \frac{1}{2}\int_{-4}^{4} y^2 dx = \frac{1}{2}\int_{-4}^{4}\left(4 - \frac{x^2}{4}\right)^2 dx = \frac{1}{2}\int_{-4}^{4}\left(16 - 2x^2 + \frac{x^4}{16}\right)dx = \frac{1}{2}\left(16 - \frac{2x^3}{3} + \frac{x^5}{5\cdot 16}\right)_{-4}^{4} = \frac{512}{15}.$$

$$M_y = \int_{-4}^{4} xy dx = \int_{-4}^{4} x\cdot\left(4 - \frac{x^2}{4}\right)dx = \int_{-4}^{4}\left(4x - \frac{x^3}{4}\right)dx = \left(2x^2 - \frac{x^4}{16}\right)_{-4}^{4} = 0$$

Bu erdan $\left(\bar{x}, \bar{y}\right) = \left(\frac{M_y}{S}, \frac{M_x}{S}\right) = \left(0, \frac{8}{5}\right)$ ekanligini topamiz.▻

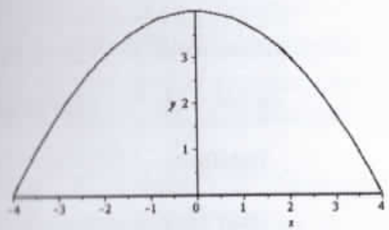
> $with(plots):$

>

> $solve(\{x^2 + 4\cdot y - 16 = 0, y = 0\}, \{x, y\})$

$$\{x = 4, y = 0\},\ \{x = -4, y = 0\}$$

> $implicitplot([x^2 + 4\cdot y - 16 = 0, y = 0], x = -4..4, y = 0..4, color$ = ["NavyBlue", "Teal"])



> $f := 4 - \frac{x^2}{4}$

$$f := 4 - \frac{1}{4}x^2$$

> $S := \int_{-4}^{4} f\, dx$

$$S := \frac{64}{3}$$

> $M_x := \frac{1}{2}\int_{-4}^{4} f^2\, dx$

$$M_x := \frac{512}{15}$$

> $M_y := \int_{-4}^{4} x\, f\, dx$

$$M_y := 0$$

> $(x_0, y_0) = \left(\frac{M_y}{S}, \frac{M_x}{S}\right)$

$$(x_0, y_0) = \left(0, \frac{8}{5}\right)$$

17.21-masala.. Xosmas integrallarni yaqinlashuvchilikka tekshiring.

a) $\int_0^{\infty}\frac{xdx}{\sqrt{x^2+4}}$; b) $\int_0^4 \frac{dx}{\sqrt{16-x^2}}$

Yechish a) Birinchi tur xosmas integrali.

$$\int_0^{\infty}\frac{xdx}{\sqrt{x^2+4}} = \left|\begin{matrix} t = x^2 + 4 \\ dt = 2x\,dx \\ xdx = \frac{1}{2}dt \\ x_1 = 0 \Leftrightarrow t_1 = 4 \\ x_2 = -\infty \Leftrightarrow t_2 = \infty \end{matrix}\right| = \int_4^{\infty}\frac{\frac{1}{2}dt}{\sqrt{t}} = \lim_{b\to\infty}\int_4^b \frac{1}{2}t^{-\frac{1}{2}}dt =$$

{Integgralni hisoblab (**2**)}

$$= \lim_{b\to\infty}\left(t^{\frac{1}{2}} \Big|_4^b \right) = = \lim_{b\to\infty}\left(\sqrt{t} \Big|_4^b \right) = \lim_{b\to\infty}\left(\sqrt{b}-2\right) = \infty$$ Uzoqlashuvchi ekanini topamiz.

b) Ikkinchi tur xosmas integrali.

$x=4$ Nuqta integral ostidagi funksiyaning uzulish nuqtasi :

$$\int_0^4 \frac{dx}{\sqrt{16-x^2}} = \lim_{\varepsilon\to 0} \int_0^{4-\varepsilon} \frac{dx}{\sqrt{16-x^2}} = \lim_{\varepsilon\to 0} \int_0^{4-\varepsilon} \frac{dx}{\sqrt{4^2-x^2}} =$$

{Integralni hisoblaymiz}

$$= \lim_{\varepsilon\to 0}\left(\arcsin\frac{x}{4} \Big|_0^{4-\varepsilon} \right) = \lim_{\varepsilon\to 0}\left(\arcsin\left(1-\frac{\varepsilon}{4}\right) - \arcsin 0 \right) = \frac{\pi}{2}$$ - Demak integral yaqinlashuvchi.

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MUNDARIJA

A.G.ABDURAXMONOV

ANIQMAS VA ANIQ INTEGRALLAR

(Matematik analiz)

Kunduzgi, kechki va sirtqi ta'lim talabalari uchun

O'quv qo'llanma

Muharrir: X. Tahirov
Texnik muharrir: S. Meliquziyeva
Musahhih: M. Yunusova
Sahifalovchi: A. Muhammad

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