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OLIY VA O‘RTA MAXSUS TA’LIM VAZIRLIGI

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FUNKSIONAL ANALIZ

(misol va masalalar yechish)

II QISM

O‘quv qo‘llanma

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Mazkur o'quv qo'llanmada "Funksional analiz"kursining II qismini tashkil etuvchi chiziqli fazolar, operatorlar nazariyasi va integral tenglamalari kabi mavzulari bayon qilingan. Har bir mavzuga oid asosiy tushunchalar ta'rifi, asosiy teoremalar va xossalar keltirilgan. Namunaviy misollar tahlil qilingan. Amaliy mashg'ulot va mavzularni mustaqil o'rganish uchun misol va masalalar berilgan. Ushbu o'quv qo'llanma universitetlarning "Matematika", "Mexanika"va "Amaliy matematika va informatika"yo'nalishlari talabalariga mo'ljallangan bo'lib, undan boshqa yo'nalish talabalari ham foydalanishlari mumkin.

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Kirish

Funksional analiz matematikaning alohida bo'limi sifatida XVIII asrning oxiri va XIX asr boshlarida shakllana boshlangandir. Funksional analizga oid dastlabki ilmiy ishlar italyan matemagi Volterra, fransuz matematigi Puankare va nemis matematigi Hilbertga taalluqlidir. Metrik fazo tushunchasi fanga fransuz matematigi Freshe tomonidan XX asr boshlarida kiritilgan, normalangan fazo tushunchasi 1922 yilda polyak matematigi Banax va unga bog'liq bo'lmagan holda amerikalik matematik Viner tomonidan kiritilgan.

Ma'lumki universitetlarning "Matematika", "Mexanika" va "Amaliy matematika va informatika" yo'nalishlari uchun tuzilgan o'quv rejada "Funksional analiz" fani ko'zda tutilgan bo'lib, ushbu fan ixtisoslik fanlar ichida asosiy o'rin tutadi.

Universitetlarda "Funksional analiz" kursi asosan ikki qismdan iborat, ular shartli ravishda quyidagicha nomlanadi:
I qism. Haqiqiy o'zgaruvchining funksiyalari nazariyasi.
II qism. Operatorlar nazariyasi.

Amaldagi "Funksional analiz" kursi fan dasturida I qism "To'plamlar nazariyasi va metrik fazolar" va "Lebeg integrali" bo'limlarini o'z ichiga oladi. II qism esa, "Normalangan fazolar", "Operatorlar nazariyasi" va "Integral tenglamalar" bo'limlarini o'z ichiga oladi. Universitetlarning matematika mutaxassisligida ta'lim oluvchi talabalar uchun "Funksional analiz" kursidan o'zbek alifbosida, birinchi marotaba akademik T.A. Sarimsoqov tomonidan darslik nashr etilgan (T.A. Sarimsoqov. Haqiqiy o'zgaruvchining funksiyalari nazariyasi; T.A. Sarimsoqov. Funksional analiz kursi). Oliy ta'lim tizimida ikki bosqichli tizimga (bakalavriatura va magistratura) o'tish munosabati bilan OTMlari barcha fan dasturlarida jiddiy o'zgarishlar yuzaga keldi. Bu o'zgarishlar o'z navbatida barcha yo'nalishlar uchun tegishli darslik va o'quv qo'llanmalarni ishlab chiqishni talab etmoqda. Ushbu [6, 7, 8] o‘quv qo‘llanmalar va [9] darslik, universitetlarning bakalavriatura o‘quv rejasidagi "Funksional analiz" kursidan yuqoridagi ehtiyojlarni qondirish maqsadida yaratilgandir.

Mazkur o‘quv qo‘llanma, universitetlarning matematika yo‘nalishi talabalari uchun "Funksional analiz" kursidan amaliy mashg‘ulotlarni olib borishda lotin yozuviga asoslangan o‘zbek alifbosida adabiyotlar tanqisligining oldini olish maqsadida yozilmoqda. Mazkur qo‘llanmada "Funksional analiz" fanining II qismini tashkil etuvchi chiziqli fazolar, operatorlar nazariyasi va integral tenglamalari mavzulari uchun misol va masalalar hamda testlar berilgan.

Birinchi bo‘lim chiziqli fazolar nazariyasiga bag‘ishlangan bo‘lib, qo‘llanmada chiziqli fazolar, normalangan fazolar, Hilbert fazolariga oid mavzular bayon qilinadi. Qo‘llanmaning bu bo‘limi chiziqli fazolar, chiziqli fazoning qism fazosi va faktor fazolari, normalangan chiziqli fazolar, Banax fazolari, Yevklid va Hilbert fazolari kabi mavzularni o‘z ichiga oladi. Har bir mavzu namunaviy misollar bilan boyitilgan. Shuningdek, mavzularni o‘zlashtirish va mustaqil o‘rganish uchun yetarlicha misol, masala va testlar berilgan.

Ikkinchi bo‘lim chiziqli operatorlar nazariyasiga bag‘ishlangan. Qo‘llanmaning bu bo‘limida, chiziqli uzluksiz operatorlar, teskari operatorlar, qo‘shma operatorlar va chiziqli operatorning spektrlariga oid mavzular bayon qilingan. Har bir mavzuga oid asosiy tushunchalar ta’rifi, asosiy teoremalar va xossalar keltirilgan. Mavzular namunaviy misollar yechimi bilan boyitilgan. Shuningdek, mavzularni o‘zlashtirish va mustaqil o‘rganish uchun yetarlicha masalalar va testlar berilgan.

Uchinchi bo‘lim kompakt operatorlar va integral tenglamalar nazariyasiga bag‘ishlangan. Qo‘llanmaning bu bo‘limida chekli o‘lchamli operatorlar, kompakt operatorlar va integral tenglamalar mavzulari bayon qilinadi. Mavzuga oid asosiy tushunchalar ta’rifi, asosiy teoremalar va xossalar keltirilgan.

Misol va masalalarni tuzishda Sh.A. Ayupov (Funksional analizdan misol va masalalar, Nukus, "BILIM", 2009) va Ю.С. Очан (Сборник задач по математическому анализу, Москва, Просвещение, 1981), Городецкий В.В., Нагнибида Н.И., Настасиев П.П. Методы решения задач по функциональному анализу. Киев 1990. Shuningdek, Abdullayev J.I., G'anixo'jayev R.N., Shermatov M.H., Egamberdiyevlarning "Funksional analiz" (Toshkent-Samarqand, 2009) o'quv qo'llanmasidan keng foydalanildi.

Ushbu o'quv qo'llanma funksional analiz fanidan namunaviy o'quv dasturga moslab tuzilgan masalalar to'plamidir. U universitetlarning matematika va mexanika bakalavriyat yo'nalishlari bo'yicha ta'lim olayotgan talabalari uchun mo'ljallab yozilgan. Bundan tashqari masalalar to'plamidan matematik tahlil va matematik fizika mutaxassisliklari bo'yicha ta'lim olayotgan magistrantlar hamda katta ilmiy xodim izlanuvchilar foydalanishlari mumkin. Masalalar to'plami funksional analizning asosiy boblarini o'z ichiga olgan. U nisbatan soddaroq misollardan tashkil topgan bo'lib, o'quvchini misol yechishga rag'batlantiradi. Masalalar to'plamida keltirilgan misollar oldingi misollar bilan aloqador. Shuning uchun misollarning barchasini yechish kerak.

O'quv qo'llanmaning asosiy maqsadi bo'lg'usi mutaxassislarni funksional analizning asosiy tushunchalari va usullari bilan tanishtirishdan iborat. O'quv qo'llanma talabalarni funksional analizga oid masalalarni yechishga o'rgatadi hamda ularda yetarli darajada texnik mahorat hosil qiladi. Ushbu to'plam O'zMU va SamDUda "Funksional analiz" fanidan ma'ruza va amaliy mashg'ulotlar olib boruvchi professor-o'qituvchilarning ko'p yillik ish tajribalari asosida tuzilgan.

O'quv qo'llanma 3 bob, 10 paragrafdan iborat. Har bir paragraf boshida qisqacha nazariy material berilgan. Har bobdan so'ng talabalar o'z bilimlarini tekshirishlari uchun test savollari javoblari bilan berilgan.

Mualliflar o'quv qo'llanmani yaxshilashda bergan foydali maslahatlari uchun mas'ul muharrir va taqrizchilarga, hamda matnni tahrir qilgani uchun B.E.Davranovga o'z minnatdorchiliklarini bildiradilar.

Masalalar to'plami birinchi marta chop qilinayotgani uchun xato va kamchiliklar bo'lishi mumkin. Xato va kamchiliklar haqidagi fikrlaringizni jabdullaev@mail.ru elektron manziliga jo'natishlaringizni so'raymiz.

I bob. Chiziqli fazolar

Bu bobda biz chiziqli fazolar, chiziqli normalangan fazolar, Evklid va Hilbert fazolarining xossalari hamda chiziqli funksionalning umumiy xossalarini o‘rganamiz. Bu bob 4 (1-4) paragrafdan iborat.

$1-\S$ da chiziqli fazo va ularga doir misollar jamlangan. Chiziqli fazo o‘lchami ta’riflanib, chekli va cheksiz o‘lchamli chiziqli fazolarga misollar keltirilgan. Bu yerda chiziqli fazoning qism fazosi va faktor fazosiga doir misollar ham bor.

$2-\S$ da chiziqli normalangan fazolarga ko‘plab misollar qaralgan.

$3-\S$ Evklid va Hilbert fazolariga bag‘ishlangan. Evklid fazolarining xarakteristik xossalari, Koshi-Bunyakovskiy tengsizligi, Bessel tengsizligi, Parseval tengliklarini tushunishga doir misollar qaralgan. Riss-Fisher, Shmidtning ortogonallashtirish jarayonini qo‘llashga doir misollar keltirilgan. Hilbert fazolarining qism fazosi, qism fazoning ortogonal to‘ldiruvchisi, ortogonal qism fazolarning to‘g‘ri yig‘indilari qaralgan. Xuddi shunday Hilbert fazolarining to‘g‘ri yig‘indilari ham qaralgan.

$4-\S$ da chiziqli funksionallar, ularning xossalariga doir misollar qaralgan. Qavariq to‘plamlar va qavariq funksionallarning xossalarini tahlil qilishga doir misollar ham shu paragrafdan joy olgan. Chiziqli funksionalni davom ettirish haqidagi Xan-Banax teoremasining qo‘llanishiga doir misollar ham shu yerda.

1-§. Chiziqli fazolar

Chiziqli fazo tushunchasi matematikada asosiy tayanch tushunchalardan hisoblanadi. Yuqoridagi kelishuvimizga ko‘ra $\mathbb{C}$ kompleks sonlar, $\mathbb{R}$ esa haqiqiy sonlar to‘plamini bildiradi. K orqali $\mathbb{C}$ yoki $\mathbb{R}$ ni belgilaymiz.

1.1-ta’rif. *Agar elementlari* $x, y, z, \ldots$ *bo‘lgan* L *to‘plamda quyidagi ikki amal aniqlangan bo‘lsa:*

I. Ixtiyoriy ikkita $x, y \in L$ *elementlarga ularning yig'indisi deb ataluvchi aniq bir* $x + y \in L$ *element mos qo'yilgan bo'lib, ixtiyoriy* $x, y, z \in L$ *elementlar uchun*

1) $x + y = y + x$ *(kommutativlik),*

2) $x + (y + z) = (x + y) + z$ *(assotsiativlik),*

3) L *da shunday* θ *element mavjud bo'lib,* $x + \theta = x$ *(nolning mavjudligi),*

4) *shunday* $-x \in L$ *element mavjud bo'lib,* $x + (-x) = \theta$ *(qarama-qarshi elementning mavjudligi) aksiomalar bajarilsa;*

II. ixtiyoriy $x \in L$ *element va ixtiyoriy* $\alpha \in K$ *uchun* x *elementning* α *songa ko'paytmasi deb ataluvchi aniq bir* $\alpha x \in L$ *element mos qo'yilgan bo'lib, ixtiyoriy* $x, y \in L$ *va barcha* $\alpha, \beta \in K$ *sonlar uchun*

5) $\alpha(\beta x) = (\alpha \beta)x,$

6) $1 \cdot x = x,$

7) $(\alpha + \beta)\, x = \alpha\, x + \beta\, x,$

8) $\alpha\,(x + y) = \alpha\, x + \alpha\, y$ *aksiomalar bajarilsa, u holda* L *to'plam* K *maydon ustidagi chiziqli fazo deyiladi.*

Ta'rifda kiritilgan I va II amallar mos ravishda *yig'indi* va *songa ko'paytirish amallari* deyiladi. Agar L ning elementlarini haqiqiy sonlarga (kompleks sonlarga) ko'paytirish aniqlangan bo'lsa, u holda L ga *haqiqiy (kompleks) chiziqli fazo* deyiladi.

1.2-ta'rif. *Agar* L *va* L^* *chiziqli fazolar o'rtasida biyektiv moslik o'rnatish mumkin bo'lib,* $x \leftrightarrow x^*$ *va* $y \leftrightarrow y^*$ $(x, y \in L,\; x^*, y^* \in L^*)$ *ekanligidan* $x + y \leftrightarrow x^* + y^*$ *va* $\alpha\, x \leftrightarrow \alpha\, x^*$ $(\alpha -$ *ixtiyoriy son) ekanligi kelib chiqsa, u holda* L *va* L^* *chiziqli fazolar o'zaro izomorf fazolar deyiladi.*

L chiziqli fazo, $x_1,\ x_2, \ldots, x_n$ uning elementlari bo'lsin.

1.3-ta'rif. *Agar* L *chiziqli fazoning* $x_1,\ x_2, \ldots, x_n$ *elementlar sistemasi uchun hech bo'lmaganda birortasi noldan farqli bo'lgan* $a_1, a_2, \ldots, a_n$ *sonlar* *mavjud bo'lib,*

$$a_1x_1 + \ a_2x_2 + \ \cdots \ + a_nx_n = 0 \tag{1.1}$$

tenglik bajarilsa, u holda $x_1, x_2, \ldots, x_n$ *elementlar sistemasi chiziqli bog'langan deyiladi. Aks holda, ya'ni (1.1) tenglikdan*

$$a_1 = \ a_2 = \ \cdots \ = a_n = 0$$

ekanligi kelib chiqsa, $x_1, x_2, \ldots, x_n$ *elementlar sistemasi chiziqli bog'lanmagan yoki chiziqli erkli deyiladi.*

1.4-ta'rif. *Agar* $x_1, x_2, \ldots, x_n, \ldots$ *cheksiz elementlar sistemasining ixtiyoriy chekli qism sistemasi chiziqli erkli bo'lsa, u holda* $\{x_n\}_{n=1}^{\infty}$ *sistema chiziqli erkli deyiladi.*

1.5-ta'rif. *Agar* L *chiziqli fazoda* n *elementli chiziqli erkli sistema mavjud bo'lib, bu fazoning ixtiyoriy* $n+1$ *ta elementdan iborat sistemasi chiziqli bog'langan bo'lsa, u holda* L *ga* n *o'lchamli chiziqli fazo deyiladi va* $\dim L = n$ *kabi yoziladi.*

1.6-ta'rif. n *o'lchamli* L *chiziqli fazoning ixtiyoriy* n *ta elementdan iborat chiziqli erkli sistemasi shu fazoning bazisi deyiladi.*

1.7-ta'rif. *Agar* L *chiziqli fazoda ixtiyoriy* $n \in \mathbb{N}$ *uchun* n *elementli chiziqli erkli sistema mavjud bo'lsa, u holda* L *cheksiz o'lchamli chiziqli fazo deyiladi va* $\dim L = \infty$ *ko'rinishda yoziladi.*

Endi biz mavzuga oid misollar qaraymiz. Quyida $\mathbb{R}^n$ to'plam va unda yig'indi va songa ko'paytirish amallari berilgan. Bu amallar uchun chiziqli fazoning 1-8 aksiomalari bajarilishini tekshiring.

1.1. $\mathbb{R}^n = \{x = (x_1,\ x_2,\ \ldots,\ x_n),\ x_i \in \mathbb{R},\ i = 1, 2, \ldots, n\}$ $-\, n$ ta haqiqiy sonlarning tartiblangan guruhlaridan iborat to'plam. Bu yerda elementlarni qo'shish va songa ko'paytirish amallari quyidagicha aniqlanadi. Ixtiyoriy $x = (x_1\, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ va $\alpha \in \mathbb{R}$ lar uchun

$$x + y = (x_1 + y_1,\, x_2 + y_2,\, \ldots,\, x_n + y_n)\ , \tag{1.2}$$

$$\alpha\, x = (\alpha\, x_1,\, \alpha\, x_2,\, \ldots,\, \alpha\, x_n)\,. \tag{1.3}$$

Yechish. Qo‘shish va songa ko‘paytirish amallari uchun chiziqli fazo aksiomalari bajarilishini tekshiramiz. Ixtiyoriy $x,\, y \in \mathbb{R}^n$ lar uchun $x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \in \mathbb{R}^n$ ekanligi ma’lum. Xuddi shunday ixtiyoriy $\alpha \in \mathbb{R}$ uchun $\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n) \in \mathbb{R}^n$ munosabat o‘rinli. Haqiqiy sonlarni qo‘shish kommutativ va assotsiativ, shuning uchun quyidagi tengliklar o‘rinli:

$$x + y = (x_1 + y_1, \ldots, x_n + y_n) = (y_1 + x_1, \ldots, y_n + x_n) = y + x,$$

$$x + (y + z) = (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), \ldots, x_n + (y_n + z_n)) =$$

$$= ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2, \ldots, (x_n + y_n) + z_n) = (x + y) + z.$$

$\mathbb{R}^n$ da nol element rolini $\theta = (0, 0, \ldots, 0)$ vektor bajaradi. Chunki ixtiyoriy $x \in \mathbb{R}^n$ uchun $x + \theta = (x_1 + 0,\, x_2 + 0, \ldots, x_n + 0) = x$ tenglik o‘rinli. $x \in \mathbb{R}^n$ elementga qarama-qarshi element $-x = (-x_1, -x_2, \ldots, -x_n)$ bo‘ladi, chunki

$$x + (-x) = (x_1 + (-x_1),\, x_2 + (-x_2),\, \ldots\, ,\, x_n + (-x_n)) = (0,\, 0, \ldots, 0) = \theta.$$

Demak, 1-4 aksiomalar o‘rinli. Endi songa ko‘paytirish amali bilan bog‘liq aksiomalarning bajarilishini tekshiramiz. Ixtiyoriy $\alpha,\, \beta \in \mathbb{R}$ lar uchun

$$\alpha(\beta\, x) = (\alpha(\beta\, x_1),\, \alpha(\beta\, x_2),\, \ldots,\, \alpha(\beta\, x_n)) =$$

$$= ((\alpha\beta)\, x_1,\, (\alpha\beta)\, x_2, \ldots,\, (\alpha\beta)\, x_n) = (\alpha\, \beta)\, x$$

tengliklar o‘rinli. Xuddi shunday

$$1 \cdot x = (1 \cdot x_1,\ 1 \cdot x_2,\, \ldots,\, 1 \cdot x_n) = (x_1,\ x_2,\, \ldots,\, x_n) = x$$

tenglik o‘rinli. Ixtiyoriy $\alpha,\ \beta \in \mathbb{R}$ va $x \in \mathbb{R}^n$ lar uchun

$$(\alpha+\beta)\,x = ((\alpha+\beta)x_1,\ (\alpha+\beta)x_2,\ \ldots,\ (\alpha+\beta)x_n) =$$

$$= (\alpha\, x_1 + \beta\, x_1,\ \alpha\, x_2 + \beta\, x_2,\ \ldots,\ \alpha\, x_n + \beta\, x_n) = (\alpha\, x_1,\ \alpha\, x_2,\ \ldots,\ \alpha\, x_n) +$$

$$+(\beta x_1, \beta x_2, \ldots, \beta x_n) = \alpha(x_1, x_2, \ldots, x_n) + \beta(x_1, x_2, \ldots, x_n) = \alpha x + \beta x$$

tengliklar o‘rinli. Ixtiyoriy $\alpha \in \mathbb{R}$ va $x,\ y \in \mathbb{R}^n$ lar uchun

$$\alpha(x+y) = (\alpha(x_1+y_1),\ \alpha(x_2+y_2),\ \ldots,\ \alpha(x_n+y_n)) =$$

$$(\alpha\, x_1 + \alpha\, y_1,\ \alpha\, x_2 + \alpha\, y_2,\ \ldots,\ \alpha\, x_n + \alpha\, y_n) = (\alpha\, x_1,\ \alpha\, x_2,\ \ldots,\ \alpha\, x_n) +$$

$$+(\alpha y_1, \alpha y_2, \ldots, \alpha y_n) = \alpha(x_1, x_2, \ldots, x_n) + \alpha(y_1, y_2, \ldots, y_n) = \alpha x + \alpha y$$

tengliklar bajariladi va $\mathbb{R}^n$ to‘plam *haqiqiy chiziqli* fazo bo‘ladi. □

1.2. $[-a,\ a]$ kesmada aniqlangan, uzluksiz va $x(a) = b$ shartni qanoatlantiruvchi funksiyalar to‘plami, b ning qanday qiymatida chiziqli fazo bo‘ladi. Funksiyalarni qo‘shish va funksiyani songa ko‘paytirish amallari mos ravishda

$$(f+g)\,(x) = f(x) + g(x) \tag{1.4}$$

va

$$(\alpha f)(x) = \alpha f(x) \tag{1.5}$$

ko‘rinishda aniqlanadi.

Yechish. Ma’lumki, (1.4) va (1.5) tengliklar yordamida aniqlangan funksiyalarni qo‘shish va songa ko‘paytirish amallari chiziqli fazo ta’rifidagi 1-8 shartlarni qanoatlantiradi. Shuning uchun berilgan to‘plamning bu amallarga nisbatan yopiqligini ko‘rsatish kifoya. $[-a,\ a]$ kesmada uzluksiz funksiyalar yig‘indisi yana uzluksiz funksiya bo‘ladi. Endi $(x+y)(a) = b$ shartning bajarilishini tekshiramiz. Shartga ko‘ra $(x+y)(a) = x(a) + y(a) = b + b = 2b$

tenglik oʻrinli. Yuqoridagilardan $b = 2b$, ya'ni $b = 0$ shartga kelamiz. Bu holda $\alpha\, x$ funksiya uchun $(\alpha\, x)(a) = 0$ tenglik oʻrinli. Demak, $[-a,\, a]$ kesmada aniqlangan, uzluksiz va $x(a) = 0$ shartni qanoatlantiruvchi funksiyalar toʻplami chiziqli fazo tashkil qiladi. Agar faqat haqiqiy qiymatlar qabul qiluvchi funksiyalar qaralsa, bu fazo haqiqiy chiziqli fazo boʻladi. Agar funksiyalar kompleks qiymatlar qabul qilsa, u holda bu fazo kompleks chiziqli fazo boʻladi.
□

Quyida keltirilgan $\ell_1(\mathbb{Z})$ toʻplam qoʻshish ((1.4) ga qarang) va songa koʻpaytirish ((1.5) ga qarang) amallariga nisbatan chiziqli fazo tashkil qiladimi?

1.3. $\ell_1(\mathbb{Z}) - \mathbb{Z}$ da aniqlangan va $\sum\limits_{n\in\mathbb{Z}} |f(n)| < \infty$ shartni qanoatlantiruvchi funksiyalar toʻplami (1.4) qoʻshish va (1.5) songa koʻpaytirish amallariga nisbatan chiziqli fazo boʻlishligini tekshiring.

Yechish. 1.3-misolning yechimida ta'kidlanganidek $\ell_1(\mathbb{Z})$ toʻplamni qoʻshish va songa koʻpaytirish amallariga nisbatan yopiqligini koʻrsatish kifoya. Faraz qilaylik, f va g lar $\ell_1(\mathbb{Z})$ ning elementlari boʻlsin. $|f(n) + g(n)| \leq |f(n)| + |g(n)|$ va $|\alpha f(n)| = |\alpha|\,|f(n)|$ munosabatlardan, quyidagilar kelib chiqadi:

$$\sum_{n\in\mathbb{Z}} |f(n) + g(n)| \leq \sum_{n\in\mathbb{Z}} |f(n)| + \sum_{n\in\mathbb{Z}} |g(n)| < \infty, \quad f + g \in \ell_1(\mathbb{Z}),$$

$$\sum_{n\in\mathbb{Z}} |\alpha f(n)| = |\alpha| \sum_{n\in\mathbb{Z}} |f(n)| < \infty, \;\; \alpha f \in \ell_1(\mathbb{Z}).$$

Demak, $\ell_1(\mathbb{Z})$ toʻplam qoʻshish va songa koʻpaytirish amallariga nisbatan yopiq. Bu toʻplam kompleks chiziqli fazo boʻladi. □

1.4. $\mathbb{R}^3$ chiziqli fazoda $x_1 = (1, 1, 1)$, $x_2 = (1, 1, 0)$, $x_3 = (1, 0, 0) \in \mathbb{R}^3$ sistema berilgan. Uni chiziqli bogʻlanganlikka tekshiring.

Yechish. $x_1,\ x_2,\ x_3$ elementlarning chiziqli kombinatsiyasini fazoning nol elementiga tenglashtiramiz, ya'ni

$$C_1\,x_1 + C_2\,x_2 + C_3\,x_3 = \theta$$

yoki

$$(C_1 \cdot 1,\, C_1 \cdot 1,\, C_1 \cdot 1) + (C_2 \cdot 1,\, C_2 \cdot 1,\, 0) + (C_3 \cdot 1,\, 0,\, 0) = (0,0,0)\,.$$

Bu yerdan quyidagi tenglamalar sistemasini olamiz:

$$\begin{cases} C_1 + C_2 + C_3 = 0 \\ C_2 + C_3 = 0 \\ C_3 = 0. \end{cases}$$

Bu sistema faqat nol yechimga ega. Shuning uchun $\{x_k\}_{k=1}^{3}$ sistema chiziqli erkli. □

1.5. $L = \mathbb{R}^3$ va $L^* = \mathbb{P}_{\leq 2}$ ($\mathbb{P}_{\leq 2}$ fazo 1.20-misolda aniqlangan) fazolarning izomorfligini isbotlang.

Isbot. Biyektiv $\varphi : \mathbb{R}^3 \to \mathbb{P}_{\leq 2}$ moslikni quyidagicha aniqlaymiz

$$\varphi(x) = \varphi((x_1, x_2, x_3)) = x_1 + x_2\,t + x_3\,t^2. \tag{1.6}$$

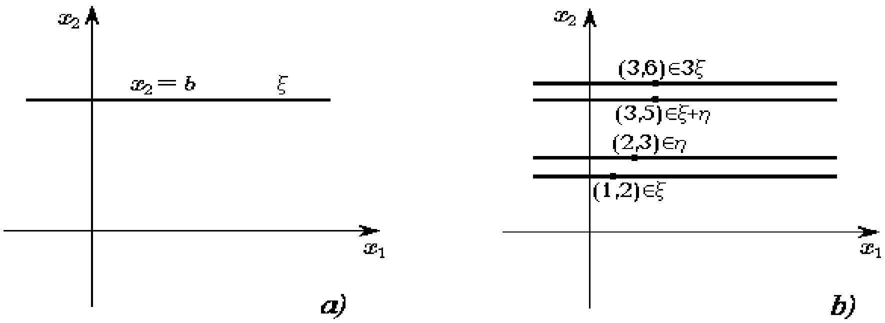
Agar $(x_1, x_2, x_3) \neq (y_1, y_2,\, y_3)$ bo'lsa, u holda $\varphi(x) = x_1 + x_2\,t + x_3\,t^2 = x^*(t)$ va $\varphi(y) = y_1 + y_2\,t + y_3\,t^2 = y^*(t)$ ko'phadlar hech bo'lmaganda bitta koeffitsiyenti bilan farq qiladi, ya'ni $\varphi(x) \neq \varphi(y)$. Bu yerdan (1.6) tenglik bilan aniqlanuvchi $\varphi : \mathbb{R}^3 \to \mathbb{P}_{\leq 2}$ akslantirishning inyektiv ekanligi kelib chiqadi. Ixtiyoriy $a^*(t) = a_1 + a_2 t + a_3 t^2 \in \mathbb{P}_{\leq 2}$ kvadrat uchhad uchun $\varphi(a) = a^*(t)$, $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ tenglik o'rinli, ya'ni $\varphi : \mathbb{R}^3 \to \mathbb{P}_{\leq 2}$ syuryektiv akslantirish. Demak, φ biyektiv akslantirish ekan. $\varphi(x + y) = x^*(t) + y^*(t)$ va $\varphi(\lambda x) = \lambda x^*$, $\lambda \in \mathbb{R}$ tengliklar (1.8) dan bevosita kelib chiqadi. □

1.6. O'zgarishi chegaralangan funksiyalar fazosi $V[a,b]$ ni qaraymiz. Ma'lumki, $[a,b]$ kesmada monoton funksiyalar to'plami $V[a,b]$ ning qism to'plami bo'ladi. Monoton funksiyalar to'plami $V[a,b]$ ning qism fazosi bo'lmaydi. Isbotlang.

Isbot. Ikki monoton funksiyaning yig'indisi har doim monoton funksiya bo'lavermaydi. Bunga quyidagi misolda ishonch hosil qilish mumkin. $x(t) = t^2+1,\ y(t) = -2t$ funksiyalarning har biri $[0,\ 2]$ kesmada monoton funksiya bo'ladi, ammo ularning yig'indisi $x(t)+y(t) = (t-1)^2$ funksiya $[0,\ 2]$ kesmada monoton emas. Demak, $[a,b]$ kesmada monoton funksiyalar to'plami $V[a,b]$ fazoning qism fazosi bo'la olmaydi. □

1.7. $L = \mathbb{R}^2$ fazoning $L' = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 = 0\}$ xos qism fazo bo'yicha L/L' faktor fazoning tavsifini bering, ya'ni L/L' fazo elementlarini tavsiflang.

Yechish. Ma'lumki, $x-y = (x_1-y_1, x_2-y_2) \in L'$ bo'lishi uchun $x_2 = y_2$ bo'lishi zarur va yetarli. Demak, L/L' faktor fazoning elementlari (qo'shni sinflar) Ox_1 o'qiga parallel bo'lgan to'g'ri chiziqlardan iborat.



1.1-chizma

Masalan, $(a,\ b) \in \mathbb{R}^2$ nuqtani o'zida saqlovchi ξ qo'shni sinf Ox_1 o'qiga parallel bo'lgan $x_2 = b$ to'g'ri chiziqdan (1.1 chizmaning a) si) iborat. Xuddi shunday, (1, 2) va (2, 3) nuqtalarni saqlovchi qo'shni sinflar yig'indisi (3, 5) nuqtani saqlovchi $x_2 = 5$ to'g'ri chiziqdan (1.1 chizmaning b) si) iborat.

$(1,2) \in \xi$ qo‘shni sinfning 3 ga ko‘paytmasi (3, 6) nuqtani saqlovchi $x_2 = 6$ to‘g‘ri chiziqdan (1.1-chizmaning b) si) iborat. □

Uy vazifalari va mavzuni o‘zlashtirish uchun masalalar

Quyida 1.8-1.20-misollarda L to‘plam va unda yig‘indi va songa ko‘paytirish amallari berilgan. Bu amallar uchun chiziqli fazoning 1-8 aksiomalari bajarilishini tekshiring.

1.8. $L = \mathbb{R}$ haqiqiy sonlar to‘plami. Haqiqiy sonlar to‘plamida odatdagi qo‘shish va ko‘paytirish amallari.

1.9. $L = \mathbb{C}$ kompleks sonlar to‘plami. Kompleks sonlar to‘plamida kompleks sonlarni qo‘shish va ko‘paytirish amallari.

1.10. $\mathbb{C}^n = \{z = (z_1,\ z_2,\ \ldots,\ z_n)\,,\ z_k \in \mathbb{C},\ k = 1, 2, \ldots, n\}$. Bu yerda ham elementlarni qo‘shish va songa ko‘paytirish amallari (1.2) va (1.3) tengliklar ko‘rinishida aniqlanadi.

1.11. $C[a,b]$ – $[a,b]$ kesmada aniqlangan uzluksiz funksiyalar to‘plami.

1.12. $\ell_2 = \left\{x = (x_1,\ x_2,\ \ldots,\ x_n, \ldots) : \sum\limits_{n=1}^{\infty} |x_n|^2 < \infty\right\}$– kvadrati bilan jamlanuvchi ketma-ketliklar to‘plami. Bu yerda elementlarni qo‘shish va songa ko‘paytirish amallari quyidagicha aniqlanadi:

$$x + y = (x_1 + y_1,\ x_2 + y_2,\ \ldots,\ x_n + y_n, \ldots)\ , \qquad (1.7)$$

$$\alpha x = \alpha\,(x_1, x_2, \ldots, x_n, \ldots) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n, \ldots)\,. \qquad (1.8)$$

Yig‘indi $x + y \in \ell_2$ ekanligi $|\,a + b\,|^2 \leq 2\,|a|^2 + 2|\,b|^2$ tengsizlikdan foydalanib isbotlanadi.

1.13. $c_0 = \{\,x = (x_1, x_2, \ldots, x_n, \ldots) : \lim\limits_{n\to\infty} x_n = 0\}$– nolga yaqinlashuvchi ketma-ketliklar to‘plami. Bu to‘plamda ham qo‘shish va songa ko‘paytirish amallari (1.7) va (1.8) tengliklar ko‘rinishida aniqlanadi.

1.14. $c = \left\{ x = (x_1,\, x_2,\, \ldots,\, x_n, \ldots) : \ \exists \lim\limits_{n \to \infty} x_n \right\} -$ barcha yaqinlashuvchi ketma-ketliklar to‘plami. Bu to‘plamda ham qo‘shish va songa ko‘paytirish amallari (1.7) va (1.8) tengliklar ko‘rinishida aniqlanadi.

1.15. $m-$ barcha chegaralangan ketma-ketliklar to‘plami. Bu to‘plamda ham qo‘shish va songa ko‘paytirish amallari (1.7) va (1.8) tengliklar ko‘rinishida aniqlanadi.

Endi Lebeg ma’nosida integrallanuvchi funksiyalar va o‘zgarishi chegaralangan funksiyalar to‘plamini qaraymiz.

1.16. Berilgan $[a,\, b]$ kesmada o‘lchovli va Lebeg ma’nosida integrallanuvchi ekvivalent funksiyalar sinflaridan iborat to‘plamni $L_1[a,\ b]$ bilan belgilaymiz. Bu to‘plamda elementlarni qo‘shish va elementni songa ko‘paytirish amallari (1.4) va (1.5) tengliklar bilan aniqlanadi.

1.17. Berilgan $[a,\ b]$ kesmada o‘lchovli va $p\,(p \geq 1) -$ darajasi Lebeg ma’nosida integrallanuvchi funksiyalar to‘plami $\widetilde{L}_p[a,\, b]$ bilan belgilanadi. Bu to‘plamda ham qo‘shish va songa ko‘paytirish amallari (1.4) va (1.5) tengliklar bilan aniqlanadi.

1.18. Berilgan $[a,\, b]$ kesmada aniqlangan va o‘zgarishi chegaralangan funksiyalar to‘plamini $V[a,b]$ bilan belgilaymiz. Bu to‘plamda ham funksiyalarni qo‘shish va songa ko‘paytirish amallari (1.4) va (1.5) tengliklar bilan aniqlanadi.

1.19. n satr va m ustundan iborat matritsalar to‘plamini M_{nm} bilan belgilaymiz. Bu to‘plamda qo‘shish va songa ko‘paytirish amallari odatdagi matritsalarni qo‘shish va matritsani songa ko‘paytirish kabi aniqlanadi.

1.20. $\mathbb{P}_{\leq n}-$darajasi n dan oshmaydigan ko‘phadlar to‘plami. Ko‘phadlarni qo‘shish va songa ko‘paytirish amallari (1.4) va (1.5) tengliklar bilan aniqlanadi.

1.21-1.36-misollarda keltirilgan to‘plamlar funksiyalarni qo‘shish ((1.4) ga qarang) va songa ko‘paytirish ((1.5) ga qarang) amallariga nisbatan chiziqli fazo tashkil qiladimi? Qaysilari haqiqiy chiziqli fazo, qaysilari kompleks chiziqli fazo bo‘ladi.

1.21. $[a,\ b]$ kesmada aniqlangan monoton funksiyalar to‘plami.

1.22. $[-a,\ a]$ kesmada aniqlangan uzluksiz va toq funksiyalar to‘plami.

1.23. $[-a,\ a]$ kesmada aniqlangan uzluksiz va juft funksiyalar to‘plami.

1.24. $\mathbb{P}-$ barcha ko‘phadlar to‘plami.

1.25. $C^{(n)}[a,\ b]-[a,b]$ kesmada aniqlangan n marta uzluksiz differensiallanuvchi funksiyalar to‘plami.

1.26. $[a,\ b]$ kesmada qisman chiziqli uzluksiz funksiyalar to‘plami.

1.27. $[-a,\ a]$ kesmada aniqlangan, uzluksiz va $\int\limits_{-a}^{a} x(t)dt = 0$ shartni qanoatlantiruvchi funksiyalar to‘plami.

1.28. $AC[a,\ b]-[a,b]$ kesmada aniqlangan absolyut uzluksiz funksiyalar to‘plami.

1.29. $V_0[a,\ b]-[a,b]$ kesmada o‘zgarishi chegaralangan va $f(a)=0$ shartni qanoatlantiruvchi funksiyalar to‘plami.

1.30. $\mathbb{R}$ da aniqlangan uzluksiz va davriy funksiyalar to‘plami.

1.31. $M(\mathbb{R})-\mathbb{R}$ da aniqlangan chegaralangan funksiyalar to‘plami.

1.32. $L[a,\ b]-[a,b]$ kesmada aniqlangan va Lipshits shartini qanoatlantiruvchi funksiyalar to‘plami.

1.33. Birlik doira $D = \{z \in C : \ |z| < 1\}$ da analitik va $\overline{D}$ da uzluksiz funksiyalar to'plami.

1.34. $[-a, \ a]$ kesmada aniqlangan uzluksiz va $T = 2a$ davriy funksiyalar to'plami.

1.35. $\ell_2(\mathbb{Z}) - \mathbb{Z}$ da aniqlangan va $\sum\limits_{n\in\mathbb{Z}} |f(n)|^2 < \infty$ shartni qanoatlantiruvchi funksiyalar to'plami.

1.36. $\widetilde{L}_2[a, \ b] \ - [a, \ b]$ kesmada o'lchovli va kvadrati Lebeg ma'nosida integrallanuvchi funksiyalar to'plami.

1.37-1.44-misollarda L chiziqli fazo va unda $\{x_k\}_{k=1}^3$ sistema berilgan. Uni chiziqli bog'langanlikka tekshiring.

1.37. $x_1(t) = \ 1, \quad x_2(t) = 1 + t, \quad x_3(t) = 1 + t + t^2 \in \mathbb{P}_{\leq 2}.$

1.38. $x_1(t) = \ 1, \quad x_2(t) = t, \quad x_3(t) = t^2 \in C[0, \ 1].$

1.39. $x_1(t) = \ 1, \quad x_2(t) = \cos t, \quad x_3(t) = \cos^2 t \in C[0, \ 2\pi].$

1.40. $x_1(t) = \ -1, \quad x_2(t) = \cos^2 t, \quad x_3(t) = \sin^2 t \in C[0, \ \pi].$

1.41. $x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ x_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ x_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_{22}.$

1.42. $x_1 = (1, 1, 1, \ldots), \quad x_2 = (1, 0, 1, 0, \ldots), \quad x_3 = (0, 1, 0, 1, \ldots) \in m.$

1.43. $x_1(t) = \ [t], \quad x_2(t) = \{t\}, \quad x_3(t) = t \in V[0, \ 4].$

1.44. $\mathfrak{D}(x)$, $\mathfrak{R}(x)$, $1(x) \equiv 1 \in \widetilde{L}_2[0, 1]$, $\mathfrak{D} -$ Dirixle, $\mathfrak{R}-$ Riman funksiyasi.

1.45. $A \subset \mathbb{R}$ to'plamni shunday tanlangki, $f_1(x) = 1$, $f_2(x) = \operatorname{sign} x$, $f_3(x) = \chi_A(x)$ elementlar $L_1[-1, \ 1]$ fazoda chiziqli bog'langan bo'lsin. f_1, f_2 va f_3 elementlar $V[-1, \ 1]$ fazoda chiziqli bog'langan bo'ladigan $A \subset (-1, 1)$ to'plam mavjudmi?

1.46. $A,\ B \subset \mathbb{R}$ to'plamlarni shunday tanlangki, $f_1(x) = \operatorname{sign} x$, $f_2(x) = \chi_A(x)$, $f_3(x) = \chi_B(x)$ elementlar:

a) $M[-2,\ 1]$ fazoda chiziqli bog'langan bo'lsin,

b) $V[-2,\ 3]$ fazoda chiziqli bog'langan bo'lsin.

1.47-1.50-misollarda berilgan L chiziqli fazoning o'lchamini toping.

1.47. $L = \mathbb{R}^5$, $\quad L = \mathbb{P}_{\leq 8}$, $\quad L = M_{33}$ ($M_{nm}-$ 1.19-misolda aniqlangan).

1.48. $L = \mathbb{C}^5$, $\quad L = m$, $\quad L = c$.

1.49. $L = C[a,\ b]$, $\quad L = V[a,\ b]$, $\quad L = c_0$.

1.50. $L = \widetilde{L}_1[a,\ b]$, $\quad L = \widetilde{L}_2[a,\ b]$, $\quad L = \ell_2$.

1.51-1.52-misollarda L va L^* fazolarning izomorfligini isbotlang.

1.51. $L = \mathbb{R}^4$, $\quad L^* = M_{22}$ ($M_{nm}-$ 1.19-misolda aniqlangan).

1.52. $L^* = \mathbb{P}_{\leq 8}$, $\quad L^* = M_{33}$.

Chiziqli fazoning qism fazosi va faktor fazosi. Bizga L chiziqli fazoning bo'sh bo'lmagan L' qism to'plami berilgan bo'lsin.

1.8-ta'rif. *Agar L' ning o'zi L da kiritilgan amallarga nisbatan chiziqli fazoni tashkil qilsa, u holda L' to'plam L ning qism fazosi deyiladi.*

Boshqacha qilib aytganda, agar ixtiyoriy $x, y \in L'$ va $a,\ b \in \mathbb{C}(\mathbb{R})$ sonlar uchun $ax + by \in L'$ bo'lsa, L' *qism fazo* bo'ladi va aksincha.

Har qanday L chiziqli fazoning faqat nol elementdan iborat $\{\theta\}$ qism fazosi bor. Ikkinchi tomondan, ixtiyoriy L chiziqli fazoni o'zining qism fazosi sifatida qarash mumkin.

1.9-ta'rif. *L chiziqli fazodan farqli va hech bo'lmaganda bitta nolmas elementni saqlovchi qism fazo xos qism fazo deyiladi.*

Bizga L fazoning bo'sh bo'lmagan $\{x_i\}$ qism to'plami berilgan bo'lsin. U holda L chiziqli fazoda $\{x_i\}$ sistemani o'zida saqlovchi minimal qism fazo mavjud. Bu qism fazoni $L(\{x_i\})$ orqali belgilaymiz. Bu qism fazo $\{x_i\}$ *"sistemadan hosil bo'lgan"* qism fazo yoki $\{x_i\}$ sistemaning *chiziqli qobig'i* deyiladi.

Bizga L chiziqli fazo va uning L' xos qism fazosi berilgan bo'lsin. L ning elementlari orasida quyidagicha munosabat o'rnatish mumkin.

1.10-ta'rif. *Agar $x, y \in L$ elementlar uchun $x-y$ ayirma L' ga tegishli bo'lsa, x va y elementlar ekvivalent deyiladi.*

Fazo elementlari orasida o'rnatilgan bu munosabat refleksivlik, simmetriklik va tranzitivlik xossalariga ega. Shuning uchun bu munosabat L ni o'zaro kesishmaydigan sinflarga ajratadi va har bir sinf o'zaro ekvivalent elementlardan tashkil topgan. Bu sinflar *qo'shni sinflar* deyiladi. Barcha qo'shni sinflar to'plami L chiziqli fazoning L' qism fazo bo'yicha *faktor fazosi* deyiladi va L/L' ko'rinishda belgilanadi.

Faktor fazoda yig'indi va songa ko'paytirish amallari tabiiy ravishda kiritiladi. Aytaylik, ξ va η lar L/L' dan olingan ixtiyoriy qo'shni sinflar bo'lsin. Bu sinflarning har biridan bittadan vakil tanlaymiz, masalan $x \in \xi,\ y \in \eta$. ξ va η sinflarning yig'indisi sifatida $x + y$ elementni saqlovchi ζ sinf qabul qilinadi. ξ qo'shni sinfning α songa ko'paytmasi sifatida $\alpha\, x$ elementni saqlovchi ζ_1 sinf qabul qilinadi. Natija $x \in \xi,\ y \in \eta$ vakillarning tanlanishiga bog'liq emas, chunki, qandaydir boshqa $x' \in \xi,\ y' \in \eta$ vakillarni olsak ham $(x+y)-(x'+y') = (x-x')+(y-y') \in L'$ va $\alpha(x-x') \in L'$ bo'lgani uchun $x'+y' \in \zeta$ va $\alpha\, x' \in \zeta_1$ bo'ladi. Bevosita tekshirish shuni ko'rsatadiki, L/L' da aniqlangan qo'shish va songa ko'paytirish amallari chiziqli fazo ta'rifidagi aksiomalarni qanoatlantiradi. Boshqacha aytganda, L/L' faktor fazo chiziqli fazo tashkil qiladi.

1.11-ta'rif. *L/L' faktor fazoning o'lchami L' qism fazoning koo'lchami deyiladi va $dim L/L' = codim L'$ shakldа yoziladi.*

1.53. $\ell_2 \subset c_0 \subset c \subset m$ fazolarning har biri o'zidan keyingilari uchun xos qism fazo bo'ladi. Isbotlang.

1.54. $\mathbb{R}^n$ fazoda $V = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \ x_1 = x_2\}$ to'plam qism fazo tashkil qilishini isbotlang, uning o'lchamini toping.

1.55. ℓ_2 fazoda $M = \{(x_1, \ldots, x_n, \ldots) \in \ell_2 : \ x_1 + x_2 + x_3 = 0\}$ to'plam qism fazo tashkil qilishini isbotlang, qism fazoning koo'lchamini toping.

1.56. $\widetilde{L}_p\,[a, b]$, $(p \geq 1)$ fazoning nolga ekvivalent funksiyalaridan tashkil topgan qism to'plamni $\widetilde{L}_p^{(0)}\,[a, b]$ ko'rinishda belgilaymiz. $\widetilde{L}_p^{(0)}\,[a, b]$ ni qism fazo bo'lishini isbotlang.

1.57. Absolyut uzluksiz funksiyalar to'plami $AC[a, b]$ o'zgarishi chegaralangan funksiyalar fazosi $V[a, b]$ ning qism fazosi bo'ladi. Isbotlang.

1.58. $V[a, b]$ fazoda $f(a) = 0$ shartni qanoatlantiruvchi funksiyalar to'plamini $V_0[a, b]$ bilan belgilaymiz. Bu to'plam $V[a, b]$ fazoning qism fazosi bo'ladi. Isbotlang.

1.59. Ma'lumki (1.56-misolga qarang), $\widetilde{L}_p\,[a,\ b]$ fazoning nolga ekvivalent funksiyalaridan tashkil topgan qism fazosi $\widetilde{L}_p^0\,[a,\ b]$ ko'rinishda belgilanadi. Endi $\widetilde{L}_p\,[a,\ b]$ chiziqli fazoning $\widetilde{L}_p^0\,[a,\ b]$ qism fazo bo'yicha faktor fazosini qaraymiz va bu faktor fazoni $L_p[a,\ b]$ bilan belgilaymiz. Bu fazo $[a,\ b]$ kesmada aniqlangan va $p-$ darajasi bilan Lebeg ma'nosida integrallanuvchi ekvivalent funksiyalar fazosi deb ataladi. Dirixle va Riman funksiyalarini bir sinfda yotishini isbotlang.

1.60. $\mathbb{R}^+ = (0, \infty)$ to'plamda x va y sonlar yig'indisi deganda ularning ko'paytmasini, x elementni $\lambda-$ haqiqiy songa ko'paytirish deganda x^λ

ni tushunamiz. U holda $\mathbb{R}^+$ to'plam unda kiritilgan amallarga nisbatan chiziqli fazo tashkil qilishini isbotlang. Bu fazoning nol elementini toping. Bu fazoning o'lchamini toping.

1.61. $\mathfrak{A}(X)$ orqali X chiziqli fazoning barcha qism to'plamlari sistemasini belgilaymiz. Ixtiyoriy M, $N \in \mathfrak{A}(X)$ lar uchun

$$M + N = \{x + y : \ x \in M, \ y \in N\}, \quad \lambda M = \{\lambda x : \ x \in M\}$$

kabi amallarni kiritamiz. Bu amallar chiziqli fazo aksiomalarini qanoatlantiradimi?

2-§. Chiziqli normalangan fazolar

Chiziqli fazolarda elementlarning bir-biriga yaqinligi degan tushuncha yo'q. Ko'plab amaliy masalalarni hal qilishda elementlarni qo'shish va ularni songa ko'paytirish amallaridan tashqari, elementlar orasidagi masofa, ularning yaqinligi tushunchasini kiritishga to'g'ri keladi. Bu bizni normalangan chiziqli fazo tushunchasiga olib keladi.

2.1-ta'rif. *L chiziqli fazoning har bir elementiga aniq bir sonni mos qo'yuvchi p akslantirishga funksional deyiladi.*

2.2-ta'rif. *Bizga L chiziqli fazo va unda aniqlangan p funksional berilgan bo'lsin. Agar p funksional quyidagi uchta shartni qanoatlantirsa, unga norma deyiladi:*

1) $p(x) \geq 0, \quad \forall x \in L; \quad p(x) = 0 \Leftrightarrow x = \theta\,;$

2) $p(ax) = |a|\, p(x), \quad \forall a \in \mathbb{C}, \ \forall x \in L\,;$ *bir jinslilik aksiomasi,*

3) $p(x + y) \leq p(x) + p(y), \quad \forall x, \, y \in L\,,$ *uchburchak tengsizligi.*

2.3-ta'rif. *Norma kiritilgan chiziqli fazo chiziqli normalangan fazo deyiladi va $x \in X$ elementning normasi $\|x\|$ orqali belgilanadi.*

Bitta chiziqli fazoda har xil normalar kiritish mumkin. Agar X chiziqli fazoda $p_1, p_2, \ldots, p_n$ normalar aniqlangan bo'lsa, u holda $(X, p_1)\,,$ $(X, p_2),$

$\ldots, (X, p_n)$ normalangan fazolar mos ravishda $X_1, X_2, \ldots, X_n$ harflari bilan belgilanadi. Bizga X chiziqli fazo va unda $\|\bullet\|_1$ va $\|\bullet\|_2$ normalar berilgan bo'lsin.

2.4-ta'rif. *Agar shunday $C_1 > 0$ va $C_2 > 0$ sonlar mavjud bo'lib, barcha $x \in X$ lar uchun*

$$C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1$$

tengsizlik o'rinli bo'lsa, $\|\bullet\|_1$ va $\|\bullet\|_2$ normalar ekvivalent deyiladi.

Har qanday normalangan fazoni metrik fazo sifatida qarash mumkin. Shuning uchun metrik fazolarda isbotlangan barcha teoremalar va tasdiqlar normalangan fazolar uchun ham o'rinli. Agar X chiziqli normalangan fazo bo'lsa, u holda $\rho : X \times X \to \mathbb{R}$, $\rho(x, y) = \|x - y\|$ akslantirish metrika shartlarini qanoatlantiradi. Xuddi metrik fazolar holidagidek yaqinlashuvchi va fundamental ketma-ketlik tushunchalarini keltirish mumkin.

Bizga $x \in X$ element va $\{x_n\} \subset X$ ketma-ketlik berilgan bo'lsin.

2.5-ta'rif. *Agar ixtiyoriy $\varepsilon > 0$ uchun shunday $n_0 = n_0(\varepsilon) > 0$ mavjud bo'lib, barcha $n > n_0$ larda $\|x_n - x\| < \varepsilon$ tengsizlik bajarilsa, $\{x_n\}$ ketma-ketlik $x \in X$ elementga yaqinlashadi deyiladi.*

2.6-ta'rif. *Agar ixtiyoriy $\varepsilon > 0$ son uchun shunday $n_0 = n_0(\varepsilon) > 0$ mavjud bo'lib, barcha $n > n_0$ va $p \in \mathbb{N}$ larda $\|x_{n+p} - x_n\| < \varepsilon$ tengsizlik bajarilsa, $\{x_n\}$ ga fundamental ketma-ketlik deyiladi.*

2.7-ta'rif. *Agar X chiziqli normalangan fazodagi ixtiyoriy fundamental ketma-ketlik yaqinlashuvchi bo'lsa, u holda X ga to'la normalangan fazo yoki Banax fazosi deyiladi.*

Bu ta'rifni quyidagicha ham aytish mumkin.

2.8-ta'rif. *Agar (X, ρ), $\rho(x, y) = \|x - y\|$ metrik fazo to'la bo'lsa, u holda X to'la normalangan fazo yoki Banax fazosi deyiladi.*

Xuddi metrik fazo holidagidek $B(x_0, r) = \{x \in X : \|x - x_0\| < r\}$ to'plam markazi x_0 da radiusi $r > 0$ bo‘lgan *ochiq shar* deyiladi. Markazi x_0 da radiusi $r \geq 0$ bo‘lgan *yopiq shar* deganda

$$B[x_0, r] = \{x \in X : \|x - x_0\| \leq r\}$$

to‘plam tushuniladi. Agar X chiziqli normalangan fazodagi M to‘plamni biror sharga joylashtirish mumkin bo‘lsa, unga *chegaralangan to‘plam* deyiladi. M to‘plamning *diametri* deb $diamM = \sup\limits_{x,y \in M} \|x - y\|$ songa aytiladi. $\rho(x, M) = \inf\limits_{y \in M} \|x - y\|$ miqdorga x nuqtadan M *to‘plamgacha bo‘lgan masofa* deyiladi. Xuddi shunday

$$\rho(A, B) = \inf_{x \in A, y \in B} \|x - y\|$$

miqdorga A va B *to‘plamlar orasidagi masofa* deyiladi. Normalangan fazolarda ham ochiq va yopiq to‘plamlar xuddi metrik fazolardagidek ta’riflanadi. M ning barcha limitik nuqtalari to‘plami M' orqali belgilanadi. Xuddi metrik fazolardagidek $M \cup M'$ to‘plam M *to‘plamning yopig‘i* deyiladi va $[M]$ yoki $\overline{M}$ orqali belgilanadi. X chiziqli normalangan fazodagi A va B to‘plamlarning *arifmetik yig‘indisi* deganda $A + B = \{a + b : a \in A, b \in B\}$ to‘plam tushuniladi.

2.9-ta’rif. *Agar L va M lar X normalangan fazoning qism fazolari bo‘lib, X ning har bir x elementi yagona usul bilan $x = u + v$, $u \in L$, $v \in M$ ko‘rinishda tasvirlansa, X normalangan fazo L va M qism fazolarning to‘g‘ri yig‘indisiga yoyilgan deyiladi va bu $X = L \oplus M$ shaklda yoziladi.*

2.1. Ushbu $p : \mathbb{R}^2 \to \mathbb{R}$, $p(x) = 2|x_1| + 3|x_2|$ funksional norma shartlarini qanoatlantiradimi?

Yechish. Bu funksional qiymatlari manfiymas va $p(x) = 0$ faqat va faqat shu holdaki, $x_1 = 0$, $x_2 = 0$ da, ya’ni $x = (0, 0)$ da bajariladi. Shunday qilib, normaning 1-sharti bajariladi. 2-shartning bajarilishini ko‘rsatamiz:

$$p(\lambda\, x) = 2\,|\lambda x_1| + 3\,|\lambda x_2| = 2\,|\lambda||x_1| + 3\,|\lambda||x_2| = |\lambda|(2|x_1| + 3|x_2|) = |\lambda| p(x).$$

Bu tenglik barcha $\lambda \in \mathbb{R}$ va $x \in \mathbb{R}^2$ lar uchun o‘rinli. Endi 3 - shartning bajarilishini ko‘rsatamiz:

$$p(x+y) = 2\,|x_1 + y_1| + 3\,|x_2 + y_2| \leq 2\,|x_1| + 3\,|x_2| + 2\,|y_1| + 3\,|y_2| = p(x) + p(y).$$

Bu tengsizlik barcha $x,\, y \in \mathbb{R}^2$ lar uchun o‘rinli. Demak, berilgan funksional normaning barcha shartlarini qanoatlantiradi. □

2.2. $C[-1,\, 1]$ fazoda $x_n(t) = t^n$ $(n \in \mathbb{N})$ ketma-ketlikni fundamentallikka tekshiring.

Yechish. $C[-1,\ 1]$ fazo to‘la normalangan fazo bo‘lganligi uchun $\{x_n\}$ ketma-ketlikning fundamentalligidan uning yaqinlashuvchi ekanligi kelib chiqadi. $C[-1,\ 1]$ fazodagi yaqinlashish tekis yaqinlashishni ifodalaganligi uchun $\{x_n\}$ ketma-ketlikning limiti (agar u mavjud bo‘lsa) ham uzluksiz bo‘lishi kerak. Qaralayotgan ketma-ketlikning "limiti" uzluksiz emas. Shuning uchun qaralayotgan ketma-ketlikning fundamental emasligini ko‘rsatishga harakat qilamiz. Buning uchun shunday $\varepsilon_0 > 0$ soni mavjud bo‘lib, istalgan $n \in \mathbb{N}$ uchun undan katta $n_0 > n$ va shunday $p_0 \in \mathbb{N}$ sonlari mavjud bo‘lib, $\|x_{n_0+p_0} - x_{n_0}\| \geq \varepsilon_0$ tengsizlik o‘rinli ekanligini ko‘rsatish kifoya. $\varepsilon_0 = \dfrac{1}{5}$ va har bir $n \in \mathbb{N}$ dan katta biror $n_0 > n$ natural son uchun $p_0 = n_0$ deb olamiz. Barcha $t \in [0,\ 1]$ lar uchun

$$\|\, x_{2n_0} - x_{n_0}\| = \max_{-1 \leq t \leq 1} \left| t^{2n_0} - t^{n_0} \right| \geq t^{n_0} - t^{2n_0}$$

tengsizlikka ega bo‘lamiz. Bu tengsizlikdan $t = \dfrac{1}{\sqrt[n_0]{2}}$ bo‘lganida ushbu

$$\|x_{2n_0} - x_{n_0}\| \geq \frac{1}{2} - \frac{1}{4} = \frac{1}{4} > \frac{1}{5}$$

tengsizlik kelib chiqadi. Bu esa $\{x_n\}$ ketma-ketlikning fundamental emasligini ko‘rsatadi. □

2.3. X normalangan fazo va $x_n,\ x,\ y_n,\ y \in X$ bo‘lsin. Quyidagiarni isbotlang:

a) agar $x_n \to x$ bo‘lsa, u holda $\|x_n\| \to ||x||$;

b) agar $x_n \to x$ bo‘lsa, u holda $\{x_n\}$ chegaralangan ketma-ketlik;

c) agar $x_n \to x,\ \lambda_n \to \lambda,\ \lambda_n \in \mathbb{C}$ bo‘lsa, u holda $\lambda_n \cdot x_n \to \lambda \cdot x$;

d) agar $x_n \to x$ va $\|x_n - y_n\| \to 0$ bo‘lsa, u holda $y_n \to x$;

e) agar $x_n \to x$ bo‘lsa, u holda $\|x_n - y\| \to \|x - y\|$;

f) agar $x_n \to x,\ y_n \to y$ bo‘lsa, u holda $\|x_n - y_n\| \to \|x - y\|$.

Isbot. Faraz qilaylik, $x_n \to x$ bo‘lsin. Modulning manfiymasligidan hamda (2.1) tengsizlikdan $0 \leq |\, \|x_n\| - \|x\| \,| \leq \|x_n - x\|$ kelib chiqadi. Bu tengsizlikda limitga o‘tib $\lim\limits_{n\to\infty} \|x_n\| = \|x\|$ tenglikni olamiz. Ya’ni a) tasdiq isbot bo‘ldi. $\{\|x_n\|\}$ sonli ketma-ketlikning yaqinlashuvchi ekanligidan uning chegaralangan ekanligi kelib chiqadi, ya’ni $\{x_n\}$ chegaralangan ketma-ketlik ekan. b) tasdiq isbot bo‘ldi. Endi c) tasdiqni isbotlaymiz. Quyidagi

$$0 \leq \|\lambda_n \cdot x_n - \lambda \cdot x\| = \|\lambda_n \cdot x_n - \lambda_n \cdot x + \lambda_n \cdot x - \lambda \cdot x\| \leq$$

$$\leq \|\lambda_n \cdot x_n - \lambda_n \cdot x\| + \|\lambda_n \cdot x - \lambda \cdot x\| = |\lambda_n|\, \|x_n - x\| + \|x\|\ |\lambda_n - \lambda|$$

tengsizlikdan $\lambda_n \cdot x_n \to \lambda \cdot x$ ekanligi kelib chiqadi. Quyidagi

$$0 \leq \|x - y_n\| = \|x - x_n + x_n - y_n\| \leq \|x - x_n\| + \|x_n - y_n\|$$

tengsizlikda limitga o‘tib d) tasdiqning isbotiga ega bo‘lamiz. (2.1) tengsizlikka ko‘ra

$$0 \leq |\, \|x_n - y\| - \|x - y\| \,| \leq \|x_n - y - (x - y)\| = \|x_n - x\|$$

tengsizlik o‘rinli. Bu yerdan limitga o‘tib e) tasdiqning isbotiga ega bo‘lamiz. Oxirgi f) tasdiq ham (2.1) tengsizlik yordamida isbotlanadi. □

Uy vazifalari va mavzuni o'zlashtirish uchun masalalar

2.4-2.21-misollarda berilgan $p : X \to \mathbb{R}$ akslantirishning norma shartlarini qanoatlantirishini tekshiring.

2.4. $p(x) = \sqrt{\sum\limits_{k=1}^{n} x_k^2}, \ x \in \mathbb{R}^n.$

2.5. $p_q(x) = \sqrt[q]{\sum\limits_{k=1}^{n} |x_k|^q}, \ x \in \mathbb{R}^n, \ q \geq 1.$

2.6. $p_\infty(x) = \max\limits_{1 \leq i \leq n} |x_i|, \ x \in \mathbb{R}^n.$

2.7. $p_1(x) = \sum\limits_{k=1}^{n} |x_k|, \ x \in \mathbb{R}^n.$

2.8. $p(x) = \sqrt{\sum\limits_{i=1}^{n} |x_i|^2}, \ x \in \mathbb{C}^n.$

2.9. $p(x) = \sqrt{\sum\limits_{n=1}^{\infty} |x_n|^2}, \ x \in \ell_2.$

2.10. $p(x) = \sqrt[q]{\sum\limits_{n=1}^{\infty} |x_n|^q}, \ x \in \ell_q, \ q \geq 1.$

2.11. $p(f) = \max\limits_{a \leq x \leq b} |f(x)|, \ f \in C[a, \ b].$

2.12. $p_1(f) = \int\limits_a^b |f(x)| \, dx, \ f \in C[a, \ b].$

2.13. $p_2(f) = \sqrt{\int\limits_a^b |f(x)|^2 \, dx}, \ f \in C[a, \ b].$

2.14. $p_q(f) = \sqrt[q]{\int\limits_a^b |f(x)|^q \, dx}, \ f \in C[a, \ b], \ q \geq 1.$

2.15. $p(x) = \sup\limits_{1 \leq n < \infty} |x_n|, \ x \in m.$

2.16. $p(x) = \sup\limits_{1 \leq n < \infty} |x_n|, \ x \in c.$

2.17. $p(x) = \sup\limits_{1 \le n < \infty} |x_n|, \ x \in c_0.$

2.18. $p(x) = |x(a)| + V_a^b[x], \ x \in V[a, \ b].$

2.19. $p(x) = |x(a)| + V_a^b[x], \ x \in AC[a, \ b].$

2.20. $p : M[a, \, b] \to \mathbb{R}, \ p(x) = \sup\limits_{a \le t \le b} |x(t)|.$

2.21. $p(x) = \max\limits_{a \le t \le b} |\, x(t) \,| + \sum\limits_{k=1}^{n} \max\limits_{a \le t \le b} \left| \, x^{(k)}(t) \, \right|, \ x \in C^{(n)}[a, \ b].$

Xuddi metrik fazo holidagidek $(\mathbb{R}^n, p_q) = \mathbb{R}_q^n, \ (\mathbb{R}^n, p_\infty) = \mathbb{R}_\infty^n,$ $(\mathbb{R}^n, p) = \mathbb{R}^n, \ (C[a, \ b], p) = C[a, \ b], \ (C[a, \ b], p_q) = C_q[a, \, b], \ q \ge 1$ belgilashlarni kiritamiz.

2.22. $\mathbb{R}^n$ fazoda kiritilgan $p, \ p_q, \ p_\infty, \ p_1$ normalarning (2.4-2.7-misollarga qarang) istalgan ikkisi ekvivalent ekanligini isbotlang.

2.23. Chekli o'lchamli chiziqli fazodagi ixtiyoriy ikki norma ekvivalentligini isbotlang.

2.24. $C[a, \ b]$ fazoda kiritilgan $p, \, p_1, \ p_2, \ p_q$ normalarning (2.11-2.14-misollarga qarang) istalgan ikkisi ekvivalent emasligini isbotlang.

2.25-2.30-misollarda keltirilgan akslantirishlar norma shartlarini qanoatlantiradimi?

2.25. $p : \mathbb{P}_{\le n} \to \mathbb{R}, \ \ p(x) = \max\{|x_0|, |x_1|, \ldots, |x_n|\},$

bu yerda $x(t) = x_0 + x_1 t + \cdots + x_n t^n.$

2.26. $p : C^{(1)}[a, \, b] \to \mathbb{R}, \quad p(x) = |x(b) - x(a)| + \max\limits_{a < t \le b} |x'(t)|.$

2.27. $p : C^{(1)}[a, \, b] \to \mathbb{R}, \ p(x) = \max\limits_{a \le t \le b} |x'(t)|.$

2.28. $p : C^{(2)}[a, \, b] \to \mathbb{R}, \ p(x) = |\, x(a) \,| + |\, x'(a) \,| + \max\limits_{a \le t \le b} |\, x''(t) \,|.$

2.29. $p: C^{(2)}[a,\, b] \to \mathbb{R},\ p(x) = |\, x(a)\,| + |\, x(b)\,| + \max\limits_{a \le t \le b} |\, x''(t)\,|\,.$

2.30. $p: \Phi_C(\mathbb{R}) \to \mathbb{R}\,,\ p(x) = \max\limits_{-\infty \le t \le \infty} |x(t)|\,.$ Bu yerda $\Phi_C(\mathbb{R})-$ sonlar o'qida aniqlangan uzluksiz va finit funksiyalar to'plami.

2.31-2.40-misollarda keltirilgan funksiyalar ketma-ketligi $\theta(t) \equiv 0$ funksiyaga ko'rsatilgan fazoda yaqinlashuvchimi?

2.31. $x_n(t) = \dfrac{nt}{1 + n^2 + t^2}\ ,\ \ C[0,\ 1].$

2.32. $x_n(t) = te^{-nt},\ \ C_1[0,\ 10].$

2.33. $x_n(t) = \dfrac{\sin\ nt}{n},\ \ C_1[-\pi,\ \pi].$

2.34. $x_n(t) = t^n - t^{2n},\ \ C_2[0,\ 1].$

2.35. $x_n(t) = \dfrac{t^{n+1}}{n+1} - \dfrac{t^{2+n}}{2+n},\ \ C[0,\ 1].$

2.36. $x_n(t) = \dfrac{t}{1 + n^2 t^2};\ \ C_1\,[0,\ 1]\,.$

2.37. $x_n(t) = \sqrt[n]{t^n + \dfrac{1}{n^2}} - t;\ \ C_1\,[0,\ 1]\,.$

2.38. $x_n(t) = t^n - t^{n+1};\ \ C_2\,[0,\ 1]\,.$

2.39. $x_n(t) = n^{-0{,}5}\sqrt{2nt}\ \cdot e^{-0{,}5nt};\ \ C_2\,[0,\ 1]\,.$

2.40. $x_n(t) = 2n \cdot t \cdot e^{-nt^2};\ \ C_1\,[0,\ 1]\,.$

2.41. $x = (1,2,2)$ va $y = (-3,0,4)$ elementlarning $\mathbb{R}^3,\ \mathbb{R}^3_1,\ \mathbb{R}^3_4,\ \mathbb{R}^3_\infty$ fazolardagi normasini hisoblang.

2.42. $f(x) = \sin x$ va $g(x) = \cos x$ elementlarning $C[-\pi,\, \pi],\ C_1[-\pi,\, \pi],$ $C_2[-\pi,\, \pi]$ fazolardagi normasini hisoblang.

2.43. $\varphi_n(x) = \sin nx$ va $\psi_n(x) = \cos nx,\, n \in \mathbb{N}$ elementlarning $C[-\pi,\, \pi],$ $C_1[-\pi,\ \pi],\ L_2[-\pi,\ \pi],\ \ M[-\pi,\ \pi],\ \ V[-\pi,\ \pi]$ fazolardagi normasini hisoblang.

2.44. Agar $p_1 : X \to \mathbb{R}$ va $p_2 : X \to \mathbb{R}$ normalar bo'lsa, u holda ixtiyoriy a_1, a_2 musbat sonlar uchun $p = a_1p_1 + a_2p_2 : X \to \mathbb{R}$ ham norma shartlarini qanoatlantiradi. Isbotlang.

2.45. Agar $\|\bullet\|_1 : X \to \mathbb{R}$ va $\|\bullet\|_2 : X \to \mathbb{R}$ lar ekvivalent normalar bo'lsa, u holda $\lim\limits_{n\to\infty} \|x_n - x\|_1 = 0$ tenglikdan $\lim\limits_{n\to\infty} \|x_n - x\|_2 = 0$ tenglik kelib chiqadi va aksincha. Isbotlang.

2.46. Agar $p_1 : X \to \mathbb{R}$ va $p_2 : X \to \mathbb{R}$ lar ekvivalent normalar bo'lsa, u holda $\{x_n\}$ ketma-ketlikning X_1 normalangan fazoda fundamentalligidan, uning X_2 da ham fundamental ekanligi kelib chiqadi. Isbotlang.

2.47. Agar $p_1 : X \to \mathbb{R}$ va $p_2 : X \to \mathbb{R}$ lar ekvivalent normalar bo'lsa, u holda $M \subset X$ ning X_1 normalangan fazoda kompakt (nisbiy kompakt) ekanligidan, uning X_2 da ham kompakt (nisbiy kompakt) ekanligi kelib chiqadi. Isbotlang.

2.48. Ixtiyoriy $x,\ y \in X$ lar uchun quyidagi tengsizlikni isbotlang

$$|\,\|x\| - \|y\|\,| \leq \ \|x - y\|. \tag{2.1}$$

2.49. Har qanday normalangan fazoda ochiq shar ochiq to'plam, yopiq shar yopiq to'plam bo'lishini isbotlang.

2.50. $[B(x_0, r)] = B[x_0, r]$ tenglikni isbotlang.

2.51. Ixtiyoriy $x,\ y \in X$ lar uchun $\|x\| \leq \max\{\|x + y\|,\ \|x - y\|\}$ tengsizlik o'rinli. Isbotlang.

2.52. Chegaralangan to'plamlarning birlashmasi yana chegaralangan to'plam bo'lishini isbotlang.

2.53. Chegaralangan to'plamlarning arifmetik yig'indisi yana chegaralangan to'plam bo'lishini isbotlang.

2.54. $M \subset X$ to‘plam chegaralangan bo‘lishi uchun $diam\, M < \infty$ tengsizlikning bajarilishi zarur va yetarli. Isbotlang.

2.55. $M \subset X$ chegaralangan to‘plam. U holda $[M]$ ham chegaralangan to‘plam, hamda $diam\, M = diam\ [M]$ tenglik o‘rinli. Isbotlang.

2.56. Har qanday $M \subset X$ to‘plam uchun M' yopiq to‘plam bo‘lishini isbotlang.

2.57. Har qanday $M \subset X$ to‘plam uchun $(M')' \subset M'$ munosabatni isbotlang. $M' \backslash (M')' \neq \emptyset$ bo‘lishi mumkinmi?

2.58. $[A] \subset [B]$ ekanligidan $A \subset B$ munosabat kelib chiqadimi?

2.59. $M \subset X$ yopiq to‘plam bo‘lsin. $\rho(x, M) = 0$ bo‘lishi uchun $x \in M$ bo‘lishi zarur va yetarli. Isbotlang.

2.60. $A,\, B \subset\ X$ ixtiyoriy to‘plamlar bo‘lsin. $\rho(A, B) = \rho(A, \overline{B}) = \rho(\overline{A}, B)$ $= \rho(\overline{A}, \overline{B})$ tengliklarni isbotlang.

2.61. $M \subset X$ ixtiyoriy to‘plam bo‘lsin. M to‘plamning chegarasi - ∂M shunday $x \in X$ nuqtalardan iboratki, markazi x da bo‘lgan har qanday shar ham M to‘plamdan, ham $X \backslash M$ dan hech bo‘lmaganda bittadan elementni o‘zida saqlaydi. $\partial M-$ yopiq to‘plam hamda $\partial M = \partial (X \backslash M)$ tenglikni isbotlang.

2.62. Shunday $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots, x_k^{(n)}, \ldots)$ ketma-ketlikka misol keltiringki:

a) m da yaqinlashuvchi, ℓ_1 da uzoqlashuvchi bo‘lsin;

b) m da yaqinlashuvchi, ℓ_2 da uzoqlashuvchi bo‘lsin;

c) ℓ_2 da yaqinlashuvchi, ℓ_1 da uzoqlashuvchi bo‘lsin;

d) c_0 da yaqinlashuvchi, ℓ_1 da uzoqlashuvchi bo‘lsin;

e) c_0 da yaqinlashuvchi, ℓ_2 da uzoqlashuvchi bo‘lsin.

2.63. $x = (1, \frac{1}{\ln 2}, \frac{1}{\ln 3}, \ldots, \frac{1}{\ln n}, \ldots)$ elementning c_0 da yotishini ko'rsating va birorta ham $p \in \mathbb{N}$ da $x \notin \ell_p$ ekanligini isbotlang.

2.64. $\mathbb{P}-$ barcha ko'phadlar to'plami $C[a,b]$ fazoda ochiq to'plam bo'ladimi?

2.65. $\mathbb{P}-$ barcha ko'phadlar to'plami $C[a,b]$ da yopiq to'plam bo'ladimi?

2.66. Qisman chiziqli uzluksiz funksiyalar to'plami $C[a,b]$ fazoning hamma yerida zich ekanligini isbotlang.

2.67. $\mathbb{P}-$ barcha ko'phadlar to'plami $C[a,b]$ fazoning hamma yerida zich ekanligini isbotlang.

2.68. ℓ_2 fazoda $\{x = (x_1, x_2, \ldots) \in \ell_2 : |x_n| < 1\}$ parallelepiped ochiq to'plam bo'lishini isbotlang.

2.69. Agar $\|x + y\| = \|x\| + \|y\|$ tenglik faqat $y = \lambda\, x,\ \lambda > 0$ ko'rinishdagi elementlar uchun o'rinli bo'lsa, u holda X normalangan fazo *qat'iy normalangan* deyiladi. Quyidagilarning qaysilari qat'iy normalangan fazo bo'ladi?

a) $\mathbb{R}^2$; b) ℓ_1; c) ℓ_2; d) m; e) $C[a,b]$; f) $C_2[a,b]$.

2.70. Agar $A,\, B \subset X$ to'plamlardan birortasi ochiq bo'lsa, u holda $A + B$ to'plam ham ochiq bo'ladi. Isbotlang.

2.71. $A,\, B \subset X$ lar hamma yerda zich to'plamlar bo'lsin. $A \cap B = \emptyset$ bo'lishi mumkinmi?

2.72. $C[-1,\ 1]$ fazoni ikkita cheksiz o'lchamli qism fazolarning to'g'ri yig'indisi shaklida yozing.

2.73. Normalangan fazoda fundamental ketma-ketlikning chegaralangan ekanligini isbotlang.

2.74. $\{x_n\} \subset X$ fundamental ketma-ketlik va uning biror x_{n_k} qismiy ketma-ketligi yaqinlashuvchi bo'lsin. U holda x_n ketma-ketlik ham yaqinlashuvchi bo'ladi. Isbotlang.

2.75. $\{x_n\} \subset X$ va $\sum\limits_{n=1}^{\infty} \|x_{n+1}-x_n\|$ qator yaqinlashuvchi bo'lsin. U holda $\{x_n\}$ fundamental ketma-ketlik bo'ladi. Isbotlang. Teskari tasdiq o'rinlimi?

2.76. Har qanday chekli o'lchamli normalangan fazo to'ladir. Isbotlang.

3-§. Evklid va Hilbert fazolari

Chiziqli fazolarda norma kiritishning sinalgan usullaridan biri, unda skalyar ko'paytma kiritishdir. L haqiqiy chiziqli fazo bo'lsin.

3.1-ta'rif. *Agar $L \times L$ dekart ko'paytmada aniqlangan p funksional quyidagi to'rtta shartni qanoatlantirsa, unga skalyar ko'paytma deyiladi:*

1) $p(x,x) \geq 0, \quad \forall x \in L; \quad p(x,x) = 0 \Leftrightarrow x = \theta,$

2) $p(x,y) = p(y,x), \quad \forall x, y \in L,$ *simmetriklik,*

3) $p(\alpha x,\, y) = \alpha p(x,\, y), \quad \forall \alpha \in \mathbb{R}, \;\; x,\, y \in L,$ *bir jinslilik,*

4) $p(x_1 + x_2,\; y) = p(x_1, y) + p(x_2, y), \quad \forall x_1,\, x_2,\; y \in L,$ *additivlik.*

Agar L kompleks chiziqli fazo bo'lsa, u holda 2) shart $p(x,\, y) = \overline{p(y,\, x)}$ bilan almashtiriladi va 3) tenglik barcha kompleks α da bajarilishi talab qilinadi.

3.2-ta'rif. *Skalyar ko'paytma kiritilgan chiziqli fazo Evklid fazosi deyiladi. x va y elementlarning skalyar ko'paytmasi (x,y) orqali belgilanadi.*

Evklid fazosida x elementning normasi

$$\|x\| = \sqrt{(x,x)} \tag{3.1}$$

formula orqali aniqlanadi. Demak, har qanday Evklid fazosini normalangan fazo sifatida qarash mumkin. Normalangan fazolarda isbotlangan barcha tasdiqlar Evklid fazosida ham o'rinli bo'ladi.

Teskari masalani qaraymiz. $E-$ normalangan fazo bo'lsin. E da aniqlangan norma qanday qo'shimcha shartlarni qanoatlantirsa, E Evklid fazosi ham bo'ladi? Boshqacha aytganda, qanday shartlarda norma orqali unga mos skalyar ko'paytma aniqlash mumkin?

3.1-teorema. *E normalangan fazo Evklid fazosi bo'lishi uchun, ixtiyoriy ikkita $f,\, g \in E$ elementlar uchun*

$$\|f+g\|^2 + \|f-g\|^2 = 2\,\|f\|^2 + 2\,\|g\|^2 \tag{3.2}$$

tenglikning bajarilishi zarur va yetarli.

(3.2) *parallelogramm ayniyati* deyiladi. (3.2) shart bajarilganda

$$p(x,\, y) = \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2\right)$$

$p: E \times E \to \mathbb{R}$ funksional skalyar koʻpaytma shartlarini qanoatlantiradi.

3.3-ta'rif. *Agar* $(x, y) = 0$ *boʻlsa, u holda* x *va* y *vektorlar ortogonal deyiladi va* $x \perp y$ *kabi belgilanadi.*

3.4-ta'rif. *Agar ixtiyoriy* $\alpha \neq \beta$ *da* $(x_\alpha,\, x_\beta) = 0$ *boʻlsa, u holda nolmas* $\{x_\alpha\}$ *vektorlar sistemasiga ortogonal sistema deyiladi. Agar bu holda har bir elementning normasi birga teng boʻlsa,* $\{x_\alpha\}$ *ortogonal normalangan sistema, qisqacha ortonormal sistema deyiladi.*

3.5-ta'rif. *Agar* $\{x_\alpha\}$ *sistemani oʻzida saqlovchi minimal yopiq qism fazo* E *fazoning oʻziga teng boʻlsa, u holda* $\{x_\alpha\}$ *sistema toʻla deyiladi.*

3.6-ta'rif. *Agar* $\{x_\alpha\}$ *ortonormal sistema toʻla boʻlsa, u holda bu sistema* E *fazodagi ortonormal bazis deyiladi.*

3.7-ta'rif. *Bizga* E *Evklid fazosi va* $\{\phi_k\}$ *ortonormal sistema berilgan boʻlsin. Agar ixtiyoriy* $f \in E$ *uchun*

$$\sum_{k=1}^{\infty} |c_k|^2 = \|f\|^2, \quad c_k = (f,\, \phi_k) \tag{3.3}$$

tenglik oʻrinli boʻlsa, $\{\phi_k\}$ *ortonormal sistema yopiq sistema deyiladi.*

(3.3) tenglik *Parseval tengligi* deyiladi. $c_k = (f,\, \phi_k)$ sonlar $f \in E$ elementning $\{\phi_k\}$ ortonormal sistemadagi *Furye koeffitsiyentlari* deyiladi.

Ixtiyoriy $f \in E$ element uchun uning Furye koeffitsiyentlari

$$\sum_{k=1}^{\infty} |c_k|^2 \leq \|f\|^2 \tag{3.4}$$

tengsizlikni qanoatlantiradi. (3.4) tengsizlik *Bessel tengsizligi* deyiladi.

3.8-ta'rif. *E Evklid fazosi (3.1) normaga nisbatan toʻla boʻlsa, u toʻla Evklid fazosi deyiladi.*

3.9-ta'rif. *Cheksiz o'lchamli to'la Evklid fazosi Hilbert fazosi deyiladi.*

3.10-ta'rif. *Agar E Evklid fazosining hamma yerida zich bo'lgan sanoqli to'plam mavjud bo'lsa, E separabel Evklid fazosi deyiladi.*

3.2-teorema *(Shmidtning ortogonallashtirish jarayoni). Bizga E Evklid fazosida chiziqli bog'lanmagan*

$$f_1,\ f_2,\ \ldots, f_n, \ldots$$

elementlar sistemasi berilgan bo'lsin. U holda E Evklid fazosida shunday

$$\phi_1,\ \phi_2,\ \ldots, \phi_n, \ldots \tag{3.5}$$

ortonormal sistema mavjudki, quyidagi tasvirlar o'rinli:

$$\phi_n = a_{n1} f_1 + a_{n2} f_2 + \ \cdots \ + a_{nn} f_n, \quad a_{nn} > 0\,;$$

va

$$f_n = b_{n1}\, \phi_1 + \, b_{n2}\, \phi_2 + \ \cdots \ + b_{nn}\, \phi_n, \quad b_{nn} > 0.$$

3.3-teorema. *To'la separabel Evklid fazosidagi $\{\phi_n\}$ ortonormal sistema to'la bo'lishi uchun, E da $\{\phi_n\}$ sistemaning barcha elementlariga ortogonal bo'lgan nolmas elementning mavjud bo'lmasligi zarur va yetarli.*

3.1. $\mathbb{C}^n = \{x = (x_1, \ldots, x_n) : \ x_k \in \mathbb{C}, \ k = 1, 2, \ldots, n\}$ chiziqli fazoni qaraylik.

$$p(x,\, y) = (x,\, y) = \sum_{k=1}^{n} x_k \overline{y_k}, \quad x,\, y \in \mathbb{C}^n \tag{3.6}$$

formula yordamida aniqlangan p funksional skalyar ko'paytma aksiomalarini qanoatlantirishini ko'rsating.

Yechish. $\mathbb{C}^n$ kompleks chiziqli fazo. Shuning uchun biz kompleks chiziqli fazoda berilgan skalyar ko'paytma shartlarini tekshiramiz.

1) $p(x,\, x) = \sum\limits_{k=1}^{n} x_k\overline{x_k} = \sum\limits_{k=1}^{n} |x_k|^2 \geq 0, \quad \forall x \in \mathbb{C}^n$
tengsizlik $|x_k| \geq 0$ ekanligidan kelib chiqadi. Endi

$$p(x,\, x) = \sum_{k=1}^{n} |x_k|^2 = 0$$

bo'lsin. Qo'shiluvchilarning manfiy emasligidan har bir k uchun $|x_k|^2 = 0$, bundan $x_k = 0$, ya'ni $x = \theta$ ekanligi kelib chiqadi. Aksincha, har bir k uchun $x_k = 0$ bo'lsa, u holda $p(x,\, x) = 0$ bo'lishi ko'rinib turibdi.

2) $p(x,\, y) = \sum\limits_{k=1}^{n} x_k\overline{y_k} = \sum\limits_{k=1}^{n} \overline{\overline{x_k} y_k} = \overline{\sum\limits_{k=1}^{n} y_k\overline{x_k}} = \overline{p(y,\, x)}.$
Bu tenglik ko'paytmaning qo'shmasi qo'shmalar ko'paytmasiga, yig'indining qo'shmasi esa qo'shmalar yig'indisiga tengligidan kelib chiqadi.

3) $p(x+y, z) = \sum\limits_{k=1}^{n} (x_k + y_k)\,\overline{z_k} = \sum\limits_{k=1}^{n} x_k\overline{z_k} = \sum\limits_{k=1}^{n} y_k\overline{z_k} =$

$$= p(x, z) + p(y, z), \quad \forall x, y, z \in \mathbb{C}^n,$$

4) $p(\lambda x,\, y) = \sum\limits_{k=1}^{n} \lambda x_k\overline{y_k} = \lambda \sum\limits_{k=1}^{n} x_k\overline{y_k} = \lambda p(x,\, y), \quad \forall x,\, y \in \mathbb{C}^n, \quad \lambda \in \mathbb{C}$. Bu tengliklarning bajarilishi kompleks sonlarni qo'shish va ko'paytirish xossalaridan kelib chiqadi. Demak, (3.6) tenglik yordamida aniqlangan p funksional skalyar ko'paytma aksiomalarini qanoatlantiradi va $\mathbb{C}^n$ kompleks Evklid fazosi bo'ladi. □

3.2. $E = \mathbb{R}^2$, $p\,(x, y) = \sqrt{(x_1^2 + x_2^2)\,(y_1^2 + y_2^2)}$ funksional uchun skalyar ko'paytmaning qaysi shartlari bajarilmasligini aniqlang.

Yechish. $p\,(x, y) = \sqrt{(x_1^2 + x_2^2)\,(y_1^2 + y_2^2)}$, $x, y \in \mathbb{R}^2$ funksional uchun skalyar ko'paytmaning 1-sharti bajariladi. Haqiqatan ham, ixtiyoriy nolmas $x \in \mathbb{R}^2$ da $p\,(x, x) = \sqrt{(x_1^2 + x_2^2)\,(x_1^2 + x_2^2)} = x_1^2 + x_2^2 > 0$ va $p(x, x) = 0 \Leftrightarrow x = (0, 0)$. Bu funksional uchun simmetriklik $p\,(x, y) = p\,(y, x)$ sharti o'rinli. Bu funksional uchun $p\,(\lambda x, y) = \lambda p\,(x, y)$ tenglik o'rinli emas. Masalan, $p\,(-2x, y) = 2p\,(x, y)$. Oson tekshirish mumkinki, bu funksional uchun $p\,(x + z, y) = p\,(x, y) + p\,(z, y)$ tenglik ham o'rinli emas. □

3.3. $\mathbb{R}^3$ fazoda $f_1 = (1, 0, 0)$, $f_2 = (1, 1, 0)$, $f_3 = (1, 1, 1)$ vektorlarning chiziqli erkliligini tekshiring, Shmidtning ortogonallashtirish jarayonini qo'llab, ortonormal sistema hosil qiling.

Yechish. Ma'lumki, $\mathbb{R}^n$ fazoda n ta vektordan iborat sistemaning chiziqli erkli bo'lishi uchun, bu vektorlarning koordinatalaridan tuzilgan determinantning noldan farqli bo'lishi zarur va yetarlidir. Berilgan vektorlar uchun bu determinant

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$$

bo'lganligi sababli, ular chiziqli erklidir. Endi bu elementlarga Shmidtning ortogonallashtirish jarayonini qo'llaymiz. $\varphi_1 = f_1 = (1,0,0)$ deb olsak, $\|\varphi_1\| = \sqrt{1^2+0^2+0^2} = 1$ bo'ladi. φ_2 elementni $\varphi_2 = f_2 - a_{21}\,\varphi_1$ ko'rinishda olib, a_{21} koeffitsiyentni $(\varphi_2, \varphi_1) = 0$ ortogonallik shartini qanoatlantiradigan qilib tanlaymiz:

$$0 = (\varphi_2, \varphi_1) = (f_2, \varphi_1) - a_{21}(\varphi_1, \varphi_1) \quad \text{yoki } a_{21} = \frac{(f_2, \varphi_1)}{\|\varphi_1\|^2} = \frac{1}{1} = 1.$$

U holda

$$\varphi_2 = (1,1,0) - (1,0,0) = (0,1,0)\,, \quad ||\varphi_2|| = 1,$$

bo'ladi. φ_3 vektorni quyidagi ko'rinishda izlaymiz:

$$\varphi_3 = f_3 - \ a_{31}\,\varphi_1 - \ a_{32}\,\varphi_2. \tag{3.7}$$

Bunda a_{31}, a_{32} koeffitsiyentlar, ortogonallik shartlaridan, ya'ni

$$(\varphi_3\,, \varphi_1) = (\varphi_3\,, \varphi_2) = 0 \tag{3.8}$$

shartlardan topiladi. Buning uchun (3.7) ni φ_1 va φ_2 ga skalyar ko'paytirib, (3.8) shartlardan foydalansak, a_{31}, a_{32} koeffitsiyentlarga nisbatan chiziqli tenglamalar sistemasi hosil bo'ladi. Bu tenglamaning yechimi:

$$a_{31} = \frac{(f_3\,, \varphi_1)}{\|\varphi_1\|^2} = \frac{1\cdot 1 + 1\cdot 0 + 0\cdot 1}{1} = 1, \quad a_{32} = \frac{(f_3, \varphi_2)}{\|\varphi_2\|^2} = \frac{1}{1} = 1.$$

Demak,

$$\varphi_3 = (1,1,1) - (1,0,0) - (0,1,0) = (0,0,1), \quad \|\varphi_3\| = 1.$$

Hosil bo'lgan φ_1 , φ_2, φ_3 vektorlar sistemasi ortonormaldir. □

3.4. $L_2(\mathbb{R}, e^{-t^2}d\mu)$ bilan $\mathbb{R}$ da aniqlangan, o'lchovli va

$$\int_{\mathbb{R}} |x(t)|^2 \, e^{-t^2} dt < \infty$$

shartni qanoatlantiruvchi funksiyalardan iborat fazoni belgilaymiz. Bu fazoda x va y elementlarning skalyar ko'paytmasini

$$(x,y) = \int_{\mathbb{R}} x(t)\, y(t)\, e^{-t^2} dt$$

formula bilan aniqlaymiz. Bu fazoda $1,\, t,\, t^2,\, t^3, \ldots, t^n, \ldots$ chiziqli bog'-lanmagan sistemadan ortonormal sistema hosil qiling. Hosil qilingan ortonormal sistema *Chebishev-Ermit ko'phadlari* deyiladi. Uning dastlabki uchta hadini toping.

Yechish. Ortonormal sistema $\psi_n = w_n(t) = a_0 + a_1 t + \cdots + a_{n-1}t^{n-1}$ shaklda izlanadi. Bu yerda $w_1(t) = a_0$ bo'lib, a_0 soni

$$\|w_n\|^2 = \int_{\mathbb{R}} a_0^2 \, e^{-t^2} dt = 1 \tag{3.9}$$

shartdan topiladi. Integral ostidagi funksiyaning juftligidan foydalanib, (3.9) ni quyidagicha yozamiz:

$$\|w_n\|^2 = a_0^2 \int_{\mathbb{R}} e^{-t^2} dt = 2a_0^2 \int_0^{\infty} e^{-t^2} dt. \tag{3.10}$$

Bu integralda $t^2 = x,\ t = \sqrt{x}$ o'zgaruvchini almashtirib, (3.10) integralni

$$\|w_n\|^2 = 2a_0^2 \int_0^{\infty} e^{-t^2} dt = a_0^2 \int_0^{\infty} x^{\frac{1}{2}-1} e^{-x} dx = a_0^2 \Gamma\left(\frac{1}{2}\right) = 1 \tag{3.11}$$

shaklda yozamiz. Gamma funksiyaning $\dfrac{1}{2}$ nuqtadagi qiymati $\sqrt{\pi}$ ekanligidan hamda (3.11) dan $a_0 = \dfrac{1}{\sqrt[4]{\pi}}$ ni olamiz. Demak, $\psi_1(t) = w_1(t) = \dfrac{1}{\sqrt[4]{\pi}}$

ekan. $\psi_2(t) = w_2(t) = b_0 + a_1 t$ shaklda izlanadi. b_0 va a_1 koeffitsiyentlar ortogonallik $(\psi_1, \psi_2) = 0$ va normallanganlik $\|\psi_2\| = 1$ shartlaridan topiladi. Ortogonallik sharti

$$(\psi_1, \psi_2) = \frac{b_0}{\sqrt[4]{\pi}} \int_{\mathbb{R}} e^{-t^2} dt + \frac{a_1}{\sqrt[4]{\pi}} \int_{\mathbb{R}} t e^{-t^2} dt = \frac{b_0}{\sqrt[4]{\pi}} \sqrt{\pi} + \frac{a_1}{\sqrt[4]{\pi}} \cdot 0 = 0$$

dan $b_0 = 0$ ni olamiz. Demak, $\psi_2(t) = a_1 t$ ekan. Normallanganlik sharti

$$\|\psi_2\|^2 = a_1^2 \int\limits_{\mathbb{R}} t^2 e^{-t^2} dt = 2a_1^2 \int\limits_0^\infty t^2 e^{-t^2} dt = a_1^2 \int\limits_0^\infty x^{\frac{3}{2}-1} e^{-x} dx = a_1^2 \Gamma\left(\frac{3}{2}\right) = 1$$

dan $a_1^2 = \dfrac{1}{\Gamma(1+\frac{1}{2})} = \dfrac{1}{\dfrac{1}{2}\,\Gamma\,(1/2)} = \dfrac{2}{\sqrt{\pi}}$ ni, ya'ni $a_1 = \dfrac{\sqrt{2}}{\sqrt[4]{\pi}}$ ni olamiz. Demak, $\psi_2(t) = \dfrac{\sqrt{2}\,t}{\sqrt[4]{\pi}}$ ekan. Navbatdagi ψ_3 element $\psi_3(t) = w_3(t) = c_0 + c_1 t + c_2 t^2$ shaklda izlanadi. c_0, c_1 va c_2 koeffitsiyentlar ortogonallik shartlari $(\psi_1, \psi_3) = (\psi_2, \psi_3) = 0$ va normallanganlik sharti $\|\psi_3\| = 1$ dan topiladi. Bu shartlardan $\psi_3(t) = \dfrac{2t^2 - 1}{\sqrt[4]{25\pi}}$ ekanligini topamiz. □

Eslatma. Faraz qilaylik, $\psi_1, \psi_2, \ldots, \psi_{n-1}$ ortonormal sistema qurilgan bo'lsin, u holda $\varphi_n = f_n - (f_n, \psi_1)\psi_1 - (f_n, \psi_2)\psi_2 - \cdots - (f_n, \psi_{n-1})\psi_{n-1}$ element $\psi_1, \psi_2, \ldots, \psi_{n-1}$ elementlarga ortogonal bo'ladi. Agar $\psi_n = \varphi_n : \|\varphi_n\|$ desak, u holda $\psi_1, \psi_2, \ldots, \psi_n$ sistema ortonormal sistema bo'ladi.

3.5. $x(t) = t(t-1) + 6^{-1}$ funksiyaning Rademaxer (Rademaxer sistemasi 3.63-misolda aniqlangan) sistemasidagi barcha $\{x_n\}_{n=0}^\infty$ larga ortogonal ekanligini ko'rsating. Demak, Rademaxer sistemasi to'la ortonormal sistema emas.

Yechish. $r_0 : [0,\ 1] \to \mathbb{R}$ funksiyani quyidagicha aniqlaymiz

$$r_0(t) = \begin{cases} 1, & t \in [0,\ \frac{1}{2}] \\ -1, & t \in (\frac{1}{2},\ 1] \end{cases}$$

va uni $\mathbb{R}$ ga 1 davrli funksiya sifatida davom ettiramiz. U holda barcha butun $m \geq 0$ lar uchun $r_m(t) = r_0(2^m t)$ tenglik o'rinli bo'ladi. Ravshanki, $\int_0^1 r_0(t)dt = 0$. Xuddi shunday

$$\int_0^1 r_m(t)dt = \int_0^1 r_0(2^m t)dt = 2^{-m}\int_0^{2^m} r_0(t)dt = \int_0^1 r_0(t)dt = 0$$

Endi $x(t) = t(t-1) + 6^{-1}$ funksiyani $r_m, m \in \mathbb{Z}_+$ larga ortogonal ekanligini ko'rsatamiz. Buning uchun

$$\int_0^1 r_m(t)x(t)dt = \int_0^1 r_0(2^m t)\left((t-\frac{1}{2})^2 + \frac{1}{6} - \frac{1}{4}\right)dt = \int_0^1 r_0(2^m t)(t-\frac{1}{2})^2 dt = 0$$

ekanligini ko'rsatish yetarli. Oxirgi integralda $t - \dfrac{1}{2} = s$ almashtirish olamiz, natijada

$$\int_0^1 r_m(t)x(t)dt = \int_{-0,5}^{0,5} r_0(2^m s)s^2 ds = 0$$

ni olamiz. Chunki, $r_0(2^m s)$ aniqlanishiga ko'ra toq funksiya, toq funksiyadan esa simmetrik $(-\dfrac{1}{2}, \dfrac{1}{2})$ oraliq bo'yicha olingan integral nolga teng. Shunday qilib, barcha butun $m \in \mathbb{Z}_+$ lar uchun $x(t)$ funksiya $r_m(t)$ ga ortogonal ekan. $\square$

Uy vazifalari va mavzuni o'zlashtirish uchun masalalar

3.6-3.10-misollarda keltirilgan $p : E \times E \to \mathbb{R}$ funksional, haqiqiy chiziqli fazoda skalyar ko'paytma shartlarini qanoatlantiradimi?

3.6. $p\,(x, y) = \sum\limits_{k=1}^{n} x_k y_k, \quad x, y \in \mathbb{R}^n$

3.7. $p\,(f, g) = \int\limits_a^b f(t)\,g(t)\,dt, \quad f, g \in C[a,\ b]$.

3.8. $p\,(x, y) = \sum\limits_{k=1}^{\infty} x_k y_k, \quad x, y \in \ell_2$.

3.9. $p\ (f, g) = \int\limits_a^b f(t)\, g(t)\, dt, \quad f, g \in L_2[a,\ b].$

3.10. $p\ (f, g) = \sum\limits_{n\in\mathbb{Z}} f(n)\, g(n), \quad f, g \in \ell_2(\mathbb{Z}).$

Eslatma. 3.7-misolda keltirilgan $(C[a,\ b], p)$ Evklid fazosi $C_2[a,\ b]$ bilan belgilanadi.

3.11-3.14-misollarda keltirilgan $p : E \times E \to \mathbb{C}$ funksional, koʻrsatilgan kompleks chiziqli fazoda skalyar koʻpaytma shartlarini qanoatlantirishini koʻrsating.

3.11. $p\ (f, g) = \int\limits_a^b f(t)\, \overline{g(t)}\, dt, \quad f, g \in C[a,\ b].$

3.12. $p\ (x, y) = \sum\limits_{k=1}^{\infty} x_k \overline{y_k}, \quad x, y \in \ell_2.$

3.13. $p\ (f, g) = \int\limits_a^b f(t)\, \overline{g(t)}\, dt, \quad f, g \in L_2[a,\ b].$

3.14. $p\ (f, g) = \sum\limits_{k\in\mathbb{Z}} f(k)\, \overline{g(k)}, \quad f, g \in \ell_2(\mathbb{Z}).$

3.15-3.23-misollarda keltirilgan $p : E \times E \to \mathbb{R}$ funksional, koʻrsatilgan haqiqiy chiziqli fazoda skalyar koʻpaytma shartlarini qanoatlantiradimi?

3.15. $E = \mathbb{R}^2,\ p(x, y) = x_1 y_1 - x_2 y_2.$

3.16. $E = \mathbb{R}^2,\ \ p(x, y) = x_1 y_1 - x_2 y_1 + 2 x_2 y_2.$

3.17. $E = \mathbb{R}^3,\ p(x, y) = x_1 y_1 + x_2 y_2 - x_3 y_3.$

3.18. $E = \mathbb{R}^3,\ \ p(x, y) = x_1 y_1 + x_2 y_2 - x_3 y_3.$

3.19. $E = \ell_2,\ p\ (x, y) = \sum\limits_{k=1}^{\infty} \lambda_k x_k y_k,\ 0 < \lambda_n < 1.$

3.20. $E = \ell_2, \quad p(x, y) = \sum\limits_{k=1}^{\infty} \dfrac{x_k y_k}{k}.$

3.21. $E = C[a,b]\,,\ p(x,y) = \int\limits_a^b e^t x(t)\ y(t)\,dt.$

3.22. $E = C[a,b]\,,\ \ p(x,y) = \int\limits_a^b x^4(t)y^4(t)dt.$

3.23. $E = C^{(1)}[a,\ b],\ p(x,y) = \int\limits_a^b x(t)\ y(t)\,dt + \int\limits_a^b x'(t)\,y'(t)\,dt.$

3.24-3.33-misollarda keltirilgan vektorlarning chiziqli erkliligini tekshiring, Shmidtning ortogonallashtirish jarayonini qo'llab, ortonormal sistema hosil qiling. $\mathbb{R}^n$, ℓ_2, $C_2[a,\ b]$, $L_2[a,\ b], \ell_2(\mathbb{Z})$ fazolardagi skalyar ko'paytmalarni 3.6-3.10-misollardan qarab oling.

3.24. $E = \mathbb{R}^3$ fazoda $x = (0,\ 0,\ 1),\ \ y = (0,\ 1,\ 1),\ \ z = (1,\ 1,\ 1)$

3.25. $E = \mathbb{R}^3, \quad x = (1,\ 1,\ 0),\ \ y = (2,\ 0,\ -1),\ \ z = (0,\ -1,\ 1).$

3.26. $E = \mathbb{R}^3, \quad x = (-1,\ 0,\ 0),\ \ y = (0,\ -1,\ 1),\ \ z = (2,\ 0,\ -1).$

3.27. $E = C_2[-1,\ 1],\ \ x_1(t) = 1,\ \ x_2(t) = t^3,\ \ \ x_3(t) = t^6.$

3.28. $E = C_2[-1,\ 1], \quad x(t) = 1, \quad y(t) = t, \quad\ z(t) = t^2.$

3.29. $E = L_2[0,\ \pi],\ \ x(t) = 1,\ \ y(t) = \cos t,\ \ z(t) = \sin t.$

3.30. $E = L_2[-1,\ 1],\ \ x_1(t) = 1,\ \ x_2(t) = t,\ \ \ x_3(t) = t^2 + 1.$

3.31. $E = \ell_2,\ x = (1,0,0,\ldots)\,,\ y = (1,1,0,0,\ldots)\,,\ z = (1,1,1,0,0,\ldots)\,.$

3.32. $E = \ell_2,\ \ x = \left(0, 1, \frac{1}{2}, \ldots, \frac{1}{2^n}, \ldots\right),\ y = \left(1, 0, \frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^n}, \ldots\right).$

3.33. $E = \ell_2,\ x = (1,1,0,0,\ldots)\,, \quad y = (0,0,1,1,0,\ldots)\,.$

3.34. E Evklid fazosida ixtiyoriy x, y, z elementlar uchun Apolloniy ayniyatini isbotlang

$$\|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\,\|x-y\|^2 + 2\,\left\|z - \frac{x+y}{2}\right\|^2.$$

3.35. E Evklid fazosida ixtiyoriy x, y, z, t elementlar uchun Ptolemey tengsizligini isbotlang

$$\|x - z\| \cdot \|y - t\| \leq \|x - y\| \cdot \|z - t\| + \|y - z\| \cdot \|x - t\|.$$

3.36. E Evklid fazosida x va y elementlar ortogonal bo‘lishi uchun

$$\|x\|^2 + \|y\|^2 = \|x + y\|^2$$

tenglikning bajarilishi zarur va yetarli. Isbotlang.

3.37. E haqiqiy normalangan fazo va ixtiyoriy x, y elementlar uchun parallelogramm ayniyati

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

bajarilsin. U holda

$$p : E \times E \to \mathbb{R}, \quad p(x, y) = \frac{1}{4}\left\{\|x + y\|^2 - \|x - y\|^2\right\}$$

funksional skalyar ko‘paytma shartlarini qanoatlantirishini ko‘rsating.

3.38. $x_1, x_2, \ldots, x_n$ lar E Evklid fazosidagi ixtiyoriy ortonormal sistema bo‘lsin. Bu sistemaning chiziqli erkli ekanligini isbotlang.

3.39. E haqiqiy Evklid fazosi. Ixtiyoriy x, y elementlar uchun Koshi-Bunyakovskiy tengsizligi $|(x, y)| \leq \|x\| \cdot \|y\|$ ni isbotlang.

3.40. E Evklid fazosidagi $x_1, x_2, \ldots, x_n \in E$ sistemaning Gram determinanti deb

$$\mathfrak{T}(x_1, x_2, \ldots, x_n) = \begin{vmatrix} (x_1, x_1) & (x_1, x_2) & \cdots & (x_1, x_n) \\ (x_2, x_1) & (x_2, x_2) & \cdots & (x_2, x_n) \\ \cdots & \cdots & \cdots & \cdots \\ (x_n, x_1) & (x_n, x_2) & \cdots & (x_n, x_n) \end{vmatrix}$$

determinant tushuniladi. $x_1, x_2, \ldots, x_n \in E$ elementlar sistemasining chiziqli erkli bo'lishi uchun, uning Gram determinanti noldan farqli bo'lishi zarur va yetarli. Isbotlang.

3.41. x_n va y_n lar H Hilbert fazosidagi yopiq birlik sharga tegishli va $\lim\limits_{n\to\infty}(x_n, y_n) = 1$ bo'lsa, u holda $\lim\limits_{n\to\infty}\|x_n - y_n\| = 0$ bo'ladi. Isbotlang.

3.42. $H-$ Hilbert fazosi, L uning qism fazosi bo'lsin. x element L qism fazoga orthogonal bo'lishi uchun istalgan $y \in L$ da $\|x\| \leq \|x - y\|$ tengsizlikning bajarilishi zarur va yetarli. Isbotlang.

3.43. $C_2[-\pi,\ \pi]$ Evklid fazosida $\varphi_n(t) = \sin nt,\ \ n \in \mathbb{N}$ sistemaning ortogonal ekanligini isbotlang. $\{\varphi_n\}$ sistemadan ortonormal sistemaga o'ting.

3.44. $L_2[-\pi,\ \pi]$ kompleks Hilbert fazosida $\varphi_n(t) = \exp\{int\},\ \ n \in \mathbb{Z},$ sistemaning ortogonal ekanligini isbotlang. $\{\varphi_n\}$ sistemadan ortonormal sistemaga o'ting.

3.45. Kompleks Hilbert fazosida quyidagi tenglikni isbotlang:

$$(x, y) = \frac{1}{4}\left\{\|x + y\|^2 - \|x - y\|^2 + i\,\|x + iy\|^2 - i\,\|x - iy\|^2\right\}.$$

3.46. $C_2[-\pi,\ \pi]$ Evklid fazosida $\varphi(t) = \cos^2 t$ elementning

$$\left\{\frac{1}{\sqrt{2\pi}},\ \varphi_n(t) = \frac{1}{\sqrt{\pi}}\cos nt,\quad n \in \mathbb{N}\right\}$$

ortonormal sistemadagi Furye koeffitsiyentlarini toping.

3.47. $H-$ Hilbert fazosi, $x_1, x_2, \ldots, x_n$ undagi ixtiyoriy ortogonal sistema va $x = x_1 + x_2 + \cdots + x_n$ bo'lsin. Pifagor tengligini isbotlang.

$$\|x\|^2 = \|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_n\|^2.$$

3.48. Hilbert fazosi qat'iy normalangan fazo ekanligini isbotlang.

3.49. H Hilbert fazosidagi x_1, x_2 elementlar uchun $Re(x_1, x_2) = \|x_1\|^2 = \|x_2\|^2$ tenglik o'rinli bo'lsin. U holda $x_1 = x_2$ ekanligini isbotlang.

3.50. Har bir natural n da $M_n = \{x \in \ell_2 : x_1 + x_2 + \cdots + x_n = 0\}$ to'plam ℓ_2 Hilbert fazosining qism fazosi bo'lishini isbotlang. M_1, M_2, M_3 qism fazolarning ortogonal to'ldiruvchilarini tavsiflang, ularning o'lchamlarini toping.

3.51. ℓ_2 Hilbert fazosida $M = \{x \in \ell_2 : x_1 + x_2 + \cdots + x_n + \cdots = 0\}$ to'plamning chiziqli ko'pxillilik ekanligini hamda ℓ_2 fazoning hamma yerida zich bo'lishini isbotlang.

3.52. $L_2^-[-1,\ 1] = \{f \in L_2[-1,\ 1] :\ f(-t) = -f(t)\}$ toq funksiyalar to'plami $L_2[-1,\ 1]$ fazoning qism fazosi bo'lishini isbotlang. Uning ortogonal to'ldiruvchisini toping. $\dim L_2^-[-1,\ 1]$ va $\dim (L_2^-[-1,\ 1])^\perp$ larni hisoblang.

3.53. $L_2^-[-1,\ 1]$ toq funksiyalar fazosida $\{\varphi_n(t) = \sin n\pi t\}_{n\in\mathbb{N}}$ sistemaning ortonormal bazis bo'lishini isbotlang.

3.54. $L_2^+[-\pi,\ \pi]$ juft funksiyalar fazosida $\left\{\varphi_n(t) = \dfrac{1}{\sqrt{\pi}} \cos nt\right\}_{n\in\mathbb{N}}$ ortonormal sistemaning to'la emasligini isbotlang.

3.55. $L_2^+[-1,\ 1]$ juft funksiyalar fazosida $\left\{\dfrac{1}{\sqrt{2}},\ \varphi_n(t) = \cos n\pi t\right\}_{n\in\mathbb{N}}$ ortonormal sistemaning to'laligini isbotlang.

3.56. *Lejandr ko'phadlari haqida to'tr (3.56-3.59) masala.*

$$P_0(x) = 1, \qquad P_n(x) = \frac{1}{2^n}\frac{1}{n!}\frac{d^n(x^2-1)^n}{dx^n},\ n \in \mathbb{N}$$

ko'phadlarga *Lejandr ko'phadlari* deyiladi. Ixtiyoriy $m < n$ uchun

$$(P_n, Q_m) = \int_{-1}^{1} P_n(x)\, Q_m(x) dx = 0$$

ekanligini isbotlang. Bu yerda Q_m bilan $m-$ darajali ko'phad belgilangan.

3.57. $L_2[-1,\ 1]$ fazoda Lejandr ko'phadlari $\{1,\ P_n\}_{n\in\mathbb{N}}$ ning ortogonal sistema ekanligini isbotlang.

3.58. Lejandr ko'phadi P_n ni $P_n(x) = c_n x^n + Q_{n-1}(x),\ n \in \mathbb{N}$ shaklda tasvirlang, c_n- koeffitsiyentni toping.

3.59. Lejandr ko'phadlari $P_n \in L_2[-1,\ 1]$ ning normasini hisoblang.

3.60. $L_2[-1,\ 1]$ fazoda $1, t, t^2, t^3, \ldots, t^n, \ldots$ ko'phadlardan ortonormal sistema hosil qiling. Hosil qilingan ortonormal sistemani Lejandr ko'phadlari bilan taqqoslang.

3.61. $L_2(\mathbb{R}_+, e^{-t}d\mu)$ bilan $\mathbb{R}_+ = [0,\ \infty)$ da aniqlangan, o'lchovli va

$$\int_0^\infty |x(t)|^2\, e^{-t}dt < \infty$$

shartni qanoatlantiruvchi funksiyalardan iborat fazoni belgilaymiz. Bu fazoda x va y elementlarning skalyar ko'paytmasini

$$(x, y) = \int_0^\infty x(t)\, y(t)\, e^{-t}dt$$

formula bilan aniqlaymiz. Bu fazoda $1, t, t^2, t^3, \ldots, t^n, \ldots$ chiziqli bog'lanmagan sistemadan ortonormal sistema hosil qiling. Hosil qilingan ortonormal sistema *Chebishev-Lagger ko'phadlari* deyiladi. Uning dastlabki uchta hadini toping.

3.62. $L_{20}^{+}[-1,\ 1] = \{f \in L_2[-1,\ 1] :\ f(t)\,\chi_{[0,\,1]}(t) \sim 0\}$ to'plam $L_2[-1,\ 1]$ fazoning qism fazosi bo'lishini isbotlang. Uning ortogonal to'ldiruvchisini toping.

3.63. $[0,\ 1]$ kesmada x_n funksional ketma-ketlikni quyidagicha aniqlaymiz: $r_0(t) = 1$ va

$$r_n(t) = (-1)^k,\ t \in \left(\frac{k}{2^n}, \frac{k+1}{2^n}\right),\ \ n \in \mathbb{N};\ k = 0, 1, 2, \ldots, 2^n - 1,$$

bu interval chekkalarida $r_n(t) = 0$ deymiz. Bu *Rademaxer sistemasi* deyiladi. Bu sistemaning $L_2[0,\ 1]$ fazoda ortonormal ekanligini isbotlang.

3.64. Agar μ_n lar $tg\mu = \mu$ tenglamaning musbat ildizlari bo‘lsa, u holda $x_n(t) = \sin \mu_n t,\ n \in \mathbb{N}$ sistema $L_2[0,\ 1]$ da to‘la ortonormal bazis bo‘lishini isbotlang.

3.65. $f(x) = 1,\ g(x) = x,\ \varphi(x) = x^2$ elementlarning $L_2[-1,\ 1]$ fazoda $\{\varphi_n^-(t) = \sin n\pi t\},\quad n \in \mathbb{N},\quad \{\varphi_n^+(t) = \cos n\pi t\},\ \ n \in \mathbb{N},$ $\left\{\psi_n(t) = 2^{-1/2} \exp\{in\pi t\}\right\},\ n \in \mathbb{Z}$ ortonormal sistemalardagi Furye koeffitsiyentlarini toping.

3.66. $f(x) = \operatorname{sign} x,\ g(x) = [x],\ \varphi(x) = \chi_{[0,\,1]}(x)$ elementlarning $L_2[-1,\ 1]$ fazoda $\{\varphi_n^-(t) = \sin n\pi t\},\ \{\varphi_n^+(t) = \cos n\pi t\},\ n \in \mathbb{N},$

$$\left\{\psi_n(t) = 2^{-1/2} \exp\{in\pi t\}\right\},\ \ n \in \mathbb{Z}$$

ortonormal sistemalardagi Furye koeffitsiyentlarini toping.

3.67. $f(t) = e^t$ funksiya uchun shunday $n-$darajali $p_n(t),\ n = 1, 2, 3$ ko‘phadlar topingki, $\|f - p_n\|$ norma $L_2[-1,\ 1]$ da minimal bo‘lsin.

3.68. $f(t) = t^4$ funksiya uchun shunday $n-$ darajali $p_n(t),\ n = 1, 2, 3$ ko‘phadlar topingki, $\|f - p_n\|$ norma $L_2[-1,\ 1]$ da minimal bo‘lsin.

3.69. $\Psi[a,\ b]$ bilan $[a,\ b]$ kesmada aniqlangan va

$$\sup_S \sum_{x \in S} |f(x)|^2 < \infty$$

shartni qanoatlantiruvchi funksiyalar to‘plamini belgilaymiz. Bu yerda aniq yuqori chegara $[a,\ b]$ da saqlanuvchi barcha chekli yoki sanoqli S to‘plamlar bo‘yicha olinadi. Bu to‘plam funksiyalarni qo‘shish va funksiyani songa ko‘paytirish amallariga nisbatan chiziqli fazo tashkil qiladi. $\Psi[a,\ b] \times \Psi[a,\ b]$ da aniqlangan

$$p(f, g) = \sum_{x \in [a,\, b]} f(x)\overline{g(x)}\, dx, \quad f, g \in \Psi[a,\ b]$$

funksional skalyar ko‘paytma shartlarini qanoatlantiradi. Hosil bo‘lgan Evklid fazosi to‘la, ammo separabel emasligini isbotlang.

3.70. $\mathbb{R}^n,\ n \in \mathbb{N}$ da aniqlangan va kvadrati integrallanuvchi bo‘lgan ekvivalent funksiyalar sinflaridan tashkil topgan vektor fazoni qaraymiz. Bu fazoda

$$p(f, g) = \int_{\mathbb{R}^n} f(t)\overline{g(t)}\, dt$$

funksionalning skalyar ko‘paytma shartlarini qanoatlantirishini tekshiring. Hosil bo‘lgan Hilbert fazosi $L_2(\mathbb{R}^n)$ bilan belgilanadi. $f(x) = \dfrac{1}{1+x^2}$ va $g(x) = \chi_{[-1,\,1]}(x)$ larni $L_2(\mathbb{R})$ ga qarashli ekanligini ko‘rsating. Bu elementlarning skalyar ko‘paytmasini toping. Ular ortogonalmi?

3.71. $f(x, y) = \exp\{-|x| - |y|\}$ va $g(x, y) = \chi_{[-1,1]\times[0,\,1]}(x,\ y)$ larni $L_2(\mathbb{R}^2)$ ga qarashli ekanligini ko‘rsating. Bu elementlarning normalarini va ularning skalyar ko‘paytmasini toping.

3.72. $\ell_2(\mathbb{Z})$ Hilbert fazosida $f(0) = 0,\ f(n) = n^{-1},\ n \in \mathbb{Z}\backslash\{0\}$ elementning normasini hisoblang.

3.73. Parametr α va β larning qanday qiymatlarida $f(n, m) = (1 + |n|^{\alpha} + |m|^{\beta})^{-1}$ funksiya $\ell_2(\mathbb{Z}^2)$ Hilbert fazosining elementi bo‘ladi.

3.74. Agar H Hilbert fazosida $\{\varphi_n\}_{n\in\mathbb{N}}$ ortonormal bazis bo‘lsa, u holda quyidagilarni isbotlang:

a) istalgan $x \in H$ uchun $x = \sum\limits_{n=1}^{\infty} (x, \varphi_n)\, \varphi_n$ tenglik o'rinli.

b) ixtiyoriy $x, y \in H$ lar uchun $(x, y) = \sum\limits_{n=1}^{\infty} (x, \varphi_n)(\varphi_n, y)$ tenglik o'rinli.

3.75. E Evklid fazosi, $\{\varphi_n\}_{n\in\mathbb{N}}$ esa E dagi ixtiyoriy ortonormal sistema. $\{\varphi_n\}$ ketma-ketlik nolga kuchsiz yaqinlashadi. Isbotlang.

4-§. Chiziqli funksionallar

Bu paragrafda biz chiziqli funksionallar, qavariq funksionallar hamda qavariq jismlarga doir masalalar qaraymiz.

4.1-ta'rif. *X chiziqli fazoda aniqlangan f sonli funksiyaga funksional deyiladi. Agar barcha $x, y \in X$ lar uchun*

$$f(x + y) = f(x) + f(y)$$

bo'lsa, f additiv funksional deyiladi. Agar barcha $x \in X$ va barcha $\alpha \in \mathbb{C}$ lar uchun $f(\alpha x) = \alpha f(x)$ bo'lsa, f bir jinsli funksional deyiladi.

Agar barcha $x \in X$ va barcha $\alpha \in \mathbb{C}$ lar uchun $f(\alpha x) = \overline{\alpha} f(x)$ bo'lsa, f ga qo'shma bir jinsli funksional deyiladi.

4.2-ta'rif. *Additiv va bir jinsli funksional chiziqli funksional deyiladi. Additiv va qo'shma bir jinsli funksionalga qo'shma chiziqli funksional deyiladi.*

4.3-ta'rif. *$Ker\, f = \{x \in X : \ f(x) = 0\}$ to'plam f chiziqli funksionalning yadrosi deyiladi.*

X haqiqiy chiziqli fazo, x va y uning ikki nuqtasi bo'lsin. U holda

$$\alpha x + \beta y, \qquad \alpha, \beta \in [0, \ 1], \quad \alpha + \beta = 1$$

shartni qanoatlantiruvchi barcha elementlar to'plami x va y *nuqtalarni tutashtiruvchi kesma* deyiladi va u $[x, y]$ bilan belgilanadi, ya'ni

$$[x, y] = \{\alpha x + \beta y : \qquad \alpha, \beta \in [0, \ 1], \quad \alpha + \beta = 1\}.$$

4.4-ta'rif. *Agar* $M \subset X$ *to'plam o'zining ixtiyoriy* $x, y \in M$ *nuqtalarini tutashtiruvchi* $[x, y]$ *kesmani ham o'zida saqlasa,* M *ga qavariq to'plam deyiladi.*

4.5-ta'rif. *Agar biror* $x \in M$ *nuqta va ixtiyoriy* $y \in X$ *uchun shunday* $\varepsilon = \varepsilon(y) > 0$ *son mavjud bo'lib, barcha* t, $|t| < \varepsilon$ *larda* $x + t\,y \in M$ *munosabat bajarilsa,* $x \in M$ *nuqta* M *to'plamning yadrosiga qarashli deyiladi.*

$M \subset X$ to'plamning yadrosi $-J(M)$ bilan belgilanadi, ya'ni

$$J(M) = \{x \in M : \forall y \in X, \, \exists \varepsilon = \varepsilon(y) > 0, \, \forall t \in \mathbb{R}, \, |t| < \varepsilon, \, x + t\,y \in M\}\,.$$

Agar X chiziqli normalangan fazo bo'lsa, u holda $M \subset X$ ning yadrosi M ning ichi bilan ustma-ust tushadi, ya'ni $J(M) = \overset{\circ}{M}$.

4.6-ta'rif. *Yadrosi bo'sh bo'lmagan qavariq to'plam qavariq jism deyiladi.*

4.7-ta'rif. X *chiziqli fazoda aniqlangan manfiymas* p *funksional*

1) $p\,(x + y) \leq p\,(x) + p\,(y)\,, \quad \forall x, y \in X,$

2) $p\,(a\,x) = a\,p\,(x)\,, \quad \forall a \geq 0$ *va* $\forall\, x \in X$ *shartlarni qanoatlantirsa,* p *ga qavariq funksional deyiladi.*

Biz bu yerda $p(x)$ miqdorni chekli deb faraz qilmaymiz, ya'ni ayrim $x \in X$ lar uchun $p(x) = \infty$ ham bo'lishi mumkin. Agar barcha $x \in X$ lar uchun $p(x)$ chekli bo'lsa, p *chekli qavariq funksional* deyiladi.

4.8-ta'rif. $L-$ *haqiqiy chiziqli fazo va* L_0- *uning biror qism fazosi bo'lsin.* L_0 *qism fazoda* f_0 *chiziqli funksional va* L *fazoda* f *chiziqli funksional berilgan bo'lsin. Agar ixtiyoriy* $x \in L_0$ *uchun* $f(x) = f_0(x)$ *tenglik bajarilsa,* f *chiziqli funksional* f_0 *funksionalning* L *fazoga davomi deyiladi.*

4.1-teorema *(Xan-Banax). Aytaylik,* $p\, -\, L$ *haqiqiy chiziqli fazoda aniqlangan qavariq funksional va* L_0 *esa* L *ning qism fazosi bo'lsin. Agar* L_0 *da aniqlangan* f_0 *chiziqli funksional*

$$f_0\,(x) \leq p\,(x)\,, \quad x \in L_0 \tag{4.1}$$

shartni qanoatlantirsa, u holda f_0 ni L da aniqlangan va L da (4.1) shartni qanoatlantiruvchi f chiziqli funksionalgacha davom ettirish mumkin.

4.9-ta'rif. *X chiziqli normalangan fazoda aniqlangan f funksional berilgan bo'lsin. Agar ixtiyoriy $\varepsilon > 0$ uchun shunday $\delta = \delta(\varepsilon) > 0$ mavjud bo'lib, $\|x - x_0\| < \delta$ tengsizlikni qanoatlantiruvchi barcha $x \in X$ lar uchun $|f(x) - f(x_0)| < \varepsilon$ tengsizlik bajarilsa, f funksional $x = x_0$ nuqtada uzluksiz deyiladi. Agar f funksional ixtiyoriy $x \in X$ nuqtada uzluksiz bo'lsa, f uzluksiz funksional deyiladi.*

4.9-ta'rifga teng kuchli bo'lgan quyidagi ta'rifni keltiramiz.

4.10-ta'rif. *Agar x_0 nuqtaga yaqinlashuvchi ixtiyoriy x_n ketma-ketlik uchun $\lim\limits_{n\to\infty} |f(x_n) - f(x_0)| = 0$ bo'lsa, u holda f funksional x_0 nuqtada uzluksiz deyiladi.*

4.2-teorema. *X chiziqli normalangan fazoda aniqlangan chiziqli funksional biror $x_0 \in X$ nuqtada uzluksiz bo'lsa, u holda bu chiziqli funksional butun X fazoda uzluksiz bo'ladi.*

Endi chegaralangan funksional ta'rifini keltiramiz.

4.11-ta'rif. *Agar biror $M > 0$ soni va barcha $x \in X$ lar uchun $|f(x)| \leq M\|x\|$ tengsizlik bajarilsa, $f : X \to \mathbb{R}$ ga chegaralangan funksional deyiladi.*

4.3-teorema. *X chiziqli normalangan fazoda aniqlangan chiziqli f funksional uzluksiz bo'lishi uchun uning chegaralangan bo'lishi zarur va yetarli.*

$|f(x)| \leq M\|x\|$ tengsizlikni qanoatlantiruvchi M sonlar to'plamining aniq quyi chegarasi f *funksionalning normasi* deyiladi va u $\|f\|$ bilan belgilanadi. Shunday qilib,

$$|f(x)| \leq \|f\| \cdot \|x\|.$$

4.4-teorema. *Chiziqli chegaralangan funksionalning normasi $\|f\|$ uchun quyidagi tenglik o'rinli:*

$$\|f\| = \sup_{\|x\|=1} |f(x)| = \sup_{x\neq 0} \frac{|f(x)|}{\|x\|}. \tag{4.2}$$

4.5-teorema *(Xan-Banax). L kompleks chiziqli normalangan fazo, L_0 esa L ning qism fazosi va $f_0 : L_0 \to \mathbb{C}$ chiziqli uzluksiz funksional bo'lsin. U holda f_0 ni normasini saqlagan holda L da aniqlangan f chiziqli funksionalgacha davom ettirish mumkin, ya'ni $f(x) = f_0(x), \quad x \in L_0$ va $\|f\| = \|f_0\|$ shartlarni qanoatlantiruvchi $f : L \to \mathbb{C}$ chiziqli funksional mavjud.*

X chiziqli normalangan fazoda aniqlangan chiziqli uzluksiz (chegaralangan) funksionallar to'plamini $L(X, \mathbb{C})$ bilan belgilaymiz.

4.12-ta'rif. *$f : X \to \mathbb{C}$ va $g : X \to \mathbb{C}$ chiziqli funksionallarning yig'indisi deb, $x \in X$ elementga $f(x) + g(x) = \varphi(x)$ sonni mos qo'yuvchi $\varphi = f + g : X \to \mathbb{C}$ funksionalga aytiladi.*

Ravshanki, $\varphi : X \to \mathbb{C}$ chiziqli funksional bo'ladi. Agar $f, g \in L(X, \mathbb{C})$ bo'lsa, u holda φ ham chegaralangan (uzluksiz) funksional bo'ladi va quyidagi tengsizlik o'rinli

$$\|f + g\| \leq \|f\| + \|g\| \, .$$

4.13-ta'rif. *$f : X \to \mathbb{C}$ chiziqli funksionalning songa ko'paytmasi x elementga $\alpha\, f(x)$ sonni mos qo'yuvchi funksional sifatida aniqlanadi, ya'ni*

$$(\alpha\, f)(x) = \alpha\, f(x).$$

$L(X, \mathbb{C})$ to'plamda kiritilgan qo'shish va songa ko'paytirish amallari chiziqli fazo ta'rifidagi 1-8 shartlarni qanoatlantiradi. Demak, $L(X, \mathbb{C})$ to'plam chiziqli fazo bo'ladi. Bu fazoda $p(f) = \|f\|$ funksional norma shartlarini qanoatlantiradi. Shunday qilib, $L(X, \mathbb{C})$ chiziqli normalangan fazo bo'ladi. Bu fazo X ga *qo'shma fazo* deyiladi va X^* bilan belgilanadi, ya'ni $X^* = L(X, \mathbb{C})$. Funksional fazolarda chiziqli uzluksiz funksionallarning umumiy ko'rinishidan foydalanib, asosiy funksional fazolarga qo'shma fazolarni izomorfizm aniqligida topish mumkin. Hozir biz $C[a,\ b]$ va $\ell_p,\ p > 1$ fazo hamda Evklid (chekli va cheksiz o'lchamli) fazolarda chiziqli uzluksiz funksionalning umumiy ko'rinishini keltiramiz.

4.7-teorema (Riss). *$C[a, b]$ fazoda berilgan ixtiyoriy f chiziqli uzluksiz funksional uchun shunday $u \in V_0[a, b]$ o'zgarishi chegaralangan funksiya mavjudki, barcha $x \in C[a, b]$ larda*

$$f(x) = \int_a^b x(t)du(t)$$

tenglik o'rinli. Bundan tashqari $\|f\| = V_a^b[u]$ tenglik ham o'rinli.

4.8-teorema. *ℓ_p, $p > 1$ fazoga qo'shma $(\ell_p)^*$ fazo ℓ_q, $\dfrac{1}{p} + \dfrac{1}{q} = 1$ fazoga izomorfdir, ya'ni har bir $f : \ell_p \to C$ chiziqli uzluksiz funksional uchun shunday $\widetilde{f} = (\widetilde{f}_1, \widetilde{f}_2, \ldots, \widetilde{f}_n, \ldots) \in \ell_q$ element mavjudki, quyidagilar o'rinli:*

$$f(x) = \sum_{n=1}^{\infty} \widetilde{f}_n x_n, \quad x \in \ell_p, \quad \|f\| = \left\|\widetilde{f}\right\|.$$

4.9-teorema *(Riss). Har bir $f \in H^*$ funksional uchun shunday yagona $y \in H$ element mavjudki, quyidagilar o'rinli:*

$$f(x) = (x, y), \quad x \in H, \quad \|f\| = \|y\|.$$

Bu yerda H Hilbert fazosi (x, y) esa x va y larning skalyar ko'paytmasi.

4.14-ta'rif. *Agar $f : H \times H \to \mathbb{C}$ akslantirish uchun*

1) $f(\alpha x + \beta y, z) = \alpha f(x, z) + \beta f(y, z)$;

2) $f(x, \alpha y + \beta z) = \overline{\alpha} f(x, y) + \overline{\beta} f(x, z)$;

3) *shunday $C > 0$ mavjud bo'lib, barcha $x, y \in H$ larda $|f(x, y)| \leq C \|x\| \cdot \|y\|$ bo'lsa, f ga bichiziqli uzluksiz funksional deyiladi.*

4.15-ta'rif. *Agar $f : H \times H \to \mathbb{C}$ bichiziqli uzluksiz funksional uchun barcha $x, y \in H$ larda $f(x, y) = \overline{f(y, x)}$ bo'lsa, f ga simmetrik bichiziqli funksional deyiladi.*

Har bir bichiziqli $f(x, y)$ funksional $\varphi(x) = f(x, x)$ kvadratik formani hosil qiladi.

4.16-ta'rif. *Agar simmetrik bichiziqli f funksional uchun barcha $x \neq 0$ larda $f(x, x) > 0$ bo'lsa, f ga qat'iy musbat bichiziqli funksional deyila-*

di. Agar barcha $x \in H$ larda $f(x,x) \geq 0$ bo'lsa, f ga musbat bichiziqli funksional deyiladi.

Bichiziqli uzluksiz f funksionalning normasi $\|f\|$ quyidagi tenglik yordamida aniqlanadi

$$\|f\| = \sup \{|f(x,y)| : \ \|x\| = \|y\| = 1\}.$$

Bizga $f \in L(X, \mathbb{C})$ va $f_n \in L(X, \mathbb{C})$ funksionallar ketma-ketligi berilgan bo'lsin.

4.17-ta'rif. *Agar $\lim\limits_{n\to\infty} \|f_n - f\| = 0$ bo'lsa, $\{f_n\}$ funksionallar ketma-ketligi f funksionalga yaqinlashadi deyiladi.*

4.18-ta'rif. *Agar har bir $x \in X$ uchun $\lim\limits_{n\to\infty} f_n(x) = f(x)$ bo'lsa, $\{f_n\}$ funksionallar ketma-ketligi f funksionalga kuchsiz yaqinlashadi deyiladi.*

4.1. $f : C[0, 1] \to \mathbb{C}, \quad f(x) = \int\limits_0^1 x(t)\, dt$ funksional chiziqli, qo'shma chiziqli, uzluksiz bo'ladimi? Tekshiring.

Yechish. Integralning additivlik va bir jinslilik xossalaridan foydalansak quyidagilarga ega bo'lamiz:

$$f(x+y) = \int_0^1 (x(t) + y(t))\, dt = \int_0^1 x(t)\ dt + \int_0^1 y(t)\ dt = f(x) + f(y),$$

$$f(\alpha x) = \int_0^1 (\alpha x(t))\, dt = \alpha \int_0^1 x(t)\ dt = \alpha f(x).$$

Demak, $f : C[0, 1] \to \mathbb{C}$ chiziqli funksional ekan. Uni uzluksizlikka tekshiramiz:

$$|f(x)| = \left| \int_0^1 x(t)\ dt \right| \leq \int_0^1 |x(t)|\, dt \leq \max_{0 \leq t \leq 1} |x(t)| \cdot \int_0^1 dt = 1 \cdot ||x||.$$

4.11-ta'rifga ko'ra $f : C[0, 1] \to \mathbb{C}$ chegaralangan funksional bo'ladi 4.3-teoremaga ko'ra f uzluksiz bo'ladi. □

4.2. $f : C[-1, 1] \to \mathbb{C}, \quad f(x) = 2[x(1) - x(0)]$ funksionalni chiziqli chegaralanganlikka tekshiring, chegaralangan bo'lsa, uning normasini toping.

Yechish. Berilgan $f(x) = 2[x(1) - x(0)]$, $x \in C[-1, 1]$ funksionalning chiziqli ekanligi oson tekshiriladi. Uning chegaralangan ekanligini ko‘rsatib normasini topamiz.

$$|f(x)| = |2\,x(1) - 2\,x(0)| \leq 2|x(1)| + 2|x(0)| \leq (2+2) \max_{0 \leq t \leq 1} |x(t)| = 4 \cdot \|x\|.$$

4.11-ta’rifga ko‘ra $f : C[-1, 1] \to \mathbb{C}$ chegaralangan funksional bo‘ladi va uning normasi uchun $\|f\| \leq 4$ tengsizlik o‘rinli. $x_0(t) = \cos \pi t$, $x_0 \in C[-1, 1]$ element uchun quyidagilar o‘rinli:

$$x_0(0) = 1, \quad x_0(1) = -1, \quad \|x_0\| = 1, \quad |f(x_0)| = 4.$$

Endi (4.2) ga ko‘ra, $\|f\| \geq |f(x_0)| = 4$ o‘rinli. $\|f\| \leq 4$ va $\|f\| \geq 4$ tengsizliklardan $\|f\| = 4$ kelib chiqadi. □

4.3. $f(x) = V_0^1[x]$, $x \in C[0, 1]$ funksional qavariq, chekli qavariq, uzluksiz bo‘ladimi? Tekshiring.

Yechish. Shuni ta’kidlaymizki, shunday $x_0 \in C[0, 1]$ funksiyalar mavjudki, ularning $[0, 1]$ kesmadagi to‘la o‘zgarishi ∞ ga teng. Masalan, $x_0(0) = 0$, $x_0(t) = t \cdot \sin \pi t^{-1}$, $t \in (0, 1]$ uzluksiz funksiya uchun $V_0^1[x_0] = \infty$. Shuning uchun $f(x) = V_0^1[x]$, $x \in C[0, 1]$ chekli funksional emas. Funksiya to‘la o‘zgarishining xossalariga ko‘ra

$$f(x+y) = V_0^1[x+y] \leq V_0^1[x] + V_0^1[y] = f(x) + f(y),$$

$$f(a\,x) = V_0^1[a\,x] = a\,V_0^1[x] = a\,f(x)$$

munosabatlar barcha $x, y \in C[0, 1]$ va $a \geq 0$ lar uchun o‘rinli. 4.7-ta’rifga ko‘ra, $f : C[0, 1] \to \mathbb{C}$ chekli bo‘lmagan qavariq funksional bo‘ladi. Bu funksionalni $\theta \in C[0, 1]$, $\theta(t) \equiv 0$ nuqtada uzluksiz emasligini ko‘rsatamiz. Nolga yaqinlashuvchi $\{x_n\} \subset C[0, 1]$ ketma-ketlikni quyidagicha tanlaymiz:

$$x_n(t) = \begin{cases} 0, & t \in [0, n^{-1}] \\ a_n^{-1}\,x_0(t), & t \in (n^{-1}, 1]. \end{cases}$$

Bu yerda $a_n = V^1_{n^{-1}}[x_0]$, $x_0(t) = t\sin \pi t^{-1}$, $t \in (0,\ 1]$. Yuqorida ta'kidlanganidek, $\lim_{n\to\infty} a_n = \infty$. x_n elementning $C[0,\ 1]$ fazodagi normasi a_n^{-1} ga teng. Shuning uchun $\lim_{n\to\infty} \|x_n\| = 0$. Ammo $|f(x_n) - f(\theta)| = |f(x_n)| \geq 1$, ya'ni $\{f(x_n)\}$ ketma-ketlik $f(\theta) = 0$ ga yaqinlashmaydi. □

4.4. Quyida berilgan to'plamlarning qaysilari qavariq to'plam, qaysilari qavariq jism bo'ladi. Agar M qavariq jism bo'lsa, uning yadrosini toping.

a) $M = \left\{x \in \mathbb{R}^3 :\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0\right\}$ birinchi oktant.

b) $M = \{x \in C[-1,\ 1] :\ x(t) \leq 0,\ \ \forall t \in [-1,\ 1]\}$.

c) $M = \{x \in X : \|x\| \leq 1\}$, $X-$ normalangan fazodagi birlik shar.

d) $M = \{x \in \ell_2 :\ |x_n| \leq 2^{-n}\}$, ℓ_2- dagi asosiy parallelepiped.

Yechish. Biz faqat a) qismini tekshirish bilan cheklanamiz. Boshqalarini javobini beramiz. a) Faraz qilaylik, x va y lar M ning ixtiyoriy ikki nuqtasi bo'lsin. U holda barcha

$$\alpha,\, \beta \geq 0, \quad \alpha + \beta = 1 \tag{4.3}$$

lar uchun $\alpha\, x + \beta\, y \in M$ munosabat o'rinli. Chunki, $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$ va (4.3) shartlardan $\alpha x_1 + \beta y_1 \geq 0$, $\alpha x_2 + \beta y_2 \geq 0$, $\alpha x_3 + \beta y_3 \geq 0$ shartlar kelib chiqadi. Ya'ni $[x, y]$ kesma M ga qarashli. 22.4-ta'rifga ko'ra M qavariq to'plam bo'ladi. Bu to'plam qavariq jism ham bo'ladi. Uning yadrosi $J(M) = \left\{x \in \mathbb{R}^3 :\ x_1 > 0,\ x_2 > 0,\ x_3 > 0\right\}$ dir.

b) M qavariq to'plam qavariq jism ham bo'ladi. Uning yadrosi

$J(M) = \{x \in C[-1,\ 1] :\ x(t) < 0,\ \ \forall t \in [-1,\ 1]\}$ to'plamdan iborat.

c) M qavariq to'plam qavariq jism ham bo'ladi. Uning yadrosi

$M = \{x \in X :\ ||x|| < 1\}$ to'plamdan iborat.

d) M qavariq to'plam qavariq jism bo'lmaydi. □

Uy vazifalari va mavzuni o'zlashtirish uchun masalalar

4.5-4.13-misollarda keltirilgan funksionallarning qaysilari chiziqli, qaysilari qo‘shma chiziqli, qaysilari uzluksiz. Tekshiring.

4.5. $f:\mathbb{R}^3\to\mathbb{R},\ f(x)=a_1x_1+a_2x_2+a_3x_3.$

4.6. $f:C[0,\,\pi]\to\mathbb{C},\quad f(x)=\int\limits_0^{\pi}(1-\cos\,t)\,\overline{x(t)}\,dt.$

4.7. $f:C\,[0,\,1]\to\mathbb{C},\quad f(x)=x(0)\,.$

4.8. $f:C[0,\,1]\to\mathbb{C},\quad f(x)=\int\limits_0^{1}t^2\overline{x(t)}\,dt\,.$

4.9. $f:C^{(1)}[0,\,1]\to\mathbb{C},\quad f(x)=x'\left(\dfrac{1}{2}\right).$

4.10. $f:L_1[0,\,1]\to\mathbb{C},\quad f(x)=\int\limits_0^{1}\overline{e^tx(t)}\,dt.$

4.11. $f:L_2[0,\,1]\to\mathbb{C},\quad f(x)=x(0)+\int\limits_0^{1}x(t)\,dt.$

4.12. $f:L_2[0,\,\pi]\to\mathbb{C},\quad f(x)=\int\limits_0^{\pi}\cos\ t\,\overline{x(t)}\,dt.$

4.13. $f:L_2[0,\,\pi]\to\mathbb{C},\quad f(x)=\int\limits_0^{\pi}\sin\ t\,x(t)\,dt.$

4.14-4.22-misollarda keltirilgan funksionallarni chiziqli chegaralanganlikka tekshiring, chegaralangan bo‘lsa, uning normasini toping.

4.14. $f:C[-1,\,1]\to\mathbb{C},\quad f(x)=\dfrac{1}{3}[x(-1)+x(1)].$

4.15. $f:C[-1,\,1]\to\mathbb{C},\ f(x)=\dfrac{1}{2\varepsilon}\left(x(\varepsilon)+x(-\varepsilon)-2x(0)\right),\ \varepsilon\in(0,1].$

4.16. $f:L_2[0,\,1]\to\mathbb{C},\quad f(x)=\int\limits_0^{1}t^{-1/3}\,x(t)\,dt.$

4.17. $f:L_2[-1,\,1]\to\mathbb{C},\quad f(x)=\int\limits_{-1}^{0}t^{1/3}\,x(t)\,dt+\int\limits_0^{1}t^{-1/3}\,x(t)\,dt.$

4.18. $f:L_2[0,\,1]\to\mathbb{C},\quad f(x)=\int\limits_0^{1}\mathrm{sign}(t-1/3)\,x(t)\,dt.$

4.19. $f : \ell_1 \to \mathbb{C},\ f(x) = \sum\limits_{k=1}^{\infty} x_k.$

4.20. $f : \ell_2 \to \mathbb{C},\ f(x) = x_1 + x_3 + x_5 + x_7.$

4.21. $f : \ell_2 \to \mathbb{C},\ f(x) = x_2 + x_4 + x_6 + \cdots + x_{200}.$

4.22. $f : \ell_{3/2} \to \mathbb{C},\ f(x) = \sum\limits_{n=1}^{\infty} \dfrac{x_n}{\sqrt[3]{3^n}}.$

4.23-4.31-misollarda keltirilgan funksionallarning qaysilari qavariq, qaysilari chekli qavariq, qaysilari uzluksiz. Tekshiring.

4.23. $f(x) = \sum\limits_{k=1}^{\infty} \dfrac{1}{k}|x_k|, \quad x = (x_1, x_2, \ldots) \in \ell_2.$

4.24. $f(x) = \sum\limits_{k=1}^{\infty} |x_k|, \quad x = (x_1, x_2, \ldots) \in \ell_1.$

4.25. $f(x) = |x_1 + x_2 + \cdots + x_{49}|, \quad x = (x_1, x_2, \ldots) \in m.$

4.26. $f(x) = \sqrt{\sum\limits_{k=1}^{\infty} 2^{1-k}\, |x_k|^2}, \quad x = (x_1, x_2, \ldots) \in c_0.$

4.27. $f(x) = \lim\limits_{n\to\infty} |x_n|, \quad x = (x_1, x_2, \ldots) \in c.$

4.28. $f(x) = \max\limits_{0 \le t \le 1} x(t), \quad x \in C[0,\ 1].$

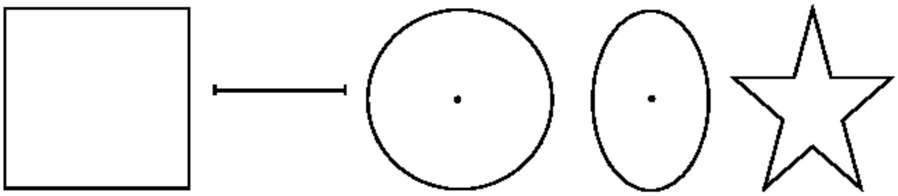
4.29. $f(x) = |\int\limits_0^1 x(t)\, dt|, \quad x \in C[0,\ 1].$

4.30. $f(x) = 2|\int\limits_0^1 t x(t^2)\, dt|, \quad x \in C[0,\ 1].$

4.31. $f(x) = V_0^1[x], \quad x \in V[0,\ 1].$

4.32. Tekislikda berilgan quyidagi to‘plamlarning qaysilari qavariq to‘plam, qaysilari qavariq jism (4.1-chizmaga qarang) bo‘ladi.

a) kvadrat, b) kesma, c) doira, d) ellips, e) besh yulduz.



4.1-chizma

4.33. $A, B \subset X$ ixtiyoriy qavariq to‘plamlar. $A \cup B$, $A \cap B$, $A + B$ to‘plamlardan qaysilari qavariq to‘plam bo‘ladi?

4.34. Normalangan fazoda qavariq to‘plamning yopig‘i qavariq bo‘ladimi?

4.35. Agar $p : L \to \mathbb{R}_+$ chekli qavariq funksional bo‘lsa, u holda

$$M = \{ x \in L \, : \, p(x) \leq 1 \}$$

to‘plam qavariq jism bo‘ladi va uning yadrosi nol elementni saqlaydi. Isbotlang.

4.36. $p : \mathbb{R}^2 \to \mathbb{R}_+, \ p(x) = 2|x_1| + 3|x_2|$ chekli qavariq funksional uchun

$$M = \left\{ x \in \mathbb{R}^2 \, : \, p(x) \leq 1 \right\}$$

to‘plamni tekislikda chizing. M to‘plamning yadrosi toping.

4.37. *Minkovskiy funksionali haqidagi masala.* $M \subset L$ qavariq jismning yadrosi nol elementni saqlasin. U holda har bir $x \in L$ ga

$$p_M(x) = \inf \left\{ r > 0 : \ \frac{x}{r} \in M \right\}$$

sonni mos qo‘yuvchi $p_M : L \to \mathbb{R}$ akslantirish qavariq funksional bo‘lishini isbotlang. Bu funksional M qavariq jism uchun *Minkovskiy funksionali* deyiladi.

4.38. $\mathbb{R}^2$ fazoda $M = \{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_1 < 2, \ \ -2 \leq x_2 < 1\}$ to‘plamning qavariq jism ekanligini isbotlang. Uning yadrosi nolni saqlashini ko‘rsating. Unga mos Minkovskiy funksionalini toping.

4.39. $C_1[0,\ 1]$ fazoning hamma yerida aniqlangan chiziqli, ammo uzluksiz bo'lmagan funksionalga misol keltiring.

4.40. $C^{(1)}[0,\ 1]-\ [0,1]$ kesmada aniqlangan uzluksiz differensiallanuvchi funksiyalar fazosi. $L=\{x\in C^{(1)}[0,\ 1]:\ \ x(0)=x(1)=0\}$ uning qism fazosi va $u,\ v,\ w\in C[0,\ 1]$ bo'lsin. Quyidagi f va g funksionallar uchun a), b) va c) tasdiqlarni isbotlang.

$$f(x)=\int_0^1 u(t)\,x'(t)\,dt,\quad g(x)=\int_0^1 [v(t)x(t)+w(t)x'(t)]\,dt.$$

a) f va g lar $C^{(1)}[0,\ 1]$ fazoda chiziqli uzluksiz.

b) agar $\forall\, x\in L$ uchun $f(x)=0$ bo'lsa, $u(t)\equiv const$ bo'ladi.

c) agar barcha $x\in L$ lar uchun $g(x)=0$ bo'lsa, $w\in C^{(1)}[0,\ 1]$ va $w'(t)=v(t)$ bo'ladi.

4.41. $L=\{x\in\mathbb{R}^n : x_1+x_2+\cdots+x_n=0\}$ to'plam $\mathbb{R}^n$ fazoning qism fazosi bo'ladi. Qism fazoning koo'lchamini toping. Shunday $f\in(\mathbb{R}^n)^*$ topingki, $L=Ker\, f$ bo'lsin.

4.42. $L=\left\{x\in C[-1,\ 1]: \int\limits_{-1}^{0} x(t)dt=\int\limits_{0}^{1} x(t)dt\right\}$ to'plam $C[-1,\ 1]$ fazoning qism fazosi bo'lishini isbotlang. Shunday $f\in C^*[-1,\ 1]$ topingki, $L=Ker\, f$ bo'lsin.

4.43. Agar $\dim\ X=n$ bo'lsa, u holda $\dim X^*=n$ bo'lishini isbotlang.

4.44. Agar $\dim\ X=\infty$ bo'lsa, u holda $\dim X^*=\infty$ bo'lishini isbotlang.

4.45. $f: C[-1,\ 1]\to C,\ \ f\,(x)=x\,(0)$ chiziqli uzluksiz funksionalni

$$f(x)=\int_{-1}^1 x(t)\,dg(t)$$

shaklda tasvirlang, ya'ni $g\in V_0[-1,\ 1]$ funksiyani toping.

4.46. $f: C[-1,\ 1] \to \mathbb{R},\quad f(x) = \dfrac{x(-1)+x(1)}{2} + \int\limits_{-1}^{1} tx\,(t)\,dt$ chiziqli uzluksiz funksionalni

$$f(x) = \int_{-1}^{1} x(t)\,dg(t)$$

shaklda tasvirlang, ya'ni $g \in V_0[-1,\ 1]$ funksiyani toping.

4.47. $\mathbb{R}^2$ fazoning $L = \{x \in \mathbb{R}^2 :\ 2x_1 - x_2 = 0\}$ qism fazosida $f(x) = x_1$ chiziqli uzluksiz funksional berilgan. Bu funksionalni normasini saqlagan holda davom ettiring. Bu davom yagonami?

4.48. $C[0,\ 1]$ fazoning $L = \{\lambda \cdot t\}$ qism fazosida f chiziqli uzluksiz funksionalning $x(t) = \lambda \cdot t$ nuqtadagi qiymatini $f(x) = \lambda$ deb aniqlaymiz. Bu funksionalni normasini saqlagan holda davom ettiring. Bu davom yagonami?

4.49. H Hilbert fazosi, $y \in H$ biror element bo'lsin, $f_y(x) = (x, y),\ x \in H$ funksionalning chiziqli uzluksiz ekanligini isbotlang. Bu yerda $(x, y) - H$ dagi skalyar ko'paytma.

4.50. $f_y : H \to \mathbb{C},\ f_y(x) = (y, x)$ funksionalning qo'shma chiziqli ekanligini isbotlang. Uning normasini toping.

4.51. $f : H \times H \to \mathbb{C},\quad f(x, y) = (x, y)$ akslantirishning bichiziqli uzluksiz funksional ekanligini isbotlang.

4.52. 4.51-misoldagi f akslantirishning simmetrik bichiziqli funksional ekanligini isbotlang. Uning normasini toping.

4.53-4.55-misollarda keltirilgan funksionallar ketma-ketligini nolga kuchsiz ma'noda yaqinlashishga tekshiring.

4.53. $f_n : L_2[-\pi,\ \pi] \to \mathbb{C},\quad f_n(x) = \int\limits_{-\pi}^{\pi} \cos nt\, x(t)\,dt.$

4.54. $f_n : L_2[-\pi,\ \pi] \to \mathbb{C}, \quad f_n(x) = \int\limits_{-\pi}^{\pi} \sin nt\, x(t)\, dt.$

4.55. $f_n : L_2[0,\ 2\pi] \to \mathbb{C}, \quad f_n(x) = \int\limits_{0}^{2\pi} \exp\{-int\}\, x(t)\, dt.$

4.56-4.58-misollarda keltirilgan funksionallar ketma-ketligini yaqinlashish xarakterini (kuchli, kuchsiz) aniqlang. Limitik funksionalni toping.

4.56. $f_n : L_2[-1,\ 1] \to \mathbb{C}, \quad f_n(x) = \sum\limits_{k=0}^{n} \int\limits_{-1}^{1} \frac{t^k}{k!}\, x(t)\, dt.$

4.57. $f_n : L_2[-1,\ 1] \to \mathbb{C}, \quad f_n(x) = \sum\limits_{k=0}^{n} \int\limits_{-1}^{1} \frac{(-1)^k t^{2k+1}}{(2k+1)!}\, x(t)\, dt.$

4.58. $f_n : L_2[-\pi,\ \pi] \to \mathbb{C}, \quad f_n(x) = \sum\limits_{k=1}^{n} \int\limits_{-\pi}^{\pi} (-1)^{k+1} \frac{2}{k} \sin k\, t\, x(t)\, dt.$

4.59-4.63 misollarda keltirilgan chiziqli normalangan fazolarga qo‘shma fazolarni toping.

4.59. $\mathbb{R}^n, \ \ \mathbb{R}^n_\infty, \ \ \mathbb{R}^n_p, \ \ p > 1$ fazolarga qo‘shma fazolarni toping.

4.60. $\mathbb{R}^n_1$ fazoga qo‘shma fazoni toping.

4.61. $\ell_2, \ \ell_p, \ p > 1, \ c, \ \ c_0, \ \ m,$ fazolarga qo‘shma fazolarni toping.

4.62. $L_2[a,\ b], \ \ L_p[a,\ b], \ \ p > 1$ fazolarga qo‘shma fazolarni toping.

4.63. $C[a,\ b]$ fazoga qo‘shma fazoni toping.

I bobni takrorlash uchun test savollari

1. Darajasi 100 dan oshmaydigan ko'phadlar fazosining o'lchamini toping.

A) 100 B) 101 C) 50 D) 200

2. Uch satr va uch ustundan iborat matritsalar fazosining o'lchamini toping.

A) 3 B) 6 C) 9 D) 27

3. Chekli o'lchamli chiziqli fazolar ko'rsatilgan javobni toping.

A) $C[a, b],\ \ell_2$ B) $C_2[a, b],\ c_0$ C) $\mathbb{C}^n,\ \mathbb{R}^3$ D) $\mathbb{C}^n,\ L_2[a, b]$

4. Cheksiz o'lchamli chiziqli fazolar ko'rsatilgan javobni toping.

A) $\mathbb{C}^n,\ C[a, b],\ \ell_2$ B) $C[a, b],\ \ell_2,\ c_0$

C) $\mathbb{C}^n,\ c,\ m$ D) $\mathbb{C}^n,\ L_2[a, b],\ \ell_p$

5. $C[0,\ 1]$ fazoda chiziqli bog'langan vektorlar sistemasini toping.

A) $1,\ t,\ t^2$ B) $t^2,\ t^3,\ t^5$

C) $1+t^2,\ 2t,\ (1-t)^2$ D) $1,\ t^2,\ t^4$

6. $f:\mathbb{R}^3\to\mathbb{R},\ \ f(x)=x_1$ chiziqli funksionalning yadrosini toping.

A) $\left\{x\in\mathbb{R}^3:\ x_1=x_2=0\right\}$ B) $\left\{x\in\mathbb{R}^3:\ x_1=0\right\}$

C) $\left\{x\in\mathbb{R}^3:\ x_2=0\right\}$ D) $\left\{x\in\mathbb{R}^3:\ x_3=0\right\}$

7. $L'=\left\{x\in\mathbb{R}^5:\ x_1=x_5=0\right\}$ qism fazoning koo'lchamini toping.

A) 1 B) 2 C) 3 D) 4

8. Faktor fazoda elementning normasi qanday aniqlanadi?

A) $\|\xi\|=\sup\limits_{x\in\xi}\|x\|$ B) $\|\xi\|=\inf\limits_{x\in\xi}\|x\|$

C) $\|\xi\|=\|x\|$ D) $\|\xi\|=\sup\limits_{y\in L'}\|x-y\|$

9. $C[0,\ 1]$ fazoda aniqlangan chiziqli bo'lmagan funksionalni toping.

A) $f(x)=\int\limits_0^1 x(t)dt$ B) $f(x)=\int\limits_0^1 x(t)\,e^t\,dt$

C) $f(x)=x(0)+x(1)$ D) $f(x)=|x(0)|$

10. $C[a,\ b]$ fazoda aniqlangan qavariq funksionalni toping.

A) $f(x) = \int\limits_a^b x(t)dt$ B) $f(x) = \int\limits_a^b x(t)\, e^t\, dt$

C) $f(x) = x(a) + x(b)$ D) $f(x) = \left|\int\limits_a^b x(t)\, dt\right|$

11. Tekislikda qavariq bo'lmagan to'plamni toping.

A) uchburchak B) kvadrat C) trapetsiya D) besh yulduz

12. Tekislikda keltirilgan quyidagi to'plamlardan qaysi biri qavariq to'plam bo'ladi, ammo qavariq jism bo'lmaydi.

A) uchburchak B) kvadrat C) doira D) kesma

13. Quyidagi to'plamlardan qaysi biri $C[a, b]$ fazoning qism fazosi bo'ladi?

A) Monoton o'suvchi funksiyalar B) Manfiymas funksiyalar

C) Monoton kamayuvchi funksiyalar D) Barcha ko'phadlar

14. Noto'g'ri tasdiqni toping.

A) ℓ_1 fazo ℓ_2 fazoning qism fazosi bo'ladi.

B) c_0 fazo c fazoning qism fazosi bo'ladi.

C) ℓ_2 fazo c_0 fazoning qism fazosi bo'ladi.

D) m fazo c fazoning qism fazosi bo'ladi.

15. To'la bo'lmagan normalangan fazoni toping.

A) $C_1[a,\ b]$ B) ℓ_2 C) $\mathbb{R}^n$ D) $C[a,\ b]$

16. E normalangan fazo. Noto'g'ri tasdiqni toping.

A) E dagi ixtiyoriy chegaralangan ketma-ketlik yaqinlashuvchidir

B) Agar $x_n \to x,\ \ y_n \to y$ bo'lsa, u holda $x_n + y_n \to x + y$

C) $x,\, y \in E$ uchun $\|x\| - \|y\| \le \|x - y\|$ tengsizlik o'rinli

D) $x_n \to x$ va $\lambda_n \to \lambda$ bo'lsa, u holda $\lambda_n x_n \to \lambda x$

17. Quyidagi ketma-ketliklardan qaysilari $C_1[0,\,1]$ fazoda nol funksiyaga yaqinlashadi.

$$1)\ x_n(t)=\frac{t}{1+n^2t^2},\qquad 2)\ x_n(t)=te^{-n},\qquad 3)\ x_n(t)=t^n.$$

A) 1, 2 B) 1, 2, 3 C) 1, 3 D) 2, 3

18. Quyidagi ketma-ketliklardan qaysilari $C[0,\,1]$ fazoda fundamental?

$$1)\ x_n(t)=\frac{nt}{1+n^2t^2},\qquad 2)\ x_n(t)=te^{-n},\qquad 3)\ x_n(t)=t^n$$

A) 1, 2 B) 1, 2, 3 C) 1, 3 D) 2

19. Quyidagi to'plamlardan qaysi biri $C[-1,\,1]$ fazoda qism fazo tashkil qilmaydi?

A) Barcha ko'phadlar to'plami

B) $x(-1)=0$ shartni qanoatlantiruvchi funksiyalar to'plami

C) Monoton funksiyalar to'plami

D) Uzluksiz differensiallanuvchi funksiyalar to'plami

20. Quyidagi to'plamlardan qaysi biri $C[-1,\,1]$ fazoda qism fazo tashkil qilmaydi?

A) Darajasi 100 dan oshmaydigan ko'phadlar to'plami

B) $x(1)=1$ shartni qanoatlantiruvchi funksiyalar to'plami

C) Toq funksiyalar to'plami

D) Juft funksiyalar to'plami

21. Quyidagi to'plamlardan qaysi biri $C[-1,\,1]$ fazoda qism fazo tashkil qiladi?

A) Monoton o'suvchi funksiyalar

B) Monoton kamayuvchi funksiyalar

C) Darajasi 2 bo'lgan ko'phadlar

D) $\{x\in C[-1,\,1]:\int\limits_{-1}^{1}x(t)dt=0\}$ shartni qanoatlantiruvchi funksiyalar

22. Noto‘g‘ri tasdiqni toping.

A) Chiziqli bog‘lanmagan sistemaning biror qism sistemasi chiziqli bo‘g-langan bo‘ladi

B) Chiziqli bog‘lanmagan sistemaning ixtiyoriy qism sistemasi ham chiz-iqli bo‘glanmagan bo‘ladi

C) Agar sistemaning biror qism sistemasi chiziqli bog‘langan bo‘lsa, beril-gan sistema ham chiziqli bo‘g‘langan bo‘ladi

D) Agar $x_1, x_2, \ldots, x_n$ vektorlar sistemasi chiziqli bog‘langan bo‘lsa, bu vektorlardan biri qolganlarining chiziqli kombinatsiyasidan iborat bo‘ladi

23. $E-$ chiziqli fazo, $x_1, x_2, \ldots, x_n \in E$ bo‘lsin. Chiziqli bog‘lanmagan vektorlar sistemasining ta’rifini ko‘rsating.

A) $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0 \quad \Leftrightarrow \quad \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$

B) $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0 \quad \Leftrightarrow \quad \alpha_1 = 1, \ \alpha_2 = \ldots = \alpha_n = 0$

C) $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0 \ \Leftrightarrow \ \alpha_1 + \alpha_2 = 0, \ \alpha_3 = \ldots = \alpha_n = 0$

D) $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0 \quad \Leftrightarrow \quad \alpha_1 + \alpha_2 + \ldots + \alpha_n = 0$

24. $f_0 : V_0[a, b] \to \mathbb{R}, \quad f_0(x) = x(b)$ funksionalning davomini toping.

A) $f(x) = x(a) - 2x(b), \, x \in V[a, b]$

B) $f(x) = x(a) + x(b), \, x \in V[a, b]$

C) $f(x) = x(a) + 2x(b), \, x \in V[a, b]$

D) $f(x) = x(a) - x(b), \, x \in V[a, b]$

25. Noto‘g‘ri tasdiqni toping.

A) $n-$ o‘lchamli chiziqli fazoda ixtiyoriy n ta chiziqli bog‘lanmagan vektorlardan iborat sistema bazis bo‘ladi

B) $\{e_k = (\underbrace{0, 0, \ldots, 1}_{k}, 0, \ldots, 0)\}_{k=1}^{n}$ vektorlar sistemasi $\mathbb{R}^n$ fazoda bazis bo‘ladi

C) $n-$ o‘lchamli chiziqli fazoda ixtiyoriy n ta vektordan iborat sistema bazis bo‘ladi

D) $\mathbb{R}^3$ fazoda ixtiyoriy to‘rtta vektor chiziqli bog‘langandir.

26. Chiziqli bog‘langan sistemani toping.

A) $x(t) = \sin^2 2t,\ \ y(t) = \cos^2 2t,\ \ z(t) = 1 \in C[0,\, \pi]$

B) $x = (0,\, 1),\ \ y = (1,\, 0) \in \mathbb{R}^2$

C) $x = (1,\, 1,\, 1),\ \ y = (0, 1,\, 1),\ \ z = (0,\, 0, 1) \in \mathbb{R}^3$

D) $x(t) = 1,\ \ y(t) = t \in C[0,\, 1]$

27. Chiziqli bog‘lanmagan sistemani toping.

A) $x(t) = t,\ \ y(t) = 2 - 3t,\ \ z(t) = 1 \in C[0,\, 1]$

B) $x = (0,\, 1),\ \ y = (2,\, 1),\ \ z = (1,\, 1) \in \mathbb{R}^2$

C) $x = (0,\, 1,\, 1),\ \ y = (1, 0,\, 0),\ \ z = (0,\, 0, 2) \in \mathbb{R}^3$

D) $x(t) = \sin^2 2t,\ \ y(t) = \cos^2 2t,\ \ z(t) = \cos 4t \in C[0,\, \pi]$

28. Quyidagi formulalar yordamida berilgan funksionallardan qaysi biri ko‘rsatilgan fazoda skalyar ko‘paytma aniqlaydi?

A) $(x, y) = \sum\limits_{k=1}^{\infty} x_k^2 y_k,\ x,\ y \in \ell_2$

B) $(x, y) = x_1 y_2 + x_2 y_1,\ x, y \in \mathbb{R}^2$

C) $(x, y) = \sum\limits_{k=1}^{n} x_k y_k,\ x, y \in \mathbb{C}^n$

D) $(x, y) = \int\limits_a^b x(t)\overline{y(t)}dt,\ x, y \in C[a, b]$

29. Qanday fazo Banax fazosi deyiladi?

A) Skalyar ko‘paytma kiritilgan chiziqli fazo

B) Har qanday normalangan fazo

C) To‘la normalangan fazo

D) Istalgan metrik fazo.

30. Qanday fazo Evklid fazosi deyiladi?

A) Skalyar ko‘paytma kiritilgan chiziqli fazo

B) Har qanday normalangan fazo

C) To‘la normalangan fazo

D) Istalgan metrik fazo

31. Evklid fazolari keltirilgan javobni toping.

A) $\mathbb{R}^n,\ C[a,\, b],\ \ell_2$ B) $\mathbb{R}^n,\ C_2[a,\, b],\ \ell_2$

C) $\mathbb{C}^n,\ C_2[a,\, b],\ \ell_1$ D) $\mathbb{C}^n,\ C_2[a,\, b],\ \ell_p$

32. Hilbert fazolari keltirilgan javobni toping.

A) $C_2[a,\, b],\, \ell_2$ B) $L_2[a,\, b],\, \ell_2$ C) $\mathbb{C}^n,\, C_2[a,\, b]$ D) $\mathbb{C}^n,\, \ell_2$

33. $L_2[a,\ b]$ kompleks Hilbert fazosidagi skalyar ko‘paytmani ko‘rsating.

A) $(x,\, y) = \int\limits_a^b x(t)\overline{y(t)}\, e^{it} dt$ B) $(x,\, y) = \int\limits_a^b x(t)\overline{y(t)}\, dt$

C) $(x,\, y) = \int\limits_a^b |x(t)\, y(t)|\ dt$ D) $(x,\, y) = \sum\limits_{n=1}^{\infty} x_n\, \overline{y_n}$

34. $C_2[a,\, b]$ haqiqiy Evklid fazosidagi skalyar ko‘paytmani ko‘rsating.

A) $(x,\, y) = \int\limits_a^b x(t)y(t)\, e^{it} dt$ B) $(x,\, y) = \int\limits_a^b x(t)\, y(t) dt$

C) $(x,\, y) = \int\limits_a^b |x(t)\, y(t)|\ dt$ D) $(x,\, y) = x(a)y(a) + x(b)y(b)$

35. Haqiqiy Evklid fazosida nolga teng bo‘lmagan x va y vektorlar orasidagi φ burchakning kosinusi qanday formula bilan aniqlanadi?

A) $\cos\varphi = \dfrac{(x,\, y)}{\|x\| \cdot \|y\|}$ B) $\cos\varphi = \dfrac{(x,\, x) - (y, y)}{\|x\| \cdot \|y\|}$

C) $\cos\varphi = \dfrac{\|x\| + \|y\|}{\|x\| \cdot \|y\|}$ D) $\cos\varphi = \dfrac{\|x + y\|}{\|x\| \cdot \|y\|}$

36. Evklid fazosida Koshi-Bunyakovskiy tengsizligini toping.

A) $|(x,\, y)| \le \|x\| \cdot \|y\|$ B) $\|x + y\| \le \|x\| + \|y\|$

C) $\|x + y\|^2 \le \|x\|^2 + \|y\|^2$ D) $\|x + y\| + \|x - y\| \le 2\,(\|x\| + \|y\|)$

37. Normalangan fazo Evklid fazosi bo‘lishi uchun quyidagi shartlardan qaysi birining bajarilishi zarur va yetarli?

A) $\|x+y\| \leq \|x\|+\|y\|$ B) $\|x+y\|^2+\|x-y\|^2 = 2\left(\|x\|^2+\|y\|^2\right)$

C) $\|\lambda x\| = |\lambda|\cdot\|x\|$ D) $\|x+y\|+\|x-y\| \leq 2\left(\|x\|+\|y\|\right)$

38. To‘la bo‘lmagan separabel Evklid fazosini toping.

A) $\mathbb{R}^n$ B) $C_2[a,\, b]$ C) $\mathbb{C}^n$ D) ℓ_2

39. E Evklid fazosidagi $\{x_n\}$ sistema uchun quyidagi shartlarning qaysi biri bajarilganda u E da ortonormal bazis deyiladi?

A) Agar $(x_n,\, x_m) = \begin{cases} 1, & agar\ \ n = m \\ 0, & agar\ \ n \neq m \end{cases}$ bo‘lsa.

B) Agar $\{x_n\}$ ortonormal sistema bo‘lib, u E da to‘la bo‘lsa.

C) Agar $\{x_n\}$ sistemani saqlovchi minimal yopiq qism fazo E ning xos qismi bo‘lsa.

D) $\{x_n\}$ sistema chiziqli bog‘lanmagan bo‘lib, $\|x_n\| = 1$ bo‘lsa.

40. Noto‘g‘ri tasdiqni toping.

A) Har qanday Evklid fazosida sanoqli ortonormal bazis mavjud.

B) Evklid fazosida yig‘indi amali uzluksizdir.

C) Evklid fazosida skalyar ko‘paytma amali uzluksizdir.

D) Evklid fazosida songa ko‘paytirish amali uzluksizdir.

41. E to‘la haqiqiy Evklid fazosi, $\{\varphi_n\}_{n=1}^{\infty}$ undagi ortonormal sistema va $f \in E,\ c_k = (f,\, \varphi_k)$ bo‘lsin. Quyidagi shartlarning qaysi biri bajarilganda berilgan sistema yopiq deyiladi?

A) $\sum\limits_{k=1}^{\infty} c_k^2 \leq \|f\|^2,\ \forall f \in E$ B) $\sum\limits_{k=1}^{\infty} c_k^2 = \|f\|^2,\ \forall f \in E$

C) $\sum\limits_{k=1}^{\infty} c_k^2 \geq \|f\|^2,\ \forall f \in E$ D) $\left\|f - \sum\limits_{k=1}^{\infty} c_k\varphi_k\right\|^2 = \|f\|^2 - \sum\limits_{k=1}^{\infty} c_k^2$

42. Quyidagi tasdiqlarning qaysi biri to‘g‘ri?

A) Separabel Evklid fazosida har qanday to‘la ortonormal sistema yopiq va aksincha.

B) Separabel Evklid fazosida har qanday ortonormal sistema to‘ladir.

C) Separabel Evklid fazosida har qanday ortonormal sistema yopiqdir.

D) To‘la Evklid fazosida har qanday ortonormal sistema yopiqdir.

43. $C_2[-1, 1]$ Evklid fazosida $f(x) = 1$ funksiyaning $\{\varphi_n(t) = \sin n\pi t\}_{n=1}^{\infty}$ ortonormal sistemadagi Furye koeffitsiyentlarini toping.

A) $c_n = 0, \ \ n \in \mathbb{N}$ B) $c_n = n^{-1}, \ \ n \in \mathbb{N}$

C) $c_n = (-1)^n/n, \ \ n \in \mathbb{N}$ D) $c_n = n^{-2}, \ \ n \in \mathbb{N}$

44. $L_2[-\pi, \pi]$ kompleks Evklid fazosida to‘la ortonormal sistemani toping.

A) $\left\{(2\pi)^{-1/2} \exp\{int\}\right\}_{n=-\infty}^{\infty}$ B) $\{\pi^{-1/2} \cos nt\}_{n=1}^{\infty}$

C) $\{\pi^{-1/2} \sin nt\}_{n=1}^{\infty}$ D) $\{t^n\}_{n=1}^{\infty}$

I bob uchun javoblar va ko‘rsatmalar

1-§. Chiziqli fazolar

8-20- misollarda chiziqli fazoning 1.8 aksiomalari bajariladi.

21. Monoton funksiyalar to‘plami chiziqli fazo tashkil qilmaydi.

22-29-misollarda keltirilgan to‘plamlar chiziqli fazo tashkil qiladi.

30. Davriy funksiyalar to‘plami chiziqli fazo tashkil qilmaydi.

31-36-misollarda keltirilgan to‘plamlar chiziqli fazo tashkil qiladi.

37, 38, 39 va **41** larda sistema chiziqli erkli.

40, 42, 43, 44 larda sistema chiziqli bog‘langan.

45. $A=[-1,1]$ yoki $A=[-1,0]$, $A=[0,1]$ desak, f_1, f_2 va f_3 elementlar $L_1[-1,\ 1]$ fazoda chiziqli bog‘langan bo‘ladi.

46. a) $A=[-2,0)$, $B=(0,1]$, b) $A=[-2,0)$, $B=(0,3]$.

47. $\dim \mathbb{R}^5 = 5$, $\dim P_{\leq 8} = 9$, $\dim M_{33} = 9$.

48. $\dim \mathbb{C}^5 = 5$, $\dim m = \dim c = \infty$.

49. Barcha fazolarning o‘lchami cheksiz.

50. Barcha fazolarning o‘lchami cheksiz.

54. O‘lchami $n-1$.

55. $co\dim M = 3$.

56. Ma’lumki, nolga ekvivalent funksiyalar yig‘indisi yana nolga ekvivalent bo‘lgan funksiya bo‘ladi. Nolga ekvivalent funksiyaning songa ko‘paytmasi ham nolga ekvivalent funksiya bo‘ladi. Demak, $\widetilde{L}_p^{(0)}[a,b]$ to‘plam $\widetilde{L}_p[a,b]$ fazoning xos qism fazosi bo‘ladi.

60. Bu fazoning nol elementi bir soni bo‘ladi. O‘lchami 1.

61. Yo‘q.

2-§. Chiziqli normalangan fazolar

4-21. Normaning barcha shartlari bajariladi. Biz **2.12-misolning** yechimini beramiz. Ixtiyoriy uzluksiz funksiya $[a,\ b]$ kesmada integrallanuvchidir. Shuning uchun p_1 funksional $C[a,\ b]$ fazoning hamma yerida aniqlangan.

$$p_1(f) = \int_a^b |f(x)|\, dx, \quad f \in C[a,\ b]$$

funksional uchun norma shartlarining bajarilishini tekshiramiz. 1-shart

$$p_1(f) = \int_a^b |f(x)|\, dx \geq 0$$

ixtiyoriy $f \in C[a,b]$ uchun $|f(x)| \geq 0, \quad \forall x \in [a,\ b]$ shartdan kelib chiqadi. Agar $p_1(f) = 0$ bo'lsa, u holda Lebeg integralining IV xossasidan f ning nolga ekvivalentligi kelib chiqadi. f ning uzluksizligidan $|f(x)| \equiv 0$ ekanligini olamiz, ya'ni $f(x) \equiv 0$. Agar $f(x) \equiv 0$ bo'lsa, u holda $p_1(f) = 0$ ekanligi integralning ta'rifidan kelib chiqadi. 2-shart

$$p_1(\alpha f) = \int_a^b |\alpha f(x)|\, dx = |\alpha| \int_a^b |f(x)|\, dx = |\alpha|\, p_1(f)$$

tenglikdan kelib chiqadi. 3-shart

$$p_1(f+g) = \int_a^b |f(x)+g(x)|dx \leq \int_a^b |f(x)|\, dx + \int_a^b |g(x)|dx = p_1(f) + p_1(g)$$

tengsizlikdan kelib chiqadi. Demak, bu funksional uchun normaning barcha shartlari bajariladi.

25 misolda normaning barcha shartlari bajariladi.

26. Bu misolda berilgan funksional uchun normaning 1-sharti bajarilmaydi. Haqiqatan ham, noldan farqli $x_0(t) \equiv 1$ element uchun

$$p(x_0) = |x_0(b) - x_0(a)| + \max_{a<t\leq b} |x_0'(t)| = |1-1| + \max_{a<t\leq b} |0| = 0$$

tenglik o'rinli. Ya'ni $p(x) = 0$ shartdan $x(t) = 0$ shart kelib chiqmaydi.

2-shart $p(\alpha x) = |\alpha| p(x)$ va 3-shart $p(x+y) \leq p(x) + p(y)$ ning bajarilishi modul hamda maksimum xossalaridan kelib chiqadi.

27- misolda normaning 1-sharti bajarilmaydi.

28-30-misollarda normaning barcha shartlari bajariladi.

31. Bu misollarning barchasida $\lim\limits_{n\to\infty}\|x_n-\theta\| = \lim\limits_{n\to\infty}\|x_n\| = 0$ shartni tekshirish kerak bo‘ladi. Chunki $\theta(t)\equiv 0$ funksiya 2.31-2.40-misollarda keltirilgan fazolarning barchasida nol elementdir. Bu misolda berilgan $x_n(t) = \dfrac{nt}{1+n^2+t^2}$ elementning normasini baholaymiz

$$\|x_n\| = \max_{0\le t\le 1}\left|\frac{nt}{1+n^2+t^2}\right| \le \frac{n}{1+n^2} \le \frac{n}{n^2} = \frac{1}{n}.$$

Demak, $\lim\limits_{n\to\infty}\|x_n\| = 0$ munosabat o‘rinli, ya’ni $\{x_n\}$ nolga yaqinlashadi.

32-39 nolga yaqinlashadi.

40. Berilgan $x_n(t) = 2n\cdot t\cdot e^{-nt^2}$ elementning normasini hisoblaymiz

$$\|x_n\| = \int_0^1 |2n\cdot t\cdot e^{-nt^2}|dt = 2n\int_0^1 t\cdot e^{-nt^2}dt = \int_0^1 e^{-nt^2}d(nt^2) =$$

$$= -\,e^{-nt^2}\Big|_0^1 = -e^{-n}+1.$$

Demak, $\lim\limits_{n\to\infty}\|x_n\| = 0$ tenglik o‘rinli emas, shuning uchun $\{x_n\}$ ketma-ketlik nol elementga yaqinlashmaydi.

41. $\|x\| = 3,\ \|x\|_1 = 5,\ \|x\|_4 = \sqrt[4]{33},\ \|x\|_\infty = 2,$

$\|y\| = 5,\ \|y\|_1 = 7,\ \|y\|_4 = \sqrt[4]{337},\ \|y\|_\infty = 4.$

42. $\|f\| = 1,\ \|f\|_1 = 4,\ \|f\|_2 = \sqrt{\pi},\ \|g\| = 1,\ \|g\|_1 = 4,\ \|g\|_2 = \sqrt{\pi}.$

43. $\|\varphi_n\|_C = \|\psi_n\|_C = 1,\ \|\varphi_n\|_{L_2} = \|\psi_n\|_{L_2} = \sqrt{\pi},$

$\|\varphi_n\|_M = \|\psi_n\|_M = 1,\ \|\varphi_n\|_V = \|\psi_n\|_V = 4n.$

48. Normaning ucburchak tengsizligiga ko‘ra quyidagilar o‘rinli

$$\|x\| = \|x-y+y\| \le \|x-y\|+\|y\| \ \Rightarrow\ \|x\|-\|y\| \le \|x-y\|,$$

$$\|y\| = \|y-x+x\| \le \|y-x\|+\|x\| \ \Rightarrow\ \|y\|-\|x\| \le \|x-y\|.$$

Bu ikki tengsizlikdan (2.1) tengsizlik kelib chiqadi.

51. $x = \dfrac{x+y+x-y}{2}$ deymiz. Bu yerdan va normaning xossalaridan

$$\|x\| \le \frac{1}{2}\left(\|x+y\| + \|x-y\|\right) \le \max\{\|x+y\|,\ \|x-y\|\}$$

ni olamiz.

57. Ha. **58.** Yo'q. Masalan, $A = [0,\,1]\backslash\mathbb{Q}$, $B = [0,\,1]\cap\mathbb{Q}$.

62. a), c) va d) lar uchun $x^{(n)} = (1+\frac{1}{n}, \frac{1}{2}+\frac{1}{n}, \ldots, \frac{1}{k}+\frac{1}{n}, \ldots)$.

b) va e) uchun $x^{(n)} = (1+\frac{1}{n!}, \frac{1}{\sqrt{2}}+\frac{1}{n!}, \ldots, \frac{1}{\sqrt{k}}+\frac{1}{n!}, \ldots)$.

64. Ochiq to'plam bo'lmaydi.

65. Yopiq to'plam bo'lmaydi.

69. $\mathbb{R}^2$, ℓ_2 va $C_2[a,\,b]$ lar qat'iy normalangan fazolar bo'ladi. ℓ_1, m va $C[a,\,b]$ lar qat'iy normalangan fazolar emas.

71. Ha. $X = \mathbb{R}$, $A = \mathbb{Q}$, $B = \mathbb{R}\backslash\mathbb{Q}$.

72. $C[-1,\,1] = C^{-}[-1,\,1] \oplus C^{+}[-1,\,1]$, bu yerda $C^{-}[-1,\,1] = \{x : x(-t) = -x(t)\}$ toq funksiyalar, $C^{+}[-1,\,1] = \{x : x(-t) = x(t)\}$ esa juft funksiyalar fazosi.

75. Yo'q. $X = \mathbb{R}$, $x_n = \dfrac{(-1)^n}{n}$.

3-§. Evklid va Hilbert fazolari

6-10 - misollardagi p funksional skalyar ko'paytma shartlarini qanoatlantiradi.

11-14 - misollardagi p funksional skalyar ko'paytma shartlarini qanoatlantiradi.

15-18 - misollardagi p funksional skalyar ko'paytmaning 1-shartini qanoatlantirmaydi.

19-21 - misollardagi p funksional skalyar ko'paytma shartlarini qanoatlantiradi.

22 misoldagi p funksional skalyar ko'paytmaning 3 va 4-shartlarini qanoatlantirmaydi.

23 - misoldagi p funksional skalyar ko'paytma shartlarini qanoatlantiradi.

24. $e_1=(0,0,1),\ \ e_2=(0,1,0),\ \ e_3=(1,0,0).$

25. $e_1=\dfrac{1}{\sqrt{2}}(1,1,0),\ \ e_2=\dfrac{1}{\sqrt{3}}(1,-1,-1),\ \ e_3=\dfrac{1}{\sqrt{6}}(1,-1,2).$

26. $e_1=(-1,0,0),\, e_2=\dfrac{1}{\sqrt{2}}(0,-1,1), e_3=\dfrac{1}{\sqrt{2}}(0,1,1).$

27. $\psi_1(t)=\dfrac{1}{\sqrt{2}},\ \psi_2(t)=\dfrac{1}{\sqrt{3,5}}\,t^3,\ \psi_3(t)=\dfrac{1}{a}\left(t^6-\dfrac{7}{10}t^3-\dfrac{1}{7}\right).$

28. $\psi_1(t)=\dfrac{1}{\sqrt{2}},\ \psi_2(t)=\sqrt{\dfrac{2}{3}}\,t,\ \ \psi_3(t)=\sqrt{\dfrac{5}{8}}\,\left(3\,t^2-1\right)$

29. $\psi_1(t)=\dfrac{1}{\sqrt{\pi}},\ \ \psi_2(t)=\sqrt{\dfrac{2}{\pi}}\cos t,\ \psi_3(t)=\sqrt{\dfrac{2\pi}{\pi^2-8}}\sin\,\dfrac{t-2}{\pi}.$

30. $\psi_1(t)=\dfrac{1}{\sqrt{2}},\ \psi_2(t)=\sqrt{\dfrac{2}{3}}\,t,\ \ \psi_3(t)=\sqrt{\dfrac{5}{8}}\,\left(3\,t^2-1\right)$

31. $e_1=(1,0,0,\ldots),\ \ e_2=(0,1,0,0,\ldots),\ \ e_3=(0,0,1,0,0,\ldots).$

32. $e_1=\sqrt{3}\left(0,\dfrac{1}{2},\ldots,\dfrac{1}{2^{n-1}},\ldots\right), e_2=\left(1,-\dfrac{1}{4},\dfrac{3}{2^3}\ldots,\dfrac{3}{2^n},\ldots\right).$

33. $e_1=\dfrac{1}{\sqrt{2}}\left(1,1,0,0,\ldots\right),\, e_2=\dfrac{1}{\sqrt{2}}\left(0,\ 0,,1,1,0,0,\ldots\right).$

43. $\psi_n(t)=\dfrac{1}{\sqrt{\pi}}\ \sin nt,\ \ n\in\mathbb{N}.$

44. $\psi_n(t)=\dfrac{1}{\sqrt{2\pi}}\ \exp\{i\,n\,t\},\, n\in\mathbb{Z}.$

46. $c_0=\dfrac{\sqrt{\pi}}{\sqrt{2}},\ \ c_1=0,\ \ c_2=\dfrac{\sqrt{\pi}}{2},\ \ c_n=0,\ \ n\geq 3.$

50. $\dim(M_n)^{\perp}=n,\ \ n=1,2,3\,.$

52. $L_2^-[-1,\ 1]$ ning ortogonal to'ldiruvchisi $(L_2^-[-1,\ 1])^{\perp}$ - $L_2[-1,\ 1]$ dagi juft funksiyalardan iborat. $\dim L_2^-[-1,\ 1]=\dim(L_2^-[-1,\ 1])^{\perp}=\infty.$

58. $c_n=\dfrac{(2n)!}{2^n\,(n!)^2},\ \ \ n\in\mathbb{N}.$

58. $\|P_n\|=\sqrt{\dfrac{2}{2n+1}},\ \ \ n\in\mathbb{N}\cup\{0\}.$

59. $\psi_0(t)=\dfrac{1}{\sqrt{2}},\qquad \psi_1(t)=\sqrt{\dfrac{3}{2}}\,t,$

$\psi_2(t)=\sqrt{\dfrac{5}{8}}\,\left(3\,t^2-1\right),\ \ \psi_3(t)=\dfrac{5\sqrt{7}}{2\sqrt{2}}\,\left(t^3-\dfrac{3}{5}t\right).$

61. Ortonormal sistema $\psi_n=x_n(t)=a_0+a_1t+\cdots+a_{n-1}t^{n-1}$ shaklda izlanadi. Demak, $x_1(t) = a_0$ bo'lib,

$$\|x_1\|^2 = \int_0^\infty a_0^2 e^{-t} dt = a_0^2 = 1$$

dan $a_0 = 1$ ni olamiz. Demak, $\psi_1(t) = x_1(t) = 1$ ekan. 3.4-misol eslatmasiga ko'ra, $\varphi_2(t) = t - (f_2, \psi_1)\psi_1 = t - 1$ bo'ladi. Chunki, $f_2(t) = t$ va

$$(f_2, \psi_1) = \int_0^\infty te^{-t} dt = \Gamma(2) = (2-1)! = 1.$$

Endi φ_2 ning normasini hisoblaymiz.

$$\|\varphi_2\|^2 = \int_0^\infty (t-1)^2 e^{-t} dt = \int_0^\infty (t^2 - 2t + 1)e^{-t} dt = \Gamma(3) - 2\Gamma(2) + \Gamma(1) = 1.$$

Biz bu yerda barcha $n \in \mathbb{N}$ larda o'rinli bo'lgan $\Gamma(n) = (n-1)!$ tenglikdan foydalandik. Demak, $\psi_2(t) = x_2(t) = t - 1$ ekan. 3.4-misol eslatmasiga ko'ra, $\varphi_3(t) = t^2 - (f_3, \psi_1)\psi_1 - (f_3, \psi_2)\psi_2$ deymiz va (f_3, ψ_1) va (f_3, ψ_2) koeffitsiyentlarni hisoblaymiz:

$$(f_3, \psi_1) = \int_0^\infty t^2 e^{-t} dt = \Gamma(3) = 2,$$

$$(f_3, \psi_2) = \int_0^\infty t^2(t-1)e^{-t} dt = \Gamma(4) - \Gamma(3) = 3! - 2! = 4.$$

Endi $\varphi_3(t) = t^2 - 2 - 4(t-1) = t^2 - 4t + 2$ ning normasini hisoblaymiz.

$$\|\varphi_3\|^2 = \int_0^\infty (t^2 - 4t + 2)^2 e^{-t^2} dt = \int_0^\infty (t^4 - 8t^3 + 20t^2 - 16t + 4)e^{-t} dt =$$

$$= \Gamma(5) - 8\Gamma(4) + 20\Gamma(3) - 16\Gamma(2) + 4\Gamma(1) = 24 - 48 + 40 - 16 + 4 = 4.$$

Bu yerdan $\psi_3(t) = \dfrac{\varphi_3(t)}{\|\varphi_3\|} = \dfrac{1}{2} t^2 - 2t + 1$ ni olamiz.

62. Nolga ekvivalent funksiyalarning yig'indisi yana nolga ekvivalent funksiya bo'ladi, nolga ekvivalent funksiyaning songa ko'paytmasi yana nolga ekvivalent funksiya bo'ladi. Shuning uchun $L_{20}^{+}[-1, 1]$ to'plam qism fazo tashkil qiladi. Uning ortogonal to'ldiruvchisi esa

$$L_{20}^{-}[-1, 1] = \{f \in L_2[-1, 1] : f(x)(1 - \chi_{[0,1]}(x)) \sim 0\}$$

to‘plamdan iborat.

65. $(f,\varphi_n^-)=(f,\varphi_n^+)=0,\ n\in\mathbb{N},\ \ (f,\psi_0)=\sqrt{2},\ (f,\psi_n)=0,\ \ n\neq 0.$
$(g,\varphi_n^-)=\dfrac{2(-1)^{n-1}}{n\pi},\ \ n\in\mathbb{N},\ \ (g,\varphi_n^+)=0,\ n\in\mathbb{N},\ \ (g,\psi_0)=0,$
$(g,\psi_n)=\dfrac{\sqrt{2}i(-1)^{n-1}}{n\pi},\ \ \ n\neq 0.$

66. $(f,\varphi_n^-)=\dfrac{2}{\pi\, n}(1+(-1)^{n-1}),\ n\in\mathbb{N},\ \ (f,\varphi_n^+)=0,\ n\in\mathbb{N},$
$(f,\psi_0)=0,\ (f,\psi_n)=\dfrac{2}{\pi\, n}(1+(-1)^{n-1}),\ n\in\mathbb{Z}\setminus\{0\}.$

67. Normallangan Lejandr ko‘phadlari

$P_0=\dfrac{1}{\sqrt{2}},\ P_1(t)=\sqrt{\dfrac{3}{2}}\,t,\ P_2(t)=\sqrt{\dfrac{5}{8}}\left(3\,t^2-1\right),\ P_3(t)=\dfrac{5\sqrt{7}}{2\sqrt{2}}\left(t^3-\dfrac{3}{5}t\right),\ldots$

$L_2[-1,\ 1]$ Hilbert fazosida ortonormal sistema tashkil qiladi. Shuning uchun $p_n(t),\ n=1,2,3$ ko‘phadni topish uchun e^t ni bu sistema bo‘yicha Furye qatoriga yoyish kifoya.

$$e^t=C_0P_0+C_1P_1(t)+C_2P_2(t)+C_3P_3(t)+\cdots$$

yoki

$$e^t=\frac{C_0}{\sqrt{2}}+C_1\sqrt{\frac{3}{2}}\,t+C_2\sqrt{\frac{5}{8}}\left(3\,t^2-1\right)+C_3\frac{5\sqrt{7}}{2\sqrt{2}}\left(t^3-\frac{3}{5}t\right)+\cdots \quad (3.1j)$$

Ma’lumki $\|f-p_n\|$ masofa minimal bo‘lishi uchun $p_n,\ f\in E$ elementning $\{\varphi_k\}$ ortonormal sistemadagi Furye qatorining qismiy yig‘indisi bo‘lishi zarur va yetarli. Shunday ekan $p_n(t)=C_0P_0+C_1P_1(t)+C_2P_2(t)+\cdots+C_nP_n(t)$, $n=1,2,3$ bo‘ladi. Demak, C_k sonlar $f(t)=e^t$ elementning $\{P_k\}$ ortonormal sistemadagi Furye koeffitsiyentlari bo‘lishi kerak. Shunday qilib, Furye koeffitsiyentlari C_k larni hisoblaymiz:

$$C_0=(f,P_0)=\int_{-1}^{1}\frac{1}{\sqrt{2}}\cdot e^t dt=\frac{1}{\sqrt{2}}e^t\Big|_{-1}^{1}=\frac{1}{\sqrt{2}}(e-e^{-1})=\sqrt{2}\,\mathrm{sh}\,1.$$

Endi C_1 ni hisoblaymiz.

$$C_1=(f,P_1)=\sqrt{\frac{3}{2}}\int_{-1}^{1}t\cdot e^t dt=\sqrt{\frac{3}{2}}\,te^t\Big|_{-1}^{1}-\sqrt{\frac{3}{2}}\int_{-1}^{1}e^t dt=\sqrt{6}(\mathrm{ch}\,1-\mathrm{sh}\,1).$$

Demak, $p_1(t) = \operatorname{sh} 1 + 3(\operatorname{ch} 1 - \operatorname{sh} 1)t$ ekan. Endi C_2 ni hisoblaymiz:

$$C_2 = (f, P_2) = \sqrt{\frac{5}{8}} \int\limits_{-1}^{1} (3t^2 - 1)e^t dt = \sqrt{\frac{5}{8}} (3t^2 - 1)e^t \Big|_{-1}^{1} - 2\sqrt{\frac{5}{8}} \int\limits_{-1}^{1} t\, e^t dt.$$

te^t funksiyaning integrali C_1 koeffitsiyentni hisoblashda topilgan edi. Bulardan $C_2 = \dfrac{4\sqrt{10}}{3} \operatorname{sh} 1 - \sqrt{10} \operatorname{ch} 1$ ni olamiz. Demak,

$$p_2(t) = \operatorname{sh} 1 + 3(\operatorname{ch} 1 - \operatorname{sh} 1)t + (\frac{10}{3} \operatorname{sh} 1 - \frac{5}{2} \operatorname{ch} 1)(3t^2 - 1)$$

ekan. Xuddi shunday $c_3 = (f, P_3)$ koeffitsiyent hisoblanib, $p_3(t)$ ko‘phad topiladi.

68. Bu misol 67-misolga o‘xshash ishlanadi. Ya’ni $p_n(t) = C_0 P_0 + C_1 P_1(t) + C_2 P_2(t) + \cdots + C_n P_n(t)$, $n = 1, 2, 3$ bo‘ladi. Bu yerda P_n, $n \in \mathbb{N}$ normallangan Lejandr ko‘phadlari. Bu misolda $f(t) = t^4$ juft funksiya bo‘lganligi uchun Furye qatorida t ning faqat juft darajalari qatnashadi, toq koeffitsiyentlar $C_{2n-1} = 0$, $n \in \mathbb{N}$ bo‘ladi. Bu yerdan $p_0(t) = p_1(t) = C_0 P_0$ va $p_2(t) = p_3(t) = C_0 P_0 + C_2 P_2(t)$ ni olamiz. $p_0 = p_1$, $p_2 = p_3$ ko‘phadlarni topish uchun, faqat C_0 va C_2 koeffitsiyentlarni hisoblash yetarli.

$$C_0 = (f, P_0) = \int_{-1}^{1} \frac{1}{\sqrt{2}} \cdot t^4 dt = \frac{1}{\sqrt{2}} t^5 \Big|_{-1}^{1} = \frac{\sqrt{2}}{5},$$

$$C_2 = (f, P_2) = \sqrt{\frac{5}{8}} \int\limits_{-1}^{1} t^4 (3t^2 - 1) dt = \sqrt{\frac{5}{8}} \left(\frac{3}{7} t^7 - \frac{1}{5} t^5 \right) \Big|_{-1}^{1} = \sqrt{\frac{5}{8}} \frac{16}{35}.$$

Bu yerdan

$$p_1(t) = \frac{1}{5}, \quad p_2(t) = p_3(t) = \frac{1}{5} + \frac{2}{7}(3t^2 - 1)$$

ni olamiz.

70. Agar $|x| \geq 1$ bo‘lsa, $\dfrac{1}{(1 + x^2)^2} \leq \dfrac{1}{x^4}$ tengsizlik bajariladi. Shuning uchun

$$\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} \frac{dx}{(1 + x^2)^2} \leq \int_{-1}^{1} \frac{dx}{(1 + x^2)^2} + \int_{|x|>1} \frac{dx}{x^4}$$

integral mavjud, ya'ni $f \in L_2(\mathbb{R})$. g funksiyaning tashuvchisi ($A = [-1,\ 1]$) chekli o'lchovli to'plam bo'lganligi uchun $g \in L_2(\mathbb{R})$ va $\int_{\mathbb{R}} |g(x)|^2 dx = 2$. Ularning skalyar ko'paytmasi

$$(f,\ g) = \int_{\mathbb{R}} f(x)\overline{g(x)}dx = \int_{-1}^{1} \frac{dx}{1+x^2} = \text{arctg} x\Big|_{-1}^{1} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

nolmas, ya'ni ular ortogonal elementlar emas.

71. $f(x,y) = e^{-|x|}\, e^{-|y|}$ ni $L_2(\mathbb{R}^2)$ qarashli ekanligini ko'rsatamiz. $e^{-|x|} \in L_2(\mathbb{R})$ bo'lganligi uchun $f \in L_2(\mathbb{R}^2)$ bo'ladi. g funksiyaning tashuvchisi chekli o'lchovli to'plam ($A = [-1,\ 1] \times [0,\ 1]$) bo'lganligi uchun $g \in L_2(\mathbb{R}^2)$ bo'ladi. Ular normasining kvadratlari

$$\|f\|^2 = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-2|x|}e^{-2|y|}dxdy = \left(\int_{\mathbb{R}} e^{-2|x|}dx\right)^2 = 4\left(\int_0^{\infty} e^{-2|x|}dx\right)^2 = 1,$$

$$\int_{\mathbb{R}^2} |g(x)|^2 dxdy = \mu(A) = 2.$$

Demak, $\|f\| = 1$, $\|g\| = \sqrt{2}$. Ularning skalyar ko'paytmasi

$$(f,\ g) = \int\limits_{\mathbb{R}^2} f(x)\overline{g(x)}dx = 2\int\limits_{-1}^{1}\int\limits_{-1}^{1} e^{-x}e^{-y}dxdy = 2\left(\int\limits_{-1}^{1} e^{-x}dx\right)^2 = 2(1-e^{-1})^2$$

nolmas, ya'ni ular ortogonal elementlar emas.

72. $\|f\|^2 = 2\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$. 65-misolda $L_2[-1,\ 1]$ fazoda $g(x) = x$ elementning $\left\{\psi_n(t) = 2^{-1/2}\exp\{in\pi t\}\right\}$, $n \in \mathbb{Z}$ to'la ortonormal sistemadagi Furye koeffitsiyentlari topilgan, ya'ni

$$g(x) = x = \sum_{n\in\mathbb{Z}} C_n\psi_n(x), \quad C_0 = 0, \quad C_n = \frac{\sqrt{2}i(-1)^{n-1}}{n\pi}, \quad n \neq 0$$

tenglik o'rinli. Parseval ayniyatiga ko'ra,

$$\|g\|^2 = \sum_{n\in\mathbb{Z}\setminus\{0\}} |C_n|^2 = 2\sum_{n=1}^{\infty}|C_n|^2 = 2\sum_{n=1}^{\infty}\left|\frac{\sqrt{2}i(-1)^{n-1}}{n\pi}\right|^2 = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

bo'ladi. Endi

$$\|g\|^2 = \int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$$

ni hisobga olsak,

$$\frac{2}{3} = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \iff 2\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{3}$$

ni olamiz. Bu yerdan $\|f\| = \dfrac{\pi}{\sqrt{3}}$ bo'ladi.

73. Barcha $n, m \in \mathbb{Z}$ lar uchun $f(n,m) > 0$ bo'lganlidan $\sum\limits_{(n,m)\in\mathbb{Z}} |f(n,m)|^2$ qator yaqinlashuvchi bo'lishi uchun

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(1+n^\alpha+m^\beta)^2} \qquad (3.2j)$$

qatorning yaqinlashuvchi bo'lishi zarur va yetarli. Shuning uchun $\alpha \le 0,5$ yoki $\beta \le 0,5$ bo'lsa, $\{f(n,m)\} \notin \ell_2(\mathbb{Z}^2)$. Shunday ekan, $\alpha > 0,5$ va $\beta > 0,5$ shartlarning bajarilishi zarur. (3.2j) qatorning yaqinlashuvchi bo'lishi uchun integral alomatga ko'ra

$$\int_1^{\infty}\int_1^{\infty}\frac{dxdy}{(x^\alpha+y^\beta)^2} \qquad (3.3j)$$

integralning yaqinlashishi zarur va yetarli. $\alpha > 0,5$ bo'lsin

$$\int_1^{\infty}\frac{dx}{(x^\alpha+y^\beta)^2}$$

integralda $x = y^{\beta/\alpha}z,\ dx = y^{\beta/\alpha}dz$ almashtirish olamiz, natijada (3.3j) integral

$$\int_1^{\infty}\int_1^{\infty}\frac{dxdy}{(x^\alpha+y^\beta)^2} = \int_1^{\infty}\frac{1}{y^{2\beta-\beta/\alpha}}\left(\int_{y^{-\beta/\alpha}}^{\infty}\frac{dz}{(z^\alpha+1)^2}\right)dy$$

ko'rinishni oladi. Bu integral yaqinlashuvchi bo'lishi uchun

$$2\beta - \frac{\beta}{\alpha} > 1 \iff \beta(2-\frac{1}{\alpha}) > 1 \iff \beta > \frac{\alpha}{2\alpha-1}$$

shart bajarilishi kerak. Demak, $\alpha > 0,5$ va $\beta > \dfrac{\alpha}{2\alpha-1}$ bo'lsa, f funksiya $\ell_2(\mathbb{Z}^2)$ fazoga qarashli bo'ladi.

75. E Evklid fazosi, $\{\varphi_k\}_{k=1}^{\infty}$ E dagi ixtiyoriy ortonormal sistema bo'lsin. U holda ixtiyoriy $f \in E$ uchun Bessel tengsizligi $\sum\limits_{k=1}^{\infty} |(\varphi_k, f)|^2 \leq \|f\|^2$ o'rinli. Bu yerdan $\sum\limits_{k=1}^{\infty} |(\varphi_k, f)|^2$ qatorning yaqinlashuvchi ekanligi kelib chiqadi. Qator yaqinlashishining zaruriy shartiga ko'ra, ixtiyoriy $f \in E$ uchun $\lim\limits_{k\to\infty} (\varphi_k, f) = 0$ bo'ladi. Demak, $\{\varphi_k\}_{k=1}^{\infty}$ ketma-ketlik nolga kuchsiz ma'noda yaqinlashadi.

4-§. Chiziqli funksionallar

5. Funksional chiziqli va uzluksiz.

6, 8, 10 lar qo'shma chiziqli funksional, ular uzluksiz.

7, 9, 11 lar chiziqli funksional, ular uzluksiz.

12-misolning yechimi. Integralning asosiy xossalaridan foydalansak, quyidagilarga ega bo'lamiz:

$$f\,(x+y) = \int_0^{\pi} \cos t\,\overline{(x\,(t) + y(t))}\,dt = \int_0^{\pi} \cos t\overline{x\,(t)}\,dt + \int_0^{\pi} \cos t\overline{y(t)}\,dt =$$

$$= f(x) + f(y),$$

$$f\,(\alpha\,x) = \int_0^{\pi} \cos t\,\overline{(\alpha\,x\,(t))}\,dt = \overline{\alpha}\,\int_0^{\pi} \cos t\,\overline{x\,(t)}\,dt = \overline{\alpha}\,f\,(x)\,.$$

Demak, $f : L_2[0,\,\pi] \to \mathbb{C}$ qo'shma chiziqli funksional ekan. Uning uzluksizligini 4.10-ta'rif yordamida tekshiramiz. Faraz qilaylik, $x_0 \in L_2[0,\,\pi]$ ixtiyoriy tayinlangan nuqta va $\{x_n\} \subset L_2[0,\,\pi]$ esa x_0 ga yaqinlashuvchi ixtiyoriy ketma-ketlik bo'lsin. U holda Koshi-Bunyakovskiy tengsizligiga ko'ra

$$|f\,(x_0) - f(x_n)\,|^2 = |\int_0^{\pi} \cos\,t\,\overline{(x_0\,(t) - x_n\,(t))}\,dt|^2 \leq$$

$$\leq \int_0^{\pi} \cos^2 t\,dt \int_0^{\pi} |x_0\,(t) - x_n\,(t)\,|^2\,dt = \frac{\pi}{2} \cdot \|x_0 - x_n\|^2$$

tengsizlik o'rinli. Bu yerdan $\lim\limits_{n\to\infty} f(x_n) = f(x_0)$ tenglik kelib chiqadi. Demak, $f : L_2[0,\,\pi] \to \mathbb{C}$ uzluksiz funksional bo'ladi.

13. Funksional chiziqli va uzluksiz.

14. $||f|| = \frac{2}{3}$. **15.** $||f|| = \frac{2}{\varepsilon}$. **16.** $||f|| = \sqrt{3}$. **17.** $||f|| = 3\sqrt{\frac{2}{5}}$.

18. Berilgan funksionalni quyidagicha yozish mumkin:

$$f(x) = (x, y), \quad y(t) = \text{sign}(t - 1/3), \ \ y \in L_2[0, \ 1].$$

Demak, 4.9-teoremaga koʻra $f : L_2[0, 1] \to \mathbb{C}$, chiziqli uzluksiz funksional boʻladi. Yana 4.9-teoremaga koʻra, uning normasi

$$\|f\| = \|y\| = \sqrt{\int_0^1 |y(t)|^2 dt} = \sqrt{\int_0^1 |\text{sign}(t - 1/3)|^2 dt} \ = \ 1.$$

19. $||f|| = 1$. **20.** $||f|| = 2$. **21.** $||f|| = 10$.

23-27 lar chekli qavariq funksionallar.

29-30 lar chekli qavariq funksionallar.

31. Chekli boʻlmagan qavariq funksional.

32. a) qavariq, qavariq jism, b) qavariq, qavariq jism emas, c) qavariq, qavariq jism, d) qavariq, qavariq jism, e) qavariq toʻplam emas.

33. $A \cap B \quad A + B$. **34.** Ha.

38. $p_M(x) = \max\{|x_1 - 1|, |x_2 + 1|\}$. **39.** $f : C_1[0, \ 1] \to \mathbb{R}, \ \ f(x) = x(0)$.

42. $f(x) = \int\limits_{-1}^{1} \text{sign}\, t \cdot x(t)\, dt$.

45. $g(t) = \begin{cases} 0, & t \in [-1, \ 0) \\ 1, & t \in [0, \ 1]. \end{cases}$ **46.** $g(t) = \begin{cases} \frac{t^2}{2}, & t = -1 \\ \frac{1 + t^2}{2}, & t \in (-1, \ 1) \\ \frac{2 + t^2}{2}, & t = 1. \end{cases}$

48. $f(x) = x(1)$, davom yagona.

50. $\|f_y\| = \|y\|$. **52.** $\|f\| = 1$.

53-55. $\{f_n\}$ ketma-ketlik 0 da kuchsiz ma’noda yaqinlashadi.

56. Kuchli, $f(x) = \int\limits_{-1}^{1} e^t x(t)\, dt$.

57. Kuchli, $f(x) = \int\limits_{-1}^{1} \sin t\, x(t)\, dt$.

58. Kuchli, $f(x) = \dfrac{1}{\sqrt{\pi}} \int\limits_{-\pi}^{\pi} t\,x(t)\,dt.$

59. $(\mathbb{R}^n)^* = \mathbb{R}^n, \ (\mathbb{R}^n_\infty)^* = \mathbb{R}^n_1, \ \left(\mathbb{R}^n_p\right)^* = \mathbb{R}^n_q, \ p^{-1} + q^{-1} = 1.$

60. $(\mathbb{R}^n_1)^* = \mathbb{R}^n_\infty.$

61. $(\ell_2)^* = \ell_2, \ (\ell_p)^* = \ell_q, \ p^{-1} + q^{-1} = 1, \ c^* = (c_0)^* = \ell_1.$

62. $(L_2[a, \ b])^* = L_2[a, \ b]\,, (L_p[a, \ b])^* = L_q[a, \ b], \ p^{-1} + q^{-1} = 1.$

63. $(C[a, \ b])^* = V_0[a, \ b].$

I bobda keltirilgan test javoblari

1-B 2-C 3-C 4-B 5-C 6-B 7-B 8-B 9-D 10-D 11-D 12-D 13-D 14-D 15-A 16-A 17-B 18-D 19-C 20-B 21-D 22-A 23-A 24-B 25-C 26-A 27-C 28-D 29-C 30-A 31-B 32-B 33-B 34-B 35-A 36-A 37-B 38-B 39-B 40-A, 41-B 42-A 43-A 44-A 45-A 46-B 47-A 48-C 49-A 50-A.

II bob. Chiziqli operatorlar

Bu bobda normalangan fazolarda chegaralangan chiziqli operatorlar, ularning normasini topish, chiziqli operatorlarning teskarisi mavjud yoki mavjud emasligini tekshirish, agar teskari operator mavjud bo‘lsa, uni aniqlash, chiziqli operatorlarga qo‘shma operatorlarni aniqlash (Banax fazolarida Banax bo‘yicha qo‘shma operatorni, Hilbert fazolarida Hilbert qo‘shmasini), chiziqli operatorlarning xos qiymatlari, xos vektorlari, spektri va rezolventasini aniqlashga doir masalalar jamlangan.

Bobning $5-\S$ da operatorlarning chiziqli chegaralanganligini tekshirib, ularning normasini topishga doir mashqlar bor. Chiziqli operatorning aniqlanish sohasini ko‘rsatib, uning chegaralanmaganligini yoki uzluksiz emasligini ko‘rsatishga doir mashqlar ham shu paragrafdan joy olgan. Chiziqli operatorlar ketma-ketligining yaqinlashishlarini tekshirishga doir mashqlar ham shu paragrafga kiritilgan. $6-\S$ da berilgan chiziqli operatorga teskari operator mavjud ekanligini ko‘rsatib, uning teskarisini topishga doir mashqlar keltirilgan. Bundan tashqari bu paragrafda teskari operatorning mavjud emasligini ko‘rsatishga doir mashqlar ham bor. $7-\S$ da esa, berilgan chiziqli operatorga mos Banax yoki Hilbert qo‘shmasini topishga doir mashqlar keltirilgan. Oxirgi $8-\S$ da esa chiziqli operatorning xos qiymatlari, xos vektorlari, spektri va rezolventasini hamda spektral yoyilmasini topishga doir mashqlar jamlangan.

5-§. Chiziqli uzluksiz operatorlar

Bu paragrafda biz normalangan fazolarda aniqlangan chiziqli operatorlarni qaraymiz. Chiziqli normalangan fazolarni X, Y va Z bilan, chiziqli operatorlarni esa A, B va C harflari bilan belgilaymiz.

5.1-ta'rif. *X chiziqli normalangan fazodan olingan har bir x elementga Y fazoning yagona y elementini mos qo‘yuvchi $Ax = y$ akslantirish operator deyiladi.*

Umuman, A operator X ning hamma yerida aniqlangan bo‘lishi shart emas. Bu holda Ax mavjud va $Ax \in Y$ bo‘lgan barcha $x \in X$ lar to‘plami A operatorning *aniqlanish sohasi* deyiladi va u $D(A)$ bilan belgilanadi, ya'ni:

$$D\,(A) = \{\, x \in X \;:\; Ax \quad \text{mavjud} \;\; \text{va} \quad Ax \in Y \;\}\,.$$

5.2-ta'rif. *Agar ixtiyoriy $x, y \in D\,(A)$ elementlar va ixtiyoriy $\alpha, \beta \in \mathbb{C}$ sonlar uchun $\alpha\, x + \beta\, y \in D(A)$ bo‘lib,*

$$A(\alpha\, x + \beta\, y) = \alpha\, A\, x + \beta\, A\, y$$

tenglik o‘rinli bo‘lsa, A ga chiziqli operator deyiladi.

5.3-ta'rif. *Agar ixtiyoriy $\varepsilon > 0$ uchun shunday $\delta = \delta\,(\varepsilon) > 0$ mavjud bo‘lib, $\|x - x_0\| < \delta$ tengsizlikni qanoatlantiruvchi barcha $x \in D\,(A)$ lar uchun $\|Ax - Ax_0\| < \varepsilon$ tengsizlik bajarilsa, A operator $x = x_0$ nuqtada uzluksiz deyiladi. Agar A operator ixtiyoriy $x \in D(A)$ nuqtada uzluksiz bo‘lsa, A ga uzluksiz operator deyiladi.*

5.4-ta'rif. *$Ax = \theta$ tenglikni qanoatlantiruvchi barcha $x \in X$ lar to‘plami A operatorning yadrosi deyiladi va u $KerA$ bilan belgilanadi.*

5.5-ta'rif. *Biror $x \in D(A)$ uchun $y = Ax$ tenglik bajariladigan barcha $y \in Y$ lar to‘plami A operatorning qiymatlar sohasi yoki tasviri deyiladi va u ImA yoki $R(A)$ bilan belgilanadi.*

Matematik simvollar yordamida operator yadrosi va qiymatlar sohasini quyidagicha yozish mumkin:

$$KerA = \{\; x \in D\,(A) : \;\; Ax = \theta \,\}\,,$$

$$R(A) := ImA = \{\, y \in Y : \;\text{biror}\;\; x \in D\,(A) \;\;\text{uchun}\;\; y = Ax \,\}\,.$$

5.6-ta'rif. *Agar shunday $C > 0$ son mavjud bo‘lib, barcha $x \in D\,(A)$ lar uchun*

$$\|Ax\| \leq C \cdot \|x\| \tag{5.1}$$

tengsizlik bajarilsa, A chegaralangan operator deyiladi.

5.7-ta'rif. (5.1) *tengsizlikni qanoatlantiruvchi C sonlar to'plami-ning aniq quyi chegarasi A operatorning normasi deyiladi va u $\|A\|$ bilan belgilanadi, ya'ni $\|A\| = \inf C$.*

5.1-teorema. *$A : X \to Y$ chiziqli chegaralangan operatorning normasi $\|A\|$ uchun quyidagi tenglik o'rinli*

$$\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|\neq 0} \frac{\|Ax\|}{\|x\|}. \tag{5.2}$$

Chiziqli operatorlarning uzluksizligi va chegaralanganligi orasida quyidagi bog'lanish mavjud.

5.2-teorema. *A chiziqli operator chegaralangan bo'lishi uchun uning uzluksiz bo'lishi zarur va yetarli.*

X chiziqli normalangan fazoni Y chiziqli normalangan fazoga akslantiruvchi chiziqli chegaralangan operatorlar to'plamini $L(X, Y)$ bilan belgilaymiz. Xususan, $X = Y$ bo'lsa $L(X, X) = L(X)$. $L(X, \mathbb{C})$ bilan X ga qo'shma fazo belgilanadi. Hilbert fazolarini H bilan belgilaymiz.

5.8-ta'rif. *$A : X \to Y$ va $B : X \to Y$ chiziqli operatorlarning yig'indisi deb, $x \in D(A) \cap D(B)$ elementga $y = Ax + Bx \in Y$ elementni mos qo'yuvchi $C = A + B$ operatorga aytiladi.*

Agar $A, B \in L(X, Y)$ bo'lsa, u holda C ham chiziqli chegaralangan operator bo'ladi va quyidagi tengsizlik o'rinli

$$\|C\| = \|A + B\| \leq \|A\| + \|B\|.$$

5.9-ta'rif. *A chiziqli operatorning α songa ko'paytmasi x element-ga αAx elementni mos qo'yuvchi operator sifatida aniqlanadi, ya'ni*

$$(\alpha A)(x) = \alpha Ax.$$

$L(X, Y)$ to'plamda kiritilgan operatorlarni qo'shish va operatorni songa ko'paytirish amallari chiziqli fazo ta'rifidagi 1-8 shartlarni qanoatlantiradi. Demak, $L(X, Y)$ to'plam operatorlarni qo'shish va operatorni songa ko'paytirish amallariga nisbatan chiziqli fazo tashkil qiladi. Bu fazoda aniqlangan $p : L(X, Y) \to \mathbb{R}$, $p(A) = \|A\|$ funksional (5.59-misolga qarang) normaning barcha shartlarini qanoatlantiradi. Demak, $L(X, Y)$ chiziqli normalangan fazodir. Bu fazoning to'laligi haqida quyidagi teorema o'rinli.

5.3-teorema. *Agar Y to'la normalangan fazo bo'lsa, u holda $L(X, Y)$ ham to'la normalangan fazo ya'ni Banax fazosi bo'ladi.*

5.10-ta'rif. *$A : X \to Y$ va $B : Y \to Z$ chiziqli operatorlar berilgan bo'lib, $R(A) \subset D(B)$ bo'lsin. B va A operatorlarning ko'paytmasi deganda, har bir $x \in D(A)$ ga Z fazoning $z = B(Ax)$ elementini mos qo'yuvchi $C = BA : X \to Z$ operatorga aytiladi.*

Agar A va B lar chiziqli chegaralangan operatorlar bo'lsa, u holda $C = BA$ ham chiziqli chegaralangan operator bo'ladi va

$$\|C\| = \|BA\| \leq \|B\| \cdot \|A\| \tag{5.3}$$

tengsizlik o'rinli.

5.11-ta'rif. *Agar $\{A_n\} \subset L(X, Y)$ operatorlar ketma-ketligi uchun shunday $A \in L(X, Y)$ operator mavjud bo'lib, $\lim_{n\to\infty} \|A_n - A\| = 0$ bo'lsa, $\{A_n\}$ operatorlar ketma-ketligi A operatorga norma bo'yicha yoki tekis yaqinlashadi deyiladi va $A_n \Longrightarrow A$ $(A_n \overset{u}{\Longrightarrow} A)$ shaklda belgilanadi.*

5.12-ta'rif. *Agar ixtiyoriy $x \in X$ uchun $\lim_{n\to\infty} \|A_n x - Ax\| = 0$ bo'lsa, $\{A_n\}$ operatorlar ketma-ketligi A operatorga kuchli yoki nuqtali yaqinlashadi deyiladi va $A_n \longrightarrow A$ $(A_n \overset{s}{\longrightarrow} A)$ shaklda belgilanadi.*

5.13-ta'rif. *Agar ixtiyoriy $f \in Y^*$ va barcha $x \in X$ lar uchun $\lim_{n\to\infty} f(A_n x) = f(Ax)$ bo'lsa, $\{A_n\}$ operatorlar ketma-ketligi A operatorga kuchsiz yoki kuchsiz ma'noda yaqinlashuvchi deyiladi va $A_n \longrightarrow A$ $(A_n \overset{w}{\longrightarrow} A)$ shaklda* *belgilanadi.*

Bu ta'rifni Hilbert fazosida quyidagicha bayon qilish mumkin.

5.14-ta'rif. *Agar ixtiyoriy* $x, y \in H$ *uchun* $\lim_{n\to\infty}(A_n x, y) = (Ax, y)$ *bo'lsa,* $\{A_n\} \subset L(H)$ *operatorlar ketma-ketligi* A *operatorga kuchsiz yaqinlashuvchi deyiladi.*

5.1. X – ixtiyoriy chiziqli normalangan fazo bo'lsin.

$$Ix = x, \ \ x \in X$$

akslantirish *birlik operator* deyiladi. Uni chiziqlilik va uzluksizlikka tekshiring. Agar chegaralangan bo'lsa, uning normasini toping.

Yechish. Bu operatorning chiziqliligi va uzluksizligi quyidagi tengliklardan bevosita kelib chiqadi:

$$I(\alpha x + \beta y) = \alpha x + \beta y = \alpha Ix + \beta Iy, \quad \|I(x - x_0)\| = \|x - x_0\|.$$

5.2-teoremadan uning chegaralangan ekanligi kelib chiqadi. 5.1-teoremaga ko'ra, $\|I\| = \sup_{\|x\|=1} \|Ix\| = \sup_{\|x\|=1} \|x\| = 1$ ekanligini olamiz. Qo'shimcha qilib aytishimiz mumkinki, uning aniqlanish sohasi, qiymatlar sohasi va yadrosi uchun quyidagilar o'rinli:

$$D(I) = X, \quad R(I) = X, \quad Ker I = \{\theta\}. \qquad \square$$

5.2. X va Y ixtiyoriy chiziqli normalangan fazolar bo'lsin.

$$\Theta : X \to Y, \ \ \Theta x = \theta$$

operator *nol operator* deyiladi. Uni chiziqlilik va uzluksizlikka tekshiring. Chegaralangan ekanligini ko'rsatib, normasini toping. Aniqlanish sohasi, qiymatlar sohasi va yadrosi haqida ma'lumot bering.

Yechish. Nol operatorning chiziqliligi va uzluksizligi bevosita ta'rifdan kelib chiqadi. Endi nol operatorning chegaralangan ekanligini ko'rsatib, uning normasini topamiz. Istalgan $x \in X$ uchun $\|\Theta x\| = \|\theta\| = 0$ tenglik o'rinli. Bundan $\|\Theta\| = \sup\limits_{\|x\|=1} \|\Theta x\| = \sup\limits_{\|x\|=1} \|\theta\| = 0$ ekanligi kelib chiqadi. Nol operator $L(X, Y)$ chiziqli normalangan fazoning nol elementi bo'ladi. Uning aniqlanish sohasi, qiymatlar sohasi va yadrosi uchun quyidagilar o'rinli:

$$D(\Theta) = X, \quad R(\Theta) = \{\theta\}, \quad Ker\,\Theta = X. \qquad \square$$

5.3. Aniqlanish sohasi $D(A) = C^{(1)}[a, b] \subset C[a, b]$ bo'lgan va $C[a, b]$ fazoni o'zini-o'ziga akslantiruvchi

$$A : C[a, b] \to C[a, b], \quad (Af)\,(x) = f'\,(x)$$

operatorni qaraymiz. Bu operator *differensial operator* deyiladi. Uni chiziqlilik va uzluksizlikka tekshiring.

Yechish. Differensial operatorni chiziqli ekanligini ko'rsatamiz. Buning uchun ixtiyoriy $f, g \in D(A)$ elementlarning chiziqli kombinatsiyasi bo'lgan $\alpha f + \beta g$ elementga A operatorning ta'sirini qaraymiz:

$$(A\,(\alpha f + \beta g))\,(x) = (\alpha f\,(x) + \beta g\,(x))' =$$

$$= \alpha f'\,(x) + \beta g'\,(x) = \alpha\ (Af)\,(x) + \beta\ (Ag)\,(x)\,.$$

Biz bu yerda yig'indining hosilasi hosilalar yig'indisiga tengligidan, hamda o'zgarmas sonni hosila belgisi ostidan chiqarish mumkinligidan foydalandik. Demak, A operator chiziqli ekan. Uni nol nuqtada uzluksizlikka tekshiramiz. Ma'lumki, $A\theta = \theta$, bu yerda $\theta - C[a, b]$ fazoning nol elementi, ya'ni $\theta(x) = 0$. Endi nolga yaqinlashuvchi $f_n \in D(A)$ ketma-ketlikni tanlaymiz. Umumiylikni buzmagan holda $a = 0,\ b = 1$ deymiz.

$$f_n(x) = \frac{x^{n+1}}{n+1}, \quad \lim_{n\to\infty} \|\, f_n\| = \lim_{n\to\infty} \max_{0\le x\le 1} \left|\frac{x^{n+1}}{n+1}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Ikkinchi tomondan,

$$(Af_n)\,(x) = x^n, \quad \lim_{n\to\infty} \|\, Af_n - A\theta\,\| = \lim_{n\to\infty} \max_{0\le x\le 1} |x^n| = \lim_{n\to\infty} 1 = 1 \neq 0.$$

Demak, A operator nol nuqtada uzluksiz emas ekan. Uning chiziqliligidan, differensial operator aniqlanish sohasining barcha nuqtalarida uzilishga ega bo‘ladi. 5.2-teoremaga ko‘ra differensial operator chegaralanmagan operator bo‘ladi. A ning qiymatlar sohasi va yadrosi uchun quyidagilar o‘rinli:

$$R(A) = C[a, b], \quad Ker A = \{const\}. \qquad \Box$$

5.4. $C[a,\, b]$ fazoni o‘zini-o‘ziga akslantiruvchi B operatorni quyidagicha aniqlaymiz:

$$(Bf)\,(x) = \int_a^b K\,(x, t)\, f\,(t)\; dt. \tag{5.4}$$

B ga *integral operator* deyiladi. Bu yerda $K : [a,\ b] \times [a,\ b] \to \mathbb{R}$ – uzluksiz funksiya, u integral operatorning *o‘zagi (yadrosi)* deyiladi. B operatorni chiziqlilik va uzluksizlikka tekshiring.

Yechish. Ma’lumki, ixtiyoriy $f \in C[a,\ b]$ uchun $K(x,\, t)\, f(t)$ funksiya x va t ning uzluksiz funksiyasidir. Matematik analiz kursidan ma’lumki,

$$\int_a^b K(x,\, t)\, f(t)\, dt$$

integral parametr $x \in [a,\, b]$ ning uzluksiz funksiyasi bo‘ladi. Bulardan B operatorning aniqlanish sohasi $D(B)$ uchun $D(B) = C[a,\ b]$ tenglik o‘rinli ekanligi kelib chiqadi. Integral operatorning chiziqli ekanligi integrallash amalining chiziqlilik xossasidan kelib chiqadi, ya’ni ixtiyoriy $f,\ g \in C[a,\ b]$ va $\alpha,\, \beta \in \mathbb{C}$ lar uchun

$$(B\,(\alpha\, f + \beta\, g))\;(x) = \int_a^b K\,(x, t)\;(\alpha\, f\,(t) + \beta\, g\,(t))\; dt =$$

$$= \alpha \int_a^b K(x,t) f(t)\, dt + \beta \int_a^b K(x,t) g(t)\, dt = \alpha\,(Bf)\,(x) + \beta\,(Bg)\,(x)$$

tengliklar o'rinli. Endi B operatorning uzluksiz ekanligini ko'rsatamiz. $f_0 \in C[a,\ b]$ ixtiyoriy tayinlangan element va $\{f_n\} \subset C[a,\ b]$ unga yaqinlashuvchi ixtiyoriy ketma-ketlik bo'lsin. U holda

$$\| Bf_n - Bf_0\| = \max_{a\le x\le b} \left| \int_a^b K(x,t)\ (f_n(t) - f_0(t))\, dt \right| \le$$

$$\le \max_{a\le t\le b} |f_n(t) - f_0(t)|\ \max_{a\le x\le b} \int_a^b |K(x,t)|\, dt = C\cdot \| f_n - f_0\|. \qquad (5.5)$$

Bu yerda

$$C = \max_{a\le x\le b} \int_a^b |K(x,t)|\, dt.$$

C ning chekli ekanligi $[a,\ b]$ kesmada uzluksiz funksiyaning chegaralangan ekanligidan kelib chiqadi. Agar (5.5) tengsizlikda $n\to\infty$ da limitga o'tsak,

$$0 \le \lim_{n\to\infty} \| Bf_n - Bf_0\| \le C\cdot \lim_{n\to\infty} \| f_n - f_0\| = 0$$

ekanligini olamiz. Demak,

$$\lim_{n\to\infty} \| B\,f_n - B\,f_0\| = 0.$$

Shunday qilib, B integral operator ixtiyoriy nuqtada uzluksiz ekan. 5.2-teoremaga ko'ra, u chegaralangan operator bo'ladi. B integral operatorning qiymatlar sohasi va yadrosi uning o'zagi K ning berilishiga bog'liq. Masalan, $K(x,\ t)\equiv 1$ bo'lsa, B operatorning qiymatlar sohasi ImB o'zgarmas funksiyalardan iborat, ya'ni $ImB = \{f\in C[a,\ b] : f(t) = const\}$, uning yadrosi $KerB$ o'zgarmasga ortogonal funksiyalardan iborat, ya'ni

$$KerB = \left\{ f\in C[a,\ b] : \int_a^b f(t)\, dt = 0 \right\}. \qquad \square$$

5.5. Quyidagi operatorning chiziqli chegaralanganligini ko'rsatib, normasini toping:

$$A : L_5[-1,\ 1] \to L_3[-1,\ 1],\quad (Ax)(t) = t^4\, x(t^3).$$

Yechish. Dastlab ixtiyoriy $x \in L_5[-1\,,\ 1]$ uchun $Ax \in L_3[-1\,,\ 1]$ ekanligini ko‘rsatamiz. Buning uchun $t^3 = r$ almashtirishdan foydalanamiz:

$$\|Ax\|_3 = \left(\int_{-1}^{1} \left|t^4 x(t^3)\right|^3 dt\right)^{\frac{1}{3}} = \left(\frac{1}{3}\int_{-1}^{1} |r|^{\frac{10}{3}} \, |x(r)|^3 \, dr\right)^{\frac{1}{3}}.$$

Oxirgi integralni baholashda quyidagi umumlashgan Gyolder tengsiz-ligidan foydalanamiz:

$$\|x \cdot y\|_s \leq \|x\|_k \cdot \|y\|_r\,,$$

bu yerda $x \in L_k[a,\ b]$, $y \in L_r[a,\ b]$, $\dfrac{1}{k} + \dfrac{1}{r} = \dfrac{1}{s}$. Biz qarayotgan holda $k = \dfrac{15}{2}$, $r = 5$, $s = 3$. Shuning uchun

$$\|Ax\|_3 = \left(\int_{-1}^{1} \left|\frac{1}{\sqrt[3]{3}} r^{\frac{10}{9}} x(r)\right|^3 dr\right)^{\frac{1}{3}} \leq$$

$$\leq \left(\int_{-1}^{1} \left|\frac{1}{\sqrt[3]{3}} r^{\frac{10}{9}}\right|^{\frac{15}{2}} dr\right)^{\frac{2}{15}} \left(\int_{-1}^{1} |x(r)|^{\,5} dr\right)^{\frac{1}{5}} = \frac{1}{\sqrt[3]{3}}\left(\frac{3}{14}\right)^{\frac{2}{15}} \cdot \|x\|_5.$$

Demak, ixtiyoriy $x \in L_5[-1,\ 1]$ uchun

$$\|Ax\|_3 \leq \frac{1}{\sqrt[3]{3}}\left(\frac{3}{14}\right)^{\frac{2}{15}} \cdot \|x\|_5 \qquad (5.6)$$

tengsizlik o‘rinli. Shunday qilib, har bir $x \in L_5[-1,\ 1]$ uchun $Ax \in L_3[-1,\ 1]$. Demak, $D(A) = L_5[-1,\ 1]$. Endi A operatorning chiziqli ekanligini ko‘rsatamiz:

$$[A(\alpha x + \beta y)](t) = t^4 (\alpha x + \beta y)\left(t^3\right) = \alpha t^4 x\left(t^3\right) + \beta t^4 y\left(t^3\right) =$$

$$= \alpha\,(Ax)\,(t) + \beta\,(Ay)\,(t) = [\alpha Ax + \beta Ay]\,(t),\ \forall t \in [-1,\ 1],$$

ya’ni $\forall x, y \in L_5[-1,\ 1]$ va $\forall \alpha, \beta \in \mathbb{C}$ uchun $A(\alpha x + \beta y) = \alpha Ax + \beta Ax$. Operatorning chegaralanganligi (5.6) dan kelib chiqadi. Uning normasini topish uchun $x_0(t) = t^{\frac{5}{3}} \in L_5[-1\,,\ 1]$ elementni qaraymiz:

$$\|x_0\|_5 = \left(\int_{-1}^{1} |\,t\,|^{\frac{25}{3}} dt\right)^{\frac{1}{5}} = \left(\frac{3}{14}\right)^{\frac{1}{5}}, \qquad (Ax_0)\,(t) = t^9,$$

$$\|Ax_0\| = \left(\int_{-1}^{1} |t|^{27}\, dt\right)^{\frac{1}{3}} = \left(\frac{1}{14}\right)^{\frac{1}{3}} = \frac{1}{\sqrt[3]{3}}\left(\frac{3}{14}\right)^{\frac{1}{3}},$$

$$\frac{\|Ax_0\|_3}{\|x_0\|_5} = \frac{\frac{1}{\sqrt[3]{3}}\,(3/14)^{\frac{1}{3}}}{(3/14)^{\frac{1}{5}}} = \frac{1}{\sqrt[3]{3}}\left(\frac{3}{14}\right)^{\frac{2}{15}}. \tag{5.7}$$

(5.6) va (5.7) dan $\|A\| = \frac{1}{\sqrt[3]{3}} \cdot \left(\frac{3}{14}\right)^{\frac{2}{15}}$ tenglikni hosil qilamiz. □

5.6. $B : AC_0[-1,\ 4] \to V_0[-1,\ 4],\ (Bx)(t) = \operatorname{sign} t\, x(t)$ operatorni chiziqli chegaralanganligini ko‘rsatib, normasini toping.

Yechish. *Chiziqliligi* 5.5-misol kabi ko‘rsatiladi. *Chegaralanganligi.* Berilgan operatorning chegaralangan ekanligini ko‘rsatishda o‘zgarishi chegaralangan funksiyalar uchun o‘rinli bo‘lgan

$$V_a^b\,[f \cdot g] \le \sup_{x\in[a,\,b]} |f(x)| \cdot V_a^b\,[g] + \sup_{x\in[a,\,b]} |g(x)| \cdot V_a^b\,[f]$$

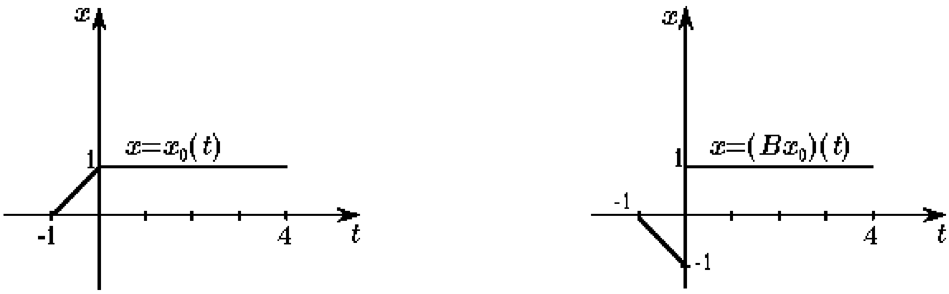
tengsizlikdan (shu kitobning I qismi 9.50-misoliga qarang) foydalanamiz:

$$\|Bx\| = \mathop{V}_{-1}^{4}[Bx] \le \sup_{-1\le t\le 4} |\operatorname{sign} t|\, \mathop{V}_{-1}^{4}[x] + \sup_{-1\le t\le 4} |x(t)|\, \mathop{V}_{-1}^{4}[\operatorname{sign}] \le 3\,\|x\|\,.$$

Bu yerda biz $\sup\limits_{-1\le t\le 4} |\operatorname{sign} t| = 1,\ \mathop{V}\limits_{-1}^{4}[\operatorname{sign}] = 2$ va barcha $x \in AC_0[-1,\ 4]$ lar uchun o‘rinli bo‘lgan $\max\limits_{-1\le t\le 4} |x(t)| \le \mathop{V}\limits_{-1}^{4}[x] = \|x\|$ tengsizlikdan foydalandik. Yuqoridagi tengsizlikdan $\|B\| \le 3$ kelib chiqadi.

$$x_0(t) = \begin{cases} t+1, & -1 \le t < 0 \\ 1, & 0 \le t \le 4 \end{cases}, \qquad x_0 \in AC_0[-1,\ 4]$$

element uchun $\|x_0\| = 1$ va $\|Bx_0\| = 3$ tengliklar o‘rinli (5.1-chizma). Bu yerdan $\|B\| \ge 3$ ekanligi kelib chiqadi. Demak, $\|B\| = 3$ ekan. □



5.1-chizma

5.7. $A : AC_0[0,\ 1] \to L_1[0,\ 1],\ \ (Ax)(t) = \dfrac{dx(t)}{dt}$ differensiallash operatorini chiziqli chegaralanganligini ko‘rsatib, normasini toping.

Yechish. $[a,\, b]$ kesmada absolyut uzluksiz F funksiyaning hosilasi $F'(x) = f(x)$ integrallanuvchidir ([9] ga qarang). Demak, operatorning aniqlanish sohasi $D(A) = AC_0[0,\ 1]$ ekan. Ma’lumki, ixtiyoriy $x \in AC_0[0,\ 1]$ uchun

$$V_0^1[Ax] = \int\limits_0^1 |x(t)|\, dt$$

tenglik o‘rinli. Bu tenglikning chap tomoni $\|Ax\|$ ga o‘ng tomoni $\|x\|$ ga teng. Demak, barcha $x \in AC_0[0,\ 1]$ uchun $\|Ax\| = \|x\|$ tenglik o‘rinli. Bu yerdan A operatorning chegaralanganligi va $\|A\| = 1$ ekanligi kelib chiqadi. $\square$

5.8. $A : \ell_1 \to \ell_1,\ \ Ax = (x_1, \ln 2 \cdot x_2, \ldots, \ln n \cdot x_n, \ldots)$ operatorni chegaralanganlikka tekshiring.

Yechish. Matematik analiz kursidan ma’lum bo‘lgan quyidagi qatorlarni qaraymiz:

$$1)\quad 1 + \sum_{n=2}^{\infty} \frac{1}{n \cdot \ln^2 n}, \qquad 2)\quad 1 + \sum_{n=2}^{\infty} \frac{1}{n \cdot \ln n}.$$

Ulardan 1)-si yaqinlashuvchi, 2)-si esa uzoqlashuvchi. Bu tasdiq integral alomat yordamida ko‘rsatiladi. Agar biz $x_0 = \left(1, \dfrac{1}{2\,\ln^2 2}, \ldots, \dfrac{1}{n\,\ln^2 n}, \ldots\right)$ desak, u holda $x_0 \in \ell_1$ bo‘ladi, $Ax_0 = \left(1, \dfrac{1}{2 \cdot \ln 2}, \ldots, \dfrac{1}{n \cdot \ln n}, \ldots\right)$ bo‘lib, uning hadlarining modullaridan tuzilgan qator (2-qator) uzoqlashuvchi, ya'ni $Ax_0 \notin \ell_1$. Demak, $D(A) \neq \ell_1$. Agar

$$x_m = \left(1, \frac{1}{2 \cdot \ln^2 2}, \ldots, \frac{1}{m \cdot \ln^2 m}, 0, 0, \ldots\right)$$

desak, u holda $x_m \in D(A)$ bo'lib, biror $C > 0$ va barcha m larda $\|x_m\| \leq C$ bo'ladi. Ammo $\lim\limits_{m\to\infty} \|Ax_m\| = \infty$. Ya'ni A chegaralanmagan operatordir. 5.2-teoremaga ko'ra, u uzluksiz ham emas. □

Uy vazifalari va mavzuni o'zlashtirish uchun masalalar

Quyidagi belgilashlarni kiritamiz. Faraz qilaylik, $G \subset \mathbb{R}^m$ biror soha bo'lsin. $C(G, \mathbb{R}^n)$ bilan $u(p) = (u_1(p), u_2(p), \ldots, u_n(p)) \in \mathbb{R}^n$, $u_j \in C(G)$, $p \in G$, $j = 1, 2, \ldots, n$ vektor funksiyalar to'plamini belgilaymiz. Xuddi shunday $L_2(G, \mathbb{R}^n)$ bilan $u(p) = (u_1(p), u_2(p), \ldots, u_n(p)) \in \mathbb{R}^n$, $u_j \in L_2(G)$, $j = 1, 2, \ldots, n$ vektor funksiyalar to'plamini belgilaymiz. Bu ikkala fazoda ham skalyar ko'paytma bir xilda kiritiladi, ya'ni

$$(u, v) = \sum_{k=1}^{n} \int_G u_k(p)\, v_k(p)\, dp.$$

5.9. Har bir $u \in C(G)$, $G \subset \mathbb{R}^3$ funksiyaga, uning $\ell(\cos\alpha, \cos\beta, \cos\gamma)$ yo'nalish bo'yicha olingan hosilasini mos qo'yib, natijada

$$A : C(G) \to C(G),\ (Au)(p) = \cos\alpha \frac{\partial u(p)}{\partial x} + \cos\beta \frac{\partial u(p)}{\partial y} + \cos\gamma \frac{\partial u(p)}{\partial z}$$

operatorga ega bo'lamiz. Bu operator chiziqli, uzluksiz bo'ladimi? Agar bu operatorni $A : C^{(1)}(G) \to C(G)$ akslantirish sifatida qarasak u uzluksiz bo'ladimi?

5.10. Har bir $u \in C^{(1)}(G)$, $G \subset \mathbb{R}^3$ funksiyaga uning divergensiyasini mos qo'yib, natijada

$$A : C^{(1)}(G) \to C(G),\ \ (Au)(p) = \frac{\partial u(p)}{\partial x} + \frac{\partial u(p)}{\partial y} + \frac{\partial u(p)}{\partial z}$$

operatorga ega bo'lamiz. Bu operator chiziqli, uzluksiz bo'ladimi?

5.11. Har bir $u \in C^{(1)}(G)$, $G \subset \mathbb{R}^3$ funksiyaga uning gradiyentini mos qo‘yib, natijada

$$A : C^{(1)}(G) \to C(G, \mathbb{R}^3), \ \ (Au)(p) = \left(\frac{\partial u(p)}{\partial x}, \ \frac{\partial u(p)}{\partial y}, \ \frac{\partial u(p)}{\partial z} \right)$$

operatorga ega bo‘lamiz. Bu operatorning tabiiy aniqlanish sohasini toping. U chiziqli, uzluksiz bo‘ladimi?

5.12. Har bir $u \in C^{(1)}(G)$, $G \subset \mathbb{R}^3$ funksiyaga uning gradiyenti uzunligini mos qo‘yib, natijada $A : C^{(1)}(G) \to C(G)$,

$$(Au)(p) = \sqrt{\left(\frac{\partial u(p)}{\partial x} \right)^2 + \left(\frac{\partial u(p)}{\partial y} \right)^2 + \left(\frac{\partial u(p)}{\partial z} \right)^2}$$

operatorga ega bo‘lamiz. Bu operator chiziqli bo‘ladimi?

5.13. Har bir $u \in C(G, \mathbb{R}^3)$, $u(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$ vektor funksiyaga uning uyurmasi (rotori) ni mos qo‘yib, natijada

$$A : C(G, \mathbb{R}^3) \to C(G, \mathbb{R}^3),$$

$$(Au)(p) = \left(\frac{\partial R(p)}{\partial y} - \frac{\partial Q(p)}{\partial z}, \ \frac{\partial P(p)}{\partial z} - \frac{\partial R(p)}{\partial x}, \ \frac{\partial Q(p)}{\partial x} - \frac{\partial P(p)}{\partial y} \right)$$

operatorga ega bo‘lamiz. Bu operatorning tabiiy aniqlanish sohasini toping. U chiziqli, uzluksiz bo‘ladimi?

5.14. Laplas (kinetik energiya) operatori $\Delta : C(\mathbb{R}^3) \to C(\mathbb{R}^3)$,

$$(\Delta u)(x, y, z) = \frac{\partial^2 u(x, y, z)}{\partial x^2} + \frac{\partial^2 u(x, y, z)}{\partial y^2} + \frac{\partial^2 u(x, y, z)}{\partial z^2}$$

ni chiziqli, uzluksizlikka tekshiring. Dastlab uning tabiiy aniqlanish sohasini toping.

5.15. Quyidagi operatorning chiziqli chegaralanganligini ko‘rsatib, normasini toping:

$$A : C^{(2)}[0, \ 1] \to C[0, \ 1], \quad (Ax)(t) = \frac{d^2 x(t)}{dt^2} := x''(t).$$

5.16-5.34-misollarda berilgan operatorlarning chiziqli, chegaralangan ekanligini ko‘rsating, ularning normalarini toping.

5.16. $A : C[-2,\ 2] \to C[-2,\ 2]\ ,\ (Ax)(t) = te^t x(t)\ .$

5.17. $A : C[-1,\ 1] \to C[-1,\ 1],\ (Ax)(t) = \int\limits_{-1}^{1} (ts + t^2 s^2)\, x(s)\, ds.$

5.18. $A : L_2[-1,\ 1] \to L_2[-1,\ 1]\ ,\ (Ax)(t) = (t^2 - t)\, x(t)\ .$

5.19. $A : L_2[0,\ 1] \to L_2[0,\ 1]\, ,\ (Ax)(t) = t^3 x(t^{1/3})\, .$

5.20. $A : L_2[0,\ 1] \to L_2[0,\ 1],\ (Ax)(t) = t^2 x\left(t^3\right)\ .$

5.21. $A : L_2[-1,\ 1] \to L_2[-1,\ 1],\ (Ax)(t) = \int\limits_{-1}^{1} (t+s)^2\, x\,(s)\, ds\, .$

5.22. $A : L_3[-1,\ 1] \to L_3[-1,\ 1],\ (Ax)(t) = t\, x\, \left(\sqrt[3]{t}\right)\ .$

5.23. $A : L_3[-1,\ 1] \to L_3[-1,\ 1]\, ,\ (Ax)(t) = \sqrt[5]{1-t}\, x(t)\, .$

5.24. $A : L_5[0,2] \to L_5[0,2]\, ,\ (Ax)(t) = (t^2 - 2t + 1)\, x(t)\, .$

5.25. $A : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}),\ (Af)(n) = f(n+1)\, .$

5.26. $A : \ell_2 \to \ell_2,\ Ax = \left(2x_1, \frac{3}{2}x_2, \ldots, \frac{n+1}{n}x_n, \ldots\right).$

5.27. $A : \ell_2 \to \ell_2,\ Ax = \left(x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \ldots, \frac{1}{n}x_n, \ldots\right).$

5.28. $A : \ell_1 \to \ell_1,\ Ax = \left(2x_1, \left(1+\frac{1}{2}\right)^2 x_2, \ldots, \left(1+\frac{1}{n}\right)^n x_n, \ldots\right).$

5.29. $A : \ell_3 \to \ell_3,\ Ax = ((1+1)x_1, (1+1/2\,)x_2, \ldots, (1+1/n)\, x_n, \ldots).$

5.30. $A : \ell_5 \to \ell_5\, ,\ Ax = \left(\frac{x_1}{5}, \frac{x_2}{5^2}, \ldots, \frac{x_n}{5^n}, \ldots\right).$

5.31. $A : \ell_4 \to \ell_4,\ Ax = \left(\sin\ \frac{\pi}{8} \cdot x_1, \sin\ \frac{2\pi}{8} \cdot x_2, \ldots, \sin\ \frac{n\pi}{8} \cdot x_n, \ldots\right).$

5.32. $A : \ell_{5/2} \to \ell_{5/2},\ Ax = \left(\frac{1}{2}x_1, \frac{1}{\sqrt{2}}x_2, \ldots, \frac{1}{\sqrt[n]{2}}x_n, \ldots\right).$

5.33. $A:\ell_{5/4}\to\ell_{5/4}\,,\ Ax=\left(0,\frac{1}{2}x_2,\frac{2}{3}x_3,\dots,(1-\frac{1}{n})x_n,\dots\right).$

5.34. $A:C[0,\ 4]\to M[0,\ 4],\ \ (Ax)(t)=[t]\,x\,(t)\,.$

5.35. X va Y chiziqli normalangan fazolar, $A:X\to Y$ chiziqli operator. A operator chegaralanmagan bo'lishi uchun $D(A)$ da $\|x_n\|=1$ va $\lim\limits_{n\to\infty}\|Ax_n\|=\infty$ shartlarni qanoatlantiruvchi ketma-ketlikning mavjud bo'lishi zarur va yetarli. Isbotlang.

Endi chegaralanmagan operatorlarga misol keltiramiz.

5.36. $I:\ell_2\to\ell_1,\quad Ix=x$ operatorning chegaralanmagan ekanligini ko'rsating.

5.37-5.49-misollarda quyidagi savollarga javob bering. $X,Y-$ haqiqiy chiziqli normalangan fazolar, $A:X\to Y$ chiziqli operator.

1) A operatorning aniqlanish sohasi $D\,(A)$ butun X fazoga tengmi?

2) Berilgan operator uzluksizmi?

5.37. $A:C[0,\ 1]\to C[0,\ 1],\quad (Ax)(t)=\dfrac{dx(t)}{dt}.$

5.38. $A:C[0,\ 1]\to V_0[0,\ 1],\quad (Ax)(t)=t\,x(t).$

5.39. $A:AC_0[0,\ 1]\to C[0,\ 1],\quad (Ax)(t)=t^{-0,1}\,x(t).$

5.40. $A:AC_0[0,\ 1]\to L_1[0,\ 1],\quad (Ax)(t)=t^{-0,3}\,x(t).$

5.41. $A:C^{(1)}[-1,\ 1]\to L_1[-1,\ 1],\ \ (Ax)=\dfrac{d^2x(t)}{dt^2}.$

5.42. $A:L_1[0,\ 1]\to L_1[0,\ 1],\quad (Ax)(t)=x(t^3)\,.$

5.43. $A:L_1(\mathbb{R}_+)\to L_1(\mathbb{R}_+),\quad (Ax)(t)=t\,x(\sqrt{t}\,)$

5.44. $A:L_1(\mathbb{R})\to L_2(\mathbb{R}),\ \ (Ax)(t)=\dfrac{1}{1+|t|}x(t)\,.$

5.45. $A:L_2(\mathbb{R}_+)\to L_2(\mathbb{R}_+),\quad (Ax)(t)=\int\limits_0^\infty(ts+1)\,x\,(s)\,ds.$

5.46. $A:\ell_3 \to \ell_1,\ \ Ax=(x_1,2x_2,x_3,2x_4\ldots,x_{2n-1},2x_{2n},\ldots)\,.$

5.47. $A:\ell_2 \to \ell_2,\ \ Ax=\left(\operatorname{ctg}1\cdot x_1,\operatorname{ctg}\frac{1}{2}\cdot x_2,\ldots,\operatorname{ctg}\frac{1}{n}\cdot x_n,\ldots\right).$

5.48. $A:\ell_2 \to \ell_1,\ \ Ax=\left(x_1,2x_2,\frac{3}{2}x_3,\ldots,\frac{n}{n-1}x_n,\ldots\right).$

5.49. $A:\ell_2 \to \ell_1,\ \ Ax=\left(x_1,\frac{x_2}{\sqrt{2}},\ldots,\frac{x_n}{\sqrt{n}},\ldots\right).$

5.50-5.54-misollarda quyidagi savollarga javob bering. $X,Y-$ kompleks chiziqli normalangan fazolar, $A:X\to Y$ chiziqli operator.

1) A operatorning aniqlanish sohasi $D(A)$ butun X fazoga tengmi?

2) Berilgan operator uzluksizmi?

5.50. $A:L_2(\mathbb{R}_+)\to L_2(\mathbb{R}_+),\quad (Ax)(t)=t\,x\,(t)\,.$

5.51. $A:L_2(\mathbb{R})\to L_2(\mathbb{R}),\quad (Ax)(t)=\frac{1}{\sqrt{t}}x(t)\,.$

5.52. $A:\ell_1(\mathbb{Z})\to \ell_2(\mathbb{Z}),\ \ (Af)(n)=\sqrt{n}f(n)\,.$

5.53. $A:\ell_2(\mathbb{Z})\to \ell_1(\mathbb{Z}),\ \ (Af)(n)=\frac{1}{\sqrt{n+0,1}}f(n)\,.$

5.54. $A:\ell_2 \to \ell_3,\quad Ax=\left(x_1,\sqrt{2}\,x_2,\ldots,\sqrt{n}\,x_n,\ldots\right).$

5.55. Shunday $A,\,B\in L(X)$ operatorlarga misol keltiringki, $AB\neq BA$ bo‘lsin.

5.56. $A,\,B\in L(X,Y)$ noldan farqli operatorlar bo‘lib, $R(A)\cap R(B)=0$ bo‘lsa, A va B larning chiziqli erkli ekanligini isbotlang.

5.57. $A,\,B\in L(X,Y)$ va $R(A)=R(B),\ \ KerA=KerB$ bo‘lishidan $A=B$ ekanligi kelib chiqadimi?

5.58. $X,\,Y$ lar normalangan fazolar, $U\subset X$ ochiq to‘plam, $V\subset X$ yopiq to‘plam hamda $A\in L(X,Y)$ bo‘lsa, $A(U)$ ochiq, $A(V)$ esa yopiq to‘plam bo‘ladimi?

5.59. $p : L(X,Y) \to \mathbb{R},\ \ p(A) = \|A\|$ funksional norma shartlarini qanoatlantirishini isbotlang.

5.60. $p : L(X,Y) \to \mathbb{R},\ \ p(A) = \|A\|$ akslantirishning uzluksiz ekanligini isbotlang.

5.61. $L(\mathbb{R}^n, \mathbb{R}^m)$ fazoning oʻlchamini toping.

5.62. X chiziqli normalangan fazo, X' uning qism fazosi boʻlsin. $M = \{A \in L(X) :\ KerA = X'\}$ toʻplam $L(X)$ ning qism fazosi boʻladimi?

5.63. X chiziqli normalangan fazo, X' uning qism fazosi boʻlsin. $M = \{A \in L(X) :\ KerA \supset X'\}$ toʻplam $L(X)$ ning qism fazosi boʻladimi?

5.64. X chiziqli normalangan fazo, $A \in L(X)$ ixtiyoriy element, $N_k = Ker\, A^k$, $k = 0, 1, 2, \ldots$ boʻlsin. Quyidagilarni isbotlang.

a) $N_0 \subset N_1 \subset \cdots \subset N_k \subset N_{k+1} \subset \cdots$ munosablar oʻrinli.

b) faraz qilaylik, biror $m \in \mathbb{N}$ soni uchun $N_m = N_{m+1}$ boʻlsin. U holda barcha $p \in \mathbb{N}$ uchun $N_{m+p} = N_m$ tenglik oʻrinli.

5.65. X chiziqli normalangan fazo, $A \in L(X)$ tayinlangan element boʻlsin. $AB = BA$ shartni qanoatlantiruvchi barcha $B \in L(X)$ lar toʻplami $L(X)$ ning qism fazosi boʻladimi?

5.66. X chiziqli normalangan fazo, $A \in L(X)$ tayinlangan element boʻlsin. $AB = 0$ shartni qanoatlantiruvchi barcha $B \in L(X)$ lar toʻplami $L(X)$ ning qism fazosi boʻladimi?

5.67. H Hilbert fazosi, $A_n \in L(H)$, $n \in \mathbb{N}$ va har bir $x, y \in H$ uchun $\sup_{n\in\mathbb{N}} |(A_n x, y)| < \infty$ tengsizlik oʻrinli boʻlsin. U holda $\sup_{n\in\mathbb{N}} \|A_n\| < \infty$ tengsizlik ham oʻrinli. Isbotlang.

5.68. $X,\ Y$ lar Banax fazolari, $A_n \in L(X,Y)$ $(n \in \mathbb{N})$ va har bir $x \in X$ da $A_n x$ ketma-ketlik fundamental bo'lsin. U holda shunday $A \in L(X,\,Y)$ operator mavjud bo'lib, $\{A_n\}$ operatorlar ketma-ketligi A operatorga kuchli ma'noda yaqinlashadi. Isbotlang.

5.69. $C[-\pi,\pi]$ Banax fazosining $M = \{x \in C[-\pi,\,\pi] : \ x(-\pi) = x(\pi)\}$ qism fazosini qaraymiz va har bir $x \in M$ uchun

$$(A_n x)\,(t) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} x(s)\,ds + \frac{1}{\pi}\int\limits_{-\pi}^{\pi}\sum_{k=1}^{n}\cos k(t-s)x(s)ds$$

deymiz. Quyidagilarni isbotlang.

a) Quyidagi tenglik o'rinli:

$$(A_n x)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin[(2n+1)(s-t)]}{\sin[(s-t)/2]}\,x(s)\,ds.$$

b) $A_n \in L(M)$ va quyidagi tenglik o'rinli

$$\|A_n\| = \frac{1}{2\pi}\max_{t\in[-\pi,\,\pi]}\int\limits_{-\pi}^{\pi}\left|\frac{\sin[(2n+1)(s-t)]}{\sin[(s-t)/2]}\right|\,ds.$$

c) $\Phi \subset C[-\pi,\pi]$ - trigonometrik ko'phadlardan iborat qism fazo bo'lsin. Φ da A_n operatorlar ketma-ketligi birlik operatorga kuchli ma'noda yaqinlashadi.

5.70. $C[0,\,1]$ fazoni o'zini-o'ziga akslantiruvchi $A,\,A_n,\,B_n$ operatorlarni quyidagicha aniqlaymiz:

$$(Ax)(t) = \int\limits_0^1 e^{st}x(s)\,ds,\quad (A_n x)(t) = \int\limits_0^1\left[\sum_{k=0}^{n}\frac{(s\,t)^k}{k!}\right]x(s)\,ds,\ \ n\in\mathbb{N},$$

$$(B_n x)(t) = \int\limits_{1/n}^{1-1/n} e^{st}x(s)ds,\quad n\in\mathbb{N}.$$

$A_n,\,B_n$ operatorlar A operatorga yaqinlashadimi? Yaqinlashish xarakterini (tekis, kuchli, kuchsiz) aniqlang.

5.71. $C[0,\ 1]$ ni o'zini-o'ziga akslantiruvchi $A_n,\ n \in \mathbb{N}$ operatorlarni

$$(A_n x)(t) = x(t^{1+1/n})$$

tenglik yordamida aniqlaymiz. Quyidagilarni isbotlang.

a) Har bir $n \in \mathbb{N}$ uchun $A_n \in L(C[0,\ 1])$;

b) $\{A_n\}$ ketma-ketlik birlik operatorga kuchli yaqinlashadi.

c) $\{A_n\}$ operatorlar ketma-ketligi birlik operatorga tekis yaqinlashmaydi.

5.72. X, Y lar Banax fazolari, $x_n, x \in X, \quad x_n \to x,\ A_n, A \in L(X,Y)$. Agar $\lim_{n\to\infty} \|A_n - A\| = 0$ bo'lsa, u holda $\lim_{n\to\infty} \|A_n x_n - Ax\| = 0$ munosabatni isbotlang.

5.73. X, Y, Z lar Banax fazolari, $A_n, A \in L(X,Y),\ \ B_n, B \in L(Y,Z)$ bo'lib, A_n operatorlar ketma-ketligi A ga, B_n operatorlar ketma-ketligi B ga kuchli ma'noda yaqinlashsin. U holda $B_n \cdot A_n$ operatorlar ketma-ketligi $B \cdot A$ operatorga kuchli ma'noda yaqinlashadi. Isbotlang.

5.74. X, Y, Z lar Banax fazolari, $A_n, A \in L(X,Y), \quad B_n, B \in L(Y,Z)$ bo'lib, A_n operatorlar ketma-ketligi A ga B_n operatorlar ketma-ketligi B ga tekis (norma bo'yicha) yaqinlashsin. U holda $B_n \cdot A_n$ operatorlar ketma-ketligi $L(X,\ Z)$ fazoda $B \cdot A$ operatorga tekis yaqinlashadi.

5.75. Shunday X normalangan fazoga va $A,\ B \in L(X)$ operatorlarga misol keltiringki, $\|A \cdot B\| < \|A\| \cdot \|B\|$ bo'lsin.

5.76. X normalangan fazo va $A \in L(X),\ \ B : X \to X$ chegaralanmagan operator bo'lsin, B ning aniqlanish sohasi $D(B)$ X ning hamma yerida zich bo'lsin. $A \cdot B$ va $B \cdot A$ larning chegaralangan, chegaralanmagan hollariga misollar keltiring.

5.77. H Hilbert fazosi, $L \subset H$ uning qism fazosi bo‘lsin. $P : H \to H$, $Px = u$, $x = u + v$, $u \in L$, $v \in L^{\perp}$ operator L ga *ortogonal proyeksiyalash* operatori deyiladi. P ning chiziqli chegaralangan ekanligini ko‘rsatib, normasini toping.

5.78. $L_2[-1,\ 1]$ Hilbert fazosida A, B operatorlarni quyidagicha aniqlaymiz:

$$(Ax)(t) = \frac{1}{2}[x(t) + x(-t)], \qquad (Bx)(t) = \frac{1}{2}[x(t) - x(-t)].$$

a) $R(A)$, $R(B)$ to‘plamlarni tavsiflang. Ular $L_2[-1,\ 1]$ ning yopiq qism fazolari bo‘ladimi?

b) A, B operatorlarning chiziqli chegaralangan ekanligini ko‘psatib, normalarini toping.

c) A^2, B^2 operatorlarni toping. A va B operatorlar ortogonal proyeksiyalash operatorlari bo‘ladimi?

d) $A \cdot B$ va $B \cdot A$ operatorlarni toping.

5.79. H Hilbert fazosi, $L_1, L_2 \subset H$ uning qism fazolari bo‘lsin. P_1, P_2 lar mos ravishda L_1, L_2 larga ortogonal proyeksiyalash operatorlari bo‘lsa, $\|P_1 - P_2\| \leq 1$ ekanligini isbotlang.

5.80. Agar $A \cdot B = 0$ bo‘lsa, A va B operatorlar *ortogonal* deyiladi. H Hilbert fazosi, $L_1, L_2 \subset H$ uning qism fazolari, P_1, P_2 lar mos ravishda L_1, L_2 larga ortogonal proyeksiyalash operatorlari bo‘lsin. $P_1 \cdot P_2 = 0$ bo‘lishi uchun L_1 va L_2 qism fazolar o‘zaro ortogonal bo‘lishi zarur va yetarli. Isbotlang.

6- §. Teskari operatorlar

Biz bu paragrafda o‘zaro bir qiymatli chiziqli akslantirishlarni qaraymiz. X va Y lar Banax fazolari, A esa X ni Y ga akslantiruvchi chiziqli operator, $D(A)$ – uning aniqlanish sohasi, ImA esa uning qiymatlar sohasi bo‘lsin.

6.1-ta'rif. *Agar ixtiyoriy* $y \in ImA$ *uchun* $Ax = y$ *tenglama yagona yechimga ega bo'lsa, u holda* A *teskarilanuvchan operator deyiladi.*

6.2-ta'rif. *Agar* A *teskarilanuvchan operator bo'lsa, u holda ixtiyoriy* $y \in ImA$ *ga* $Ax = y$ *tenglamaning yechimi bo'lgan yagona* $x \in D(A)$ *element mos keladi. Bu moslikni o'rnatuvchi operator* A *operatorga teskari operator deyiladi va* A^{-1} *bilan belgilanadi.*

Teskari operator ta'rifidan quyidagilar kelib chiqadi:

$$A^{-1} : Y \to X, \ \ D(A^{-1}) = ImA, \ ImA^{-1} = D(A).$$

Bundan tashqari teskari operator uchun

$$A^{-1}Ax = x, \ x \in D(A), \quad AA^{-1}y = y, \ \ y \in D(A^{-1}) \tag{6.1}$$

tengliklar o'rinli.

$A : X \to X$ chiziqli operator bo'lsin. Agar biror $B \in L(X)$ operator uchun $BA = I$ bo'lsa, u holda B operator A operatorga *chap teskari operator* deyiladi. Xuddi shunday, $AC = I$ tenglik bajarilsa, C operator A ga *o'ng teskari operator* deyiladi. Adabiyotlarda A ga chap teskari operator A_l^{-1}, o'ng teskari operator esa A_r^{-1} orqali belgilanadi.

6.1-lemma. *Agar* A *operator uchun ham chap teskari, ham o'ng teskari operatorlar mavjud bo'lsa, u holda ular o'zaro teng.*

Isbot. A uchun A_l^{-1} chap teskari, A_r^{-1} o'ng teskari operatorlar bo'lsin, u holda

$$A_l^{-1} = A_l^{-1}I = A_l^{-1}(A\,A_r^{-1}) = (A_l^{-1}A)A_r^{-1} = IA_r^{-1} = A_r^{-1}. \ \Box$$

Ma'lumki, agar A uchun bir vaqtda ham o'ng teskari, ham chap teskari operatorlar mavjud bo'lsa, u holda A teskarilanuvchan operator bo'ladi va $A^{-1} = A_l^{-1} = A_r^{-1}$ tenglik o'rinli.

6.1-teorema. *Chiziqli operatorga teskari operator chiziqlidir.*

6.2-teorema *(Teskari operator haqida Banax teoremasi). A operator X Banax fazosini Y Banax fazosiga biyektiv akslantiruvchi chiziqli chegaralangan operator bo'lsin. U holda A^{-1} operator mavjud va chegaralangan.*

Hozir biz chiziqli operator teskarilanuvchan bo'lishligining zarur va yetarli shartini keltiramiz.

6.3-teorema. *$A : X \to Y$ chiziqli operator teskarilanuvchan bo'lishi uchun $Ax = \theta$ tenglama faqat $x = \theta$ yechimga ega bo'lishi zarur va yetarli.*

Endi chegaralangan teskari operator mavjud bo'lishligining zarur va yetarli shartini keltiramiz.

6.4-teorema. *ImA da A ga chegaralangan teskari operator mavjud bo'lishi uchun, shunday $m > 0$ son mavjud bo'lib, barcha $x \in D(A)$ larda*

$$\|Ax\| \geq m\,\|x\| \tag{6.2}$$

tengsizlikning bajarilishi zarur va yetarli.

Teskari operatorni topishda foydali bo'lgan quyidagi ikki teoremani keltiramiz.

6.5-teorema. *X – Banax fazosi va $A \in L(X)$. Agar $\|A\| < 1$ bo'lsa, u holda $I - A$ operator uchun chegaralangan teskari operator mavjud.*

6.2-lemma. *Agar $A, B \in L(X)$ bo'lib, $A^{-1}, B^{-1} \in L(X)$ bo'lsa, u holda AB operatorga chegaralangan teskari operator mavjud va $(AB)^{-1} = B^{-1}A^{-1}$ tenglik o'rinli.*

Lemmaning isboti $ABB^{-1}A^{-1} = I, \quad B^{-1}A^{-1}AB = I$ tengliklardan hamda 6.1-lemmadan kelib chiqadi.

6.6-teorema. *$A \in L(X)$ operatorga chegaralangan teskari operator mavjud bo'lsin. Agar $A' : X \to X$ operatorning normasi*

$$\|A'\| < \frac{1}{\|A^{-1}\|}$$

tengsizlikni qanoatlantirsa, u holda $B = A - A'$ operatorga chegaralangan teskari operator mavjud.

6.1. $A : \ell_2 \to \ell_2,\ Ax = (0, x_1, x_2, \ldots, x_{n+1}, \ldots)$ operatorga chap teskari operatorni toping. A *o'ngga siljitish operatori* deyiladi.

Yechish. $B : \ell_2 \to \ell_2$ bilan *chapga siljitish operatorini* belgilaymiz:

$$Bx = (x_2, x_3, \ldots, x_{n+1}, \ldots).$$

Endi BA operatorning $x \in \ell_2$ elementga ta'sirini qaraymiz.

$$BAx = B(Ax) = B(0, x_1, x_2, \ldots, x_{n-1}, \ldots) = (x_1, x_2, \ldots, x_n, \ldots) = Ix.$$

Demak, B operator A ga chap teskari operator ekan, ya'ni $B = A_l^{-1}$. □

6.2. 6.1-misolda keltirilgan o'ngga siljitish operatori $A : \ell_2 \to \ell_2$ ga o'ng teskari operator mavjudmi?

Yechish. Faraz qilaylik, A ga o'ng teskari operator mavjud bo'lsin, u holda 6.1-lemmaga ko'ra $A_r^{-1} = B = A_l^{-1}$ bo'ladi, ya'ni

$$A_r^{-1}x = (x_2, x_3, \ldots, x_{n+1}, \ldots).$$

Endi AA_r^{-1} operatorning nolmas $x \in \ell_2$ elementga ta'sirini qaraymiz:

$$AA_r^{-1}x = A(A_r^{-1}x) = A(x_2, \ldots, x_{n+1}, \ldots) = (0, x_2, x_3, \ldots, x_n, \ldots) \neq Ix.$$

Demak, A uchun o'ng teskari operator mavjud emas ekan. □

6.3. $A : C[0,\ 1] \to C[0,\ 1],\quad (Ax)(s) = \int_0^1 e^{s+t}x(t)\,dt + x(s)$ operator berilgan. Operator teskarilanuvchanmi? Agar teskarilanuvchan bo'lsa, teskari operatorni toping.

Yechish. Dastlab berilgan operatorning teskarilanuvchanligini tekshiramiz. 6.3-teoremaga ko'ra, A operator teskarilanuvchan bo'lishi uchun $Ax = 0$

tenglama faqat nol yechimga ega bo‘lishi zarur va yetarli. $Ax = 0$ tenglamani qaraymiz, ya’ni

$$\int_0^1 e^{s+t} x(t)\, dt + x(s) = 0 \quad \text{yoki} \quad x(s) = -c(x)\, e^s, \tag{6.3}$$

bu yerda

$$c(x) = \int_0^1 e^t x(t)\, dt. \tag{6.4}$$

Endi (6.3) ni (6.4) ga qo‘ysak,

$$c(x) = -c(x) \int_0^1 e^{2t} dt = -\frac{1}{2} c(x) \left(e^2 - 1\right) \quad \text{yoki} \quad \frac{1}{2} \left(e^2 + 1\right) c(x) = 0$$

tenglikka ega bo‘lamiz. Bundan $c(x) = 0$. (6.3) dan esa $x(s) \equiv 0$ ekanligiga ega bo‘lamiz. Demak, $Ax = 0$ tenglama faqat $x = 0$ yechimga ega, shuning uchun A teskarilanuvchan operator. A^{-1} ni topish maqsadida ixtiyoriy $y \in C[0,\ 1]$ element uchun $Ax = y$ tenglamani, ya’ni

$$\int_0^1 e^{s+t} x(t)\ dt + x(s) = y(s) \qquad \text{yoki} \quad x(s) = y(s) - c(x)\, e^s \tag{6.5}$$

tenglamani yechamiz. Bu yerda $c(x)$ (6.4) ko‘rinishga ega. $x(s)$ uchun olingan (6.5) ifodani (6.4) ga qo‘ysak,

$$c(x) = \int_0^1 e^t y(t)\, dt - c(x) \int_0^1 e^{2t}\, dt = \int_0^1 e^t y(t)\, dt - \frac{1}{2} c(x) \left(e^2 - 1\right)$$

ni olamiz. Bundan

$$c(x) = \frac{2}{e^2 + 1} \int_0^1 e^t y(t)\, dt \tag{6.6}$$

ni olamiz. $c(x)$ uchun olingan (6.6) ifodani (6.4) ga qo‘ysak, $Ax = y$ tenglama yechimi quyidagi ko‘rinishni oladi:

$$x(s) = y(s) - \frac{2}{e^2 + 1} \int_0^1 e^{s+t} y(t)\, dt.$$

Demak, har bir $y \in C[0, 1]$ elementga $Ax = y$ tenglama yechimini mos qo'yuvchi A^{-1} operator quyidagi formula yordamida aniqlanar ekan:

$$A^{-1} : C[0,\ 1] \to C[0,\ 1]\,, \left(A^{-1}x\right)(s) = x\,(s) - \frac{2}{e^2+1}\int_0^1 e^{s+t}x\,(t)\,dt. \quad \square$$

6.4. Quyidagi operatorning teskarilanuvchan emasligini ko'rsating:

$$A : C[0,\ 1] \to C[0,\ 1], \quad (Ax)(t) = x\,(0)\ \ t + x\,(1)\ \ t^2. \tag{6.7}$$

Yechish. 6.3-teoremaga ko'ra, A chiziqli operator teskarilanuvchan bo'lishi uchun $Ax = 0$ tenglama faqat $x = 0$ yechimga ega bo'lishi zarur va yetarli. (6.7) formula bilan berilgan operator uchun $x_0\,(t) = t\,(t-1) \neq 0$ funksiyani olsak, $x_0\,(0) = x_0\,(1) = 0$ bo'lganligi uchun

$$(Ax_0)\ (t) = x_0\,(0)\ t + x_0\,(1)\ t^2 = 0, \quad \forall t \in [0,\ 1]\,.$$

Demak, $Ax = 0$ tenglama nolmas yechimga ega. Shunday ekan 6.3-teoremaga ko'ra, A teskarilanuvchan operator emas. $\square$

Uy vazifalari va mavzuni o'zlashtirish uchun masalalar

6.5. $\ell_2(\mathbb{Z})$ Hilbert fazosida o'ngga siljitish operatori

$$A : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}), \quad (Af)(n) = f(n-1)$$

ni qaraymiz. Uning uchun o'ng va chap teskari operatorlar mavjudligini isbotlang.

6.6. $L_2[0,\ 1]$ fazoda x ga ko'paytirish operatorini, ya'ni

$$A : L_2[0,\ 1] \to L_2[0,\ 1], \quad (Af)(x) = xf(x) \tag{6.8}$$

operatorni qaraymiz. Bu operator 6.3-teorema shartlarini qanoatlantiradimi? A teskarilanuvchan operator bo'ladimi?

6.7. (6.8) tenglik bilan aniqlangan operator 6.4-teorema shartlarini qanoatlantiradimi? A ga chegaralangan teskari operator mavjudmi?

6.8. Hilbert fazosi $L_2[-1,\ 1]$ ni oʻzini-oʻziga akslantiruvchi

$$A : L_2[-1,\ 1] \to L_2[-1,\ 1], \quad (Af)(x) = \cos x\, f(x)$$

operatorni qaraymiz. A operator 6.4-teorema shartlarini qanoatlantiradimi? A ga chegaralangan teskari operator mavjudmi?

6.9-6.28-misollarda berilgan operatorlarning teskarilanuvchan ekanligini koʻrsating va ularga teskari operatorlarni toping.

6.9. $A : \mathbb{R}^3 \to \mathbb{R}^3, \quad Ax = (x_1 + x_3,\ x_1 + x_2, x_2 + x_3).$

6.10. $A : \mathbb{R}^3 \to \mathbb{R}^3, \quad Ax = (x_1, x_1 - x_2, x_2 - x_3)\,.$

6.11. $A : \mathbb{R}^4 \to \mathbb{R}^4, \quad Ax = (x_2, x_3, x_4, x_1)\,.$

6.12. $A : \mathbb{R}^5 \to \mathbb{R}^5, \quad Ax = (x_1 + x_2 + x_3, x_1 - 2x_2, x_3, x_4, x_5)\,.$

6.13. $A : \mathbb{R}^7 \to \mathbb{R}^7, \quad Ax = (x_1 - x_2,\ x_1 + x_2, x_2 + x_3, x_4, x_5, x_6, x_7)\,.$

6.14. $A : \ell_2 \to \ell_2, \ Ax = (x_1, x_2 + x_3,\ x_1 - x_2 + x_3, x_4, x_5, \ldots)\,.$

6.15. $A : \ell_2 \to \ell_1, \quad Ax = \left(x_1, \frac{1}{2}\,x_2, \ldots, \frac{1}{n}\,x_n, \ldots\right).$

6.16. $A : \ell_1 \to \ell_1, \ Ax = (x_1, x_2, x_2 + x_3, x_3 + x_4, x_3 + x_4 + x_5, x_6, x_7, \ldots)\,.$

6.17. $A : m \to \ell_2, \quad Ax = \left(x_1, \frac{1}{\sqrt{2}}\,x_2, \ldots, \frac{1}{\sqrt{n}}\,x_n, \ldots\right).$

6.18. $A : \ell_2 \to \ell_2, \quad Ax = \left(x_1, \frac{1}{2}\,x_2, \frac{2}{3}\,x_3, \ldots, \frac{n-1}{n}\,x_n, \ldots\right).$

6.19. $A :\ \ell_1 \to C[0,\ 1]\,,\ (Ax)(t) = \sum\limits_{k=1}^{\infty} x_k \sin 2\pi\, k\, t.$

6.20. $A : m \to C[0,\ 1]\,,\ (Ax)(t) = \sum\limits_{k=1}^{\infty} \frac{x_k}{k^2} \cos 2\pi k t.$

6.21. $A : C_0^{(1)}[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = x'(t)\,.$

6.22. $A : C[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = \int\limits_0^t s\,x(s)\,ds.$

6.23. $A : C[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = (t+2)x(t) + \int\limits_0^1 s\,x(s)\,ds.$

6.24. $A : C[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = (1+t)\,x(t) + \int\limits_0^1 tsx(s)\,ds.$

6.25. $A : C[0,\ \pi] \to C[0,\ \pi],\ (Ax)(t) = (\sin t + 1)x(t)\,.$

6.26. $A : C[0,\ 2] \to C[0,\ 2],\ (Ax)(t) = (t+1)x(t) + x(1)\,t + x(0)\,.$

6.27. $A : C[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = t^2 x(t) + x(1)\,.$

6.28. $A : C[0,\ \pi] \to C[0,\ \pi],\quad (Ax)\ (t) = x(t) + \int\limits_0^\pi \sin t \cdot \sin s\, x(s)\,ds.$

6.29-6.48-misollarda keltirilgan operatorlarning teskarilanuvchan emasligini ko‘rsating.

6.29. $A : \mathbb{R}^4 \to \mathbb{R}^4,\quad Ax = (x_1 + x_2, x_2 + x_3, x_3 + x_4, x_1 + x_4)\,.$

6.30. $A : \ell_2 \to \ell_2,\quad Ax = (x_1, 0, x_3,\ 0, x_5, 0, x_7, x_8, x_9, \dots).$

6.31. $A : \ell_1 \to \ell_1,\quad Ax = (0, x_2, 0, x_4,\ 0, x_6, x_7, x_8, x_9, \dots).$

6.32. $A : \ell_1 \to \ell_1,\quad Ax = (x_1 + x_2, x_2 + x_3, x_3 + x_4, x_1 + x_4, x_5, x_6, \dots)\,.$

6.33. $A : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}),\quad (Af)(n) = \chi_{\{-1,\,0,\,1\}}(n)f(n).$

6.34. $A : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}),\quad (Af)(n) = (1 - \chi_{\mathbb{N}}(n))f(n).$

6.35. $A : m \to m,\quad Ax = (x_1, x_2, 0, 0, 0, x_6, x_7, x_8, \dots)\,.$

6.36. $A : AC[0,\ 1] \to L_1[0,\ 1],\quad (Ax)(t) = x'(t)\,.$

6.37. $A : C^{(2)}[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = x''(t)\,.$

6.38. $A: C[-1,\ 1] \to C[-1,\ 1],\ (Ax)(t) = \int\limits_{-1}^{1} ts\, x(s)\, ds.$

6.39. $A: C[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = x(0) \cdot \left(t^2+1\right) + x(1)\left(t^2+3t+2\right).$

6.40. $A: C^{(1)}[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = x(0) + x'(t).$

6.41. $A: C[0,\ 1] \to C[0,\ 1],\quad (Ax)(t) = (t^2+t+1)\int\limits_{0}^{1} s\, x(s)\ ds.$

6.42. $A: C[0,\ 2] \to C[0,\ 2],\ (Ax)(t) = x(0) + x(1)\, t + x(2)\, t^2.$

6.43. $A: C^{(1)}[0,\ 1] \to C^{(1)}[0,\ 1],\ (Ax)(t) = \int\limits_{0}^{t} x'(s)\, ds + [x(0) - x(1)]\, t.$

6.44. $A: C^{(1)}[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = x'(t) - 2x(t).$

6.45. $A: C[0,\ \pi] \to C[0,\ \pi],\ (Ax)(t) = (\sin t + \cos t) x(0) - \cos 2t\, x(\pi).$

6.46. $A: C^{(1)}[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = x'(t) - \dfrac{x(0)}{t^2+1}.$

6.47. $A: L_2[-1,\ 1] \to L_2[-1,\ 1],\ (Ax)(t) = \int\limits_{-1}^{1} t^2(s^2+1)\, x(s)\, ds.$

6.48. $A: L_2[-1,\ 1] \to L_2[-1,\ 1],\ (Ax)\ (t) = \cos t \int\limits_{-1}^{1} \sin\ s\, x(s)\ ds.$

6.49. X, Y chiziqli normalangan fazolar, $A: X \to Y$ teskarilanuvchan chiziqli operator bo‘lsin. U holda $x_1, x_2, \ldots, x_n \in D(A)$ va $Ax_1, Ax_2, \ldots, Ax_n$ elementlar sistemasi bir vaqtda yo chiziqli bog‘langan, yo chiziqli bog‘lanmagan bo‘ladi. Isbotlang.

6.50. X chiziqli normalangan fazo, $A: X \to X$ chiziqli operator bo‘lsin. Agar biror $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$ uchun $I + \lambda_1 \cdot A + \lambda_2 \cdot A^2 + \cdots + \lambda_n \cdot A^n = 0$ shart bajarilsa A ga teskari operator mavjudligini isbotlang.

6.51. X chiziqli normalangan fazo, $A,\ B: X \to X$ chiziqli operatorlar bo‘lib, $D(A) = D(B) = X$, hamda $(A\, B)^{-1}$, $(B\, A)^{-1}$ operatorlar mavjud bo‘lsin. A va B larga teskari operatorlar mavjudmi?

6.52. X chiziqli normalangan fazo, $A : X \to X$ chiziqli operator va $D(A)$ da $||x_n|| = 1$ va $\lim\limits_{n\to\infty} ||A\,x_n|| = 0$ shartni qanoatlantiruvchi ketma-ketlik mavjud bo'lsin. U holda A ga chegaralangan teskari operator mavjud emasligini isbotlang.

6.53. $C^{(1)}[0,\ 1]$ Banax fazosi, $L = \{x \in C^{(1)}[0,\ 1] : \ \ x(0) = 0\}$ uning qism fazosi va $A : L \to C[0,\ 1]$ chiziqli operatorni

$$(A\,x)(t) = \frac{dx(t)}{dt} + u(t)\,x(t), \quad u \in C[0,\ 1]$$

tenglik bilan aniqlaymiz. A operatorning chegaralangan teskarisi mavjudligini isbotlang.

6.54. $A : C^{(1)}[0, 1] \to C[0, 1]$, $(A\,x)(t) = \dfrac{dx(t)}{dt}$ chiziqli operatorga o'ng teskari operator mavjud, chap teskari operator mavjud emas. Isbotlang.

6.55. $A : C[0, 1] \to C[0, 1]$ operatorni

$$(A\,x)(t) = \int\limits_0^t x(s)\,ds$$

tenglik bilan aniqlaymiz. Uning qiymatlar sohasi $R(A)$ qanday shartlarni qanoatlantiruvchi funksiyalardan iborat? $R(A)$ da A ga teskari operator mavjudmi? Agar mavjud bo'lsa, u chegaralanganmi?

6.56. $A : C[0,\ 1] \to C[0,\ 1]$, $(A\,x)(t) = \int\limits_0^t x(s)\,ds + x(t)$ operatorni qaraymiz:

a) $Ker A = \{\theta\}$ tenglikni isbotlang.

b) A ga chegaralangan teskari operator mavjudligini isbotlang. A^{-1} ni toping.

6.57. $A : C[0,\ \pi] \to C[0,\ \pi]$, $(A\,x)(t) = x(t) + \int\limits_0^\pi \cos(t-s)\,x(s)\,ds$ operatorga teskari operator mavjudligini ko'rsating va uni toping.

6.58. $A : C[0,\ 1] \to C[0,\ 1]$ operatorni quyidagicha aniqlaymiz:

$$(A\,x)(t) = \frac{d^2x(t)}{dt^2} + x(t),\ D(A) = \{x \in C^{(2)}[0,\,1] :\ x(0) = x'(0) = 0\}.$$

a) A chegaralanmagan operator. Isbotlang.

b) A ga chegaralangan teskari operator mavjudligini isbotlang va uni toping.

6.59. H Hilbert fazo, $A \in L(H)$, $R(A) = H$ va A ga chegaralangan o‘ng teskari A_r^{-1} operator mavjud bo‘lsin. U holda A ga chegaralangan teskari operator mavjud. Isbotlang.

7-§. Qo‘shma operatorlar

Bu paragrafda bizning asosiy maqsadimiz Banax yoki Hilbert fazolarida aniqlangan operatorlarga qo‘shma operatorlarni topish va ularning asosiy xossalarini o‘rganishdir. Bundan tashqari biz musbat, normal, unitar, izometrik va giponormal operator tushunchalarini kiritamiz va ularga doir misollar qaraymiz.

X va $Y-$ chiziqli normalangan fazolar, $A : X \to Y$ chiziqli chegaralangan operator bo‘lsin. $f : X \to \mathbb{C}$ funksionalning x nuqtadagi qiymatini (f, x) deb belgilaymiz.

7.1-ta’rif. *Agar biror* $A^* : Y^* \to X^*$ *chiziqli chegaralangan operator va barcha* $x \in X,\ g \in Y^*$ *lar uchun*

$$(g,\, Ax) = (A^*g,\ x)$$

tenglik o‘rinli bo‘lsa, A^* *operator* A *ga qo‘shma operator deyiladi.*

Hilbert fazosida qo‘shma operator quyidagicha ta’riflanadi.

7.2-ta’rif. H *Hilbert fazosi va* $A \in L(H)$ *operator berilgan bo‘lsin. Agar biror* $A^* : H \to H$ *operator va ixtiyoriy* $x, y \in H$ *lar uchun*

$$(Ax, y) = (x, A^*y)$$

tenglik o'rinli bo'lsa, A^* operator A ga qo'shma operator deyiladi.

7.3-ta'rif. *Agar $A = A^*$ bo'lsa, $A \in L(H)$ o'z-o'ziga qo'shma operator deyiladi.*

7.4-ta'rif. *$A : H \to H$ chiziqli operator va $H_0 \subset H$ qism fazo berilgan bo'lsin. Agar ixtiyoriy $x \in H_0$ uchun $Ax \in H_0$ bo'lsa, u holda H_0 qism fazo A operatorga nisbatan invariant qism fazo deyiladi.*

Endi Hilbert fazosida chegaralanmagan $A : H \to H$ chiziqli operator berilgan va uning aniqlanish sohasining yopig'i $\overline{D(A)} = H$ bo'lsin. $y \in H$ shunday elementki, biror $y^* \in H$ va barcha $x \in D(A)$ lar uchun $(Ax, y) = (x, y^*)$ tenglik o'rinli bo'lsin. $D(A)$ ning H da zichligidan $y^* \in H$ element bir qiymatli aniqlanadi. Bu $y^* = A^*y$ moslikni o'rnatuvchi $A^* : H \to H$ operator A *ga qo'shma operator* deyiladi.

7.5-ta'rif. *Agar $A : H \to H$ va $A_1 : H \to H$ operatorlar uchun $D(A_1) \subset D(A)$ bo'lib $Ax = A_1x$, $x \in D(A_1)$ bo'lsa, u holda A operator A_1 operatorning davomi deyiladi, A_1 esa A operatorning $D(A_1)$ dagi qismi deyiladi, bu holat $A_1 \subset A$ shaklda yoziladi.*

A operatorning grafigi deb $Gr(A) = \{(x, Ax) : x \in D(A)\} \subset H \times H$ to'plamga aytiladi. Agar A operatorning grafigi $Gr(A)$ yopiq bo'lsa, A chiziqli operator *yopiq* deyiladi.

7.6-ta'rif. *Agar $A \subset A^*$ bo'lib, $\overline{D(A)} = H$ bo'lsa, u holda $A : H \to H$ chiziqli operator simmetrik deyiladi. Agar $A = A^*$ bo'lsa, u holda A chiziqli operator o'z-o'ziga qo'shma deyiladi.*

7.7-ta'rif. *O'z-o'ziga qo'shma A operator uchun barcha $x \in D(A)$ larda $(Ax, x) \geq 0$ bo'lsa, A ga musbat operator deyiladi va bu $A \geq 0$ shaklda yoziladi.*

O'z-o'ziga qo'shma A va B operatorlar uchun $A \geq B$ yozuvi $A - B \geq 0$ ekanligini anglatadi.

7.8-ta'rif. *Agar* $A \geq 0$ *operator uchun shunday* $B \geq 0$ *operator mavjud bo'lib,* $B^2 = A$ *bo'lsa,* B *operator* A *operatorning musbat kvadrat ildizi deyiladi va* $B = A^{\frac{1}{2}}$ *shaklda belgilanadi.*

Hilbert fazolarida aniqlangan o'z-o'ziga qo'shma operatorlarning muhim sinfi bo'lgan proyeksiyalash operatorlariga ta'rif beramiz. H Hilbert fazosi L uning biror qism fazosi bo'lsin. U holda har bir $x \in H$ element yagona usul bilan quyidagicha tasvirlanadi,

$$x = y + z, \quad \text{bu yerda} \quad y \in L, \ \ z \in L^{\perp}.$$

7.9-ta'rif. *Har bir* $x \in H$ *ga* $Px = y$ *ni mos qo'yib,* H *ning hamma yerida aniqlangan* P *operatorni hosil qilamiz. Uning qiymatlar sohasi* L *bo'ladi. Shuni ta'kidlash kerakki, agar* $x \in L$ *bo'lsa* $x = y$ *va* $z = 0$ *bo'ladi. Bu operator proyeksiyalash operatori yoki* L *ning ustiga ortogonal proyeksiyalash operatori deyiladi. Zarurat bo'lgan holda* P_L *ko'rinishida ham belgilanadi.*

7.10-ta'rif. *Agar ikkita* P_1 *va* P_2 *proyeksiyalash operatorlari uchun* $P_1 P_2 = 0$ *bo'lsa, ular o'zaro ortogonal deyiladi.*

$P_1 P_2 = 0$ shart $P_2 P_1 = 0$ shartga teng kuchli, chunki $(P_1 P_2)^* = P_2 P_1 = 0$ bo'ladi va teskarisi ham o'rinli.

7.11-ta'rif. *Agar* P_1 *va* P_2 *proyeksiyalash operatorlari uchun* $P_1 P_2 = P_2$ *bo'lsa,* P_2 *proyeksiyalash operatori* P_1 *proyeksiyalash operatorining qismi deyiladi.*

Bu ta'rifdan bevosita kelib chiqadiki, P_{L_2} proyeksiyalash operatori P_{L_1} proyeksiyalash operatorining qismi bo'lishi uchun L_2 qism fazo L_1 qism fazoning qismi bo'lishi zarur va yetarli.

7.12-ta'rif. *Agar* $AB = BA$ *bo'lsa,* A *va* B *operatorlar o'rin almashinuvchi operatorlar deyiladi.* $[A, B] = AB - BA$ *operatorga* A *va* B *operatorlarning kommutatori deyiladi.*

Demak, o'rin almashinuvchi A va B operatorlarning kommutatori nolga teng bo‘ladi va aksincha.

7.13-ta’rif. *Agar* $[A,\ A^*] = 0$ *bo‘lsa,* A *ga normal operator deyiladi.*

7.14-ta’rif. *Agar barcha* $x \in H$ *lar uchun* $\|A^*x\| \leq \|Ax\|$ *bo‘lsa,* A *ga giponormal operator deyiladi.*

7.15-ta’rif. *Agar* $A\,A^* = A^*A = I$ *bo‘lsa,* A *unitar, agar barcha* $x \in H$ *uchun* $\|Ax\| = \|x\|$ *bo‘lsa,* A *izometrik operator deyiladi. Agar* $A : H \to H$ $(H = L \oplus L^{\perp})$ *operator* L *da izometrik bo‘lib, barcha* $x \in L^{\perp}$ *lar uchun* $Ax = 0$ *bo‘lsa,* A *qisman izometrik operator deyiladi.*

Unitar va izometrik operator ta’riflarini, H_1 Hilbert fazosini H_2 Hilbert fazosiga akslantiruvchi $U : H_1 \to H_2$ operatorlar uchun ham keltirish mumkin.

7.16-ta’rif. *Agar* $U : H_1 \to H_2$ *biyektiv akslantirish bo‘lib, barcha* $x \in H$ *uchun* $\|Ux\| = \|x\|$ *bo‘lsa,* U *ga unitar operator deyiladi. Agar* $U : H_1 \to H_2$ *inyektiv akslantirish bo‘lib, barcha* $x \in H$ *uchun* $\|Ux\| = \|x\|$ *bo‘lsa,* U *ga izometrik operator deyiladi.*

7.17-ta’rif. *Agar shunday* C *teskarilanuvchan operator mavjud bo‘lib,* $A = C^{-1}BC$ *bo‘lsa,* A *va* B *operatorlar o‘xshash deyiladi.*

7.1. $T \in L(\ell_1)$ o‘ngga siljitish operatori, ya’ni

$$Tx = T\left(x_1, x_2, \ldots, x_n, \ldots\right) = \left(0, x_1, x_2, \ldots, x_n, \ldots\right), \quad x \in \ell_1$$

bo‘lsin. T ga qo‘shma T^* operatorni toping.

Yechish. Ma’lumki, $T \in L\left(X, Y\right)$ operatorning Banax qo‘shmasi hamma $x \in X$ va $f \in Y^*$ lar uchun

$$\left(T^* f\right)(x) = f\left(Tx\right) \tag{7.1}$$

tenglikni qanoatlantiruvchi va Y^* fazoni X^* fazoga akslantiruvchi T^* operatordan iborat. Bizga ma’lumki, $\ell_1^* \cong m$, boshqacha aytganda har qanday

$f \in \ell_1^*$ uchun shunday yagona $y \in m$ mavjudki,

$$f(x) = \sum_{k=1}^{\infty} x_k y_k, \quad y = (y_1, y_2, \dots, y_n, \dots) \in m, \tag{7.2}$$

tenglik ixtiyoriy $x \in \ell_1$ lar uchun o‘rinli bo‘ladi. Xuddi shuningdek, shunday $\xi \in m$ mavjudki,

$$j(T^* f)(x) = \sum_{k=1}^{\infty} x_k \xi_k, \quad \xi = (\xi_1, \xi_2, \dots, \xi_n, \dots) \in m, \tag{7.3}$$

tenglik ixtiyoriy $x \in \ell_1$ lar uchun bajariladi. (7.2) va (7.3) tengliklarni hisobga olsak, berilgan T operator uchun (7.1) shart quyidagi ko‘rinishga keladi:

$$j \sum_{k=1}^{\infty} x_k \xi_k = \sum_{k=2}^{\infty} x_{k-1} y_k = \sum_{k=1}^{\infty} x_k y_{k+1}. \tag{7.4}$$

Bu tenglik barcha $x \in \ell_1$ lar uchun bajariladi. Xususiy holda, $e_k \in \ell_1$, $k = 1, 2, 3, \dots$ elementlar uchun (7.4) tenglik $\xi_k = y_{k+1}$, $k = 1, 2, 3, \dots$ tengliklarga aylanadi. Shunday qilib, $T^* : m \to m$ operator

$$T^* y = T^*(y_1, y_2, \dots, y_n, \dots) = (y_2, y_3, \dots, y_{n+1}, \dots)$$

formula bilan aniqlanar ekan.

Ma’lumki, agar $T \in L(X, Y)$ bo‘lsa, $T^* \in L(Y^*, X^*)$ bo‘ladi va

$$\|T^*\| = \|T\| \tag{7.5}$$

tenglik o‘rinli. Qaralayotgan misolda bu tasdiqning bajarilishini tekshirib ko‘ramiz. T^* operatorning chiziqli ekanligi uning aniqlanishidan ko‘rinib turibdi. (7.5) tenglik ham bajariladi. Haqiqatan ham,

$$\|T\| = \sup_{\|x\| \le 1} \|Tx\| = \sup_{\|x\| \le 1} \left(\sum_{k=1}^{\infty} |x_k| \right) = 1,$$

$$\|T^*\| = \sup_{\|y\| \le 1} \|T^* y\| = \sup_{\|y\| \le 1} \sup_{2 \le k < \infty} |y_k| = 1. \qquad \square$$

7.2. $\ell_2 = \left\{ x = (x_1, x_2, \ldots, x_n, \ldots) : \sum\limits_{k=1}^{\infty} |x_k|^2 < \infty \right\}$ kompleks Hilbert fazosida $(Tx)_n = x_{n+1}$, ya'ni $Tx = (x_2, x_3, \ldots, x_{n+1}, \ldots)$ operator berilgan bo‘lsin. T^* operatorni toping.

Yechish. ℓ_2 fazo quyidagi $(x, y) = \sum\limits_{k=1}^{\infty} x_k \overline{y_k}$ skalyar ko‘paytmaga nisbatan Hilbert fazosi bo‘ladi. Shuning uchun 7.2-ta'rifdan foydalanamiz.

$$(Tx, y) = \sum_{k=1}^{\infty} (Tx)_k \, \overline{y_k} = \sum_{k=1}^{\infty} x_{k+1} \overline{y_k} =$$

$$= \sum_{k=2}^{\infty} x_k \overline{y_{k-1}} = (x, T^* y) = \sum_{k=1}^{\infty} x_k \overline{(T^* y)_k}.$$

Bu tenglik barcha $x \in \ell_2$ lar uchun o‘rinli. Bundan esa T^* operatorning

$$(T^* y)_1 = 0, \quad (T^* y)_k = y_{k-1}, \quad k = 2, 3, \ldots,$$

yoki

$$T^* y = T^* (y_1, y_2, \ldots, y_n, \ldots) = (0, y_1, y_2, \ldots, y_n, \ldots)$$

formula bilan aniqlanishini ko‘ramiz.

$$(Tx)_n = x_{n+1} = x_{n-1} = (T^* x)_n, \quad n = 1, 2, 3, \ldots$$

tenglik ℓ_2 fazoning faqat nol vektori uchun bajariladi. Shu sababli T operator o‘z-o‘ziga qo‘shma operator bo‘la olmaydi. □

7.3. $L_2[a,\ b]$ kompleks Hilbert fazosida, uzluksiz u funksiyaga ko‘paytirish operatori, ya'ni

$$(Tx)(t) = u(t)\, x(t), \quad x \in L_2[a,\ b]$$

operatorni qaraymiz. T ga qo‘shma operatorni toping.

Yechish. 7.2-ta'rifga ko‘ra $T \in L(H)$ operatorning Hilbert qo‘shmasi barcha $x, y \in H$ lar uchun

$$(Tx, y) = (x, T^* y) \tag{7.6}$$

tenglikni qanoatlantiruvchi $T^* \in L(H)$ operatordan iborat. $L_2[a,\ b]$ fazo

$$(x,y) = \int_a^b x(t)\overline{y(t)}\,dt \tag{7.7}$$

skalyar ko'paytmaga nisbatan Hilbert fazosi bo'ladi. Shunday ekan misolda berilgan T operator uchun (7.6) tenglik

$$\int_a^b u(t)\,x(t)\,\overline{y(t)}\,dt = \int_a^b x(t)\ \overline{(T^*y)\ (t)}\,dt$$

ko'rinishda bo'ladi. Bu tenglikni quyidagicha ham yozish mumkin:

$$\int_a^b x(t)\ \overline{\overline{u(t)}\,y(t)}\,dt = \int_a^b x(t)\ \overline{(T^*y)\ (t)}\,dt. \tag{7.8}$$

Agar $z(t) = \overline{u(t)}\,y(t)$ belgilashni kiritsak, (7.6) ga ko'ra (7.8) tenglik $(x,z) = (x, T^*y)$ yoki $(x,T^*y) - (x,z) = (x, T^*y - z) = 0$ ko'rinishga keladi. Oxirgi tenglik barcha $x \in L_2[a,b]$ lar uchun o'rinli bo'ladi. Xususiy holda, $x = T^*y - z$ elementni olsak, $(T^*y - z,\ T^*y - z) = 0$ tenglik hosil bo'ladi. Skalyar ko'paytmaning ta'rifiga asosan oxirgi tenglik o'rinli bo'lishi uchun $T^*y - z = 0$, ya'ni $T^*y = z$ bo'lishi kerak. Shunday qilib, qo'shma $T^* : L_2[a,b] \to L_2[a,b]$ operator

$$(T^*y)\ (t) = \overline{u(t)}\,y(t)\,, \quad y \in L_2[a,b]$$

formula yordamida aniqlanar ekan. Ma'lumki, $T^* = T$ bo'lsa, T operator o'z-o'ziga qo'shma operator deyiladi. Shuning uchun 7.21-misoldagi T operator u funksiya faqat haqiqiy qiymatlar qabul qilgandagina (ya'ni, $\overline{u(t)} \equiv u(t)$ bo'lganda) o'z-o'ziga qo'shma operator bo'ladi. □

7.4. $T : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})\,,\quad (Tf)(n) = v(n)f(n),\quad |v(n)| \le M,\quad \forall n \in \mathbb{N}$ operatorning o'z-o'ziga qo'shma bo'lish shartini toping.

Yechish. 7.3-misoldagidek ko'rsatish mumkinki, $\ell_2(\mathbb{Z})$ fazoda ham T ko'paytirish operatoriga qo'shma operator $(T^*f)(n) = \overline{v(n)}\,f(n)$ tenglik bilan aniqlanadi. Demak, $T = T^*$ bo'lishi uchun, barcha $n \in \mathbb{Z}$ larda $v(n) = \overline{v(n)}$ tenglikning bajarilishi zarur va yetarli. □

7.5. $(Tx)(t) = \int\limits_{-\pi}^{\pi} \exp\{\alpha\, ts + i\,\beta t^2 s^2\}\, x(s)ds, \quad x \in L_2[-\pi,\ \pi]$ operator uchun $\alpha,\ \beta \in \mathbb{C}$ parametrlarni shunday tanlangki, natijada $T \in L(H)$ o'z-o'ziga qo'shma operator bo'lsin.

Yechish. 7.23-misolning b) bandiga ko'ra $T = T^*$ bo'lishi uchun

$$K(t,s) = \exp\{\alpha\, ts + i\,\beta t^2 s^2\} = \overline{\exp\{\alpha\, st + i\,\beta s^2 t^2\}} =$$

$$= \exp\{\overline{\alpha}\, ts - i\overline{\beta}\, t^2 s^2\} = \overline{K(s,t)}$$

tenglikning bajarilishi zarur va yetarli. Bu tenglik $\alpha = \overline{\alpha}$ va $\beta = -\overline{\beta}$ tengliklarga teng kuchli. Bu yerdan α haqiqiy, β sof mavhum son ekanligi, ya'ni $Im\alpha = 0$ va $Re\beta = 0$ shartlar kelib chiqadi. □

7.6. Agar $A \geq 0$ bo'lsa, u holda barcha $n \in \mathbb{N}$ da $A^n \geq 0$ bo'ladi. Isbotlang.

Isbot. $A \geq 0$ ekanligidan barcha $x \in H$ larda $(Ax, x) = (x, Ax) \geq 0$ ekanligi ma'lim. Bu yerdan barcha $x \in H$ larda $(A^{2n+1}x, x) = (AA^n x, A^n x) \geq 0$ ekanligi kelib chiqadi, ya'ni A^{2n+1} musbat operator. $(A^{2n}x, x) = (A^n x, A^n x) = \|A^n x\|^2 \geq 0$. Demak, barcha $n \in \mathbb{N}$ larda $A^n \geq 0$ ekan. □

Uy vazifalari va mavzuni o'zlashtirish uchun masalalar

7.7-7.16-misollarda Banax fazosida berilgan $T \in L(X, Y)$ operatorga qo'shma T^* operatorni toping.

7.7. $T : \mathbb{C}^4 \to \mathbb{C}^3,\ \ Tx = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3 + \lambda_4 x_4)$.

7.8. $T : \mathbb{C}^4 \to \mathbb{C}^4,\ \ Tx = (0, x_1, 2x_2, (2+i)x_3)$.

7.9. $T : c_0 \to c_0,\ Tx = (\lambda_2 x_2, \lambda_3 x_3, \ldots, \lambda_n x_n, \ldots),\ |\lambda_n| \le 5,\ \forall n \in \mathbb{N}.$

7.10. $T : \ell_1 \to c_0,\ Tx = (0, \mu_1 x_1, \mu_2 x_2, \ldots, \mu_n x_n, \ldots),\ |\mu_n| \le 1,\ \forall n \in \mathbb{N}.$

7.11. $T : c_0 \to \ell_1, \quad Tx = (x_1, x_2, \ldots, x_n, 0, 0, \ldots).$

7.12. $T : \ell_1 \to \ell_1, \quad Tx = (\underbrace{0, \ldots, 0,}_{n-1} x_1, 0, 0, \ldots).$

7.13. $T : \ell_1 \to c_0,\ Tx = \left(e^i x_1, e^{2i} x_2, \ldots, e^{in} x_n, \ldots\right).$

7.14. $T : \ell_1 \to \ell_2,\ Tx = \left(0, \frac{1}{2} x_1, \frac{2}{3} x_2, \ldots, \frac{n-1}{n} x_{n-1}, \ldots\right).$

7.15. $T : m \to m, \quad Tx = (0, 0, x_1, 2\,x_2, \ldots, 50\,x_{50}, 0, 0, \ldots).$

7.16. $T : \ell_3 \to \ell_3, \quad Tx = (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n, \ldots),\ \ |\lambda_n| \le 2,\ \ n \in \mathbb{N}.$

7.17-7.22-misollarda kompleks Hilbert fazosida berilgan $T \in L(H)$ operatorga qo‘shma T^* operatorni toping.

7.17. $T : \ell_2 \to \ell_2,\ Tx = (2\,x_1, i\,x_2, (1+i)x_3, 0, 0, \ldots).$

7.18. $T : \ell_2 \to \ell_2, \quad Tx = (\lambda_1 x_1, \ldots, \lambda_n x_n, \ldots),\ \ \lambda = \{\lambda_n\} \in \ell_2.$

7.19. $T : \ell_2 \to \ell_2,\ Tx = (x_1 + x_3, x_2 + x_4, \ldots, x_n + x_{n+2}, \ldots).$

7.20. $T : \ell_2 \to \ell_2,\ Tx = (x_1 + 2x_2 + x_3, \ldots, x_n + 2x_{n+1} + x_{n+2}, \ldots).$

7.21. $T : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}), \quad (Tf)(n) = f(n-1) + f(n+1).$

7.22. $T : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}), \quad (Tf)(n) = f(n-1) - 2f(n) + f(n+1).$

7.23. $K(s,t)$ funksiya kvadrati $[a,\ b] \times [a,\ b]$ da integrallanuvchi bo‘lsin. Quyidagilarni isbotlang.

a) $T : L_2[a,\ b] \to L_2[a,\ b],\ (Tx)(t) = \int_a^b K(t,s) f(s) ds$ operatorga qo‘shma operator

$$T^* : L_2[a,\ b] \to L_2[a,\ b],\ \ (T^* x)(t) = \int_a^b \overline{K(s,t)}\, f(s) ds$$

formula bilan aniqlanadi.

b) $T = T^*$ bo'lishi uchun deyarli barcha $s, t \in [a, b]$ larda $K(t, s) = \overline{K(s, t)}$ tenglikning bajarilishi zarur va yetarli.

7.24. Kompleks Hilbert fazosida berilgan operatorlarning qo'shmasi quyidagi xossalarga ega. Isbotlang.

a) $(A + B)^* = A^* + B^*$. b) $(\alpha A)^* = \bar{\alpha} A^*$. c) $(AB)^* = B^* A^*$.

7.25-7.31-misollarda kompleks Hilbert fazosida berilgan $T \in L(H)$ operatorga qo'shma T^* operatorni toping.

7.25. $(Tx)(t) = \int_0^1 [t\,s + i\cos(t + s)]\, x(s)\, ds, \quad x \in L_2[0,\ 1].$

7.26. $(Tx)(t) = \int_0^1 \left(t^2 + t + s\right) x(s)\, ds, \quad x \in L_2[0,\ 1].$

7.27. $(Tx)(t) = \int_0^t s\, x(s)\, ds, \quad x \in L_2[0,\ 1].$

7.28. $(Tx)(t) = (\cos t + i\sin t)\, x(t) + \int_{-1}^{1} \left(ts - it^2 s^2\right) x(s)\, ds, \quad x \in L_2[-1,\ 1]$.

7.29. $(Tx)(t) = \left(t + it^2\right) x(t) + \int_0^1 (t + is)\, x(s)\, ds, \quad x \in L_2[0,\ 1]$.

7.30. $(Tx)(t) = x(t + h),\ h > 0, \quad x \in L_2(\mathbb{R})$.

7.31. $(Tx)(t) = u(t)\, x(t + h),\ h > 0,\ u \in M(\mathbb{R}),\ x \in L_2(\mathbb{R})$.

7.32-7.34-misollarda berilgan $T \in L(H)$ operatorning o'z-o'ziga qo'shma bo'lish shartini toping.

7.32. $T : \ell_2 \to \ell_2$, $Tx = (\mu_1 x_1, \ldots, \mu_n x_n, \ldots)$, $\mu_n \in \mathbb{C}$, $|\mu_n| \leq 1$, $\forall n \in \mathbb{N}$.

7.33. $T : L_2[a,\ b] \to L_2[a,\ b]$, $(Tx)(t) = \int_a^b K(t, s)\, x(s)\, ds,\ K \in L_2\left([a,\ b]^2\right)$.

7.34. $T : L_2[a,\ b] \to L_2[a,\ b]$, $(Tx)(t) = (u(t) + iv(t))x(t),\ u, v \in C[a,\ b]$.

Quyidagi 7.35-7.38-misollarda $\alpha,\ \beta \in \mathbb{C}$ parametrlarni shunday tanlangki, natijada $T \in L(H)$ o'z-o'ziga qo'shma operator bo'lsin.

7.35. $(Tx)\ (t) = \int\limits_{-\pi}^{\pi} [\alpha \sin(s-t) + \beta \cos(s-t)]\, x(s)\ ds, \quad x \in L_2[-\pi,\ \pi].$

7.36. $(Tx)(t) = [\alpha\, u(t) + i\beta\, v(t)]x(t), \quad x \in L_2[a,\, b], \ \ u, v \in C[a,\, b].$

7.37. $T = \alpha A + \beta A^*, \quad A \in L(H)\,.$

7.38. $T = A + \alpha\, \beta A^*, \quad A \in L(H)\,.$

7.39-7.55-misollarda keltirilgan tasdiqlarni isbotlang.

7.39. $T \in L(H)$ o'z-o'ziga qo'shma operator bo'lishi uchun bichiziqli $f(x, y) = (Tx, y)$ funksionalning simmetrik bo'lishi zarur va yetarli.

7.40. $T \in L(H)$ o'z-o'ziga qo'shma operator bo'lishi uchun $f(x, x) = \varphi(x) = (Tx, x)$ kvadratik formaning barcha $x \in H$ larda haqiqiy bo'lishi zarur va yetarli.

7.41. Agar A va B operatorlar o'xshash bo'lsa, u holda ixtiyoriy $n \in \mathbb{N}$ uchun A^n va B^n lar ham o'xshash bo'ladi.

7.42. Agar A va B operatorlar o'xshash bo'lsa, u holda A^* va B^* lar ham o'xshash bo'ladi.

7.43. Agar A va B operatorlardan hech bo'maganda biri teskarilanuvchan bo'lsa, u holda AB va BA lar o'xshash bo'ladi.

7.44. Ixtiyoriy $T \in L(H)$ uchun $W(T) = \{(Tx, x) : \ \ \|x\| = 1\}$ to'plam $\mathbb{C}$ da qavariq to'plam bo'ladi. $W(T)$ to'plam T *operatorning sonli aksi* deyiladi.

7.45. $w(T) = \sup\{|(Tx, x)| : \ \ \|x\| = 1\}$ son T *operatorning sonli radiusi* deyiladi. Quyidagi $\|T\| \leq 2\, w(T)$ tengsizlik o'rinli.

7.46. Agar $T \in L(H)$ o'z-o'ziga qo'shma operator bo'lsa, u holda $\|T\| = w(T)$ tenglik o'rinli.

7.47. O'z-o'ziga qo'shma A va B operatorlarning ko'paytmasi AB o'z-o'ziga qo'shma bo'lishi uchun $[A, B] = 0$ bo'lishi zarur va yetarli.

7.48. Agar o'z-o'ziga qo'shma A operatorga teskari operator mavjud bo'lsa, u holda A^{-1} ham o'z-o'ziga qo'shma bo'ladi.

7.49. Ixtiyoriy $T \in L(H)$ operator yagona usulda $T = A + i\,B$ ko'rinishda tasvirlanadi. Bu yerda A va B lar o'z-o'ziga qo'shma operatorlar.

7.50. Agar A o'z-o'ziga qo'shma operator bo'lsa, u holda $I + iA$ ga chegaralangan teskari operator mavjud.

7.51. Agar A o'z-o'ziga qo'shma bo'lsa, u holda barcha $n \in \mathbb{N}$ da $A^{2n} \geq 0$ bo'ladi.

7.52. Agar $[A, B] = 0$ va $A \geq 0,\ B \geq 0$ bo'lsa, u holda $AB \geq 0$ bo'ladi.

7.53. $u : H \to H$ izometrik operator uchun $u^*u = I,\ \ uu^* = I - P$ tengliklar o'rinli. Bu yerda P ortogonal proyeksiyalash operatori.

7.54. Ortogonal proyeksiyalash operatori $P : H \to H$ o'z-o'ziga qo'shma operator bo'ladi.

7.55. Ortogonal proyeksiyalash operatori $P : H \to H$ musbat operator bo'ladi.

7.56-7.63-misollarda berilgan $U \in L(H_1, H_2)$ yoki $F \in L(H_1, H_2)$ akslantirishning unitar operator ekanligini isbotlang.

7.56. Istiyoriy $s \in \mathbb{Z}$ uchun $U_s : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}), \quad (U_s f)(n) = f(n+s)$.

7.57. Barcha $s \in \mathbb{Z}^n$ uchun $U_s : \ell_2(\mathbb{Z}^n) \to \ell_2(\mathbb{Z}^n), \quad (U_s f)(n) = f(n+s)$.

7.58. Furye akslantirishi

$$F : \ell_2(\mathbb{Z}) \to L_2[-\pi,\ \pi], \quad (Ff)(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \exp\{inx\} f(n).$$

7.59. Furye akslantirishi

$$F : \ell_2(\mathbb{Z}^n) \to L_2\left([-\pi,\ \pi]^n\right), \quad (Ff)(x) = \frac{1}{(2\pi)^{n/2}} \sum_{n\in\mathbb{Z}^n} \exp\{i(n,x)\} f(n).$$

7.60. Furye akslantirishi

$$F : L_2(\mathbb{R}) \to L_2(\mathbb{R}), \quad (Ff)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\{ixy\} f(y) dy.$$

7.61. Furye akslantirishi

$$F : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n), \quad (Ff)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp\{i(x,y)\} f(y) dy.$$

7.62. $U : \ell_2 \to \ell_2, \quad Ux = \left(e^i x_1, e^{2i} x_2, \ldots, e^{ni} x_n, \ldots\right).$

7.63. Ixtiyoriy $A \in L(H),\ A = A^*$ uchun $U = e^{iA}$ unitar operator bo‘ladi.

7.64. $U : \ell_2 \to \ell_2, \quad Ux = \left(0, e^i x_1, e^{2i} x_2, \ldots, e^{ni} x_n, \ldots\right)$ izometrik, lekin unitar emasligini isbotlang.

7.65. $U : \ell_2 \to \ell_2, \quad Ux = (x_2, x_3, \ldots, x_{n+1}, \ldots)$ ning qisman izometrik operator ekanligini isbotlang.

7.66. Shunday A va B o‘z-o‘ziga qo‘shma operatorlarga misol keltiringki, $A \geq B$ va $A \leq B$ munosabatlarning hech biri bajarilmasin.

7.67. Shunday $0 \leq A \leq B$ operatorlarga misol keltiringki, $A^2 \leq B^2$ tengsizlik bajarilmasin.

7.68. Agar A va B lar o‘z-o‘ziga qo‘shma chegaralangan operatorlar bo‘lib, $[A, B] = 0$ bo‘lsa, $T = A + iB$ normal operator bo‘ladi va aksincha.

7.69. $U_s : \ell_2(\mathbb{Z}^n) \to \ell_2(\mathbb{Z}^n), \quad (U_s f)(n) = f(n+s)$ operatorlar oilasini ko‘paytirish amaliga nisbatan Abel gruppasi bo‘lishini isbotlang.

7.70-7.74-misollarda berilgan operatorlarning giponormal ekanligini isbotlang.

7.70. Ixtiyoriy unitar operator.

7.71. Ixtiyoriy izometrik operator.

7.72. Ixtiyoriy normal operator.

7.73. Ixtiyoriy ortogonal proyeksiyalash operatori.

7.74. Ixtiyoriy o‘z-o‘ziga qo‘shma operator.

7.75. Giponormal operatorga qo‘shma operator giponormal bo‘lmasligi mumkin. Misol keltiring.

7.76. $T : L_2[-\pi,\ \pi] \to L_2[-\pi,\ \pi], \quad (Tx)(t) = e^{it}x(t)$ operatorning sonli aksi $W(T) = \{(Tx, x) : \ \|x\| = 1\}$ to‘plamni va T operatorning sonli radiusi $w(T) = \sup\{|(Tx, x)| : \ \|x\| = 1\}$ sonlarni toping.

7.77. $T : L_2[-1,\ 1] \to L_2[-1,\ 1], \quad (Tx)(t) = t\,x(t)$ operatorning sonli aksi $W(T) = \{(Tx, x) : \ \|x\| = 1\}$ to‘plamni va T operatorning sonli radiusi $w(T) = \sup\{|(Tx, x)| : \ \|x\| = 1\}$ sonlarni toping.

7.78-7.86-misollarda keltirilgan tasdiqlarni isbotlang.

7.78. Agar T normal operator bo‘lsa, u holda T^* ham normal operator bo‘ladi.

7.79. $A : L(H) \to L(H),\ A(T) = T^*$ operator additiv va qo‘shma bir jinsli bo‘ladi.

7.80. Nolmas $L \subset H$ qism fazoning ustiga ortogonal proyeksiyalash operatori P o‘z-o‘ziga qo‘shma, normasi birga teng bo‘lgan va $P^2 = P$ shartni qanoatlantiruvchi operator bo‘ladi.

7.81. $P^2 = P$ shartni qanoatlantiruvchi o‘z-o‘ziga qo‘shma P chiziqli operator, biror $L \subset H$ qism fazoga ortogonal proyeksiyalash operatori bo‘ladi.

7.82. Faraz qilaylik, P_1 va P_2 lar mos ravishda L_1 va L_2 qism fazolarga ortogonal proyeksiyalash operatorlari bo‘lsin. P_1 va P_2 proyeksiyalash operatorlari o‘zaro ortogonal bo‘lishi uchun ularga mos L_1 va L_2 qism fazolar o‘zaro ortogonal bo‘lishi zarur va yetarli.

7.83. Ikkita P_{L_1} va P_{L_2} proyeksiyalash operatorlarining yig‘indisi proyeksiyalash operatori bo‘lishi uchun, bu operatorlar o‘zaro ortogonal bo‘lishi zarur va yetarli. Agar bu shart bajarilgan bo‘lsa, u holda $P_{L_1} + P_{L_2} = P_{L_1 \oplus L_2}$ tenglik o‘rinli.

7.84. Ikkita P_{L_1} va P_{L_2} proyeksiyalash operatorlarining ko‘paytmasi proyeksiyalash operatori bo‘lishi uchun, bu operatorlar o‘rin almashinuvchi (kommutativ) bo‘lishi zarur va yetarli. Agar bu shart bajarilgan bo‘lsa, u holda $P_{L_1} P_{L_2} = P_{L_2} P_{L_1} = P_{L_1 \cap L_2}$ tenglik o‘rinli.

7.85. Ikkita P_{L_1} va P_{L_2} proyeksiyalash operatorlarining ayirmasi proyeksiyalash operatori bo‘lishi uchun, P_{L_2} ning P_{L_1} uchun qism bo‘lishi zarur va yetarli. Agar bu shart bajarilgan bo‘lsa, u holda $P_{L_1} - P_{L_2} = P_{L_1 \ominus L_2}$ tenglik o‘rinli.

7.86. Ortogonal proyeksiyalash operatori $P : H \to H$ musbat operator bo‘ladi.

8-§. Chiziqli operator spektri

Chiziqli operator nazariyasida eng muhim tushunchalardan biri bu spektr tushunchasidir. Spektrning asosida esa xos qiymat tushunchasi yotadi. Spektr haqida masala qaralayotganda X fazoni o‘zini-o‘ziga akslantiruvchi chiziqli A operatorlar nazarda tutiladi. Shunday qilib $A : X \to X$ chiziqli operatorning xos qiymati ta’rifiga kelamiz.

8.1-ta’rif. *Agar biror $\lambda \in \mathbb{C}$ soni uchun $(A-\lambda I)x = 0$ tenglama nolmas $(x \neq 0)$ yechimga ega bo‘lsa, λ soni A operatorning xos qiymati deyiladi, nolmas yechim x esa xos vektor deyiladi.* $\dim Ker(A - \lambda I) = n$ *soni λ xos qiymatning karraligi deyiladi. Agar $n = 1$ bo‘lsa, λ soni A operatorning oddiy xos qiymati, $n \geq 2$ bo‘lsa, λ soni A operatorning karrali xos qiymati, $n = \infty$ bo‘lsa, λ soni A operatorning cheksiz karrali xos qiymati deyiladi.*

8.2-ta’rif. *Agar $\lambda \in \mathbb{C}$ kompleks soni uchun $A - \lambda I$ ga teskari operator mavjud bo‘lib, u X ning hamma yerida aniqlangan bo‘lsa, λ soni A operatorning regulyar nuqtasi deyiladi,*

$$R_\lambda(A) = (A - \lambda I)^{-1}$$

operator esa A operatorning λ nuqtadagi rezolventasi deyiladi. Barcha regulyar nuqtalar to‘plami $\rho(A)$ orqali belgilanadi.

8.3-ta’rif. *A operatorning regulyar bo‘lmagan nuqtalari uning spekrti deyiladi, ya’ni $\sigma(A) = \mathbb{C} \backslash \rho(A)$ to‘plamga A operatorning spektri deyiladi.*

Demak, $\lambda \in \mathbb{C}$ soni A operatorning spektriga qarashli bo‘lsa, u holda yo $A - \lambda I$ ga teskari operator mavjud emas, yo mavjud bo‘lganda ham, u butun X fazoda aniqlanmagan bo‘ladi. Agar $\lambda \in \mathbb{C}$ soni A operatorning xos qiymati bo‘lsa, $(A-\lambda I)x = 0$ tenglama nolmas yechimga esa, ikkinchidan bu bir jinsli tenglama nol yechimga ham ega. Demak, 6.3-teoremaga ko‘ra, $A-\lambda I$ ga teskari operator mavjud emas. Shunday qilib operatorning xos qiymatlari, uning spektriga qarashli bo‘ladi.

Chekli o‘lchamli fazolarda berilgan chiziqli operatorlarning spektri aniq tavsiflanadi. Unga qisqacha to‘xtalamiz. Faraz qilaylik, $A : \mathbb{C}^n \to \mathbb{C}^n$ chiziqli operator berilgan bo‘lsin. Ma’lumki, har bir $A : \mathbb{C}^n \to \mathbb{C}^n$ chiziqli operatorga $\{a_{ij}\} - n \times n$ matritsa mos keladi va aksincha. Bu $A - \lambda I$ matritsa determinanti $det(A - \lambda I)$, parametr λ ning $n-$darajali ko'phadi bo'ladi.

8.1-teorema. *$A : \mathbb{C}^n \to \mathbb{C}^n$ chiziqli operatorning spektri chekli sondagi chekli karrali xos qiymatlardan iborat. Bu xos qiymatlar $det(A - \lambda I)$ ko'phadning nollari bo'ladi va aksincha.*

Agar $A : X \to X$ chiziqli operator bo'lib, dim $X = \infty$ bo'lsa, uning spektri ixtiyoriy tabiatli yopiq to'plam bo'lishi mumkin. Odatda spektr quyidagi qismlarga ajratiladi.

8.4-ta'rif. *a) A operatorning barcha chekli karrali yakkalangan xos qiymatlari to'plami nuqtali spektr deyiladi va $\sigma_{pp}(A)$ bilan belgilanadi.*

b) Agar $\lambda \in \sigma(A)$ xos qiymat bo'lmasa va $\overline{Im(A - \lambda I)} \neq X$, ya'ni $A - \lambda I$ operatorning qiymatlar sohasi X ning hamma yerida zich emas. Bunday λ lar to'plami A operatorning qoldiq spektri deyiladi va $\sigma_{qol}(A)$ bilan belgilanadi.

c) Agar $\lambda \in \mathbb{C}$ xos qiymatlar to'plamining limitik nuqtasi bo'lsa, yoki $\lambda \in \mathbb{C}$ operatorning cheksiz karrali xos qiymati bo'lsa, yoki $\lambda \in \sigma(A)$ uchun $\overline{Im(A - \lambda I)} = X$ bo'lsa, bunday λ lar A operatorning muhim spektriga qarashli deyiladi. Operatorning muhim spektri $\sigma_{ess}(A)$ bilan belgilanadi.

Asosan o'z-o'ziga qo'shma operatorlarning spektri o'rganiladi. Endi o'z-o'ziga qo'shma operatorlar uchun muhim spektr ta'rifini keltiramiz.

8.5-ta'rif. *Agar biror $\lambda \in \mathbb{C}$ soni uchun nolga kuchsiz yaqinlashuvchi $\{f_n\} \subset H$ birlik vektorlar ketma-ketligi mavjud bo'lib,*

$$\lim_{n \to \infty} \|(A - \lambda I) f_n\| = 0$$

bo'lsa, λ soni $A = A^*$ operatorning muhim spektriga qarashli deyiladi.

Chegaralangan operatorlarning spektri haqida quyidagi tasdiq o'rinli.

8.2-teorema. *Agar $A \in L(X)$ bo'lsa, u holda $\sigma(A)$ chegaralangan yopiq to'plam bo'ladi.*

8.6-ta'rif. *Agar* $A \in L(H, H_1)$ *va* $ImA \subset H_1$ *qism fazo chekli o'lchamli bo'lsa, u holda* A *operator chekli o'lchamli operator deyiladi.*

8.7-ta'rif. *Agar* $A: H \to H_1$ *operator* H *dagi har qanday chegaralan-gan to'plamni* H_1 *dagi nisbiy kompakt to'plamga akslantirsa, u holda* A *kompakt yoki to'la uzluksiz operator deyiladi.*

8.8-ta'rif. *Agar biror* U *unitar operator uchun* $B = UAU^{-1} = UAU^*$ *tenglik o'rinli bo'lsa, u holda* B *operator* A *operatorga unitar ekvivalent deyiladi.*

8.9-ta'rif. *Agar* $P \in L(H)$ *uchun* $P^* = P$ *va* $P^2 = P$ *bo'lsa,* P *ga proyektor yoki proyeksiyalash operatori deyiladi.*

8.3-teorema. *Agar* $A \in L(X)$ *va* $|\lambda| > \|A\|$ *bo'lsa, u holda* λ *regulyar nuqta bo'ladi.*

Bu teoremadan chegaralan operatorning spektri markazi koordina-talar boshida, radiusi $r = \|A\|$ bo'lgan yopiq doirada saqlanishi kelib chiqadi.

8.4-teorema. $A \in L(H)$ *o'z-o'ziga qo'shma operator bo'lsin, u holda:*

a) $\sigma_{qol}(A)$ *– bo'sh to'plam,*

b) $\sigma(A)$ *to'plam* $\mathbb{R}$ *ning qismi, ya'ni* $\sigma(A) \subset \mathbb{R}$,

c) A *operatorning har xil xos qiymatlariga mos keluvchi xos vektorlari o'zaro ortogonaldir.*

O'z-o'ziga qo'shma $A \in L(H)$ operator uchun (Ax, x) barcha $x \in H$ larda haqiqiy (7.39-misol) bo'ladi. Quyidagi belgilashlarni kiritamiz:

$$\inf_{\|x\|=1} (Ax, x) = m, \qquad \sup_{\|x\|=1} (Ax, x) = M.$$

m soni o'z-o'ziga qo'shma $A \in L(H)$ operatorning *quyi chegarasi,* M esa uning *yuqori chegarasi* deyiladi.

8.5-teorema. *O'z-o'ziga qo'shma* $A \in L(H)$ *operatorning spektri* $\sigma(A) \subset [m, M]$ *bo'ladi.*

8.6-teorema (spektral teorema). *A H Hilbert fazosida aniqlangan chegaralangan, o'z-o'ziga qo'shma operator bo'lib, m uning quyi chegarasi, M esa uning yuqori chegarasi bo'lsin. U holda quyidagi shartlarni qanoatlantiruvchi proyektorlar oilasi mavjud:*

1) agar $\lambda < m$ bo'lsa, $E_\lambda = 0$ va $M < \lambda$ bo'lsa, $E_\lambda = I$;

2) ixtiyoriy $\lambda \in \mathbb{R}$ uchun $\lim\limits_{\lambda \to \lambda_0 + 0} E_\lambda = E_{\lambda_0}$;

3) ixtiyoriy $\lambda < \mu$ uchun $E_\lambda \leq E_\mu$;

4) $A = \int\limits_{-\infty}^{\infty} \lambda \, dE_\lambda \Longleftrightarrow \forall x, \, y \in H$ uchun $(Ax, y) = \int\limits_{-\infty}^{\infty} \lambda \, d(E_\lambda x, y)$ tenglik o'rinli. E_λ larga A operatorning spektral proyektorlari deyiladi.

Dastlab chekli o'lchamli fazolarda berilgan operatorlarning spektriga oid misollar qaraymiz.

8.1. $A : \mathbb{C}^3 \to \mathbb{C}^3, \;\; Ax = (x_1 + x_2, x_2 - x_1, 2x_3)$ operatorning xos qiymatlari va xos vektorlarini toping.

Yechish. 8.1-teoremaga ko'ra, bu operatorning spektri faqat xos qiymatlardan iborat. Shuning uchun xos qiymatga nisbatan tenglama, ya'ni

$$Ax = \lambda \, x \Longleftrightarrow (x_1 + x_2, x_2 - x_1, 2x_3) = (\lambda x_1, \lambda x_2, \lambda x_3) \qquad (8.1)$$

tenglamani qaraymiz. Agar (8.1) tenglama biror $\lambda \in \mathbb{C}$ da nolmas yechimga ega bo'lsa, $\lambda \in \mathbb{C}$ ning bu qiymati xos son bo'ladi. (8.1) tenglama quyidagi sistemaga teng kuchli

$$\begin{cases} x_1 + x_2 = \lambda x_1 \\ x_2 - x_1 = \lambda x_2 \\ 2x_3 = \lambda x_3 \end{cases} \Longleftrightarrow \begin{cases} (1-\lambda)x_1 + x_2 = 0 \\ -x_1 + (1-\lambda)x_2 = 0 \\ (2-\lambda)x_3 = 0. \end{cases} \qquad (8.2)$$

Bu sistema nolmas yechimga ega bo'lishi uchun, uning determinanti

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ -1 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)(2 - 2\lambda + \lambda^2)$$

nolga teng bo'lishi zarur va yetarli. Uning nollari $\lambda_1 = 2, \ \ \lambda_2 = 1+i, \ \lambda_3 = 1-i$ lardir. Endi bu xos qiymatlarga mos xos vektorlarni topamiz. Masalan, $\lambda = 2$ bo'lsin. λ ning bu qiymatida (8.2) sistema $x^{(1)} = (0,0,1)$ nolmas yechimga ega. Xuddi shunday $\lambda_2 = 1+i, \ \lambda_3 = 1-i$ xos sonlariga mos xos vektorlar $x^{(2)} = (1,i,0), \ x^{(3)} = (1,-i,0)$ bo'ladi. □

8.2. Hilbert fazosi $L_2[a, \ b]$ da

$$(Ax)(t) = u(t)\int_a^b u(s)\,x(s)ds, \ \ x\in L_2[a, \ b]$$

formula vositasida aniqlangan A operatorning xos qiymatlari va xos funksiyalarini toping. Bu yerda $u : [a, \ b] \to \mathbb{R}$ berilgan uzluksiz funksiya.

Yechish. Ta'rifga ko'ra, nol bo'lmagan biror $x \in L_2[a, \ b]$ funksiya va $\lambda \in \mathbb{C}$ soni uchun

$$(Ax)(t) = \lambda x(t) \tag{8.3}$$

tenglik bajarilsa, x funksiya A operatorning xos funksiyasi, λ son esa unga mos keluvchi xos qiymat deyiladi. Qaralayotgan operator uchun (8.3) tenglik quyidagi ko'rinishga ega bo'ladi:

$$u(t)\int_a^b u(s)\,x(s)ds = \lambda\,x(t). \tag{8.4}$$

Bu yerda $x \neq 0$. Faraz qilaylik, $\lambda \neq 0$ bo'lsin. U holda $x \neq 0$ bo'lgani uchun

$$\alpha_x = \int_a^b u(s)x(s)ds \neq 0 \tag{8.5}$$

tengsizlik bajarildi. (8.4) tenglikda (8.5) ni e'tiborga olsak,

$$x(t) = \lambda^{-1}\alpha_x\,u(t)$$

tenglikni olamiz. (8.5) tenglikka x funksiyaning bu ifodasini qo‘yib,

$$\alpha_x = \alpha_x \lambda^{-1} \int_a^b u^2(t)dt \quad \text{yoki} \quad \alpha_x \left(\lambda - \int_a^b u^2(t)\, dt \right) = 0$$

tenglikka kelamiz. Bunda $\alpha_x \neq 0$ bo‘lgani uchun $\lambda = \int_a^b u^2(t)dt$ son A operatorning xos qiymati va $x(t) = u(t)$ esa unga mos xos vektor ekanligi kelib chiqadi. Yana shuni ta’kidlash kerakki, agar

$$\int_a^b u(t)x(t)dt = 0 \tag{8.6}$$

shartni qanoatlantiruvchi nolmas x funksiya mavjud bo‘lsa, u holda $\lambda = 0$ soni uchun ham (8.3) tenglik bajariladi. Albatta (8.6) shartni qanoatlantiruvchi nolmas x funksiya mavjud. Demak, A operator ikkita $\lambda = 0$ va $\mu = \int_a^b u^2(t)dt$ xos qiymatlarga ega. (8.6) shartni qanoatlantiruvchi funksiyalar $\lambda = 0$ xos qiymatga mos keluvchi xos funksiyalar bo‘ladi. □

8.3. $L_2[a, b]$ Hilbert fazosida erkin o‘zgaruvchi x ga ko‘paytirish operatori, ya’ni

$$A : L_2[a, b] \to L_2[a, b], \quad (Af)(x) = xf(x)$$

operatorni qaraymiz. Uning nuqtali, qoldiq va muhim spektrini toping.

Yechish. 7.3-misol natijasi va $u(x) = x = \overline{x} = \overline{u(x)}$ tenglikka ko‘ra, berilgan operator o‘z-o‘ziga qo‘shma, ya’ni $A = A^*$ dir. 8.4-teoremaning a) tasdig‘iga ko‘ra, $\sigma_{qol}(A) = \emptyset$. Ma’lumki,

$$(Af)(x) = \lambda f(x) \quad \text{ya'ni} \quad (x - \lambda)f(x) = 0 \tag{8.7}$$

tenglama ixtiyoriy $\lambda \in \mathbb{C}$ uchun yagona nol yechimga ega. Demak, A operator xos qiymatlarga ega emas, ya’ni $\sigma_{pp}(A) = \emptyset$. (8.7) tenglama faqat nol yechimga ega ekanligidan 6.3-teoremaga ko‘ra, $(A - \lambda I)f(x) = g(x)$ tenglamaning ixtiyoriy $g \in Im\, A$ da yagona yechimga ega ekanligi kelib chiqadi. Koʻrsatish mumkinki $A - \lambda I$ operatorga teskari operator

$$(A - \lambda I)^{-1} g(x) = \frac{g(x)}{x - \lambda} \tag{8.8}$$

formula bilan aniqlanadi. Agar $\lambda \notin [a,\ b]$ boʻlsa, u holda $x - \lambda \neq 0$, natijada $(A - \lambda I)^{-1}$ operator $L_2[a,\ b]$ fazoning hamma yerida aniqlangan. 8.2-ta'rifga koʻra, $\lambda \notin [a,\ b]$ regulyar nuqta, ya'ni $\sigma(A) \subset [a,\ b]$. Lekin (8.8) formula bilan aniqlangan teskari operator $\lambda \in [a,\ b]$ boʻlganda $L_2[a,\ b]$ fazoning hamma yerida aniqlanmagan. Demak, $[a,\ b] \subset \sigma(A)$. Bulardan, $\sigma(A) = [a,\ b]$. Endi A operatorning spektridagi ixtiyoriy nuqta uning muhim spektriga qarashli ekanligini koʻrsatamiz. Ixtiyoriy $\lambda \in [a,\ b)$ uchun

$$f_n(x) = \sqrt{n(n+1)}\, \chi_{A_n}(x), \ \ A_n = \left[\lambda + \frac{1}{n+1},\ \lambda + \frac{1}{n}\right) \tag{8.9}$$

deymiz. Ma'lum nomerdan boshlab $\lambda + \dfrac{1}{n} < b$ boʻladi va bunday nomerlar uchun $\|f_n\| = 1$ tenglik oʻrinli. Bundan tashqari har xil n va m larda $A_n \cap A_m = \emptyset$ boʻlgani uchun $(f_n, f_m) = 0$ tenglik oʻrinli, ya'ni $\{f_n\}$ ortonormal sistema ekan. Ma'lumki, (3.75-misol) ixtiyoriy ortonormal sistema nolga kuchsiz ma'noda yaqinlashadi, shuning uchun $\{f_n\}$ ketma-ketlik ham nolga kuchsiz ma'noda yaqinlashadi. Endi $\|(A - \lambda I) f_n\|$ norma kvadratini hisoblaymiz:

$$\|(A - \lambda I) f_n\|^2 = n(n+1) \int\limits_{\lambda + \frac{1}{n+1}}^{\lambda + \frac{1}{n}} (t - \lambda)^2 dt = \frac{3n^2 + 3n + 1}{3\, n^2 (n+1)^2} \to 0, n \to \infty.$$

8.5-ta'rifga koʻra, $\lambda \in [a, b)$ son A operatorning muhim spektriga qarashli ekan. Agar $\lambda = b$ boʻlsa, u holda nolga kuchsiz yaqinlashuvchi

$$g_n(x) = \sqrt{n(n+1)}\, \chi_{B_n}(x), \ \ B_n = \left[b - \frac{1}{n},\ b - \frac{1}{n+1}\right), \ \ (g_n, g_m) = \delta_{nm}$$

ketma-ketlik uchun $\lim\limits_{n\to\infty} \|(A - bI) g_n\| = 0$ shart bajariladi. Bu yerdan, $\lambda = b$ nuqta ham A operatorning muhim spektriga qarashli ekanligi kelib chiqadi.

Shunday qilib, A operatorning spektri faqat muhim spektrdan iborat bo‘lib, u $[a,\ b]$ kesma bilan ustma-ust tushadi. Xulosa

$$\sigma_{qol}(A) = \sigma_{pp}(A) = \emptyset, \quad \sigma_{ess}(A) = \sigma(A) = [a,\ b]. \qquad \square$$

8.4. 8.3-misolda qaralgan A operatorni $C[a,\ b]$ Banax fazosida qaraymiz, ya’ni

$$A : C[a,\ b] \to C[a,\ b], \quad (Af)(x) = xf(x)$$

operatorni qaraymiz. Uning nuqtali va qoldiq spektrini toping.

Yechish. Ma’lumki, ((8.7) ga qarang)

$$(Af)(x) = \lambda f(x) \quad \text{ya'ni} \quad (x-\lambda)f(x) = 0, \quad f \in C[a,\ b] \qquad (8.10)$$

tenglama ixtiyoriy $\lambda \in \mathbb{C}$ uchun yagona nol yechimga ega. Demak, A operator xos qiymatlarga ega emas, ya’ni $\sigma_{pp}(A) = \emptyset$. (8.10) tenglama faqat nol yechimga ega ekanligidan 6.3-teoremaga ko‘ra, $A - \lambda I$ operatorga teskari operator mavjud va u (8.8) formula bilan aniqlanadi. Xuddi 8.3-misoldagi kabi ko‘rsatishimiz mumkinki, $\sigma(A) = [a,\ b]$ tenglik o‘rinli. Haqiqatan ham, agar $\lambda \notin [a,\ b]$ bo‘lsa, u holda (8.8) ning o‘ng tomoni ixtiyoriy $g \in C[a,\ b]$ da uzluksiz funksiya bo‘ladai, ya’ni $D((A-\lambda I)^{-1}) = C[a,\ b]$. 8.2-ta’rifga ko‘ra λ regulyar nuqta, ya’ni $\sigma(A) \subset [a,\ b]$. Agar $\lambda \in [a,\ b]$ bo‘lsa, u holda (8.8) formula bilan aniqlangan $(A-\lambda I)^{-1}$ operator $C[a,\ b]$ fazoning hamma yerida aniqlanmagan, bundan $[a,\ b] \subset \sigma(A)$. Bulardan, $\sigma(A) = [a,\ b]$ ekanligi kelib chiqadi. Endi $\sigma(A) = \sigma_{qol}(A)$ ekanligini ko‘rsatamiz. Ixtiyoriy $\lambda \in [a,\ b]$ uchun $A - \lambda I$ operatorning qiymatlar sohasi

$$Im(A-\lambda I) = \{g \in C[a,\ b] : g(x) = (x-\lambda)f(x)\}$$

$C[a,\ b]$ fazoda zich emas. Haqiqatan ham, $Im(A-\lambda I)$ chiziqli ko‘pxillilikdagi ixtiyoriy g uchun $g(\lambda) = 0$ shart bajariladi. Agar biz $f_0(x) \equiv 1$ desak, u holda ixtiyoriy $g \in Im(A - \lambda I)$ uchun

$$\|g - f_0\| = \max_{x \in [a,b]} |g(x) - f_0(x)| \ \geq \ |g(\lambda) - f_0(\lambda)| = 1$$

tengsizlik o'rinli. Demak, $A - \lambda I$ operatorning qiymatlar sohasi $Im(A - \lambda I)$ dan $f_0(x) \equiv 1$ elementga yaqinlashuvchi ketma-ketlik ajratish mumkin emas, ya'ni $\overline{Im(A - \lambda I)} \neq C[a, \ b]$. 8.4-ta'rifga ko'ra, barcha $\lambda \in [a, \ b]$ lar uchun $\lambda \in \sigma_{qol}(A)$ munosabat o'rinli. Bundan $\sigma(A) \subset \sigma_{qol}(A)$ kelib chiqadi. Teskari munosabat $\sigma(A) \supset \sigma_{qol}(A)$ doim o'rinli. Demak, $\sigma(A) = \sigma_{qol}(A) = [a, \ b]$.

8.3 va 8.4-misollarda bir xil qonuniyat bo'yicha ta'sir qiluvchi A operator har xil $L_2[a, \ b]$ va $C[a, \ b]$ fazolarda qaralgan. Har ikki holda ham A operatorning spektri $[a, \ b]$ kesma bilan ustma-ust tushgan, lekin spektrning qismlarida (strukturasida) o'zgarish bo'ladi (javoblarga qarang). Birinchi holda (8.3-misolda) $\sigma_{qol}(A) = \emptyset$ edi, ikkinchi holda $\sigma_{qol}(A) = [a, \ b]$. □

8.5. Kompleks Hilbert fazosi $L_2[0, \ 1]$ da aniqlangan

$$(Ax)(t) = t\,x(t) + \int_0^1 t\,s\,x(s)\,ds, \quad x \in L_2[0,1]$$

operatorning spektri va rezolventasini toping.

Yechish. A operatorni 8.3 va 8.2-misollarda spektri va xos qiymati o'rganilgan B va C operatorlarning yig'indisi ko'rinishida tasvirlash mumkin:

$$(Bx)(t) = t\,x(t), \quad (Cx)(t) = \int_0^1 t\,s\,x(s)\,ds, \quad x \in L_2[0,1].$$

7.3-misolda $L_2[a, \ b]$ fazoda ko'paytirish operatorining o'z-o'ziga qo'shmalik shartlari topilgan. $u(t) = t = \overline{t} = \overline{u(t)}$ tenglikdan B operatorning o'z-o'ziga qo'shma ekanligi kelib chiqadi. 7.2-misolda $L_2[a, \ b]$ fazoda integral operatorining o'z-o'ziga qo'shmalik shartlari keltirilgan. $K(t,s) = t \cdot s = \overline{s \cdot t} = \overline{K(s,t)}$ tenglikdan C operatorning o'z-o'ziga qo'shma ekanligi kelib chiqadi.

$A^* = (B+C)^* = B^* + C^* = B + C = A$ tenglik A operatorning oʻz-oʻziga qoʻshma ekanligini bildiradi. 8.4-teoremaning b) qismiga koʻra, A operatorning spektri $\mathbb{R}$ ning qism toʻplami boʻladi. Xususan, uning xos qiymatlari ham haqiqiy boʻladi. A operator spektrini ikki qismga ajratib topamiz: a) qismida uning xos qiymatlarini, b) qismida esa uning muhim spektrini topamiz.

a) A operatorning xos qiymatlarini topish uchun quyidagi tasdiqdan foydalanamiz.

8.1-tasdiq. $\lambda \in \mathbb{R} \setminus [0,\ 1]$ soni A operatorning xos qiymati boʻlishi uchun

$$\Delta(\lambda) := 1 + \int\limits_0^1 \frac{s^2}{s-\lambda} ds = 0$$

tenglikning bajarilishini zarur va yetarli.

Isbot. *Zaruriyligi.* Aytaylik, $\lambda \in \mathbb{R} \setminus [0,\ 1]$ soni A operatorlarning xos qiymati boʻlsin, ya'ni biror nolmas $x \in L_2[0,\ 1]$ element uchun

$$(Ax)(t) = \lambda x(t) \Longleftrightarrow tx(t) + \int\limits_0^1 tsx(s)ds = \lambda x(t)$$

tenglik bajarilsin. U holda

$$(t-\lambda)x(t) + t \cdot \alpha_x = 0, \tag{8.11}$$

bunda

$$\alpha_x = \int\limits_0^1 sx(s)ds. \tag{8.12}$$

Agar $\alpha_x = 0$ boʻlsa, (8.11) tenglik $(t-\lambda)x(t) = 0$ tenglikka aylanadi. Bundan x ning nol ekanligiga kelamiz. Farazimizga koʻra x xos vektor, ya'ni $x \neq 0$. Shunday ekan, $\alpha_x \neq 0$. Yuqoridagi (8.11) tenglikdan

$$x(t) = -\frac{\alpha_x \cdot t}{t-\lambda}$$

ni topamiz va buni (8.12) ga qo‘yib,

$$\alpha_x = -\alpha_x \int_0^1 \frac{s^2}{s-\lambda} ds \qquad \Longleftrightarrow \qquad \alpha_x \left(1 + \int_0^1 \frac{s^2}{s-\lambda} ds\right) = 0$$

tenglikka ega bo‘lamiz. $\alpha_x \neq 0$ bo‘lganligi uchun

$$1 + \int_0^1 \frac{s^2}{s-\lambda} ds = 0 \quad \text{ya'ni} \quad \Delta(\lambda) = 0$$

tenglikni hosil qilamiz.

Yetarliligi. Aytaylik, $\lambda \in \mathbb{R} \backslash [0,\ 1]$ soni uchun $\Delta(\lambda) = 0$ tenglik bajarilsin. U holda $x(t) = t\,(t-\lambda)^{-1}$ funksiyani olsak,

$$(A - \lambda I)x(t) = \frac{(t-\lambda)t}{t-\lambda} + t \int_0^1 \frac{s \cdot s}{s-\lambda} ds = t\left(1 + \int_0^1 \frac{s^2 ds}{s-\lambda}\right) = t\Delta(\lambda) = 0$$

tenglik bajariladi. Bundan λ soni A operator uchun xos qiymat bo‘lishi va $x(t) = t\,(t-\lambda)^{-1}$ unga mos xos funksiya bo‘lishi kelib chiqadi. □

A operatorning $[0,\ 1]$ kesmadan tashqaridagi xos qiymatlarini topamiz. Barcha manfiy λ lar uchun

$$\Delta(\lambda) = 1 + \int_0^1 \frac{s^2}{s-\lambda} ds \geq 1$$

tengsizlik o‘rinli. 8.1-tasdiqqa ko‘ra, A operatorning manfiy xos qiymatlari yo‘q. Endi A operatorning 1 dan katta xos qiymatlari mavjudmi degan savolga javob beramiz. Aytaylik, $\lambda > 1$ bo‘lsin. U holda

$$\Delta'(\lambda) = \int_0^1 \frac{s^2\ ds}{(s-\lambda)^2} > 0, \qquad \lim_{\lambda \to 1} \Delta(\lambda) = -\infty, \qquad \lim_{\lambda \to +\infty} \Delta(\lambda) = 1$$

munosabatlar o‘rinli bo‘ladi. Hosilaning musbatligidan $\Delta(\cdot)$ ning $(1, \infty)$ intervalda o‘suvchi ekanligi kelib chiqadi. Limitik munosabatlardan biror ε va N uchun $\Delta(1+\varepsilon)\,\Delta(N) < 0$ tengsizlik kelib chiqadi. Ya’ni $\Delta(\cdot)$ funksiya

$[1+\varepsilon,\ N]$ kesmaning chetki nuqtalarida har xil ishorali qiymatlar qabul qiladi. Bolsana-Koshi teoremasiga ko'ra, shunday $\lambda_0 \in (1+\varepsilon,\ N)$ nuqta mavjudki, $\Delta(\lambda_0) = 0$ bo'ladi. $\Delta(\cdot)$ funksiyaning qat'iy monotonligidan $\lambda_0 \in (1, \infty)$ nuqtaning yagonaligi kelib chiqadi. Shunday qilib, $[0,\ 1]$ kesmadan tashqarida A operatorning yagona xos qiymati bor ekan. Agar $\lambda \in \mathbb{C}\backslash[0,\ 1]$ soni A operatorning xos qiymati bo'lmasa, $A-\lambda I$ operator teskarilanuvchan bo'ladi. Rezolventa uchun tenglama

$$(A-\lambda I)x(t) = y(t) \quad \Longleftrightarrow \quad (t-\lambda)x(t) + t\int_0^1 sx(s)ds = y(t)$$

dan (8.12) ni e'tiborga olgan holda $x(t)$ ni topamiz:

$$(t-\lambda)x(t) + t\cdot\alpha_x = y(t),$$

bundan

$$x(t) = \frac{y(t)}{t-\lambda} - \alpha_x\frac{t}{t-\lambda}. \tag{8.13}$$

$x(t)$ uchun olingan (8.13) ifodani (8.12) ga qo'ysak, α_x uchun

$$\alpha_x = \int_0^1 \frac{sy(s)\,ds}{s-\lambda} - \alpha_x\int_0^1 \frac{s^2\,ds}{s-\lambda} \Longleftrightarrow \alpha_x\left(1+\int_0^1 \frac{s^2ds}{s-\lambda}\right) = \int_0^1 \frac{sy(s)\,ds}{s-\lambda}$$

tenglamaga ega bo'lamiz. Bu yerdan $\Delta(\lambda) \neq 0$ bo'lganligi uchun

$$\alpha_x = \frac{1}{\Delta(\lambda)}\int_0^1 \frac{sy(s)}{s-\lambda}\,ds$$

ni hosil qilamiz. Demak,

$$x(t) = \frac{y(t)}{t-\lambda} - \frac{t}{t-\lambda}\frac{1}{\Delta(\lambda)}\int_0^1 \frac{sy(s)}{s-\lambda}\,ds.$$

Shunday qilib, A operatorning rezolventasi quyidagi formula bilan aniqlanadi:

$$R_\lambda(A)y(t) = \frac{y(t)}{t-\lambda} - \frac{t}{t-\lambda}\frac{1}{\Delta(\lambda)}\int_0^1 \frac{sy(s)}{s-\lambda}\,ds\,, \qquad y \in L_2[0,\ 1].$$

Olingan natijalardan ko'rinib turibdiki, agar $\lambda \in \mathbb{C} \setminus ([0,\ 1] \cup \{\lambda_0\})$ bo'lsa, u holda $D(R_\lambda(A)) = L_2[0,\ 1]$, ya'ni λ regulyar nuqta. Bu yerdan $\rho(A) \supset \mathbb{C} \setminus ([0,\ 1] \cup \{\lambda_0\})$ ekanligi kelib chiqadi.

b) 8.3-misol xulosasidan kelib chiqadiki, $(Bx)(t) = tx(t),\ x \in L_2[0,\ 1]$ operatorning spektri faqat muhim spektrdan iborat bo'lib, u $[0,\ 1]$ kesma bilan ustma-ust tushadi. Hozir biz ko'rsatamizki, $A = B + C$ operatorning ham muhim spektri $[0,\ 1]$ kesma bilan ustma-ust tushadi. Aytaylik, $\lambda \in [0,\ 1)$ va $\{f_n\} -$ (8.9) tenglik bilan aniqlanuvchi nolga kuchsiz yaqinlashuvchi ketma-ketlik bo'lsin. Ko'rsatamizki,

$$\lim_{n\to\infty} \|(A - \lambda I)f_n\| = \lim_{n\to\infty} \|(B - \lambda I)f_n + Cf_n\| = 0 \qquad (8.14)$$

bo'ladi. $\|(A - \lambda I)f_n\| \le \|(B - \lambda I)f_n\| + \|Cf_n\|$ tengsizlikni inobatga olsak, faqat $\lim\limits_{n\to\infty} \|Cf_n\| = 0$ ekanligini ko'rsatish kifoya, chunki

$$\lim_{n\to\infty} \|(B - \lambda I)f_n\| = 0$$

shart 8.3-misolda ko'rsatilgan. Dastlab, $(Cf_n)(t)$ ni hisoblaymiz.

$$(Cf_n)(t) = \int_0^1 t\, s\, f_n(s) ds = \sqrt{n(n+1)}t \int_{\lambda+\frac{1}{n+1}}^{\lambda+\frac{1}{n}} s\, ds =$$

$$= \frac{\sqrt{n(n+1)}}{2n(n+1)} \left(2\lambda + \frac{2n+1}{n(n+1)}\right) t.$$

Agar $g(t) = t$ desak, $\|g\| = \dfrac{1}{\sqrt{3}}$ bo'lib,

$$\lim_{n\to\infty} \|Cf_n\| = \lim_{n\to\infty} \frac{\sqrt{n(n+1)}}{2n(n+1)} \left(2\lambda + \frac{2n+1}{n(n+1)}\right) \frac{1}{\sqrt{3}} = 0$$

bo'ladi. 8.4-ta'rifga ko'ra $[0,\ 1) \subset \sigma_{ess}(A)$ bo'ladi. Agar $\lambda = 1$ bo'lsa, nolga kuchzis yaqinlashuvchi ketma-ketlik sifatida $\{g_n\}$ ni olish mumkin:

$$g_n(x) = \sqrt{n(n+1)}\, \chi_{B_n}(x), \quad B_n = \left[\frac{n-1}{n},\ \frac{n}{n+1}\right), \quad (g_n, g_m) = \delta_{nm}.$$

Bu hol uchun ham $\lim\limits_{n\to\infty}\|(A-I)g_n\|=0$ shart bajariladi. Bu yerdan, $\lambda=1$ soni A operatorning muhim spektriga qarashli ekanligi kelib chiqadi. Demak,

$$\sigma_{ess}(A)=[0,\ 1],\quad \sigma_{pp}(A)=\{\lambda_0\},\quad \sigma(A)=[0,\ 1]\cup\{\lambda_0\}. \qquad \Box$$

8.6. Banax fazosi ℓ_1 da berilgan

$$A:\ell_1\to\ell_1,\ \ Ax=(x_1+x_2, x_1+2x_2, 4x_3, x_4, x_5, x_6,\ldots)$$

operatorning spektri va rezolventasini toping.

Yechish. a) A operatorning xos qiymatlarini topish uchun $\lambda\in\mathbb{C}$ songa mos $Ax=\lambda\,x$ yoki

$$(x_1+x_2, x_1+2x_2, 4x_3, x_4, x_5,\ldots)=(\lambda x_1, \lambda x_2, \lambda x_3,\ldots) \tag{8.15}$$

tenglamani yechamiz. (8.15) tenglama quyidagi tenglamalar sistemaga teng kuchli:

$$x_1+x_2=\lambda x_1,\ \ x_1+2x_2=\lambda x_2,\ \ 4x_3=\lambda x_3,\ \ \ x_n=\lambda x_n,\ \ n\geq 4. \tag{8.16}$$

Agar $\lambda\notin\{1,\ 4\}$ bo‘lsa, (8.16) tenglik bajarilishi uchun $x_3=x_4=x_5=\ldots=0$ bo‘lishi kerak. U holda (8.16) tenglamalar sistemasi

$$\begin{cases}(1-\lambda)x_1+x_2=0\\ x_1+(2-\lambda)x_2=0\end{cases}$$

sistemaga teng kuchli. Bu sistema nolmas yechimga ega bo‘lishi uchun uning determinanti $(1-\lambda)(2-\lambda)-1=0$ yoki

$$\lambda^2-3\lambda+1=0 \tag{8.17}$$

tenglik bajarilishi kerak. (8.17) tenglama $\lambda_1=2^{-1}(3-\sqrt{5}),\quad \lambda_2=2^{-1}(3+\sqrt{5})$ ildizlarga ega. Bu yerdan kelib chiqadiki, (8.15) tenglama $\lambda_1=2^{-1}(3-\sqrt{5})$ songa mos nolmas

$$x^{(1)}=\left(1,\frac{1-\sqrt{5}}{2},0,0,\ldots\right) \tag{8.18}$$

yechimga va $\lambda_2 = 2^{-1}(3+\sqrt{5})$ soniga mos nolmas

$$x^{(2)} = \left(1, \frac{1+\sqrt{5}}{2}, 0, 0, \dots\right) \tag{8.19}$$

yechimga ega. Shunday qilib, $\lambda_1 = 2^{-1}(3-\sqrt{5})$, $\lambda_2 = 2^{-1}(3+\sqrt{5})$ sonlari A operatorning xos qiymatlari bo‘ladi. (8.18) va (8.19) ko‘rinishdagi $x^{(1)}$ va $x^{(2)}$ elementlar A operatorning λ_1 va λ_2 xos qiymatlariga mos xos vektorlari bo‘ladi. Agar $\lambda_3 = 1$ bo‘lsa $x^{(3)} = e_n$, $n \geq 4$ nolmas elementlar $Ax = 1 \cdot x$ tenglikni qanoatlantiradi, ya’ni 1 soni A operatorning cheksiz karrali xos qiymati va $x^{(3)} = e_n$, $n \geq 4$ ko‘rinishdagi elementlar unga mos xos vektorlar bo‘ladi. Xuddi shunday ko‘rsatish mumkinki, $\lambda_4 = 4$ soni ham A operatorning xos qiymati bo‘ladi va $x^{(4)} = e_3$, esa unga mos xos vektor bo‘ladi. Shunday qilib, A operator $2^{-1}(3-\sqrt{5})$, $2^{-1}(3+\sqrt{5})$, 1, 4 xos qiymatlarga ega va uning boshqa xos qiymatlari yo‘q. $2^{-1}(3-\sqrt{5})$, $2^{-1}(3+\sqrt{5})$, 4 xos qiymatlar operatorning oddiy xos qiymatlari bo‘ladi.

b) Endi $\lambda \notin \left\{2^{-1}(3-\sqrt{5}),\ 2^{-1}(3+\sqrt{5}),\ 1,\ 4\right\}$ holni qaraymiz. 8.2-ta’rifga ko‘ra, A operatorning λ nuqtadagi rezolventasi $A - \lambda I$ operatorning teskarisi sifatida aniqlanadi. $(A - \lambda I)x = y$, ya’ni

$$((1-\lambda)x_1 + x_2, x_1 + (2-\lambda)x_2, (4-\lambda)x_3, (1-\lambda)x_4, \dots) =$$

$$= (y_1, y_2, y_3 \dots) \tag{8.20}$$

tenglikdan x ni topamiz. Buning uchun

$$\begin{cases} (1-\lambda)x_1 + x_2 = y_1, \\ x_1 + (2-\lambda)x_2 = y_2, \\ (4-\lambda)x_3 = y_3, \\ (1-\lambda)x_n = y_n, \ \ n \geq 4, \end{cases}$$

tenglamalar sistemasini yechib,

$$\begin{cases} x_1 = \dfrac{(2-\lambda)y_1 - y_2}{\lambda^2 - 3\lambda + 1}, \\ x_2 = \dfrac{-y_1 + (1-\lambda)y_2}{\lambda^2 - 3\lambda + 1}, \\ x_3 = (4-\lambda)^{-1} y_3, \\ x_n = (1-\lambda)^{-1} y_n, \quad n \geq 4 \end{cases}$$

munosabatlarni olamiz, ya'ni (8.20) tenglama yagona

$$x = \left(\frac{(2-\lambda)y_1 - y_2}{\lambda^2 - 3\lambda + 1}, \frac{-y_1 + (1-\lambda)y_2}{\lambda^2 - 3\lambda + 1}, \frac{y_3}{4-\lambda}, \frac{y_4}{1-\lambda}, \frac{y_5}{1-\lambda}, \ldots \right)$$

yechimga ega. Shunday qilib, A operatorning rezolventasi quyidagi formula bilan aniqlanadi:

$$R_\lambda(A)x = \left(\frac{(2-\lambda)x_1 - x_2}{\lambda^2 - 3\lambda + 1}, \frac{-x_1 + (1-\lambda)x_2}{\lambda^2 - 3\lambda + 1}, \frac{x_3}{4-\lambda}, \frac{x_4}{1-\lambda}, \frac{x_5}{1-\lambda}, \ldots \right).$$

Koʻrinib turibdiki, agar $\lambda \notin \left\{ 2^{-1}(3-\sqrt{5}),\ 2^{-1}(3+\sqrt{5}),\ 1,\ 4 \right\}$ boʻlsa, $D(R_\lambda(A)) = \ell_1$. 8.2-ta'rifga koʻra, barcha

$$\lambda \notin \left\{ \frac{3-\sqrt{5}}{2},\ \frac{3+\sqrt{5}}{2},\ 1,\ 4 \right\}$$

lar, A operator uchun regulyar qiymat boʻladi. Bundan

$$\sigma(A) = \left\{ \frac{3-\sqrt{5}}{2},\ \frac{3+\sqrt{5}}{2},\ 1,\ 4 \right\}$$

tenglik kelib chiqadi. □

8.7. $\hat{\Delta} : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$, $(\hat{\Delta}\hat{f})(n) = \hat{f}(n+1) + \hat{f}(n-1) - 2\hat{f}(n)$ operator *ayirmali Laplas operatori* deyiladi. $\Delta = F\hat{\Delta}F^{-1}$ operatorning $f \in L_2[-\pi,\ \pi]$ elementga ta'sirini toping.

Yechish. Ayirmali Laplas operatori $\hat{\Delta}$ uchun $\hat{\Delta} = \hat{U}_1 + \hat{U}_{-1} - 2\hat{U}_0$ tenglik oʻrinli. Demak, $\Delta = F\hat{\Delta}F^{-1} = U_1 + U_{-1} - 2\,U_0$. 8.89-misol natijasiga koʻra Δf, $f \in L_2[-\pi,\ \pi]$ uchun

$$(\Delta f)(p) = (U_1 + U_{-1} - 2\,U_0)f(p) = (e^{ip} + e^{-ip} - 2)f(p) = 2(\cos p - 1)f(p).$$

tenglikni olamiz. □

8.8. Aniqlanish sohasi $L_2[0,\ 1]$ Hilbert fazosi bo‘lgan o‘z-o‘ziga qo‘shma $(Af)(t) = tf(t)$ operatorning spekrtal proyektorlarini toping.

Yechish. Spektral proyektorlar sifatida $(E_\lambda f)(t) = \chi_{A_\lambda}(t)\, f(t)$ larni olamiz, bu yerda $A_\lambda = [0,\ \lambda] \cap [0,\ 1]$. Osongina tekshirish mumkinki, E_λ uchun spektral proyektorlarning 1-3 shartlari bajariladi. $A = \int\limits_{-\infty}^{\infty} \lambda d\, E_\lambda$ tenglikni tekshiramiz. Ixtiyoriy $f,\ g \in L_2[0,\ 1]$ elementlarni olaylik. U holda

$$(Af, g) = \int\limits_0^1 tf(t)\overline{g(t)}\, dt = \int\limits_0^1 \lambda\, f(\lambda)\, \overline{g(\lambda)} d\lambda = \int\limits_0^1 \lambda\, d \int\limits_0^\lambda f(t)\, \overline{g(t)} dt =$$

$$= \int\limits_0^1 \lambda d \left(\int\limits_0^1 \chi_{[0,\, \lambda]}(t) f(t)\, \overline{g(t)} dt \right) = \int\limits_{-\infty}^{\infty} \lambda\, d\, (E_\lambda f(t), g(t)).$$ □

Uy vazifalari va mavzuni o‘zlashtirish uchun masalalar

8.9-8.14-misollarda berilgan operatorlarning xos qiymatlari va xos vektorlarini toping. O‘z-o‘ziga qo‘shma bo‘lgan holda (8.9 va 8.12-misollarda) xos qiymatlarning haqiqiyligini va har xil xos qiymatlarga mos xos vektorlarni ortogonalligini tekshiring:

8.9. $A : \mathbb{C}^3 \to \mathbb{C}^3,\ \ Ax = (x_1 + x_2, x_2 + x_1, 3x_3)\,.$

8.10. $A : \mathbb{C}^3 \to \mathbb{C}^3,\ \ Ax = (x_1, x_2 + x_3, x_3 - x_2)\,.$

8.11. $A : \mathbb{C}^3 \to \mathbb{C}^3,\ \ Ax = (x_1 + x_2, x_2 + x_3, x_3 + x_1)\,.$

8.12. $A : \mathbb{C}^4 \to \mathbb{C}^4,\ \ Ax = (x_1, 2x_2, 3x_3, 4x_4)\,.$

8.13. $A : \mathbb{C}^4 \to \mathbb{C}^4,\ \ Ax = (x_1 + x_2, x_2 + x_3, x_3 + x_1, 5x_4)\,.$

8.14. $A : \mathbb{C}^4 \to \mathbb{C}^4,\ \ Ax = (x_1 + x_2, x_2 + x_3, x_3 + x_4, x_4 + x_1)\,.$

8.15-8.20-misollarda berilgan operatorlarning xos qiymatlari va xos funksiyalarini toping. 8.17-8.21-misollarda keltirilgan operatorlarning oʻz-oʻziga qoʻshma boʻlishini koʻrsating. Ular uchun 8.4-teoremaning b) va c) tasdiqlarining bajarilishi tekshiring.

8.15. $(Ax)(t) = t\int\limits_{-1}^{1} s\,x(s)ds, \;\; x \in C[-1,\ 1].$

8.16. $(Ax)(t) = \int\limits_{-1}^{1} (1+ts)\,x(s)ds, \;\; x \in C[-1,\ 1].$

8.17. $(Ax)(t) = \int\limits_{-\pi}^{\pi} (1+\sin t\, \sin s)\,x(s)ds, \;\; x \in L_2[-\pi,\ \pi].$

8.18. $(Ax)(t) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} \cos(t-s)\,x(s)ds, \;\; x \in L_2[-\pi,\ \pi].$

8.19. $(Ax)(t) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} \sin(t+s)\,x(s)ds, \;\; x \in L_2[-\pi,\ \pi].$

8.20. $(Ax)(t) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} (1+\cos(t-s))\; x(s)ds, \;\; x \in L_2[-\pi,\ \pi].$

8.21. $(Ax)(t) = x(t) + \frac{1}{\pi}\int\limits_{-\pi}^{\pi} (\cos(t-s) - \sin t\, \sin s)\; x(s)ds, \;\; x \in L_2[-\pi, \pi].$

8.22-8.28-misollarda berilgan operatorlarning spektri (nuqtali, qoldiq va muhim) va rezolventasini toping.

8.22. $(Ax)(t) = (t^2+1)x(t), \quad x \in C[-1,\ 1].$

8.23. $(Ax)(t) = t\,x(t) + x(0)\,t^2, \quad x \in C[-1,\ 1].$

8.24. $(Ax)(t) = \sin t\, x(t) + x(0)\,\cos t, \quad x \in C[-\pi,\ \pi].$

8.25. $(Ax)(t) = (t^2+1)\,x(t), \quad x \in L_2[0,\ \infty)\,.$

8.26. $(Ax)(t) = (1-\cos\, t)\,x(t), \quad x \in L_2[-\pi,\ \pi]\,.$

8.27. $(Ax)(t) = (1-4\cos\, t + 3\sin\, t)x(t), \quad x \in L_2[-\pi,\ \pi]\,.$

8.28. $(Ax)(t) = t^2\, x(t), \quad x \in L_2(\mathbb{R})$

8.29-8.33-misollarda keltirilgan operatorlarning o‘z-o‘ziga qo‘shma ekanligini ko‘rsating, xos qiymatlari mavjudligini tekshiring, muhim spektri va rezolventasini toping.

8.29. $(Ax)(t) = \cos t\, x(t) + \int\limits_{-\pi}^{\pi} \sin t \sin s\, x(s)\, ds, \quad x \in L_2[-\pi,\ \pi].$

8.30. $(Ax)(t) = \cos 2t\, x(t) - \int\limits_{-\pi}^{\pi} \sin t \sin s\, x(s)\, ds, \quad x \in L_2[-\pi,\ \pi].$

8.31. $(Ax)(t) = \cos 4t\, x(t) + \int\limits_{-\pi}^{\pi} \cos t \cos s\, x(s)\, ds, \quad x \in L_2[-\pi,\ \pi].$

8.32. $(Ax)(t) = x(t) - \dfrac{1}{\pi} \int\limits_{-\pi}^{\pi} (1 + \sin t \sin s)\, x(s)\, ds, \quad x \in L_2[-\pi,\ \pi].$

8.33. $(Ax)(t) = x(t) - \dfrac{1}{\pi} \int\limits_{-\pi}^{\pi} (1 + \cos(t - s))\, x(s)\, ds, \quad x \in L_2[-\pi,\ \pi].$

8.34-8.48-misollarda keltirilgan operatorlarning xos qiymatlari va xos vektorlarini toping.

8.34. $A : \mathbb{C}^4 \to \mathbb{C}^4,\ Ax = (x_1,\ x_2 + x_3,\ x_2 - x_3,\ x_4).$

8.35. $A : C[0,\ 2] \to C[0,\ 2],\ (Ax)(t) = x(0)t^2 + x(t)t\ + x(2)\,.$

8.36. $A : C[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = \int\limits_0^1 (t^2 s + ts^2) x(s)\, ds.$

8.37. $A : C[0,\ 2\pi] \to C[0,\ 2\pi],\ (Ax)(t) = \dfrac{1}{2\pi} \int\limits_0^{\pi} \cos(t + s) x(s)\, ds.$

8.38. $A : C[-1,\ 1] \to C[-1,\ 1],\ (Ax)(t) = \int\limits_{-1}^{1} (1 + ts) x(s)\, ds.$

8.39. $A : C[-1,\ 1] \to C[-1,\ 1]\,,\ (Ax)(t) = 2\, x(-1)\, t + 3\, x(1)\, t^2.$

8.40. $A : L_2[-1,\ 1] \to L_2[-1,\ 1],\ (Ax)(t) = \int\limits_{-1}^{1} (t + s + st) x(s)\, ds.$

8.41. $A : \ell_2 \to \ell_2,\ Ax = (x_1 + x_2, x_2 + x_3, x_3 + x_1, x_4, \ldots, x_n, \ldots)\,.$

8.42. $A:\ell_2\to\ell_2,\ \ Ax=(3x_1,4x_2,-2x_3,5x_4,x_5,x_6,\ldots)\,.$

8.43. $A:\ell_1\to\ell_1\,,\ \ Ax=(\frac{1}{2}\,x_1,\frac{2}{3}\,x_2,\ldots,\frac{n}{n+1}\,x_n,\ldots)\,.$

8.44. $A:\ell_1\to\ell_1\,,\ \ Ax=(\frac{1}{4}\,x_1,\frac{1}{2}\,x_2,\ldots,\frac{2n-1}{4n}\,x_{2n-1},\frac{1}{2n}\,x_{2n},\ldots)\,.$

8.45. $A:\ell_1\to\ell_1,\ \ Ax=(x_1+x_2,x_1-x_2,x_2-x_3,x_4,\ 0,\ 0,\ldots)\,.$

8.46. $A:\ell_3\to\ell_3,\ \ Ax=(x_1,2x_2,x_3,3x_4,x_5,x_6,\ldots)\,.$

8.47. $A:m\to m,\ \ Ax=(x_1,2x_2+x_3,x_2+3x_3,x_4,\ 0,\ 0,\ldots)\,.$

8.48. $A:m\to m,\ \ Ax=(6\,x_1,5\,x_2,4\,x_3,3\,x_4,2\,x_5,\ x_6,x_7,\ldots)\,.$

8.49-8.66-misollarda keltirilgan operatorlarning spektri va rezolventasini toping.

8.49. $A:C[1,\ 3]\to C[1,\ 3],\ \ (Ax)(t)=x(2)\,t+x(3)\,t^2.$

8.50. $A:C[0,\ 2]\to C[0,\ 2]\,,\ (Ax)(t)=x(1)+x(t)t\ +x(2)t\,.$

8.51. $A:C[0,\ 1]\to C[0,\ 1],\ \ (Ax)(t)=t^2\,x(t)+x(0).$

8.52. $A:C[0,\ 1]\to C[0,\ 1],\ \ (Ax)(t)=t\,x(t)+x(0)t^2+x(1)t^3.$

8.53. $A:C[-2,\ 2]\to C[-2,\ 2],\ \ (Ax)(t)=|t|\,x(t).$

8.54. $A:L_2[0,\ 1]\to L_2[0,\ 1],\ \ (Ax)(t)=(t+2)\,x(t).$

8.55. $A:L_2(-\infty,\infty)\to L_2(-\infty,\infty)\,,\ (Ax)(t)=\mathrm{arctg}t\cdot x(t)\,.$

8.56. $A:C[0,\ 1]\to C[0,\ 1],\ \ (Ax)(t)=t^2\,x(t)+t\int\limits_0^1 s\,x(s)\,ds.$

8.57. $A:L_2[0,\infty)\to L_2[0,\infty)\,,\ (Ax)(t)=e^{-t}x(t)+\int\limits_0^\infty 2^{-t-s}x(s)ds.$

8.58. $A:L_2[-1,\ 1]\to L_2[-1,\ 1],\ \ (Ax)(t)=t\,x(t)+\int\limits_{-1}^1 t\,s\,x(s)\,ds.$

8.59. $A:\ell_2\to\ell_2,\ \ Ax=(-x_1,x_2,-x_3,\ldots,-x_{2n-1},x_{2n},\ldots).$

8.60. $A:\ell_2\to\ell_2\,,\ \ Ax=\left(\dfrac{1}{3}\,x_1,\dfrac{1}{4}\,x_2,\ldots,\dfrac{1}{n+2}\,x_n,\ldots\right).$

8.61. $A:\ell_2\to\ell_2,\ \ Ax=(0,\,x_2,\dfrac{x_3}{2},\ldots,\dfrac{x_n}{n-1},\ldots).$

8.62. $A:\ell_1\to\ell_1\,,\ \ Ax=(x_1,\dfrac{1}{2}x_2,\ldots,\dfrac{1}{n}x_n,\ldots)\,.$

8.63. $A:\ell_1\to\ell_1,\ \ Ax=(0,x_1,x_2,\ldots,x_n,\ldots).$

8.64. $A:\ell_1\to\ell_1\,,\ \ Ax=(x_2,x_3,\ldots,x_n,\ldots).$

8.65. $A:m\to m\,,\ \ Ax=(x_1+x_2,x_2+x_1,x_3,x_4,x_5,\ldots).$

8.66. $A:m\to m,\ \ Ax=(2\,x_1,\dfrac{3}{2}\,x_2,\dfrac{4}{3}\,x_3,\ldots,\dfrac{n+1}{n}\,x_n,\ldots).$

8.67. O‘z-o‘ziga qo‘shma $A:\mathbb{C}^n\to\mathbb{C}^n$ chiziqli operatorning xos qiymatlari soni (karraligi bilan qo’shib hisoblanganda) n ga tengligini isbotlang.

8.68. Shunday $A:\mathbb{C}^3\to\mathbb{C}^3$ chiziqli operatorga misol keltiringki, uning yagona oddiy xos qiymati bo‘lsin.

8.69. Shunday $A:\mathbb{C}^3\to\mathbb{C}^3$ chiziqli operatorga misol keltiringki, uning bitta ikki karrali xos qiymati bo‘lsin.

8.70. Faraz qilaylik $A:X\to X$ chiziqli operator va A^{-1} mavjud bo‘lsin. A va A^{-1} operatorlar bir xil xos vektorlarga ega. Isbotlang.

8.71. Faraz qilaylik $A\in L(X)$ va A^2 ning xos vektori mavjud bo‘lsin, u holda A ham xos vektorga ega. Isbotlang.

8.72. A va $R_\lambda(A)$ lar o‘rin almashinuvchi operatorlardir. Isbotlang.

8.73. Faraz qilaylik $\lambda,\,\mu\in\rho(A)$ bo‘lsin, u holda Hilbert ayniyati

$$R_\lambda(A)-R_\mu(A)=(\lambda-\mu)R_\lambda(A)R_\mu(A)=(\lambda-\mu)R_\mu(A)R_\lambda(A)$$

ni isbotlang.

8.74. Faraz qilaylik $A,\ B \in L(X), \quad \lambda \in \rho(A) \cap \rho(B)$ bo‘lsin.

$$R_\lambda(A) - R_\lambda(B) = R_\lambda(A)(B - A)R_\lambda(B)$$

tenglikni isbotlang.

8.75. Faraz qilaylik $A \in L(X)$ bo‘lsin. Biror $\lambda \in \rho(A)$ uchun $R_\lambda(A)$ to‘la uzluksiz (kompakt) bo‘lishi mumkinmi?

8.76. $C[0,\ 2\pi]$ fazoni o‘zini-o‘ziga akslantiruvchi $(Ax)(t) = e^{it}\,x(t)$ operatorni qaraymiz. $\sigma(A) = \{\lambda \in C:\ |\lambda| = 1\}$ tenglikni isbotlang.

8.77. $\ell_2(\mathbb{Z})$ fazoni o‘zini-o‘ziga akslantiruvchi $(Ax)(n) = e^{in}\,x(n)$ operatorni unitar ekanligini ko‘rsating, uning spektri $\sigma(A)$ ni toping.

8.78. $\ell_2(\mathbb{Z})$ fazoda shunday unitar operatorga misol keltiringki, uning spektri $\sigma(A)$ ikki nuqtali to’plam bo‘lsin.

8.79. Istalgan unitar operatorning spektrini saqlovchi minimal to‘plamni toping.

8.80. Shunday o‘z-o‘ziga qo‘shma operatorga misol keltiringki uning spektri $\sigma(A) = [m,\ M]$ bo‘lsin. $m = \inf\limits_{\|x\|=1} (Ax, x),\ M = \sup\limits_{\|x\|=1} (Ax, x)$.

8.81. Shunday o‘z-o‘ziga qo‘shma operatorga misol keltiringki uning spektri $\sigma(A) = \{m,\ M\}$ bo‘lsin.

8.82. $C[0,\ 1]$ fazoni o‘zini-o‘ziga akslantiruvchi $(Ax)(t) = x(0) + t\,x(1)$ operatorni qaraymiz. $\sigma(A)$ va $R_\lambda(A)$ larni toping.

8.83. L to‘plam H Hilbert fazosining qism fazosi, P esa L ga ortogonal proyeksiyalash operatori bo‘lsin. P operatorning spektrini toping, $R_\lambda(P)$ ni P orqali ifodalang.

8.84. H Hilbert fazosi, $\{e_n\}$, $n \in \mathbb{N}$ undagi ixtiyoriy ortonormal bazis bo'lsin. $A : H \to H$ operatorni quyidagicha aniqlaymiz:

$Ae_1 = 0, \quad Ae_{k+1} = e_k, \quad k \in \mathbb{N}$. Quyidagilarni isbotlang.

a) A chiziqli chegaralangan operator;

b) $A^* e_k = e_{k+1}$ tenglik o'rinli;

c) $\sigma(A) = \{\lambda \in C : \ |\lambda| \leq 1\}$ tenglik o'rinli;

d) $\sigma(A) = \{\lambda \in C : \ |\lambda| < 1\}$ to'plamning ixtiyoriy nuqtasi A operatorning xos qiymati bo'ladi;

e) $\sigma(A^*) = \{\lambda \in C : \ |\lambda| \leq 1\}$ tenglik o'rinli;

f) A^* operator xos qiymatlarga ega emas, ya'ni $\sigma_{pp}(A^*) = \emptyset$.

8.85. $C[0,\ 1]$ fazoda differensiallash operatori $(Ax)(t) = \dfrac{dx(t)}{dt}$ ni qaraymiz. Quyidagilarni isbotlang.

a) agar $D(A) = \{x \in C^{(1)}[0,1] : \ x(0) = 0\}$ bo'lsa, $\sigma(A) = \emptyset$.

b) agar $D(A) = C^{(1)}[0,1]$ bo'lsa, u holda $\sigma(A) = \mathbb{C}$ tenglik o'rinli, hamda istalgan kompleks son A operatorning xos qiymati bo'ladi.

c) agar $D(A) = \{x \in C^{(1)}[0,\ 1] : \ x(0) = x(1)\}$ bo'lsa, u holda $\sigma(A)$ faqat $2\pi\, i\, n$, $n \in \mathbb{Z}$ ko'rinishdagi xos qiymatlardan iborat.

8.86. Faraz qilaylik $A, B \in L(X)$ bo'lsin. $\sigma(A \cdot B)$ va $\sigma(B \cdot A)$ to'plamlarning noldan farqli elementlari bir xil ekanligini isbotlang.

8.87. Faraz qilaylik, $A \in L(X)$ bo'lsin. $\lambda \in \mathbb{C}$ soni A operatorning spektriga qarashli bo'lishi uchun, shunday $x_n \in D(A)$, $n \in \mathbb{N}$, $\|x_n\| = 1$ ketma-ketlik mavjud bo'lib, $\|Ax_n - \lambda\, x_n\| \to 0$ munosabatning bajarilishi zarur va yetarli. Isbotlang.

8.88. Faraz qilaylik $A \in L(X)$, $\lambda \in \sigma(A)$ bo'lsin. Istalgan $n \in \mathbb{N}$ uchun $\lambda^n \in \sigma(A^n)$ ekanligini isbotlang.

8.89. Faraz qilaylik, $A \in L(X)$ uchun uzluksiz teskari operator mavjud bo'lsin. $\lambda \in \sigma(A^{-1})$ bo'lishi uchun $\lambda^{-1} \in \sigma(A)$ bo'lishi zarur va yetarli. Isbotlang.

8.90. Unitar ekvivalent $\hat{A}$ va $A = U\hat{A}U^{-1}$ operatorlar uchun quyidagilarni isbotlang.

a) $\sigma(A) = \sigma(\hat{A})$, b) $\sigma_{pp}(A) = \sigma_{pp}(\hat{A})$, c) $\sigma_{ess}(A) = \sigma_{ess}(\hat{A})$.

8.91. Har bir $s \in \mathbb{Z}$ uchun $\hat{U}_s : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$, $(\hat{U}_s \hat{f})(n) = \hat{f}(n+s)$ deymiz. $F : \ell_2(\mathbb{Z}) \to L_2[-\pi, \pi]$ esa 3.58-misolda aniqlangan Furye almashtirishi. $U_s = F\hat{U}_s F^{-1}$ uchun $(U_s f)(p) = e^{-isp} f(p)$, $f \in L_2[-\pi, \pi]$ tenglikni isbotlang.

8.92. $(\hat{\Delta}\hat{f})(n) = \sum\limits_{|s|=1} (\hat{f}(n+s) - \hat{f}(n))$, $\hat{f} \in \ell_2(\mathbb{Z}^\nu)$ operatorga unitar ekvivalent bo'lgan $\Delta = F\hat{\Delta}F^{-1}$ operatorning $f \in L_2([-\pi, \pi]^\nu)$ elementga ta'sirini toping.

8.93. $(\hat{V}\hat{f})(n) = \hat{v}(n)\hat{f}(n)$, $\hat{f} \in \ell_2(\mathbb{Z})$ operatorga unitar ekvivalent bo'lgan $V = F\hat{V}F^{-1}$ operatorning $f \in L_2[-\pi, \pi]$ elementga ta'sirini toping.

8.94. O'z-o'ziga qo'shma $A : H \to H$ operator uchun H_1 qism fazo invariant bo'lsin. U holda $H_2 = H \ominus H_1$ qism fazoning A uchun invariant ekanligini hamda $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$, $A_i = A|_{H_i}$, $i = 1, 2$, tenglikni isbotlang.

8.95. Agar H Hilbert fazosi uchun $H = H_1 \oplus H_2 \oplus \cdots \oplus H_n$ yoyilma o'rinli bo'lib, H_k, $k \in \{1, 2, \ldots, n\}$ qism fazolar o'z-o'ziga qo'shma A operator uchun invariant bo'lsa, $\sigma(A) = \bigcup\limits_{k=1}^{n} \sigma(A_k)$ tenglikni isbotlang.

8.96. $(Af)(p) = (2 - 2\cos p) f(p) - \dfrac{\mu}{2\pi} \int\limits_{-\pi}^{\pi} \cos(p - s) f(s)\, ds$, $f \in L_2[-\pi, \pi]$ operator uchun juft funksiyalardan iborat qism fazo $L_2^+[-\pi, \pi]$ va toq funksiyalardan iborat qism fazo $L_2^-[-\pi, \pi]$ ning invariant ekanligini ko‘rsating. $A^{+} = A|_{L_2^{+}[-\pi,\ \pi]}$ operatorning $f^{+} \in L_2^{+}[-\pi,\ \pi]$ elementga, $A^{-} = A|_{L_2^{-}[-\pi,\ \pi]}$ operatorning $f^{-} \in L_2^{-}[-\pi,\ \pi]$ elementga ta’sirini toping.

8.97. $(Vf)(p) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\sum\limits_{n=0}^{\infty} e^{-n}\cos n(p-s)f(s)\,ds,\quad f \in L_2[-\pi,\ \pi]$ operator uchun 8.95-misolda aniqlangan $L_2^{+}[-\pi,\ \pi]$ va $L_2^{-}[-\pi,\ \pi]$ qism fazolarning invariant ekanligini ko‘rsating. Quyidagilarni toping:

a) $V^{+} = V|_{L_2^{+}[-\pi,\ \pi]}$, $V^{-} = V|_{L_2^{-}[-\pi,\ \pi]}$ operatorlarning xos qiymatlari va xos funksiyalarini toping.

b) V operatorning barcha xos qiymatlari va xos funksiyalarini toping. Oddiy va karrali xos qiymatlarini ajrating.

8.98. $(Af)(t) = \cos t\, f(t),\ f \in L_2[-\pi,\ \pi]$ operatorga mos spektral proyektorlarini toping.

8.99. Agar φ funksiya $[-\pi,\ \pi]$ da o‘lchovli va chegaralangan bo‘lsa, u holda $(Af)(t) = \varphi(t)\, f(t),\ f \in L_2[-\pi,\ \pi]$ operatorga mos spektral proyektorlarni $(E_\lambda f)(t) = \chi_{\varphi^{-1}(-\infty,\lambda]}(t)\, f(t)$ shaklda tanlash mumkinligini isbotlang.

II bobni takrorlash uchun test savollari

1. $A : C[-1,\ 1] \to C[-1,\ 1],\ (Af)(x) = \int\limits_{-1}^{1} xyf(y)dy$ operator yadrosini toping.

A) $KerA = \{f : f(x) = const\}$ B) $KerA = \{f : f(x) = \alpha + \beta x\}$

C) $KerA = \{f : \int\limits_{-1}^{1} yf(y)dy = 0\}$ D) $KerA = \{0\}$

2. $A : C[-1,\ 1] \to C[-1,\ 1],\ (Af)(x) = \int\limits_{-1}^{1} (1 + xy)f(y)dy$ operatorning qiymatlar sohasini toping.

A) $ImA = \{f : f(x) = const\}$ B) $ImA = \{f : f(x) = \alpha + \beta x\}$

C) $ImA = \{f : \int\limits_{-1}^{1} yf(y)dy = 0\}$ D) $ImA = \{0\}$

3. $A : C[a,\ b] \to C[a,\ b],\ (Af)(x) = f'(x)$ differensial operator yadrosini toping.

A) $KerA = \{f : f(x) = const\}$ B) $KerA = \{f : f(x) = \alpha + \beta x\}$

C) $KerA = \{f : \int\limits_{a}^{b} f(x)dx = 0\}$ D) $KerA = \{0\}$

4. $A : C[a,\ b] \to C[a,\ b],\ (Af)(x) = f'(x)$ differensial operatorning aniqlanish sohasini toping.

A) $D(A) = C[a,\ b]$ B) $D(A) = \{f : f(x) = const\}$

C) $D(A) = C^{(1)}[a,\ b]$ D) $D(A) = \{f : \int_a^b f(x)dx = 0\}$

5. $A : C[0,\ 1] \to C[0,\ 1],\ (Af)(x) = (x + 1)f(x)$ operatorning kvadratini toping.

A) $(A^2 f)(x) = (x + 1)^2 f^2(x)$ B) $(A^2 f)(x) = (x + 1)^2 f(x)$

C) $(A^2 f)(x) = (x^2 + 1)f(x)$ D) $(A^2 f)(x) = (x + 1)^2 f(x^2)$

6. Qachon $\lambda \in \mathbb{C}$ soni A operator uchun regulyar nuqta deyiladi?

A) Shunday C operator mavjud boʻlib, $A = \lambda C$ boʻlsa.

B) Agar $(A - \lambda I)^{-1}$ operator mavjud va chegaralangan boʻlsa.

C) Agar $Ax = \lambda\, x$ tenglama nolmas yechimga ega bo‘lsa.

D) Agar $Ax = \lambda\, x$ tenglama yagona $x = 0$ yechimga ega bo‘lsa.

7. Qachon $\lambda \in \mathbb{C}$ soni A operatorning xos qiymati deyiladi?

A) Shunday C operator mavjud bo‘lib, $A = \lambda C$ bo‘lsa.

B) Agar $(A - \lambda I)^{-1}$ operator mavjud va chegaralangan bo‘lsa.

C) Agar $Ax = \lambda x$ tenglama noldan farqli yechimga ega bo‘lsa.

D) Agar $Ax = \lambda x$ tenglama yagona $x = 0$ yechimga ega bo‘lsa.

8. $A : X \to X$ operator spektri ta’rifini keltiring.

A) Barcha xos qiymatlar to‘plami operatorning spektri deyiladi.

B) Regulyar bo‘lmagan barcha $\lambda \in \mathbb{C}$ lar to‘plami operatorning spektri deyiladi.

C) Barcha regulyar nuqtalar to‘plami A operatorning spektri deyiladi.

D) Barcha $|\lambda| > \|A\|$ lar to‘plami A operatorning spektri deyiladi.

9. $\{A_n\} \subset L(X, Y)$ operatorlar ketma-ketligi va $A \in L(X, Y)$ operator berilgan bo‘lsin. Agar $\ldots$ bo‘lsa, $\{A_n\}$ ketma-ketlik A ga tekis yaqinlashadi deyiladi.

A) $\lim\limits_{n\to\infty} \|A_n - A\| = 0$

B) ixtiyoriy $x \in X$ uchun $\lim\limits_{n\to\infty} \|A_n x - Ax\| = 0$

C) ixtiyoriy $f \in Y^*$ va barcha $x \in X$ lar uchun $\lim\limits_{n\to\infty} f(A_n x) = f(Ax)$

D) ixtiyoriy $x, y \in H = X = Y$ uchun $\lim\limits_{n\to\infty} (A_n x, y) = (Ax, y)$

10. $\{A_n\} \subset L(X, Y)$ operatorlar ketma-ketligining $A \in L(X, Y)$ operatorga kuchli yaqinlashish ta’rifini toping.

A) agar $\lim\limits_{n\to\infty} \|A_n - A\| = 0$ bo‘lsa.

B) ixtiyoriy $x \in X$ uchun $\lim\limits_{n\to\infty} \|A_n x - Ax\| = 0$ bo‘lsa.

C) $\forall\, f \in Y^*$ va barcha $x \in X$ lar uchun $\lim\limits_{n\to\infty} f(A_n x) = f(Ax)$ bo‘lsa.

D) ixtiyoriy $x, y \in H = X = Y$ uchun $\lim\limits_{n\to\infty} (A_n x, y) = (Ax, y)$ bo‘lsa.

11. $\{A_n\} \subset L(X,Y)$ operatorlar ketma-ketligining $A \in L(X,Y)$ operatorga kuchsiz yaqinlashish ta'rifini toping.

A) agar $\lim\limits_{n\to\infty} \|A_n - A\| = 0$ bo'lsa.

B) ixtiyoriy $x \in X$ uchun $\lim\limits_{n\to\infty} \|A_n x - Ax\| = 0$ bo'lsa.

C) $\forall\, f \in Y^*$ va barcha $x \in X$ lar uchun $\lim\limits_{n\to\infty} f(A_n x) = f(Ax)$ bo'lsa.

D) ixtiyoriy $x, y \in H = X = Y$ uchun $\lim\limits_{n\to\infty} (A_n x, y) = |(Ax, y)|$ bo'lsa.

12. Nol operatorga kuchsiz ma'noda yaqinlashuvchi, lekin kuchli ma'noda yaqinlashmaydigan operatorlar ketma-ketligini ko'rsating.

A) $A_n : \ell_2 \to \ell_2,\ A_n x = (\underbrace{0, 0, \ldots, 0}_{n}, x_1, x_2, x_3, \ldots)$

B) $Q_n : \ell_2 \to \ell_2,\ Q_n x = (0, 0, \ldots, 0, x_{n+1}, x_{n+2}, x_{n+3}, \ldots)$

C) $A_n : L_2[-1,\ 1] \to L_2[-1,\ 1],\ (A_n f)(x) = \sin^n x f(x)$

D) $P_n : \ell_2 \to \ell_2,\ P_n x = (x_{n+1}, x_{n+2}, x_{n+3}, \ldots)$

13. Nol operatorga kuchli ma'noda yaqinlashuvchi, lekin tekis yaqinlashmaydigan operatorlar ketma-ketligini ko'rsating.

A) $A_n : \ell_2 \to \ell_2,\ A_n x = (\underbrace{0, 0, \ldots, 0}_{n}, x_1, x_2, x_3, \ldots)$

B) $Q_n : \ell_2 \to \ell_2,\ Q_n x = (0, 0, \ldots, 0, x_{n+1}, x_{n+2}, x_{n+3}, \ldots)$

C) $A_n : L_2[-1,\ 1] \to L_2[-1,\ 1],\ (A_n f)(x) = \sin^n x f(x)$

D) To'g'ri javob keltirilmagan.

14. Nol operatorga tekis yaqinlashuvchi operatorlar ketma-ketligini ko'rsating.

A) $A_n : \ell_2 \to \ell_2,\ A_n x = (\underbrace{0, \ldots, 0}_{n}, x_1, x_2, x_3, \ldots)$

B) $Q_n : \ell_2 \to \ell_2,\ Q_n x = (0, \ldots, 0, x_{n+1}, x_{n+2}, x_{n+3}, \ldots)$

C) $A_n : L_2[-1,\ 1] \to L_2[-1,\ 1],\ (A_n f)(x) = \sin^n x f(x)$

D) $P_n : \ell_2 \to \ell_2,\ P_n x = (x_{n+1}, x_{n+2}, x_{n+3}, \ldots)$

15. Noto'g'ri tasdiqni toping.

A) Agar A chiziqli operator bo‘lsa, A^{-1} ham chiziqli operator bo‘ladi.

B) Agar $A \in L(X,Y)$ bo‘lsa, u holda $A^{-1} \in L(Y,X)$ bo‘ladi.

C) Agar A chiziqli operator bo‘lsa, u holda A^* ham chiziqlidir.

D) Agar $A \in L(X,Y)$ bo‘lsa, u holda $A^* \in L(Y^*, X^*)$ bo‘ladi.

16. $A : X \to Y$ chiziqli operator teskarilanuvchan bo‘lishi uchun quyidagi shartlardan qaysi birining bajarilishi zarur va yetarli.

A) $\|A\| \le q < 1$ B) $Ax = 0 \Longleftrightarrow x = 0$ C) $\dim KerA = 1$

D) biror $m > 0$ va barcha $x \in D(A)$ larda $\|Ax\| \ge m \| x \|$ bo‘lishi

17. $A : X \to Y$ operatorga chegaralangan teskari operator mavjud bo‘lishining zarur va yetarli shartini keltiring.

A) $\|A\| \le q < 1$ B) $Ax = 0 \Longleftrightarrow x = 0$ C) $\dim KerA = 1$

D) biror $m > 0$ va barcha $x \in D(A)$ larda $\|Ax\| \ge m \| x \|$ bo‘lishi

18. $I - A : X \to X$ operatorga chegaralangan teskari operator mavjud bo‘lishining yetarli shartini keltiring.

A) $\|A\| < 1$ B) $Ax = 0 \Longleftrightarrow x = 0$ C) $\dim KerA = 1$

D) biror $m > 0$ va barcha $x \in D(A)$ larda $\|Ax\| \ge m \| x \|$ bo‘lishi

19. $A - A' : X \to X$ operatorga chegaralangan teskari operator mavjud bo‘lishining yetarli shartini keltiring.

A) $\|A\| < 1$ B) $\|A'\| < \left\|A^{-1}\right\|^{-1}$

C) $KerA = \{0\}$ D) $\|A'\| < \|A\|^{-1}$

20. $A : \mathbb{R}^3 \to \mathbb{R}^3$, $Ax = (x_1, 2x_2, 3x_3)$ operatorga teskari operatorni toping.

A) $A^{-1}x = (3x_3, 2x_2, x_1)$ B) $A^{-1}x = (x_1, 2^{-1}x_2, 3^{-1}x_3)$

C) $A^{-1}x = (x_1, 2^{-2}x_2, 3^{-2}x_3)$ D) $A^{-1}x = (x_1, 2x_2^{-1}, 3x_3^{-1})$

21. $A : X \to Y$ chiziqli operator teskarilanuvchan bo‘lishi uchun quyidagi shartlardan qaysi birining bajarilishi zarur va yetarli.

A) $\|A\| < 1$ B) $KerA = \{0\}$ C) $\dim KerA = 1$ D) $\|A\| \geq 1$

22. A operator chiziqli bo'lishini ta'minlaydigan shartlarni ajrating:

1) $A(x+y) = Ax + Ay$, 2) $A(\alpha x) = \alpha Ax$, 3) $A(\alpha x) = \overline{\alpha} Ax$.

A) 1, 2 B) 2, 3 C) 1, 2, 3 D) 1, 3

23. Chiziqli bo'lmagan $A : C[a,\ b] \to C[a,\ b]$ operatorni toping.

A) $(Af)(x) = f'(x)$ B) $(Af)(x) = f(x)$

C) $(Af)(x) = 0$ D) $(Af)(x) = f(x) + 1$

24. $C[a,\ b]$ ni $C[a,\ b]$ ga akslantiruvchi birlik operatorni toping.

A) $(Af)(x) = f'(x)$ B) $(Af)(x) = f(x)$

C) $(Af)(x) = 0$ D) $(Af)(x) = f(x) + 1$

25. $C[a,\ b]$ ni $C[a,\ b]$ ga akslantiruvchi nol operatorni toping.

A) $(Af)(x) = f'(x)$ B) $(Af)(x) = f(x)$

C) $(Af)(x) = 0$ D) $(Af)(x) = f(x) + 1$

26. $C[a,\ b]$ ni $C[a,\ b]$ ga akslantiruvchi chegaralanmagan operatorni toping.

A) $(Af)(x) = f'(x)$ B) $(Af)(x) = f(x)$

C) $(Af)(x) = 0$ D) $(Af)(x) = f(x) + 1$

27. Quyidagilar ichidan A chiziqli chegaralangan operator normasini hisoblash formulalarini ajrating:

1) $\|A\| = \sup\limits_{\|x\|=1} \|Ax\|$, 2) $\|A\| = \sup\limits_{x \neq 0} \dfrac{\|Ax\|}{\|x\|}$, 3) $\|A\| = \inf\limits_{\|x\|=1} \|Ax\|$.

A) 1, 2 B) 2, 3 C) 1, 2, 3 D) 1, 3

28. Quyidagilar ichidan to'g'ri tasdiqlarni ajrating:

1) Operatorlarni qo'shish kommutativ.

2) Operatorlarni ko'paytirish kommutativ.

3) Operatorlarni ko'paytirish assotsiativ.

A) 1, 2   B) 2, 3   C) 1, 2, 3   D) 1, 3

29. Quyidagilar ichidan to‘g‘rilarini ajrating:

1) $\|A+B\| \leq \|A\| + \|B\|$,   2) $\|A \cdot B\| \leq \|A\| \cdot \|B\|$,

3) $\|A \cdot B\| = \|A\| \cdot \|B\|$.

A) 1, 2   B) 2, 3   C) 1, 2, 3   D) 1, 3

30. $A : X \to Y$ - chiziqli operator. Teng kuchli tasdiqlarni ajrating:

1) A operator biror x_0 nuqtada uzluksiz.

2) A operator uzluksiz. 3) A operator chegaralangan.

A) 1, 2   B) 2, 3   C) 1, 2, 3   D) 1, 3

31. $C[-1,\ 1]$ fazoda normasi 1 bo‘lgan operatorlarni ko‘rsating.

1) $(Af)(x) = xf(x)$,   2) $(Bf)(x) = f(x)$,   3) $(Cf)(x) = 0$.

A) 1, 2   B) 2, 3   C) 1, 2, 3   D) 1, 3

32. $\mathbb{R}^n$ fazoga qo‘shma fazoni ko‘rsating.

A) $\mathbb{R}^n$ B) $\mathbb{R}^n_\infty$ C) $\mathbb{R}^n_q$ D) $\mathbb{R}^n_p$

33. $\mathbb{R}^n_p$, $p > 1$ fazoga qo‘shma fazoni ko‘rsating.

A) $\mathbb{R}^n$ B) $\mathbb{R}^n_\infty$ C) $\mathbb{R}^n_q$, $p^{-1} + q^{-1} = 1$ D) $\mathbb{R}^n_p$

34. $\mathbb{R}^n_1$ va $\mathbb{R}^n_\infty$ fazolarga qo‘shma fazolarni ko‘rsating.

A) $\mathbb{R}^n$ va $\mathbb{R}^n_1$ B) $\mathbb{R}^n_\infty$ va $\mathbb{R}^n_1$ C) $\mathbb{R}^n_1$ va $\mathbb{R}^n_\infty$ D) $\mathbb{R}^n_1$ va $\mathbb{R}^n$

35. $C[a,\ b]$ fazoga qo‘shma fazoni ko‘rsating.

A) $C[a,\ b]$ B) $V_0[a,\ b]$ C) $L_q[a,\ b]$ D) $L_2[a,\ b]$

36. $L_p[a,\ b]$, $p > 1$ fazoga qo‘shma fazoni ko‘rsating.

A) $C[a,\ b]$ B) $V_0[a,\ b]$ C) $L_q[a,\ b]$, $p^{-1} + q^{-1} = 1$ D) $L_2[a,\ b]$

37. $L_2[a,\ b]$ fazoga qo'shma fazoni ko'rsating.

A) $C[a,\ b]$ B) $V_0[a,\ b]$ C) $L_q[a,\ b],\, p^{-1}+q^{-1}=1$ D) $L_2[a,\ b]$

38. ℓ_2 va ℓ_1 fazolarga qo'shma fazolarni ko'rsating.

A) ℓ_2 va m B) ℓ_1 va ℓ_2 C) ℓ_1 va m D) ℓ_2 va c

39. c va c_0 fazolarga qo'shma fazolarni ko'rsating.

A) ℓ_2 va ℓ_1 B) ℓ_1 va ℓ_1 C) m va m D) ℓ_1 va c

40. $\ell_p, \quad p>1$ fazoga qo'shma fazoni ko'rsating.

A) ℓ_p B) ℓ_∞ C) $\ell_q,\ p^{-1}+q^{-1}=1$ D) ℓ_1

41. $T:\ell_1\to\ell_1,\ Tx=(0,x_1,x_2,x_3,\ldots,x_{n-1},\ldots)$ ga qo'shma operatorni toping.

A) $T^*:m\to m,\ T^*x=(x_2,x_3,x_4,\ldots,x_{n+1},\ldots)$

B) $T^*:\ell_2\to\ell_2,\ T^*x=(x_2,x_3,x_4,\ldots,x_{n+1},\ldots)$

C) $T^*:\ell_p\to\ell_q,\ T^*y=(y_2,y_3,y_4,\ldots,y_{n+1},\ldots)$

D) $T^*:\ell_2\to m,\ T^*y=(y_2,y_3,y_4,\ldots,y_{n+1},\ldots)$

42. $T:\ell_2\to\ell_2,\ Tx=(0,x_1,x_2,x_3,\ldots,x_{n-1},\ldots)$ ga qo'shma operatorni toping.

A) $T^*:m\to m,\ T^*x=(x_2,x_3,x_4,\ldots,x_{n+1},\ldots)$

B) $T^*:\ell_1\to\ell_1,\ T^*x=(x_2,x_3,x_4,\ldots,x_{n+1},\ldots)$

C) $T^*:\ell_p\to\ell_q,\ T^*y=(y_2,y_3,y_4,\ldots,y_{n+1},\ldots)$

D) $T^*:\ell_2\to\ell_2,\ T^*y=(y_2,y_3,y_4,\ldots,y_{n+1},\ldots)$

43. $T:\ell_2\to\ell_2,\ Tx=(a_1x_1,a_2x_2,a_3x_3,\ldots,a_nx_n,\ldots)$ operatorga qo'shma operatorni toping.

A) $T^*:m\to m,\ T^*x=(a_1x_1,a_2x_2,a_3x_3,\ldots,a_nx_n,\ldots)$

B) $T^*:\ell_2\to\ell_2,\ T^*x=(\overline{a_1}x_1,\overline{a_2}x_2,\overline{a_3}x_3,\dots,\overline{a_n}x_n,\dots)$

C) $T^*:\ell_2\to\ell_2,\ T^*x=(x_2,x_3,x_4,\dots,x_{n+1},\dots)$

D) $T^*:\ell_2\to\ell_2,\ T^*x=(\frac{1}{a_1}x_1,\frac{1}{a_2}x_2,\frac{1}{a_3}x_3,\dots,\frac{1}{a_n}x_n,\dots)$

44. $T : L_2[a,\ b]\to L_2[a,\ b],\ (Tf)(x)=u(x)f(x)$ operatorga Hilbert ma'nosidagi qo'shma operatorni toping.

A) $(T^*f)(x)=u(x)f(x)$ B) $(T^*f)(x)=\overline{u(x)}f(x)$

C) $(T^*f)(x)=\overline{u(x)f(x)}$ D) $(T^*f)(x)=\dfrac{f(x)}{u(x)}$

45. $T:L_2[a,\ b]\to L_2[a,\ b],\ (Tf)(x)=\int\limits_a^b K(x,y)f(y)dy$ operatorga Hilbert ma'nosidagi qo'shma operatorni toping.

A) $(T^*f)(x)=\int\limits_a^b \overline{K(y,x)}f(y)dy$ B) $(T^*f)(x)=\int\limits_a^b K(y,x)f(y)dy$

C) $(T^*f)(x)=\int\limits_a^b \overline{K(x,y)}f(y)dy$ D) $(T^*f)(x)=\int\limits_a^b K^*(y,x)f(y)dy$

46. $A:\mathbb{C}^n\to\mathbb{C}^n$ chiziqli operator spektri haqidagi tasdiqlarning qaysi biri to'g'ri.

A) $\sigma(A)$ faqat chekli sondagi chekli karrali xos qiymatlardan iborat.

B) A ning spektri biror kesmani to'la to'ldiradi.

C) A ning spektri $(-\infty;\infty)$ to'plamning qismi.

D) $\sigma(A)$ doim nolni saqlaydi.

47. $A:\ell_2\to\ell_2,\ Ax=(a_1x_1,a_2x_2,\dots,a_nx_n,\dots)$ operatorning spektrini toping.

A) $\sigma(A)=\{a_1,a_2,\dots,a_n,\dots\}$ B) $\sigma(A)=\overline{\{a_1,a_2,\dots,a_n,\dots\}}$

C) $\sigma(A)=\{\overline{a_1},\overline{a_2},\dots,\overline{a_n},\dots\}$ D) $\sigma(A)=\{1/a_1,\ 1/a_2,\dots,\ 1/a_n,\dots\}$

48. $A:\ell_2\to\ell_2,\ Ax=(a_1x_1,a_2x_2,\dots,a_nx_n,\dots)$ operatorning barcha xos qiymatlarini toping.

A) $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ B) $\overline{\{a_1, a_2, a_3, \ldots, a_n, \ldots\}}$

C) $\{\overline{a_1}, \overline{a_2}, \overline{a_3}, \ldots, \overline{a_n}, \ldots\}$ D) $\{1/a_1,\ 1/a_2,\ 1/a_3, \ldots,\ 1/a_n, \ldots\}$

49. $A : L_2[a,b] \to L_2[a,b], \quad (Af)(x) = xf(x)$ operatorning spektri haqida to'liq ma'lumotni toping.

A) $\sigma(A) = [a,\ b], \quad \sigma_{pp}(A) = \emptyset, \quad \sigma_{qol}(A) = \emptyset, \quad \sigma_{ess}(A) = [a,\ b]$

B) $\sigma(A) = [a,\ b], \quad \sigma_{pp}(A) = \emptyset, \quad \sigma_{qol}(A) = [a,\ b], \quad \sigma_{ess}(A) = \emptyset$

C) $\sigma(A) = [a,\ b], \quad \sigma_{pp}(A) = [a,\ b], \quad \sigma_{qol}(A) = \emptyset, \quad \sigma_{ess}(A) = \emptyset$

D) $\sigma(A) = [a,\ b], \quad \sigma_{pp}(A) = [a,\ b], \ \sigma_{qol}(A) = \emptyset, \ \sigma_{ess}(A) = [a,\ b]$

50. $A : L_2[0,\ 1] \to L_2[0,\ 1], \quad (Af)(x) = xf(x)$ operatorning $\lambda \in C \backslash [0,\ 1]$ nuqtadagi rezolventasini toping.

A) $R_\lambda(A)f(x) = (x - \lambda)f(x)$ B) $R_\lambda(A)f(x) = (x - \lambda)^{-1}f(x)$

C) $R_\lambda(A)f(x) = (x - \overline{\lambda})^{-1}f(x)$ D) $R_\lambda(A)f(x) = |x - \lambda|^{-1}f(x)$

II bob uchun javoblar va ko'rsatmalar

5-§. Chiziqli uzluksiz operatorlar

9. Bu operator chiziqli, lekin uzluksiz emas.

Agar $A : C^{(1)}(G) \to C(G)$ sifatida qaralsa, u uzluksiz bo'ladi.

10. Bu operator chiziqli ham uzluksiz.

11. Bu operator chiziqli ham uzluksiz.

12. Bu operator chiziqli emas.

13. $D(A) = C^{(1)}(G, \mathbb{R}^3)$. Bu operator chiziqli, lekin uzluksiz emas.

14. Bu operator chiziqli, lekin uzluksiz emas.

15. Berilgan operatorning aniqlanish sohasi $D(A) = C^{(2)}[0,\ 1]$. Operator chiziqli, chunki ixtiyoriy $x, y \in C^{(2)}[0,\ 1]$ va α, β sonlar uchun

$$[A(\alpha x + \beta y)]\ (t) = (\alpha x + \beta y)''(t) = \alpha x''(t) + \beta y''(t) =$$

$$= \alpha\ (Ax)\ (t) + \beta\ (Ay)\ (t) = (\alpha Ax + \beta Ay)\ (t)$$

tengliklar o'rinli. Endi operatorning chegaralanganligini ko'rsatamiz:

$$\|Ax\| = \max_{0 \le t \le 1} |(Ax)(t)| = \max_{0 \le t \le 1} |x''(t)| \le \|x\|,$$

ya'ni ixtiyoriy $x \in C^{(2)}[0,\ 1]$ uchun

$$\|Ax\| \le \|x\|. \tag{5.1j}$$

(5.1j) tengsizlikdan $\|A\| \le 1$ ni olamiz. A operator normasini 5.1-teorema yordamida topamiz. Quyidagi funksiyalar ketma-ketligini qaraymiz:

$$x_n(t) = e^{-nt} \in D(A) = C^{(2)}[0,\ 1].$$

U holda $\|x_n\|$ va $\|Ax_n\|$ lar uchun

$$\|x_n\| = \max_{0 \le t \le 1} \left| e^{-nt} \right| + \max_{0 \le t \le 1} \left| -ne^{-nt} \right| + \max_{0 \le t \le 1} \left| n^2 e^{-nt} \right| = 1 + n + n^2,$$

$$\|Ax_n\| = \max_{0\le t\le 1} \left| n^2 e^{-nt} \right| = n^2$$

tengliklar o‘rinli bo‘ladi. (5.2) tenlikka ko‘ra,

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} \ge \sup_n \frac{\|Ax_n\|}{\|x_n\|} = \sup_n \frac{n^2}{1+n+n^2} = 1.$$

Olingan $\|A\| \le 1$ va $\|A\| \ge 1$ tengsizliklardan $\|A\| = 1$ ekanligi kelib chiqadi.

16. $\|A\| = 2e^2$.

17. $\|A\| = \dfrac{4}{\sqrt{15}}$.

18. *Chiziqliligi.* Ixtiyoriy $x \in L_2[-1, 1]$ uchun $Ax \in L_2[-1, 1]$ ekanligidan $D(A) = L_2[-1, 1]$ ekanligi kelib chiqadi.

$$(A(\alpha x + \beta y))(t) = (t^2 - t)\,(\alpha x + \beta y)(t) = (t^2 - t)\,(\alpha x(t) + \beta y(t)) =$$

$$= \alpha(t^2 - t)\,x(t) + \beta(t^2 - t)\,y(t) = \alpha(A\,x)(t) + \beta(A\,y)(t)$$

tenglikdan esa berilgan operatorning chiziqli ekanligi kelib chiqadi.

Chegaralanganligi.

$$\|Ax\|^2 = \int_{-1}^{1} |(t^2 - t)\,x(t)|^2\,dt \le \max_{-1\le t\le 1} |t^2 - t|^2 \int_{-1}^{1} |\,x(t)|^2\,dt = 4\,\|x\|^2.$$

Bu yerdan berilgan operatorning chegaralanganligi va $\|A\| \le 2$ ekanligi kelib chiqadi.

Normasi. Berilgan operatorning normasini topish uchun quyidagicha yo‘l tutamiz. $B_n = [-1, -1 + \frac{1}{n}]$ va $x_n(t) = \sqrt{n}\,\chi_{B_n}(t)$ deymiz. U holda

$$\|x_n\|^2 = \int_{-1}^{1} |x_n(t)|^2\,dt = \int_{-1}^{-1+\frac{1}{n}} n\,dt = 1 \iff \|x_n\| = 1.$$

Xuddi shunday $\|Ax_n\|^2$ ni hisoblaymiz:

$$\|Ax_n\|^2 = \int_{-1}^{1} |(t^2 - t)\,x_n(t)|^2\,dt = \int_{B_n} |\sqrt{n}(t^2 - t)|^2 dt = n \int_{-1}^{-1+\frac{1}{n}} (t^4 - 2t^3 + t^2)dt.$$

Bu jadval integrali bo'lib, uning qiymati

$$4 - \frac{6}{n} + \frac{7}{3n^2} - \frac{3}{2n^3} + \frac{1}{5n^4} = \|Ax_n\|^2$$

dir. Endi

$$\|A\| = \sup_{\|x\|=1} \|Ax\| \geq \sup_{n\geq 1} \|Ax_n\| = \sup_{n\geq 1} \sqrt{4 - \frac{6}{n} + \frac{7}{3n^2} - \frac{3}{2n^3} + \frac{1}{5n^4}} = 2$$

munosabatdan $\|A\| \geq 2$ ni olamiz. Yuqorida $\|A\| \leq 2$ tengsizlik ko'rsatilgan edi. Bulardan $\|A\| = 2$ kelib chiqadi.

19. $\|A\| = \sqrt{3}$.

17. $\|A\| = \dfrac{1}{\sqrt{3}}$. **18**. $\|A\| = 2$. **19.** $\|A\| = \sqrt{3}$. **20.** $\|A\| = \sqrt[5]{2}$.

21. $\|A\| = 1$. **22**. $\|A\| = 1$. **23.** $\|A\| = 2$. **24.** $\|A\| = 1$.

25. $\|A\| = 1$. **26.** $\|A\| = 2$. **27.** $\|A\| = 1$. **28.** $\|A\| = e$.

29. $\|A\| = 2$. **30.** $\|A\| = \dfrac{1}{5}$. **31.** $\|A\| = 1$. **32.** $\|A\| = 1$.

36. Bu operatorning aniqlanish sohasi $D(A) = \ell_1$ fazodan iborat. Quyidagi ketma-ketlikni qaraymiz: $x_n = (1, 2, \ldots, n, 0, 0, \ldots) \in D(A)$. U holda

$$\|x_n\|_{\ell_2} = \sqrt{1^2 + 2^2 + \cdots + n^2} = \sqrt{\frac{n(n+1)(2n+1)}{6}},$$

$$\|Ix_n\|_{\ell_1} = \sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Bulardan

$$\sup_{n\geq 1} \frac{\|Ix_n\|}{\|x_n\|} = \sup_{n\geq 1} \frac{n(n+1)}{2} \cdot \sqrt{\frac{6}{n(1+n)(2n+1)}} = \infty.$$

Bu esa $I : \ell_2 \to \ell_1$ operatorning chegaralanmagan ekanligini ko'rsatadi.

37. $D(A) = C^{(1)}[0, 1] \neq C[0, 1]$. 2) uzluksiz emas, chegaralangan emas.

38-42- misollarda $D(A) \neq X$. 2) uzluksiz emas, chegaralangan emas.

43. Agar $x_0(t) = \left(1 + t^2\right)^{-1}$ desak, u holda $x_0 \in L_1(\mathbb{R}_+)$ bo'ladi, $(Ax_0)(t) = t\,(1 + t\,)^{-1}$ bo'lib, u integrallanuvchi emas, ya'ni $Ax_0 \notin L_1(\mathbb{R}_+)$. Demak,

$D(A) \neq L_1(\mathbb{R}_+)$. Agar $x_n(t) = \chi_{[n,\, n+1)}(t)$ desak, u holda $x_n \in D(A)$, $\|x_n\| = 1$ bo‘lib,

$$\|Ax_n\| = \int\limits_0^\infty t\, \chi_{[n,\, n+1)}(\sqrt{t})dt = \int\limits_0^\infty 2\, s^3\, \chi_{[n,\, n+1)}(s)\, ds = 2\int\limits_n^{n+1} s^3\, ds.$$

Bu integralning qiymati $\|Ax_n\| = 2n^3 + 3n^2 + 2n + 0,5$ ga teng. 5.35-misolga ko‘ra, A ning chegaralanmagan operator ekanligi kelib chiqadi. 5.2-teoremaga ko‘ra, u uzluksiz ham emas.

44-54 - misollarda $D(A) \neq X$. 2) uzluksiz emas, chegaralangan emas.

55. $X = C[0,1]$, $(Af)(x) = xf(x)$, $(Bf)(x) = \int\limits_0^1 f(t)dt$.

57. Yo‘q. Masalan, $X = Y = \mathbb{R}^2$, $Ax = (0, x_2)$, $Bx = (0, 2x_2)$.

58. Umuman olganda yo‘q. $A : C^{(1)}[0,1] \to C[0,1]$, $Ax(t) = x(t)$ misoldan foydalaning.

61. $\dim L(\mathbb{R}^n, \mathbb{R}^m) = n \cdot m$.

62. Yo‘q. **63.** Ha. **65.** Ha. **66.** Ha.

70. A_n va B_n operatorlar ketma-ketligi A operatorga tekis yaqinlashadi. Demak ular A ga kuchli va kuchsiz ma’noda ham yaqinlashadi.

75. Istalgan noldan farqli ortogonal P va Q operatorlarni olish mumkin. Masalan: $P, Q : \ell_2 \to \ell_2$, $Px = (x_1, x_2, 0, 0, \ldots)$, $Qx = (0, 0, x_3, x_4, \ldots)$.

76. $A \in L(\ell_2)$, $Ax = (a_1x_1, a_2x_2, \ldots)$, $a = (a_1, a_2, \ldots, a_n, \ldots) \in m$, $B : \ell_2 \to \ell_2$, $Bx = (x_1, 2x_2, \ldots, n\, x_n, \ldots)$. Agar $\{na_n\}$ chegaralangan bo‘lsa, u holda $A \cdot B$ va $B \cdot A$ operatorlar ham chegaralangan bo‘ladi, agar $\{na_n\}$ chegaralanmagan bo‘lsa, u holda $A \cdot B$ va $B \cdot A$ operatorlar ham chegaralanmagan bo‘ladi. Masalan, $a_n = \dfrac{1}{n}$ va $a_n = \dfrac{n-1}{n}$ hollarni qarang.

77. $\|P\| = 1$.

78. a) $R(A) = L_1^+[-1,\ 1]$ - juft funksiyalar to‘plami, $R(B) = L_1^-[-1,\ 1]$ - toq funksiyalardan iborat to‘plami. Ikkalasi ham qism fazo tashkil qiladi. b) $\|A\| = \|B\| = 1$. c) $A^2 = A$, $B^2 = B$, A va B lar ortogonal proeksiyalash operatorlari bo'ladi. d) $A \cdot B = 0, \quad B \cdot A = 0.$

6-§. Teskari operatorlar

6. Operator 6.3-teorema shartlarini qanoatlantirmaydi. A^{-1} operator mavjud ammo, chegaralanmagan.

7. Operator 6.4-teorema shartlarini qanoatlantirmaydi.

8. Operator 6.4-teorema shartlarini qanoatlantiradi. A^{-1} operator mavjud va chegaralangan.

9. $A^{-1}y = \frac{1}{2}\left(y_1 + y_2 - y_3, -y_1 + y_2 + y_3, y_1 - y_2 + y_3\right).$

10. Ma'lumki,

$$Ax = 0 \iff (x_1, x_1 - x_2, x_2 - x_3) = (0, 0, 0)$$

tenglama yagona $x = 0$ yechimga ega. 24.3-teoremaga ko'ra, A^{-1} operator mavjud. Teskari operatorni topish maqsadida

$$Ax = y \iff (x_1, x_1 - x_2, x_2 - x_3) = (y_1, y_2, y_3)$$

tenglamadan $x = (x_1, x_2, x_3)$ ni topamiz. Bu tenglama

$$\begin{cases} x_1 = y_1 \\ x_1 - x_2 = y_2 \\ x_2 - x_3 = y_3 \end{cases}$$

sistemaga teng kuchli. Uning yechimi $x_1 = y_1$, $x_2 = y_1 - y_2$, $x_3 = y_1 - y_2 - y_3$ dan iborat. Shunday qilib, teskari operator

$$A^{-1}y = (y_1, y_1 - y_2, y_1 - y_2 - y_3)$$

tenglik bilan aniqlanar ekan.

11. $A^{-1}y = (y_4, y_1, y_2, y_3).$

13. $A^{-1}y = \left(\frac{1}{2}(y_1 + y_2), \frac{1}{2}(y_2 - y_1), y_3 - \frac{1}{2}(y_2 - y_1), y_4, y_5, y_6, y_7\right).$

14. $A^{-1}y = \left(y_1, \frac{1}{2}(y_1+y_2-y_3), \frac{1}{2}(-y_1+y_2+y_3), y_4, y_5, \ldots\right).$

15. $A^{-1}y = (y_1, 2y_2, \ldots, ny_n, \ldots).$

16. $A^{-1}y = (y_1, y_2, y_3-y_2, y_4-y_3+y_2, y_5-y_4, y_6, y_7, \ldots).$

17. $A^{-1}y = (y_1, \sqrt{2}\,y_2, \ldots, \sqrt{n}\,y_n, \ldots).$

18. $A^{-1}y = (y_1, 2y_2, \frac{3}{2}y_3, \ldots, \frac{n}{n-1}y_n, \ldots).$

19. $(A^{-1}y)_n = 2\int\limits_0^1 y(t)\sin 2\pi n\, t\, dt, \quad A^{-1}: C[0,1] \to \ell_1$

20. $(A^{-1}y)_n = n^2\int\limits_{-1}^1 y(t)\cos \pi\, n\, t\, dt, \quad A^{-1}: C[-1,1] \to m.$

21. $\left(A^{-1}y\right)(s) = \int\limits_0^s y(t)\, dt.$

22. $A^{-1}y(t) = \frac{1}{t}y'(t), \quad y(0)=0.$

23. $A^{-1}y(t) = \frac{y(t)}{t+2} - \frac{1}{2t+4}\frac{1}{1-\ln\frac{3}{2}}\int\limits_0^1 \frac{s\, y(s)ds}{s+2}.$

24. $A^{-1}y(t) = \frac{y(t)}{t+1} - \frac{t}{t+1}\frac{2}{1+2\ln 2}\int\limits_0^1 \frac{sy(s)ds}{s+1}.$

25. $A^{-1}y(t) = \frac{y(t)}{1+\sin t}.$

26. $A^{-1}y(t) = \frac{y(t)}{t+1} - \frac{3-t}{6(t+1)}y(0) - \frac{t}{3(t+1)}y(1).$

27. $A^{-1}y(t) = \frac{y(t)}{t^2} - \frac{y(1)}{2t^2}.$

28. $A^{-1}y(t) = y(t) - \frac{2\sin t}{2+\pi}\int\limits_0^\pi y(s)\sin s\, ds.$

29-48. Yuqorida keltirilgan 6.29-6.48-misollarni 6.3-teoremadan foydalanib yechish qulay. Masalan 29-misolda nolmas $x_0 = (1,\ -1,\ 1,\ -1)$ uchun $Ax_0 = 0$ tenglik oʻrinli. 6.3-teoremaga koʻra, A operatorga teskari operator mavjud emas. 30-misol uchun $e_2 = (0,\ 1,\ 0,\ \ldots, 0, \ldots)$. 31-misolda $e_1 = (1, 0, 0, \ldots, 0, \ldots)$ noldan farqli element uchun $Ae_1 = 0$ tenglik oʻrinli. 6.3-teoremaga koʻra, berilgan operatorga teskari operator mavjud emas.

38. Bu misolni yechishda ham 6.3-teoremadan foydalanamiz. $x_0(s) \equiv 1$ noldan farqli element uchun

$$(Ax_0)(t)=\int_{-1}^{1} ts\,ds = t\,\frac{s^2}{2}\bigg|_{-1}^{1}=\frac{t}{2}(1^2-(-1)^2)=0$$

tenglik o‘rinli. Demak, $Ax=0$ tenglama nolmas $x_0\in C[-1,\ 1]$ yechimga ega. 6.3-teoremaga ko‘ra, A ga teskari operator mavjud emas.

51. Mavjud.

55. Mavjud, lekin chegaralanmagan.

56. b) $(A^{-1}x)(t)=x(t)-\int\limits_0^t e^{s-t}x(s)ds$.

57. $\left(A^{-1}y\right)(t)=y(t)-\dfrac{2}{2-\pi}\int\limits_0^{\pi}\cos(t-s)y(s)\,ds.$

58. b). $A^{-1}x(t)=\int\limits_0^t x(s)\sin(t-s)ds.$

7-§. Qo‘shma operatorlar

7. $T^*y=(\lambda_1y_1,\ \lambda_2y_2,\ \lambda_3y_3,\ \lambda_4y_3)$.

8. $T^*y=(y_2,2y_3,(2+i)y_4,0)$.

9. $T^*y=(0,\lambda_2y_1,\ \lambda_3y_2,\dots,\lambda_ny_{n-1},\dots)$.

10. $T^*y=(\mu_1y_2,\ \mu_2y_3,\dots,\mu_ny_{n+1},\dots)$.

11. $T^*x=(x_1,x_2,\dots,x_n,0,0,\dots)$.

12. $T^*y=(y_n,0,0,\dots)$.

13. $T^*x=\left(e^{-i}x_1,e^{-2i}x_2,\dots,e^{-ni}x_n,\dots\right)$.

14. $T^*x=(\frac{1}{2}x_2,\frac{2}{3}x_3,\dots,\frac{n}{n-1}x_n,\dots)$.

15. $T^*x=(x_3,2x_4,\dots,50x_{52},0,0,\dots)$.

16. $T^*x=(\lambda_1x_1,\lambda_2x_2,\dots,\lambda_nx_n,\dots)$.

17. $T^*x=(2\,x_1,-i\,x_2,(1-i)x_3,0,0,\dots)$

18. $T^*x=(\bar\lambda_1x_1,\bar\lambda_2x_2,\dots,\bar\lambda_nx_n,\dots)$.

19. $T^*x=(x_1,x_2,x_3+x_1,x_4+x_2,\dots,x_n+x_{n-2},\dots)$.

20. $T^*x=(x_1,x_2+2x_1,x_3+2x_2+x_1,\dots,x_n+2x_{n-1}+x_{n-2},\dots)$.

21. $T^* = T.$

22. $T^* = T.$

25. $(T^*y)(t) = \int\limits_0^1 [ts - i\cos(t+s)]x(s)ds.$

26. $(T^*x)(t) = \int\limits_0^1 \left(s^2 + s + t\right) x(s)ds.$

27. $(T^*y)(t) = \int\limits_t^1 t\, x(s)ds.$

28. $(T^*y)(t) = (\cos t - i\sin t)\, x(t) + \int\limits_{-1}^1 (ts + it^2s^2)\, x(s)ds.$

29. Berilgan operatorni $T = A + B$ yig'indi shaklda yozamiz. Bu yerda

$$(Ax)(t) = (t + it^2)x(t), \quad (Bx)(t) = \int\limits_0^1 (t + is)x(s)\, ds, \;\; x \in L_2[0, \; 1].$$

7.3-misol tasdig'iga ko'ra, A operatorga qo'shma operator

$$(A^*x)(t) = (t - it^2)x(t), \quad x \in L_2[0, \; 1]$$

formula yordamida, 7.23-misolga ko'ra, B operatorga qo'shma operator

$$(B^*x)(t) = \int\limits_0^1 (s - it)x(s)\, ds, \;\; x \in L_2[0, \; 1]$$

formula yordamida aniqlanadi. 7.24-misolning a) tasdig'iga ko'ra, ularning yig'indisi bo'lgan $T = A + B$ operatorga qo'shma bo'lgan T^* operator

$$(T^*x)(t) = ((A^* + B^*)x)(t) = (t - it^2)x(t) + \int_0^1 (s - it)x(s)\, ds, \;\; x \in L_2[0, \; 1]$$

tenglik bilan aniqlanadi.

30. $(T^*y)(t) = y(t-h).$

31. $(T^*y)(t) = u(t-h)y(t).$

32. $T^*x = (\overline{\mu_1}x_1, \ldots, \overline{\mu_n}x_n, \ldots)$, o'z-o'ziga qo'shmalik sharti $\mu_n = \overline{\mu_n}.$

33. O'z-o'ziga qo'shmalik sharti $\overline{K(s,t)} = K(t,s).$

34. $(T^*x)(t) = \left(\overline{u(t)} - i\overline{v(t)}\right) x(t).$ u haqiqiy qiymatli funksiya, $v = 0.$

35. $\beta \in \mathbb{R},\ \alpha = ia,\ a \in \mathbb{R}$.

36. u va v lar haqiqiy funksiyalar bo‘lgan holda $\alpha \in \mathbb{R}$, $\beta = ib,\ b \in \mathbb{R}$.

37. $\overline{\alpha} = \beta$.

38. $\alpha\beta = 1$.

71. Bizga U izometrik operator berilgan bo‘lsin. Izometrik operator ta’rifiga ko‘ra $\|Ux\| = \|x\|$ va $\|U^*x\| \leq \|x\|$ tengliklar ixtiyoriy $x \in H$ uchun o‘rinli. Bu yerdan $\|U^*x\| \leq \|Ux\|$ tengsizlik kelib chiqadi. Ya’ni U giponormal operator bo‘ladi.

75. 7.64-misoldagi $U : \ell_2 \to \ell_2$ operator.

8-§. Chiziqli operatorning spektri

9. $\lambda_1 = 0,\ x^{(1)} = (1, -1, 0),\ \lambda_2 = 2,\ x^{(2)} = (1, 1, 0),\ \lambda_3 = 3,\ x^{(3)} = (0, 0, 1)$.

10. $\lambda_1 = 1,\ x^{(1)} = (1, 0, 0),\ \lambda_2 = 1 + i,\ x^{(2)} = (0, 1, i),\ \lambda_3 = 1 - i,\ x^{(3)} = (0, 1, -i)$,

11. $\lambda_1 = 2,\ x^{(1)} = (1,\ 1,\ 1),\ \lambda_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2},\ x^{(2)} = \left(1 + i\sqrt{3},\ -2,\ 1 - i\sqrt{3}\right)$,
$\lambda_3 = \frac{1}{2} - i\frac{\sqrt{3}}{2},\ x^{(3)} = \left(1 - i\sqrt{3},\ -2,\ 1 + i\sqrt{3}\right)$.

12. $\lambda_1 = 1,\ x^{(1)} = (1, 0, 0, 0),\ \lambda_2 = 2,\ x^{(2)} = (0, 1, 0, 0)$,
$\lambda_3 = 3,\ x^{(3)} = (0, 0, 1, 0),\ \lambda_4 = 4,\ x^{(4)} = (0, 0, 0, 1)$.

13. $\lambda_1 = 2,\ x^{(1)} = (1, 1, 1, 0),\ \lambda_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2},\ x^{(2)} = \left(1 + i\sqrt{3},\ -2,\ 1 - i\sqrt{3}, 0\right)$,
$\lambda_3 = \frac{1}{2} - i\frac{\sqrt{3}}{2},\ x^{(3)} = \left(1 - i\sqrt{3},\ -2,\ 1 + i\sqrt{3}, 0\right)$,
$\lambda_4 = 5,\ x^{(4)} = (0, 0, 0, 1)$.

14. $\lambda_1 = 0,\quad x^{(1)} = (1, -1, 1, -1),\ \lambda_2 = 2,\ x^{(2)} = (1, 1, 1, 1),\ \lambda_3 = 1 - i$,
$x^{(3)} = (1, -i, -1, i),\ \lambda_4 = 1 + i,\ x^{(4)} = (1, i, -1, -i)$.

15. $\lambda_0 = 0$ cheksiz karrali xos qiymat, unga mos $x_0(t)$ xos funksiyalar $\int_{-1}^{1} s\, x_0(s)\, ds = 0$ shartni qanoatlantiradi. $\lambda_1 = \frac{2}{3}$ oddiy xos qiymat, unga mos xos funksiya $x_1(t) = t$.

16. $\lambda_0 = 0$ cheksiz karrali xos qiymat, unga mos $x_0(t)$ xos funksiyalar,

$\int\limits_{-1}^{1} x_0(s)\,ds = \int\limits_{-1}^{1} s\,x_0(s)\,ds = 0$ shartni qanoatlantiradi. $\lambda_1 = \dfrac{2}{3}$ va $\lambda_2 = 2$ lar oddiy xos qiymatlar bo‘lib, ularga mos xos funksiyalar $x_1(t) = t$ va $x_2(t) = 1$ dir.

17. $\lambda_0 = 0$ cheksiz karrali xos qiymat, unga mos $x_0(t)$ xos funksiyalar: $x_0(t) = \cos nt,\ n \in \mathbb{N},\ x_0(t) = \sin nt,\ n \geq 2.$ $\lambda_1 = 2\pi$ va $\lambda_2 = \pi$ lar oddiy xos qiymatlar bo‘lib, ularga mos xos funksiyalar $x_1(t) = 1$ va $x_2(t) = \sin t$ dir.

18. $\lambda_0 = 0$ cheksiz karrali xos qiymat, unga mos $x_0(t)$ xos funksiyalar: $x_0(t) = \alpha\cos nt + \beta\sin nt,\ n \geq 2,\ x_0(t) = 1.$ $\lambda_1 = 1$ ikki karrali xos qiymat bo‘lib, unga mos xos funksiyalar: $x_1(t) = \alpha\cos t + \beta\sin t.$

19. $\lambda_0 = 0$ cheksiz karrali xos qiymat, unga mos $x_0(t)$ xos funksiyalar: $x_0(t) = \alpha\cos nt + \beta\sin nt,\ n \geq 2,\ x_0(t) = 1.$ $\lambda_1 = 1$ va $\lambda_2 = -1$ lar oddiy xos qiymatlar bo‘lib, ularga mos xos funksiyalar $x_1(t) = \sin t + \cos t$ va $x_2(t) = \sin t - \cos t.$

20. $\lambda_0 = 0$ cheksiz karrali xos qiymat, unga mos $x_0(t)$ xos funksiyalar: $x_0(t) = \alpha\cos nt + \beta\sin nt,\ n \geq 2.$ $\lambda_1 = 1$ ikki karrali xos qiymat, unga mos xos funksiyalar: $x_1(t) = \alpha\cos t + \beta\sin t.$ $\lambda_2 = 2$ oddiy xos qiymat bo‘lib, unga mos xos funksiya $x_2(t) = 1$ dir.

21. $\lambda_0 = 1$ cheksiz karrali xos qiymat, unga mos $x_0(t)$ xos funksiyalar: $x_0(t) = 1,\ x_0(t) = \sin nt,\ n \in \mathbb{N},\ x_0(t) = \cos nt,\ n \geq 2.$ $\lambda_1 = 2$ oddiy xos qiymat bo‘lib, unga mos xos funksiya $x_1(t) = \cos t$ dir.

22. $\sigma(A) = \sigma_{qol}(A) = [1,\, 2]\,,\quad R_\lambda(A)\,x(t) = \dfrac{x(t)}{1 + t^2 - \lambda}.$

23. $\sigma(A) = \sigma_{qol}(A) = [-1,\, 1]\,,\quad R_\lambda(A)\,x(t) = \dfrac{x(t)}{t - \lambda} + \dfrac{t\,x(0)}{\lambda(t - \lambda)}.$

24. $\sigma(A) = \sigma_{qol}(A) = [-1,\, 1]\,,\ R_\lambda(A)\,x(t) = \dfrac{x(t)}{\sin t - \lambda} - \dfrac{x(0)\cos t}{(1 - \lambda)(\sin t - \lambda)}.$

25. $\sigma(A) = \sigma_{ess}(A) = [1,\, \infty)\,,\quad R_\lambda(A)x(t) = \dfrac{x(t)}{1 + t^2 - \lambda}.$

26. $\sigma(A)=\sigma_{ess}(A)=[0,\ 2]\,,\ R_\lambda(A)x(t)=\dfrac{x(t)}{1-\cos t-\lambda}.$

27. $\sigma(A)=\sigma_{ess}(A)=[-4,\,6]\,,\ R_\lambda(A)\,x(t)=\dfrac{x(t)}{1-2\cos t+3\cos 2t-\lambda}.$

28. $\sigma(A)=\sigma_{ess}(A)=[0,\,\infty)\,,\quad R_\lambda(A)\,x(t)=\dfrac{x(t)}{t^2-\lambda}\,,$

29. Operatorning 1 dan katta yagona oddiy xos qiymati bo'lib, u $\Delta(\lambda)=1+\int\limits_{-\pi}^{\pi}\dfrac{\sin^2 s}{\cos s-\lambda}\,ds$ funksiyaning noli. $\sigma_{ess}(A)=[-1,\,1]\,,$

$$R_\lambda(A)\,x(t)=\frac{x(t)}{\cos t-\lambda}-\frac{\sin t}{\cos t-\lambda}\,\frac{1}{\Delta(\lambda)}\int\limits_{-\pi}^{\pi}\frac{\sin s\,x(s)}{\cos s-\lambda}\,ds.$$

30. Operatorning -1 dan kichik yagona oddiy xos qiymati bo'lib, u $\Delta(\lambda)=1-\int\limits_{-\pi}^{\pi}\dfrac{\sin^2 s}{\cos 2s-\lambda}\,ds$ funksiyaning noli, $\sigma_{ess}(A)=[-1,\,1]\,,$

$$R_\lambda(A)\,x(t)=\frac{x(t)}{\cos 2t-\lambda}+\frac{\sin t}{\cos 2t-\lambda}\,\frac{1}{\Delta(\lambda)}\int\limits_{-\pi}^{\pi}\frac{\sin s\,x(s)}{\cos 2s-\lambda}\,ds.$$

31. Operatorning 1 dan katta yagona oddiy xos qiymati bo'lib, u $\Delta(\lambda)=1+\int\limits_{-\pi}^{\pi}\dfrac{\cos^2 s}{\cos 4s-\lambda}\,ds$ funksiyaning noli, $\sigma_{ess}(A)=[-1,\,1]\,,$

$$R_\lambda(A)\,x(t)=\frac{x(t)}{\cos 4t-\lambda}-\frac{\cos t}{\cos 4t-\lambda}\,\frac{1}{\Delta(\lambda)}\int\limits_{-\pi}^{\pi}\frac{\cos s\,x(s)}{\cos 4s-\lambda}\,ds.$$

32. $\lambda_1=-1$ va $\lambda_2=0$ sonlari operatorning oddiy xos qiymatlari, $\lambda_3=1$ cheksiz karrali xos qiymat, shuning uchun $\sigma_{ess}(A)=\{1\}.$

$$R_\lambda(A)\,x(t)=\frac{x(t)}{1-\lambda}-\frac{1}{\pi(1-\lambda^2)}\int\limits_{-\pi}^{\pi}x(s)\,ds-\frac{\sin t}{\pi\,\lambda(1-\lambda)}\int\limits_{-\pi}^{\pi}\sin\ s\,x(s)\,ds.$$

33. $\lambda_1=-1$ oddiy, $\lambda_2=0$ esa ikki karrali xos qiymatdir, $\lambda_3=1$ cheksiz karrali xos qiymat, shuning uchun $\sigma_{ess}(A)=\{1\}.$

$$R_\lambda(A)\,x(t)=\frac{x(t)}{1-\lambda}-\frac{1}{\pi(1-\lambda^2)}\int\limits_{-\pi}^{\pi}x(s)\,ds-\frac{1}{\pi\,\lambda(1-\lambda)}\int\limits_{-\pi}^{\pi}\cos(t-s)x(s)\,ds.$$

34. $\lambda_1=\lambda_2=1,\ x^{(1)}=(1,0,0,0)\,,\ x^{(2)}=(0,0,0,1);\ \lambda_3=\sqrt{2},\ \ x^{(3)}=(0,\,1,\sqrt{2}-1,\,0)\,;\ \lambda_4=-\sqrt{2},\ x^{(4)}=(0,\,1,-\sqrt{2}-1,\,0)\,.$

35. $\lambda_1=-1,\quad x_1(t)=1-t,\quad \lambda_2=4,\quad x_2(t)=\dfrac{t^2+4}{4-t}.$

36. $\lambda_1=\dfrac{15-4\sqrt{15}}{60},\quad x_1(t)=60t^2-12\sqrt{15}t;\ \lambda_2=\dfrac{15+4\sqrt{15}}{60};\ x_2(t)=60t^2+12\sqrt{15}t\,,\ \lambda=0$ cheksiz karrali xos qiymat, $\int\limits_0^1 s\,x(s)\,ds=\int\limits_0^1 s^2\,x(s)\,ds=0$ shartni qanoatlantiruvchi barcha funksiyalar $\lambda=0$ xos qiymatga mos keluvchi xos vektorlar bo'ladi.

37. $\lambda_1 = 1$ va $\lambda_2 = -1$ lar oddiy xos qiymatlar, ularga mos xos funksiyalar $x_1(t) = \cos t$ va $x_2(t) = \sin t$. $\lambda_3 = 0$ cheksiz karrali xos qiymat, unga mos keluvchi xos funksiyalar $x_3(t) = 1$, $x_3(t) = \sin nt$, $y_3(t) = \cos nt$, $n \geq 2$.

38. $\lambda_1 = 2$ va $\lambda_2 = \dfrac{2}{3}$ lar oddiy xos qiymatlar, ularga mos xos funksiyalar $x_1(t) = 1$ va $x_2(t) = t$. $\lambda = 0$ cheksiz karrali xos qiymat, unga mos keluvchi xos vektorlar $\int\limits_{-1}^{1} x(s)\,ds = \int\limits_{-1}^{1} s\,x(s)\,ds = 0$ shartni qanoatlantiruvchi funksiyalardir.

39. $\lambda_1 = -3$ va $\lambda_2 = 4$ lar oddiy xos qiymatlar, ularga mos xos funksiyalar $x_1(t) = 2t - t^2$ va $x_2(t) = \dfrac{1}{2}t + \dfrac{3}{2}t^2$. $\lambda_3 = 0$ cheksiz karrali xos qiymat, unga mos xos funksiyalar $x_3(t) = (t^2 - 1)\,y(t)$, $y \in C[-1,\ 1]$.

40. $\lambda_1 = 2$ va $\lambda_2 = \dfrac{2}{3}$ lar oddiy xos qiymatlar, ularga mos xos funksiyalar $x_1(t) = 1$ va $x_2(t) = t$. $\lambda_3 = 0$ cheksiz karrali xos qiymat, unga mos xos funksiyalar $x_1(t) = 1$, $x_2(t) = t$ larga ortogonal bo'lgan funksiyalardir.

41. $\lambda_1 = 2$, $x^{(1)} = (1,\ 1,\ 1, 0, \ldots)$, $\lambda_2 = \dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$,
$x^{(2)} = \left(1 + i\sqrt{3},\ -2,\ 1 - i\sqrt{3}, 0, \ldots\right)$, $\lambda_3 = \dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$,
$x^{(3)} = \left(1 - i\sqrt{3},\ -2,\ 1 + i\sqrt{3}, 0, \ldots\right)$. $\lambda_4 = 1$ cheksiz karrali xos qiymat, unga mos keluvchi xos vektorlar $x^{(4)} = e_n$, $n \geq 4$.

42. $\lambda_1 = 3$, $x^{(1)} = e_1$, $\lambda_2 = 4$, $x^{(2)} = e_2$, $\lambda_3 = -2$, $x^{(3)} = e_3$, $\lambda_4 = 5$, $x^{(4)} = e_4$. $\lambda_5 = 1$ cheksiz karrali xos qiymat, unga mos keluvchi xos vektorlar $x^{(n)} = e_n$, $n \geq 5$.

43. Istalgan $n \in \mathbb{N}$ uchun $\lambda_n = \dfrac{n}{n+1}$ oddiy xos qiymat bo'lib, unga mos xos vektor e_n, $n \in \mathbb{N}$.

44. Istalgan $n \in \mathbb{N}$ uchun $\lambda_{2n-1} = \dfrac{2n-1}{4n}$ va $\lambda_{2n} = \dfrac{1}{2n}$ lar oddiy xos qiymatlar bo'lib, ularga mos xos vektorlar $x^{(2n-1)} = e_{2n-1}$ va $x^{(2n)} = e_{2n}$, $n \in \mathbb{N}$.

45. $\lambda_1 = -1$, $x^{(1)} = (0, 0, 1, 0, 0, \ldots)$; $\lambda_2 = -\sqrt{2}$, $x^{(2)} = (1, -\sqrt{2} - 1,\ 3 + 2\sqrt{2}, 0, 0, \ldots)$; $\lambda_3 = \sqrt{2}$, $x^{(3)} = (1,\ \sqrt{2} - 1,\ 3 - 2\sqrt{2}, 0, 0, \ldots)$;

$\lambda_4 = 1$, $x^{(4)} = e_4$; $\lambda_5 = 0$ cheksiz karrali xos qiymat, unga mos keluvchi xos vektorlar $x^{(5)} = e_n$, , $n \geq 5$.

46. $\lambda_1 = 1$ cheksiz karrali xos qiymat, unga mos keluvchi xos vektorlar $x^{(1)} = e_n$, $n = 1, 3, n \geq 5$, $\lambda_2 = 2$ va $\lambda_3 = 3$ lar oddiy xos qiymatlar bo‘lib, ularga mos xos vektorlar $x^{(2)} = e_2$ va $x^{(3)} = e_4$.

47. $\lambda_1 = 1$ ikki karrali xos qiymat, unga mos keluvchi xos vektorlar $x^{(1)} = \alpha e_1 + \beta e_4$, $\lambda_2 = \frac{5}{2} - \frac{\sqrt{5}}{2}$, $x^{(2)} = (0,\, 2,\, 1 - \sqrt{5},\, 0, 0, \ldots)$; $\lambda_3 = \frac{5}{2} + \frac{\sqrt{5}}{2}$, $x^{(3)} = (0,\, 2,\, 1 + \sqrt{5},\, 0, 0, \ldots)$; $\lambda_4 = 0$ cheksiz karrali xos qiymat, unga mos keluvchi xos vektorlar $x^{(4)} = e_n$, $n \geq 5$.

48. $\lambda_1 = 6$, $\lambda_2 = 5$, $\lambda_3 = 4$, $\lambda_4 = 3$, $\lambda_5 = 2$ lar oddiy xos qiymatlar bo‘lib, ularga mos xos vektorlar $x^{(1)} = e_1$, $x^{(2)} = e_2$, $x^{(3)} = e_3$, $x^{(4)} = e_4$, $x^{(5)} = e_5$. $\lambda_6 = 1$ cheksiz karrali xos qiymat, unga mos keluvchi xos vektorlar $x^{(6)} = e_n$, $n \geq 6$.

49. $\sigma(A) = \left\{0\,,\, \frac{11 - \sqrt{97}}{2}\,,\, \frac{11 + \sqrt{97}}{2}\right\}$; $R_\lambda(A)\, x(t) =$

$$= \frac{1}{\lambda}\left[-x(t) + \frac{(9 - \lambda)\, x(2) - 4x(3)}{\lambda^2 - 11\lambda + 6}\, t - \frac{3x(2) + (\lambda - 2)x(3)}{\lambda^2 - 11\lambda + 6}\, t^2\right].$$

50. $\sigma(A) = [0,\, 2] \bigcup \{3 + \sqrt{2}\}$; $R_\lambda(A)\, x(t) =$

$$= \frac{1}{t - \lambda}\left[x(t) + \frac{x(1) + (\lambda - 2)x(2)}{\lambda^2 - 6\lambda + 7}\, t + \frac{(\lambda - 4)x(1) + x(2)}{\lambda^2 - 6\lambda + 7}\right].$$

51. $\sigma(A) = [0,\, 1]$, $R_\lambda(A)\, x(t) = \frac{x(t)}{t^2 - \lambda} - \frac{x(0)}{(1 - \lambda)(t^2 - \lambda)}$.

52. $\sigma(A) = [0\,,\, 1] \bigcup \{2\}$,

$$R_\lambda(A)\, x(t) = \frac{x(t)}{t - \lambda} + \frac{x(0)}{\lambda(\,t - \lambda)}\, t^2 - \frac{\lambda\, x(1) + x(0)}{\lambda(2 - \lambda)(t - \lambda)}\, t^3.$$

53. $\sigma(A) = [0\,,\, 2]$, $R_\lambda(A)\, x(t) = \frac{x(t)}{|\,t\,| - \lambda}$

54. $\sigma(A) = [2\,,\, 3]$, $R_\lambda(A)\, x(t) = \frac{x(t)}{t\, + 2 - \lambda}$.

55. $\sigma(A) = [\frac{-\pi}{2},\, \frac{\pi}{2}]$, $R_\lambda(A)\, x(t) = \frac{x(t)}{\mathrm{arctg}\, t - \lambda}$.

56. $\sigma(A) = [0\,,\, 1] \bigcup \{\lambda :\ \Delta(\lambda) = 0\}$, $\Delta(\lambda) = 2 + \frac{1}{2}\ln\left|\frac{1 - \lambda}{1 + \lambda}\right|$,

$$R_\lambda(A)\,x(t)=\frac{x(t)}{t^2-\lambda}-\frac{t}{\Delta(\lambda)(t^2-\lambda)}\int_0^1\frac{s\,x(s)}{s^2-\lambda}\,ds.$$

57. $\sigma(A)=[0\,,\,1]\bigcup\{\lambda:\ \ \Delta(\lambda)=0\},\quad \Delta(\lambda)=1+\int_0^\infty\frac{2^{-2s}}{e^{-s}-\lambda}\,ds,$

$$R_\lambda(A)\,x(t)=\frac{x(t)}{e^{-t}-\lambda}-\frac{2^{-t}}{\Delta(\lambda)(e^{-t}-\lambda)}\int_0^\infty\frac{2^{-s}\,x(s)}{e^{-s}-\lambda}\,ds\,.$$

58. $\sigma(A)=[-1\,,\,1]\bigcup\{\lambda:\ \Delta(\lambda)=0\},\ \Delta(\lambda)=1+2\lambda+$
$+\lambda^2\ln\left|\dfrac{1-\lambda}{1+\lambda}\right|,\ \ R_\lambda(A)\,x(t)=\dfrac{x(t)}{t-\lambda}-\dfrac{t}{\Delta(\lambda)(t-\lambda)}\displaystyle\int_{-1}^1\frac{s\,x(s)}{s-\lambda}\,ds\,.$

59. $\sigma(A)=\{\,-1\,,\,1\},$

$$R_\lambda(A)x=\left(-\frac{x_1}{1+\lambda},\frac{x_2}{1-\lambda},\ldots,-\frac{x_{2n-1}}{1+\lambda},\frac{x_{2n}}{1-\lambda},\ldots\right).$$

60. $\sigma(A)=\left\{0,\,\frac{1}{3},\,\frac{1}{4},\ldots,\frac{1}{n},\ldots\right\},$

$$R_\lambda(A)\,x=\left(\frac{3\,x_1}{1-3\lambda}\,,\,\frac{4\,x_2}{1-4\lambda},\ldots,\frac{(n+2)\,x_n}{1-(n+2)\lambda}\,,\ldots\right).$$

61. $\sigma(A)=\left\{0,\,1\,,\ \frac{1}{2}\,,\,\frac{1}{3}\,,\,\frac{1}{4},\ \ldots\ ,\ \frac{1}{n},\ \ldots\right\},$

$$R_\lambda(A)x=\left(-\frac{1}{\lambda}x_1,\ \frac{1}{1-\lambda}x_2,\ \frac{2}{1-2\lambda}\,x_3,\ \ldots,\ \frac{n}{1-n\lambda}x_{n+1},\ \ldots\right).$$

62. $\sigma(A)=\left\{0,\,1\,,\ \frac{1}{2}\,,\,\frac{1}{3}\,,\,\frac{1}{4},\ \ldots\ ,\ \frac{1}{n},\ \ldots\right\},$

$$R_\lambda(A)x=\left(\frac{1}{1-\lambda}x_1,\ \frac{2}{1-2\lambda}x_2,\ \frac{3}{1-3\lambda}\,x_3,\ \ldots,\ \frac{n}{1-n\lambda}x_n,\ \ldots\right).$$

63. $\sigma(A)=\{\,\lambda\in\mathbb{C}:\ |\lambda|\le 1\,\},\quad R_\lambda(A)\,x=$

$$=\left(-\frac{x_1}{\lambda},\ -\frac{x_1}{\lambda^2}-\frac{x_2}{\lambda},\ldots,-\frac{x_1}{\lambda^n}-\frac{x_2}{\lambda^{n-1}}-\ \ldots\ -\frac{x_n}{\lambda},\ldots\right).$$

64. $\sigma(A)=\{\lambda\in\mathbb{C}:\,|\lambda|\le 1\},\ \ (R_\lambda(A)\,x)_n=-\sum\limits_{k=0}^{\infty}\dfrac{x_{n+k}}{\lambda^{k+1}},\ \ n\in\mathbb{N}.$

65. $\sigma(A)=\{0,\,1,\,2\},$

$$R_\lambda(A)\,x=\left(\frac{x_2-(1-\lambda)x_1}{2\lambda-\lambda^2},\frac{x_1-(1-\lambda)x_2}{2\lambda-\lambda^2},\frac{x_3}{1-\lambda},\ldots,\frac{x_n}{1-\lambda},\ldots\right).$$

66. $\sigma(A)=\left\{1,\,2,\,\frac{3}{2},\,\frac{4}{3},\ldots,\ \frac{n+1}{n},\ldots\right\},$

$$R_\lambda(A)\,x=\left(\frac{1}{2-\lambda}x_1,\ \frac{2}{3-2\lambda}x_2,\ldots,\ \frac{n}{n+1-n\lambda}x_n,\ldots\right).$$

68. $A:\mathbb{C}^3\to\mathbb{C}^3,\ \ Ax=(x_1+x_2,\,x_2+x_3,\,x_3).$

69. $A:\mathbb{C}^3\to\mathbb{C}^3,\ \ Ax=(x_1+x_2,\,x_2,\,x_3).$

75. Agar $\dim X<\infty$ bo‘lsa mumkin, $\dim X=\infty$ bo‘lsa, mumkin emas.

77. $\sigma(A) = \overline{\{e^i, e^{2i}, \ldots, e^{ni}, \ldots\}}$.

78. $(Af)(n) = (-1)^n f(n), \;\; f \in \ell_2(\mathbb{Z})$.

79. Istalgan U unitar operatorning spektri $\sigma(U) \subset \{z \in \mathbb{C} : |z| = 1\}$.

80. $(Af)(x) = xf(x), \;\; f \in L_2[a, \, b]$.

81. Birlik operatordan farqli istalgan P proyeksiyalash operatori uchun $\sigma(P) = \{0, \, 1\} = \{m, \, M\}$ tenglik o‘rinli.

82. $\sigma(A) = \{0, \, 1\}, \;\; R_\lambda(A)\, x(t) = -\dfrac{x(t)}{\lambda} + \dfrac{(1-\lambda-t)x(0)}{\lambda(1-\lambda)^2} + \dfrac{x(1)\, t}{\lambda(1-\lambda)}$.

83. $\sigma(A) = \{0, \, 1\}, \;\; R_\lambda(P) = \dfrac{1}{\lambda(1-\lambda)}\, P - \dfrac{1}{\lambda}\, I$.

92. $(\Delta f)(p) = 2 \sum\limits_{j=1}^{\nu} (\cos p_j - 1) f(p), \;\; f \in L_2([-\pi, \, \pi]^\nu)$.

93. $(Vf)(p) = \dfrac{1}{\sqrt{2\,\pi}} \int\limits_{-\pi}^{\pi} v(p-q) f(q)\, dq, \;\; v = F\hat{v}$.

96. $(A^+ f^+)(p) = (2 - 2\,\cos p) f^+(p) - \dfrac{\mu}{2\pi} \int\limits_{-\pi}^{\pi} \cos p \cos s f^+(s)\, ds,$

$(A^- f^-)(p) = (2 - 2\,\cos p) f^-(p) - \dfrac{\mu}{2\pi} \int\limits_{-\pi}^{\pi} \sin p \sin s f^-(s)\, ds.$

97. a) V^+ operatorning xos qiymatlari $\lambda_0 = 1$ va $\lambda_n = e^{-n}, \;\; n \in \mathbb{N}$ bo‘lib, unga mos xos funksiya $f_0^+(p) = 1, \;\; f_n^+(p) = \cos np$, V^- operatorning xos qiymatlari $\lambda_n = e^{-n}, \;\; n \in \mathbb{N}$ bo‘lib, unga mos xos funksiya $f_n^-(p) = \sin np$. b) V operator uchun $\lambda_0 = 1$ oddiy xos qiymat, $\lambda_n = e^{-n}, \;\; n \in \mathbb{N}$ larning har biri ikki karrali xos qiymat bo‘ladi.

98. Agar $\lambda < -1$ bo‘lsa, $E_\lambda = 0$ ga, agar $\lambda > 1$ bo‘lsa, $E_\lambda = I$ ga, agar $\lambda \in [-1, \, 1]$ bo‘lsa, $(E_\lambda f)(t) = \chi_{A(\lambda)}(t) f(t)$, bu yerda $A(\lambda) = \{t \in [-\pi, \, \pi] : \cos t \leq \lambda\}$.

II bobda keltirilgan test javoblari

1-C 2-B 3-A 4-C 5-B 6-B 7-C 8-B 9-A 10-B 11-C 12-A 13-B 14-C 15-B 16-B 17-D 18-A 19-B 20-B 21-B 22-A 23-D 24-B 25-C 26-A 27-A 28-D 29-A 30-C 31-A 32-A 33-C 34-B 35-B 36-C 37-D 38-A 39-B 40-C, 41-A 42-D 43-B 44-B 45-A 46-A 47-B

48-A 49-A 50-B.

III bob. Kompakt operatorlar va integral tenglamalar

Bu bob ikki paragrafdan iborat. $9-\S$ kompakt operatorlarga, $10-\S$ integral tenglamalarga oid masalalarni o'z ichiga oladi.

9-§. Kompakt operatorlar

Metrik va normalangan fazolarda kompakt, nisbiy kompakt to'plam ta'riflari shu nomli kitob I qismining $17-\S$ da berilgan. Bu ta'riflarni esga olamiz. Chunki kompakt operatorlar shu tushunchalar asosida ta'riflanadi. Bizga $X-$ Banax fazosi va $M \subset X$ to'plam berilgan bo'lsin. Agar M to'plamdan olingan ixtiyoriy $\{x_n\}$ ketma-ketlikdan M da yaqinlashuvchi qismiy ketma-ketlik ajratish mumkin bo'lsa, M ga *kompakt to'plam* deyiladi. Agar N to'plamning yopig'i $[N]$ kompakt to'plam bo'lsa, u holda N *nisbiy kompakt to'plam* deyiladi. *To'plam nisbiy kompakt bo'lishi uchun uning to'la chegaralangan bo'lishi zarur va yetarli* ([8] ning 17.1-teoremasi). *Chekli o'lchamli fazolarda to'plam kompakt bo'lishi uchun uning chegaralangan va yopiq bo'lishi zarur va yetarlidir* ([8] ning 17.3-teoremasi). Asosiy funksional fazolardan biri $C[a, \ b]$ fazodir. Bu fazodagi to'plamning nisbiy kompaktlik kriteriysi Arsela teoremasi yordamida bayon qilingan ([8] ning 17.2-teoremasi). Agar $A \in L(X, Y)$ va $\dim ImA < \infty$ bo'lsa, u holda A ga *chekli o'lchamli operator* (8.6-ta'rif) deyiladi. Agar $\dim ImA = n$ bo'lsa, u holda A ga n *o'lchamli operator* deyiladi. Agar A operator X dagi har qanday chegaralangan to'plamni Y dagi nisbiy kompakt to'plamga akslantirsa, u holda A *kompakt operator yoki to'la uzluksiz operator* (8.7-ta'rif) deyiladi. Bu ta'rifga teng kuchli bo'lgan quyidagi ta'riflarni keltiramiz.

9.1-ta'rif. *Agar $A : X \to Y$ chiziqli operator X fazodagi birlik sharni Y fazodagi nisbiy kompakt to'plamga akslantirsa, u holda A kompakt operator deyiladi.*

9.2-ta'rif. *$A \in L(X,Y)$ $(X,\ Y-$ Banax fazolari) operator va ixtiyoriy $\{x_n\} \subset X$ chegaralangan ketma-ketlik berilgan bo'lsin. Agar $\{A\,x_n\}$ ketma-ketlikdan yaqinlashuvchi qismiy ketma-ketlik ajratish mumkin bo'lsa, u holda A ga kompakt operator deyiladi.*

9.3-ta'rif. *Agar H Hilbert fazosida aniqlangan A operator har qanday kuchsiz yaqinlashuvchi ketma-ketlikni kuchli yaqinlashuvchi ketma-ketlikka akslantirsa, u holda A kompakt operator deyiladi.*

Agar X Banax fazosini Y Banax fazosiga akslantiruvchi barcha kompakt operatorlar to'plamini $K(X,Y)$ orqali belgilasak, u holda $K(X,Y)$ Banax fazosi bo'ladi.

9.1-teorema. *Chekli o'lchamli operator kompaktdir.*

9.2-teorema. *Kompakt operatorga qo'shma operator kompaktdir.*

9.3-teorema. *Agar $\{A_n\}$ kompakt operatorlar ketma-ketligi A operatorga tekis yaqinlashsa, u holda A ham kompakt operator bo'ladi.*

9.4-teorema *(Hilbert-Shmidt). H Hilbert fazosida kompakt, o'z-o'ziga qo'shma, chiziqli A operator berilgan bo'sin. U holda H fazoda shunday $\{\phi_n\}$ to'la ortonormal sistema mavjudki, $A\phi_n = \lambda_n\phi_n$ va $\lim\limits_{n\to\infty} \lambda_n = 0$ tengliklar o'rinli.*

9.5-teorema. *Cheksiz o'lchamli fazodagi $A : X \to Y$ kompakt operatorning chegaralangan teskarisi mavjud emas.*

9.6-teorema. *A kompakt operatorning spekrtiga qarashli noldan farqli λ soni A uchun chekli karrali xos qiymat bo'ladi.*

Kompakt (to'la uzluksiz) operatorlar sinfi bir qator ajoyib xossalarga ega. Bu paragrafda shu xossalarga doir misollar qaraladi.

Endi operatorlarning kompakt yoki kompakt emasligini tekshirishga doir bir nechta misollar qaraymiz.

9.1. $A : \ell_p \to \ell_p (p \geq 1)$ operator $Ax = (a_1x_1, a_2x_2, \ldots, a_nx_n, \ldots,)$ tenglik bilan aniqlangan. Bu operator kompakt bo'lishi uchun $\lim_{n\to\infty} a_n = 0$ munosabatning bajarilishi zarur va yetarli. Isbotlang.

Isbot. *Yetarliligi.* $\lim_{n\to\infty} a_n = 0$ bo'lsin. Quyidagi $A_n, \quad n \in \mathbb{N}$ operatorlar ketma-ketligini qaraymiz:

$$A_n x = (a_1 x_1, a_2 x_2, \ldots, a_n x_n, 0, 0, \ldots), \qquad x = (x_1, x_2, \ldots) \in \ell_p .$$

Istalgan n natural son uchun A_n chekli o'lchamli operatordir (chunki $\dim Im A_n \leq n$). 9.1-teoremaga ko'ra A_n kompakt operator bo'ladi. Bu $\{A_n\}$ ketma-ketlikning berilgan A operatorga tekis yaqinlashishini ko'rsatamiz. Buning uchun $n \to \infty$ da $\|A_n - A\| \to 0$ bo'lishini ko'rsatish kifoya. Har bir $x \in \ell_p$ element uchun

$$\|A_n x - Ax\| = \left(\sum_{k=n+1}^{\infty} |a_k|^p \, |x_k|^p \right)^{\frac{1}{p}} \leq$$

$$\leq \sup_{n+1 \leq k < \infty} |a_k| \left(\sum_{k=n+1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} \leq \|x\| \cdot \sup_{n+1 \leq k < \infty} |a_k| .$$

Bu yerdan $\|A_n - A\| \leq \sup_{n+1 \leq k < \infty} |a_k|$ tengsizlik kelib chiqadi. $\lim_{n\to\infty} a_n = 0$ shartdan

$$\lim_{n\to\infty} \|A_n - A\| \leq \lim_{n\to\infty} \sup_{n+1 \leq k < \infty} |a_k| = 0$$

ekanligi kelib chiqadi. U holda $\lim_{n\to\infty} \|A_n - A\| = 0$. 9.3-teoremaga ko'ra, limitik operator A kompakt bo'ladi.

Zaruriyligi. Teskarisini faraz qilaylik, ya'ni A kompakt operator bo'lsin, ammo $\lim_{n\to\infty} a_n = 0$ shart bajarilmasin. U holda shunday $a > 0$ soni va n_k $(n_k \to \infty)$ natural sonlar ketma-ketligi mavjud bo'lib, $|a_{n_k}| \geq a$ tengsizliklar bajariladi. $M = \left\{ x^{(k)} \in \ell_p : x^{(k)} = e_{n_k}, \;\; k \in \mathbb{N} \right\}$ to'plamni qaraymiz. Bu to'plam chegaralangan to'plamdir, chunki ixtiyoriy $x^{(k)} \in M$ uchun $\left\| x^{(k)} \right\|$

$= 1$ tenglik o'rinli. Bu to'plamning A akslantirishdagi tasviri $A(M)$ ning nisbiy kompakt emasligini ko'rsatamiz. Haqiqatan ham,

$$y^{(k)} = Ax^{(k)} = (0, 0.., 0, 0, a_{n_k}, 0, \ldots) = a_{n_k}\, e_{n_k}$$

tenglikdan, hamda $|a_{n_k}| \geq a$ dan $i \neq j$ da,

$$\left\| y^{(i)} - y^{(j)} \right\| = \left\| a_{n_i} e_{n_i} - a_{n_j} e_{n_j} \right\| \geq \sqrt[p]{2}\, a \tag{9.1}$$

tengsizlik kelib chiqadi. (9.1) tengsizlik $A(M)$ to'plamdan olingan $\left\{y^{(k)}\right\}$ ketma-ketlikdan yaqinlashuvchi qismiy ketma-ketlik ajratib olish mumkin emasligini ko'rsatadi, ya'ni $A(M)$ nisbiy kompakt to'plam emas. A operator chegaralangan M to'plamni nisbiy kompakt bo'lmagan $A(M)$ to'plamga akslantirgani uchun A operator kompakt emas. Bu ziddiyat $\lim\limits_{n\to\infty} a_n = 0$ munosabatni isbotlaydi. $\square$

9.2. $A : C[a,\ b] \to C[a,\ b]$, $(Ax)(s) = \int\limits_a^b K(s,t)\, x(t) dt$ operator berilgan. Bu yerda $K(s,t)$, $T = [a,\ b] \times [a,\ b]$ kvadratda uzluksiz bo'lgan biror funksiya. Shu operatorning kompakt ekanligini ko'rsating.

Yechish. $K(s,\ t)$ funksiya T yopiq to'plamda uzluksiz bo'lganligi sababli Veyershtrass teoremasiga ko'ra ixtiyoriy $\varepsilon > 0$ soni uchun shunday n natural son va

$$P_n(s,t) = \sum_{k=1}^{n} \sum_{l=1}^{n} C_{kl}\, s^k t^l$$

ko'phad mavjud bo'lib,

$$\|K - P_n\| \leq \max_{a \leq s,\, t \leq b} |\, K(s,t) - P_n(s,t)\,| < \varepsilon$$

tengsizlik bajariladi. Aytaylik, $k, l -$ ixtiyoriy natural sonlar bo'lsin. $C[a,\ b]$ ni bir o'lchamli $c\,t^k$ ko'rinishdagi funksiyalar fazosiga akslantiruvchi

$$(A_{k,l}x)(s) = s^k \int_a^b t^l\, x(t)\, dt$$

operatorni qaraymiz. $A_{k,l}$ chegaralangan va bir oʻlchamli operator boʻlganligi uchun 9.1-teoremaga koʻra u kompakt operator boʻladi. Ixtiyoriy n uchun

$$(A_n x)(s) = \int_a^b P_n(s,t)\, x(t)\, dt = \sum_{k=1}^{n} \sum_{l=1}^{n} C_{kl} (A_{k,l} x)(s)$$

operator $A_{k,l}$ koʻrinishdagi kompakt operatorlarning chekli chiziqli kombinatsiyasidan iborat boʻlganligi uchun kompakt operatordir. Bundan tashqari,

$$\|A_n x - Ax\| = \max_{a \le s \le b} \left| \int_a^b P_n(s,t)\, x(t)\, dt - \int_a^b K(s,t)\, x(t)\, dt \right| \le$$

$$\max_{a \le s \le b} \int_a^b |\, P_n(s,t) - K(s,t)| \cdot |x(t)| dt \le (b-a) \cdot \|P_n - K\| \cdot \|x\| \,.$$

Bu yerdan

$$\|A_n - A\,\| \le (b-a) \cdot \|P_n - K\| \le \varepsilon\, (b-a) \qquad (9.2)$$

tengsizlik kelib chiqadi. (9.2) dagi $\varepsilon > 0$ ixtiyoriy boʻlganligi sababli shunday $P_n(s,t)$ koʻphadlar ketma-ketligini tanlash mumkinki,

$$\lim_{n \to \infty} \|\, A_n - A\| = 0 \,.$$

Shunday qilib, $\{A_n\}$ kompakt operatorlar ketma-ketligi A operatorga tekis yaqinlashar ekan. U holda 9.3-teoremaga koʻra, A ning kompakt operator ekanligini olamiz. □

Izoh. 9.2-misolda keltirilgan A operatorning kompaktligini $K(s,t)$ ning $[a,\ b] \times [a,\ b]$ kvadratda tekis uzluksizligidan va Arsela teoremasidan ham keltirib chiqarish mumkin.

9.3. $C[0,\ 1]$ Banax fazosida

$$(Ax)(t) = \begin{cases} \frac{1}{t} \int_0^t x(s) ds, & t \ne 0, \\ (Ax)(0) = x(0) & \end{cases}$$

formula bilan aniqlangan A operatorning chegaralangan, ammo kompakt emasligini koʻrsating.

Yechish. a) A operator uchun $D(A) = C[0,\ 1]$ va $ImA \subset C[0,\ 1]$ ekanligini, ya'ni ixtiyoriy $x \in C[0,\ 1]$ uchun $Ax \in C[0,\ 1]$ ekanligini ko'rsatamiz. Ixtiyoriy $x \in C[0,\ 1]$ uchun $y(t) = (Ax)(t)$ funksiyaning $t = 0$ nuqtada o'ngdan uzluksizligi x funksiyaning $t = 0$ nuqtada uzluksizligidan va

$$|y(t) - y(0)| = \left| \frac{1}{t} \int_0^t x(s)\, ds - \frac{1}{t} \int_0^t x(0)\, ds \right| \leq \frac{1}{t} \int_0^t |x(s) - x(0)|\, ds$$

tengsizlikdan kelib chiqadi. Endi $t_0 \neq 0$ bo'lsin deb faraz qilaylik. U holda ixtiyoriy $0 < t \leq 1$ uchun quyidagiga ega bo'lamiz:

$$y(t) - y(t_0) = \frac{1}{t} \int_0^t x(s)\, ds - \frac{1}{t_0} \int_0^{t_0} x(s)\, ds = \frac{1}{t} \int_{t_0}^t x(s)\, ds + (\frac{1}{t} - \frac{1}{t_0}) \int_0^{t_0} x(s)\, ds$$

yoki

$$|\, y(t) - y(t_0)\, | \leq \frac{\|x\|}{t}\ |\, t - t_0\, | + \left| \frac{1}{t} - \frac{1}{t_0} \right| \|x\|\,. \qquad (9.3)$$

(9.3) dan $\lim\limits_{t \to t_0} y(t) = y(t_0)$ tenglik kelib chiqadi. Shunday qilib, y funksiya $[0,\ 1]$ da uzluksiz ekan, ya'ni $y = Ax \in C[0,\ 1]$.

b) Endi A ning chegaralangan operator ekanligini ko'rsatamiz.

$$\|Ax\| = \max_{0 \leq t \leq 1} \left| \frac{1}{t} \int_0^t x(s)\, ds \right| \leq \|x\| \max_{0 \leq t \leq 1} \frac{1}{t} \int_0^t ds = \|x\|\,.$$

Demak, $\|A\| \leq 1$, ya'ni A chegaralangan operator ekan.

c) A ning kompakt operator emasligini ko'rsatamiz. Uzluksiz funksiyalar ketma-ketligi $x_n(t),\ \ n = 0, 1, 2, \ldots$ ni quyidagicha tanlaymiz:

$$x_n(t) = \begin{cases} 0, \quad agar \ \ t \notin (2^{-n-1},\ 2^{-n}), \\ 1 - 2^{n+2} \left| t - 3 \cdot 2^{-n-2} \right|, \quad agar \ \ t \in (2^{-n-1},\ 2^{-n})\,. \end{cases}$$

$\{x_n\}$ chegaralangan ketma-ketlikdir, chunki ixtiyoriy n uchun

$$\|x_n\| = \max_{0 \leq t \leq 1} |x_n(t)| = 1.$$

$y_n(t) = (Ax_n)(t), \quad n = 1, 2, \ldots,$ funksiyalarni topamiz:

$$y_n(t) = \begin{cases} 0, & t \in \left[0,\ 2^{-n-1}\right], \\ \dfrac{1}{t}\int\limits_0^t \left(1 - 2^{n+2}\left|s - 3\cdot 2^{-n-2}\right|\right) ds, & t \in \left(2^{-n-1}, 2^{-n}\right), \\ \dfrac{1}{2^{n+2}\,t}, & t \in \left[2^{-n}, 1\right]. \end{cases}$$

U holda $y_n(2^{-n}) = 2^n \cdot 2^{-n-2} = 2^{-2}$, $y_{n-m}(2^{-n}) = 0$, $n = 2, 3, 4, \ldots$, $m = 1, 2, 3, \ldots$, $n \neq m$ chunki $y_{n-m}(t) = 0$, $t \in [0,\ 2^{-n+m-1}]$. Bu tengliklardan $n \neq m$ bo‘lganda

$$\|y_n - y_m\| \geq \frac{1}{4} \tag{9.4}$$

kelib chiqadi. (9.4) tengsizlikdan ko‘rinadiki, $\{y_n = Ax_n\}$ ketma-ketlikdan yaqinlachuvchi qismiy ketma-ketlik ajratib olish mumkin emas. Bundan A kompakt operator emas degan xulosaga kelamiz. □

9.4. $A : C[0,\ 1] \to C[0,\ 1]$, $(Ax)(t) = x(t^4)$ operatorning kompakt emasligini ko‘rsating.

Yechish. Dastlab A teskarilanuvchan operator ekanligini ko‘rsatamiz. $Ax = 0$ yoki $x(t^4) = 0$ tenglama $t^4 = s$ almashtirishdan keyin $x(s) = 0$ tenglamaga keladi. Shuning uchun, $Ax = 0$ tenglama faqat $x = 0$ yechimga ega, shunday ekan, A teskarilanuvchan operator. Ixtiyoriy $y \in C[0,\ 1]$ uchun $Ax = y$ tenglama yoki $x(t^4) = y(t)$ tenglamani yechamiz. Agar $s = t^4$ almashtirishni olsak, $0 \leq t \leq 1$ bo‘lganda $0 \leq s \leq 1$ bo‘ladi va $x(t^4) = y(t)$ tenglama $x(s) = y(\sqrt[4]{s})$ ko‘rinishni oladi, ya’ni $Ax = y$ tenglama yechimi $x(t) = y(\sqrt[4]{t})$ ko‘rinishga ega. Bu yerdan A^{-1} operator $C[0,\ 1]$ fazoning hamma yerida aniqlanganligi va $(A^{-1}y)(t) = y(\sqrt[4]{t})$ formula vositasida ta’sir qilishi kelib chiqadi. Endi ixtiyoriy $y \in C[0,\ 1]$ uchun

$$\left\|A^{-1}y\right\| = \max_{0\leq t\leq 1}\left|y(\sqrt[4]{t})\right| = \max_{o\leq \sqrt[4]{t}\leq 1}\left|y(\sqrt[4]{t})\right| = \max_{0\leq s\leq 1}|\,y(s)\,| = \|y\|$$

munosobatlar o‘rinli bo‘lgani uchun A^{-1} chegaralangan bo‘ladi. 9.5-teoremaga ko‘ra, cheksiz o‘lchamli fazoda kompakt operatorning chegaralangan teskarisi mavjud emas. Demak, A kompakt operator emas. □

Masalani $\{x_n(t) = t^n\}$ chegaralangan ketma-ketlikning tasviri nisbiy kompakt emasligini ko‘rsatish orqali ham yechish mumkin.

Uy vazifalari va mavzuni o‘zlashtirish uchun masalalar

9.5-9.24-misollarda keltirilgan operatorlarning kompaktligini ko‘rsating.

9.5. $A : C[0,\,1] \to C[0,\,1]\,,\ (Ax)(t) = \int\limits_0^1 (e^{s+t} + s\,t)\,x(t)\,dt.$

9.6. $A : C[0,\,\pi] \to C[0,\,\pi]\,,\ (Ax)(s) = \int\limits_0^{\pi} \cos(s+t)\,x(t)\,dt.$

9.7. $A : C[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = x(0)\,t + x(1)\,t^2.$

9.8. $A : C[0,\ 2\pi] \to C[0,\ 2\pi],\ (Ax)(s) = \int\limits_0^{2\pi} \sin(s+\ t)\,x(t)\,dt.$

9.9. $A : C[0,\ \pi] \to C[0,\ \pi],\ (Ax)(s) = \int\limits_0^{\pi} \cos(s-\ t)\,x(t)\,dt.$

9.10. $A : C[0,\ 3] \to C[0,\ 3],\ (Ax)(t) = x(0) + x(1)\,t + x(2)\,t^2 + x(3)\,t^3.$

9.11. $A : C[0,\ 1] \to C[0,\ 1],\ (Ax)(s) = \int\limits_0^1 \dfrac{1}{1+s\,t}\ x(t)\,dt.$

9.12. $A : C[-1,\ 1] \to C[-1,\ 1],\ (Ax)(s) = \int\limits_{-1}^1 \dfrac{1}{9-s^2t^2}\ x(t)\,dt.$

9.13. $A : L_2[0,\,1] \to L_2[0,\,1],\ (Ax)(s) = \int\limits_0^1 (s^2t + s\,t^2)\,x(t)\,dt.$

9.14. $A : L_2[0,\ \pi] \to L_2[0,\ \pi],\ (Ax)(s) = \int\limits_0^{\pi} (s\ \cos t + t\ \cos\ s)\,x(t)\,dt.$

9.15. $A : L_2[0,\ 1] \to L_2[0,\ 1],\ (Ax)(s) = \int\limits_0^1 \dfrac{1}{2-s\,t}\ x(t)\,dt.$

9.16. $A : \ell_1 \to \ell_1\,,\ Ax = (0, \dfrac{x_2}{2}, 0, \dfrac{x_4}{4}, \ldots, 0, \dfrac{x_{2n}}{2n}, \ldots).$

9.17. $A : \ell_1 \to \ell_1\,,\ Ax = (x_1, 2x_2, 3x_3, \dots, 9x_9, 10x_{10}, 0, 0, \dots).$

9.18. $A : \ell_2 \to \ell_2\,,\ Ax = (x_1, \dfrac{x_2}{\ln 2}, \dfrac{x_3}{\ln 3}, \dots, \dfrac{x_n}{\ln n}, \dots).$

9.19. $A : \ell_2 \to \ell_2,\ \ Ax = (5x_1, 4x_2, 3x_3, 2x_4, x_5, \dfrac{1}{2}x_6, \dots, \dfrac{1}{n-4}x_n, \dots).$

9.20. $A : \ell_2 \to \ell_2,\ \ Ax = (100x_1, 99x_2, \dots, 2x_{99}, x_{100}, \dfrac{1}{101}x_{101}, \dots, \dfrac{1}{n}x_n, \dots).$

9.21. $A : \ell_3 \to \ell_3\,,\ Ax = \big(\ln 2 \cdot x_1,\ \ln(1 + \tfrac{1}{2}) \cdot x_2, \dots,\ \ln(1 + \tfrac{1}{n}) \cdot x_n, \dots\big)\,.$

9.22. $A : \ell_4 \to \ell_4,\ Ax = (0, 0, 0, 0, x_1, \dfrac{1}{2}\,x_2, \dfrac{1}{3}\,x_3, \dfrac{1}{4}\,x_4, \dots, \dfrac{1}{n}\,x_n, \dots).$

9.23. $A : \ell_5 \to \ell_5\,,\ Ax = \big(\mathrm{arctg}1 \cdot x_1,\ \mathrm{arctg}\tfrac{1}{2} \cdot x_2, \dots,\ \mathrm{arctg}\tfrac{1}{n} \cdot x_n, \dots\big)\,.$

9.24. $A : m \to m\,,\ Ax = (x_1, x_1 + x_2, x_2 + x_3, x_3 + x_4, x_5, x_6, 0, 0, 0, \dots).$

9.25-9.44-misollarda keltirilgan operatorlarning kompakt emasligini koʻrsating.

9.25. $A : C[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = (t+1)\,x(t).$

9.26. $A : C[-1, 1] \to C[-1, 1],\ (Ax)(t) = \big(t^2 + 1\big)\ x(t).$

9.27. $A : C[0, 1] \to C[0,\ 1],\ (Ax)(t) = \sqrt{t+1}x(t).$

9.28. $A : C[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = (1 + 2t)\,x(t).$

9.29. $A : C[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = x(t^2)\,.$

9.30. $A : C[-1,\ 1] \to C[-1,\ 1],\ (Ax)(t) = x(t^3)\,.$

9.31. $A : C[-1, 1] \to C[-1, 1],\ (Ax)(t) = \dfrac{1}{2}[x(t) + x(-t)].$

9.32. $A : L_2[0,\ 1] \to L_2[0,\ 1],\ (Ax)(t) = (\sin t + \cos t)x(t).$

9.33. $A : L_2[-1,\ 1] \to L_2[-1,\ 1],\ (Ax)(t) = (t^2 + 2t + 3)\,x(t).$

9.34. $A : L_2[0,\ \infty) \to L_2[0,\ \infty),\ (Ax)(t) = \dfrac{t+3}{t+4}\,x(t).$

9.35. $A:\ell_1\to\ell_1,\ Ax=(0,x_2,0,x_4,\ldots,0,x_{2n},\ldots)$.

9.36. $A:\ell_1\to\ell_1,\ Ax=\left(\sin\frac{\pi}{4}\cdot x_1,\ \sin\frac{2\pi}{4}\cdot x_2,\ldots,\ \sin\frac{n\pi}{4}\cdot x_n,\ldots\right)$.

9.37. $A:\ell_1\to\ell_1,\ Ax=(x_1,(1+\frac{1}{4})^2x_2,(1+\frac{1}{9})^3x_3,\ldots,(1+\frac{1}{n^2})^nx_n,\ldots)$.

9.38. $A:\ell_2\to\ell_2,\ Ax=(2x_1,0,2x_3,0,\ldots,2x_{2n-1},0,\ldots)$.

9.39. $A:\ell_2\to\ell_2,\ Ax=\left(\frac{1}{5}x_1,\frac{2}{9}x_2,\frac{3}{13}x_3,\ldots,\frac{n}{4n+1}x_n,\ldots\right)$.

9.40. $A:\ell_2\to\ell_2,\ Ax=(x_1,\frac{1}{2}x_2,\ldots,\frac{1}{10}x_{10},x_{11},x_{12},\ldots)$.

9.41. $A:\ell_3\to\ell_3,\ Ax=(x_1,\frac{1}{2}\,x_2,x_3,\frac{1}{4}\,x_4,\ldots,x_{2n+1},\frac{1}{2n}\,x_{2n},\ldots)$.

9.42. $A:\ell_4\to\ell_4,\ Ax=\left(2x_1,(1+\frac{1}{2})^2\,x_2,\ldots,(1+\frac{1}{n})^n\,x_n,\ldots\right)$.

9.43. $A:\ell_5\to\ell_5,\ Ax=(x_1,0,0,0,x_5,0,0,0,x_9,\ldots,x_{4n+1},0,0,0,\ldots)$.

9.44. $A:\ell_5\to\ell_5,\ Ax=(0,0,0,0,0,x_1,x_2,x_3,\ldots,x_n,\ldots)$.

9.45. $\varphi\in C[0,\,1]$ nolmas funksiyaga qanday shartlar qo‘yilganda $A:C[0,\,1]\to C[0,\,1]$, $(A\,x)(t)=\varphi\,(t)\,x(t)$ operator kompakt bo‘ladi.

9.46. $A:C[0,\,1]\to C[0,\,1]$, $(A\,x)(t)=\int\limits_0^1 K(t,s)\,x(s)\,ds+\sum\limits_{k=1}^{n}\varphi_k(t)\,x(t_k)$ operator berilgan, bu yerda $K(s,t)$, $T=\{(s,t):\ 0\le s,t\le 1\}$ birlik kvadratda uzluksiz bo‘lgan biror funksiya. $\varphi_k\in C[0,1]$, $t_k\in[0,\,1]$, $k=1,2,\ldots,n$. Bu operatorning kompaktligini isbotlang.

9.47. $(A\,x)(t)=\,x'(t)$ differensial operator

a) $C^{(1)}[0,\ 1]$ ni $C[0,\ 1]$ ga;

b) $C^{(2)}[0,\ 1]$ ni $C^{(1)}[0,\ 1]$ ga;

c) $C^{(2)}[0,1]$ ni $C[0,1]$ ga, akslantiruvchi operator sifatida kompakt bo‘ladimi?

9.48. $A : L_2[a,\, b] \to L_2[a,\, b],\ (A\,x)(t) = \int\limits_a^t x(\tau)\, d\tau$ operatorning kompakt ekanligini isbotlang.

9.49. Quyidagi operatorlardan qaysilari kompakt operator bo'ladi?

a) $A : \ell_2 \to \ell_2\,, \quad Ax = (0, x_1, x_2, \ldots)\,;$

b) $A : \ell_2 \to \ell_2\,, \quad Ax = \Big(x_1, \dfrac{x_2}{2}, \dfrac{x_3}{3}, \ldots\Big)\;;$

c) $A : \ell_2 \to \ell_2\,, \quad Ax = \Big(0, x_1, \dfrac{x_2}{2}, \dfrac{x_3}{3}, \ldots\Big)\,.$

9.50. Quyidagi ichiga joylashtirish operatorlarining kompakt bo'lishini isbotlang:

a) $I :\ C^{(1)}[a,\, b] \to C[a,\, b]\,, \quad Ix = x,$

b) $I :\ H^{(1)}[a,\, b] \to C[a,\, b]\,, \quad Ix = x.$

Bu yerda $H^{(1)}[a,\, b] - [a,\, b]$ kesmada uzluksiz differensiallanuvchi funksiyalar fazosi bo'lib, unda skalyar ko'paytma

$$(x, y) = \int\limits_a^b [x(t)\,\overline{y(t)} + x'(t)\,\overline{y'(t)}\,]\, dt$$

tenglik bilan aniqlanadi.

9.51. $A :\ H^{(1)}[a, b] \to L_2[a, b]\,, \quad (A\,x)(t) = x'(t)$ operatorning kompakt emasligini isbotlang.

9.52. $A : L_2[-1, 1] \to L_2[-1, 1]\,, \ (A\,x)(t) = \int\limits_{-1}^{1} t^2 s\, x(s)\, ds$ operatorning kompakt ekanligini isbotlang va spektrini toping.

9.53. $A : L_2[0, 1] \to L_2[0, 1]\,, \ \ (A\,x)(t) = \int\limits_0^1 t\, s\,(1 - ts)\, x(s)\, ds$ operatorning kompakt ekanligini isbotlang va spektrini toping.

9.54. O'z-o'ziga qo'shma operatorning har xil xos qiymatlariga mos keluvchi xos vektorlari o'zaro ortogonal ekanligini isbotlang.

9.55. Cheksiz o‘lchamli H Hilbert fazosida berilgan o‘z-o‘ziga qo‘shma kompakt A operator cheklita xos qiymatlarga ega bo‘lsin. U holda $\lambda = 0$ son A operatorning xos qiymati bo‘lishini isbotlang.

9.56. H separabel Hilbert fazosida A kompakt operator berilgan bo‘lsin.

a) A^*A ning o‘z-o‘ziga qo‘shma kompakt operator bo‘lishini isbotlang.

b) $A^*Ax = \sum_n \mu_n (x, h_n) h_n$ tasvirda barcha n larda $\mu_n > 0$ ekanligini isbotlang.

c) $\lambda_n = \sqrt{\mu_n}$, $e_n = \frac{1}{\lambda_n} Ah_n$ bo‘lsin. $\{e_n\}$ lar ortonormal sistema tashkil qilishini va ixtiyoriy $x \in H$ uchun

$$A x = \sum_n \lambda_n (x, h_n) e_n$$

tasvir o‘rinli ekanligini isbotlang. λ_n sonlar A operatorning *singulyar sonlari* deyiladi.

9.57. $V : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$, $(Vf)(n) = v(n)f(n)$ operatorning singulyar sonlarini toping. Bu yerda barcha $n \in \mathbb{Z}$ lar uchun $v(n) > 0$ deb faraz qilinadi.

10-§. Integral tenglamalar

Funksional fazoda tenglama berilgan bo‘lib, noma’lum element funksiyadan iborat bo‘lsa, bunday tenglama *funksional tenglama* deyiladi. Agar funksional tenglamada noma’lum funksiya integral ostida bo‘lsa, u holda tenglama *integral tenglama* deyiladi.

$$\int_a^b K(s,t)\phi(t)dt + f(s) = 0, \tag{10.1}$$

$$\phi(s) = \int_a^b K(s,t)\phi(t)dt + f(s). \tag{10.2}$$

Bu yerda $\phi-$ noma'lum funksiya, $K(s,t)$ va $f(s)$ berilgan funksiyalar. (10.1) va (10.2) tenglamalar mos ravishda *birinchi va ikkinchi tur Fredholm tenglamalari* deyiladi. Xususan, $K(s,t)$ funksiya $t > s$ qiymatlar uchun $K(s,t) = 0$ shartni qanoatlantirsa, u holda (10.1) va (10.2) tenglamalar mos ravishda *birinchi va ikkinchi tur Volterra tenglamalari* deyiladi. (10.2) tenglamaning *yadrosi* deb nomlanuvchi $K(s,t)$ funksiyadan quyidagilar talab qilinadi, u – o'lchovli va

$$\int_a^b \int_a^b |K(s,t)|^2 dsdt < \infty \tag{10.3}$$

shartni qanoatlantiradi. Agar $K(s,t) = \sum\limits_{k=1}^{n} a_k(s)\, b_k(t)$ ko'rinishda bo'lsa (10.2) tenglama *ajralgan yadroli* integral tenglama deyiladi. $L_2[a,\ b]$ Hilbert fazosida aniqlangan

$$(T\phi)(s) = \int_a^b K(s,t)\phi(t)dt \tag{10.4}$$

operator K *yadroli Fredholm operatori* deyiladi.

10.1-ta'rif. *Agar biror $\lambda \in \mathbb{C}$ uchun*

$$\phi(s) = \lambda \int_a^b K(s,t)\phi(t)dt \Longleftrightarrow \phi(s) = \lambda(T\phi)(s)$$

tenglama noldan farqli yechimga ega bo'lsa, λ integral tenglamaning xarakteristik soni deyiladi. Tenglamaning nolmas yechimi esa λ xarakteristik songa mos xos funksiya deyiladi.

Agar $\lambda \neq 0$ integral tenglamaning xarakteristik soni bo'lsa, u holda $\dfrac{1}{\lambda}$ soni T operatorning xos qiymati bo'ladi.

10.1-teorema. *Agar $K(s,t)$ yadro (10.3) shartni qanoatlantirsa, u holda $L_2[a,\ b]$ fazoda (10.4) tenglik bilan aniqlanuvchi T operator chiziqli, kompakt va uning normasi uchun quyidagi tengsizlik o'rinli*

$$\| T \| \leq \sqrt{\int_a^b \int_a^b |K(s,t)|^2 dsdt}. \tag{10.5}$$

$L_2[a, \ b]$ fazoda (10.4) tenglik bilan aniqlanuvchi T operator o'z-o'ziga qo'shma (3.19-misolga qarang) bo'lishi uchun, deyarli barcha $s, t \in [a, \ b]$ larda $K(s,t) = \overline{K(t,s)}$ tenglikning bajarilishi zarur va yetarli.

10.2-teorema. *Agar* 1 *soni* $T = T^*$ *operator uchun xos qiymat bo'lmasa, u holda (10.2) tenglama ixtiyoriy* f *uchun yagona yechimga ega. Agar* 1 *soni* T *operator uchun xos qiymat bo'lsa, u holda (10.2) tenglama yechimga ega bo'lishi uchun* f *funksiya* 1 *soniga mos keluvchi barcha xos funksiyalarga ortogonal bo'lishi zarur va yetarli.*

X Banax fazosida biror T kompakt (to'la uzluksiz) operatorni olib,

$$x - Tx = y \tag{10.6}$$

ko'rinishdagi tenglamani qaraymiz. (10.6) tenglama bilan bir qatorda bir jinsli bo'lgan

$$x - Tx = 0 \tag{10.7}$$

tenglamani va ularga qo'shma bo'lgan

$$f - T^* f = g \tag{10.8}$$

$$f - T^* f = 0 \tag{10.9}$$

tenglamalarni qaraymiz.

Quyida keltiriladigan Fredholm teoremalari shu to'rt tenglamaning yechimlari orasidagi bog'lanishlarni ifodalaydi.

10.3-teorema *(10.6) tenglama berilgan* $y \in X$ *da yechimga ega bo'lishi uchun (10.9) bir jinsli tenglamaning yechimi bo'lgan har bir* $f \in X^*$ *da* $f(y) = 0$ *shartning bajarilishi zarur va yetarli.*

10.4-teorema *(Fredholm alternativasi). Yo (10.6) tenglama ixtiyoriy* $y \in X$ *da yagona yechimga ega, yo (10.7) bir jinsli tenglama noldan farqli yechimga ega.*

10.5-teorema. *Bir jinsli (10.7) va (10.9) tenglamalarning chiziqli erkli yechimlari soni chekli va o'zaro teng. Boshqacha qilib aytganda,*

$$\dim\,Ker(I-T) = \dim\,Ker(I-T^*) < \infty.$$

Integral tenglamalarga oid topshiriqlarni bajarish uchun Fredholm integral tenglamasi, uning turlari va Fredholm teoremalari haqida qo'shimcha ma'lumotlarni [9] ning $37-40-\S\S$ laridan qarab olish mumkin.

Integral tenglamalarni yechishga doir bir nechta misollar qaraymiz.

10.1. Ushbu

$$x(s) - \int_{-1}^{1} (s\,t + s^2)\, x(t)\, dt = 1$$

ajralgan yadroli integral tenglamani yeching.

Yechish. Berilgan integral tenglamani quyidagicha yozib olamiz:

$$x(s) - s\int_{-1}^{1} t\, x(t)\, dt - s^2\int_{-1}^{1} x(t)\, dt = 1. \tag{10.10}$$

Agar

$$\alpha_1 = \int_{-1}^{1} t\, x(t)\, dt, \qquad \alpha_2 = \int_{-1}^{1} x(t)\, dt \tag{10.11}$$

belgilashlarni kiritsak, (10.10) dan $x(s)$ uchun

$$x(s) = 1 + \alpha_1\, s + \alpha_2 s^2 \tag{10.12}$$

ifodani hosil qilamiz. Agar (10.12) dagi α_1 va α_2 o'zgarmaslar aniqlansa, (10.12) tenglik bilan aniqlangan x funksiya berilgan integral tenglamaning yechimi bo'ladi. α_1 va α_2 o'zgarmaslarni aniqlash uchun (10.12) ni (10.11) ga qo'yib, quyidagi chiziqli tenglamalar sistemasini hosil qilamiz:

$$\begin{cases} \alpha_1 = \int\limits_{-1}^{1} t\ (1+\alpha_1 t\ +\ \alpha_2 t^2)\, dt\ = \dfrac{2}{3}\alpha_1 \\ \alpha_2 = \int\limits_{-1}^{1} (1+\alpha_1 t\ +\ \alpha_2 t^2)\, dt\ = 2 + \dfrac{2}{3}\alpha_2. \end{cases} \tag{10.13}$$

Biz bu yerda

$$\int_{-1}^{1} dt = 2, \quad \int_{-1}^{1} t\,dt = 0, \quad \int_{-1}^{1} t^2 dt = \frac{2}{3}, \quad \int_{-1}^{1} t^3 dt = 0$$

tengliklardan foydalandik. (10.13) sistemani quyidagicha yozish mumkin:

$$\begin{cases} \frac{1}{3}\alpha_1 = 0, \\ \frac{1}{3}\alpha_2 = 2. \end{cases}$$

Bu yerdan $\alpha_1 = 0, \quad \alpha_2 = 6$ ni olamiz. Demak, berilgan tenglama yechimi $x(s) = 1 + 6s^2$ funksiyadan iborat bo'ladi. □

10.2. $C[a,\ b]$ fazoda bir jinsli

$$x(s) - \lambda \int_{-\pi}^{\pi} \sin(3s+t)x(t)dt = 0$$

ajralgan yadroli integral tenglamani yeching.

Yechish. Agar $\sin(3s+t) = \sin 3s \cdot \cos t + \cos 3s \cdot \sin t$ ayniyatni hisobga olsak, berilgan integral tenglamani quyidagicha yozish mumkin:

$$x(s) = \lambda \sin 3s \int_{-\pi}^{\pi} \cos t\ x(t)\,dt + \lambda \cos 3s \int_{-\pi}^{\pi} \sin t\ x(t)\,dt. \tag{10.14}$$

Bu yerda

$$\alpha_1 = \int_{-\pi}^{\pi} \cos t\, x(t)dt, \qquad \alpha_2 = \int_{-\pi}^{\pi} \sin t\, x(t)dt \tag{10.15}$$

belgilashlarni kiritsak, (10.14) dan $x(s)$ uchun

$$x(s) = \lambda\alpha_1 \sin 3s + \lambda\alpha_2 \cos 3s \tag{10.16}$$

ifodani olamiz. Endi α_1 va α_2 o'zgarmaslarni topish uchun (10.16) ni (10.15) tengliklarga qo'yib,

$$\begin{cases} \alpha_1 = \lambda\alpha_1 \int_{-\pi}^{\pi} \sin 3t \cdot \cos t\,dt + \lambda\alpha_2 \int_{-\pi}^{\pi} \cos 3t \cdot \cos t\,dt \\ \alpha_2 = \lambda\alpha_1 \int_{-\pi}^{\pi} \sin 3t \cdot \sin t\,dt + \lambda\alpha_2 \int_{-\pi}^{\pi} \sin t \cdot \cos 3t\,dt \end{cases} \tag{10.17}$$

algebraik tenglamalar sistemasini hosil qilamiz. Agar (10.17) da

$$\int_{-\pi}^{\pi} \sin 3t \cdot \cos t \, dt = 0, \quad \int_{-\pi}^{\pi} \cos 3t \cdot \cos t \, dt = 0, \quad \int_{-\pi}^{\pi} \sin 3t \cdot \sin t \, dt = 0,$$

$$\int_{-\pi}^{\pi} \sin t \cdot \cos 3t \, dt = 0, \quad \int_{-\pi}^{\pi} \sin t \cdot \cos t \, dt = 0$$

ekanligini e'tiborga olsak, $\alpha_1 = 0$, $\alpha_2 = 0$ larni hosil qilamiz. Demak, tekshirilayotgan integral tenglama λ parametrning barcha qiymatlari uchun yagona $x(s) = \lambda\alpha_1 \sin 3s + \lambda\alpha_2 \cos 3s = 0$ nol yechimga ega. □

10.3. Agar

a) $a = 0, \quad b = 1, \quad K(t,s) = \begin{cases} t, & agar \ \ 0 \le t \le s \le 1, \\ s, & agar \ \ 0 \le s < t \le 1; \end{cases}$

b) $a = 0, \ b = \pi/2, \ K(t,s) = \begin{cases} \sin t \, \cos s, & agar \ \ 0 \le t \le s \le \dfrac{\pi}{2}, \\ \sin s \, \cos t, & agar \ \ 0 \le s < t \le \dfrac{\pi}{2}; \end{cases}$

c) $a = 0, \ b = \pi, \ K(t,s) = \begin{cases} \sin t \, \cos s, & agar \ \ 0 \le t \le s \le \pi, \\ \sin s \, \cos t, & agar \ \ 0 \le s < t \le \pi; \end{cases}$

d) $a = 0, \ b = 1, \ K(t,s) = \begin{cases} s(t+1), & agar \ \ 0 \le t \le s \le 1, \\ t(s+1), & agar \ \ 0 \le s < t \le 1; \end{cases}$

e) $a = 0, \quad b = 1, \quad K(t,s) = e^{-|t-s|}$;

f) $a = 0, \ b = 1, \ K(t,s) = \begin{cases} (t+1)(s-2), & agar \ \ 0 \le t \le s \le 1, \\ (s+1)(t-2), & agar \ \ 0 \le s < t \le 1 \end{cases}$

bo'lsa, $L_2[a,b]$ kompleks Hilbert fazosida

$$x(t) - \lambda \int_a^b K(t,s) x(s) ds = 0$$

integral tenglamaning λ_n xarakteristik sonlari ($\dfrac{1}{\lambda_n}$ lar T operatorning xos qiymatlari) va φ_n xos funksiyalarini toping.

Yechish. Biz misolning c) qismining yechimini keltiramiz. Maqsadimiz

$$x(t) - \lambda \int_0^\pi K(t,s)x(s)ds = 0 \Longleftrightarrow x(t) = \lambda(Tx)(t)$$

tenglamaning xarakteristik sonlari λ_n va ularga mos φ_n xos funksiyalarini topishdan iborat. Integral operatorning yadrosi $K(t,s)$ haqiqiy qiymatli va simmetrik $K(t,s) = K(s,t)$ boʻlganligi uchun, $K(t,s)$ yordamida (10.4) tenglik bilan aniqlangan T operator oʻz-oʻziga qoʻshma kompakt operator boʻladi. Bu yerdan va 8.4-teoremaning b) bandiga koʻra, λ_n lar haqiqiy boʻladi. $K(t,s)$ ning berilishidan foydalanib, integral tenglamani quyidagicha yozib olamiz:

$$x(t) = \lambda \cos t \int_0^t \sin s\, x(s)ds + \lambda \sin t \int_t^\pi \cos s\, x(s)ds. \qquad (10.18)$$

Agar $x \in L_2[0,\ \pi]$ boʻlsa, $\int_0^t \sin s\, x(s)ds$ va $\int_t^\pi \cos s\, x(s)ds$ lar absolyut uzluksiz funksiyalar boʻladi. Agar $x \in AC[0,\ \pi]$ funksiya (10.18) tenglamaning yechimi boʻlsa, u differensiallanuvchi boʻladi. Xuddi shunday koʻrsatish mumkinki $x \in C^{(2)}[0,\ \pi]$ boʻladi va (10.18) tenglikda $t = 0$ deb, $x(0) = 0$ ni olamiz. (10.18) tenglikni t boʻyicha differensiallab va oʻxshash hadlarni ixchamlab

$$x'(t) = \lambda \cos t \int_t^\pi \cos s\, x(s)ds - \lambda \sin t \int_0^t \sin s\, x(s)ds \qquad (10.19)$$

ni olamiz. (10.19) tenglikda $t = \pi$ deb, $x'(\pi) = 0$ ni olamiz. (10.19) tenglikdan t boʻyicha hosila olib,

$$x''(t) = -\lambda \left(\sin t \int_t^\pi \cos s\, x(s)ds + \cos t \int_0^t \sin s\, x(s)ds \right) -$$

$$-\lambda(\cos^2 t + \sin^2 t)x(t) \Longleftrightarrow x''(t) + (\lambda + 1)x(t) = 0 \qquad (10.20)$$

ni hosil qilamiz. Shunday qilib,

$$\begin{cases} x''(t) + (\lambda + 1)x(t) = 0 \\ x(0) = x'(\pi) = 0 \end{cases} \tag{10.21}$$

chegaraviy masalaga keldik. Agar $\lambda+1 < 0$ bo‘lsa, (10.20) differensial tenglamaning umumiy yechimi $x(t) = C_1 \operatorname{sh} \sqrt{\lambda+1}\, t + C_2 \operatorname{ch} \sqrt{\lambda+1}\, t$ bo‘ladi. Chegaraviy shartlardan foydalanib, $x(0) = C_2 = 0$ va $x'(\pi) = C_1 \operatorname{ch} \sqrt{\lambda+1}\, \pi = 0$ ni, ya’ni $x(t) = 0$ ni olamiz. Bu yerdan (10.18) integral tenglama -1 dan kichik xarakteristik sonlarga ega emas degan xulosaga kelamiz. Xuddi shunday $\lambda + 1 = 0$ bo‘lsa, (10.20) differensial tenglamaning umumiy yechimi $x(t) = C_1 t + C_2$ dan chegaraviy shartlarni qanoatlantiruvchi yechimni ajratsak, $x(t) = 0$ ni olamiz. Endi $\lambda + 1 > 0$ bo‘lsin. Agar $\lambda + 1 = w^2\ (w \in \mathbb{R})$ desak, (10.20) differensial tenglamaning umumiy yechimi $x(t) = C_1 \cos w\, t + C_2 \sin w\, t$ bo‘ladi. Chegaraviy shartlardan foydalanib, $x(0) = C_1 = 0$ va $x'(\pi) = C_2\, w \cos w\pi = 0$ ni olamiz. Bu yerdan

$$w_k\, \pi = \frac{\pi}{2} + k\pi \Longleftrightarrow w_k = \frac{1}{2} + k, \quad k \in \mathbb{Z}_+$$

ni olamiz. Shunday qilib, $x_k(t) = C \sin(\frac{1}{2} + k)\, t$ funksiya (10.21) chegaraviy masalaning nolmas yechimi bo‘ladi. Demak, (10.18) integral tenglamaning xarakteristik sonlari $\lambda_k = (\frac{1}{2} + k)^2 - 1$, $k \in \mathbb{Z}_+$ lar, ularga mos xos funksiyalar $x_k(t) = C \sin(\frac{1}{2} + k)\, t$ lar bo‘ladi. Shunday qilib, $\frac{1}{\lambda_k}$, $k \in \mathbb{Z}_+$ lar T operatorning xos qiymatlari $\varphi_k(t) = \sqrt{\frac{2}{\pi}} \sin(\frac{1}{2} + k)\, t$ lar normasi bir bo‘lgan xos funksiyalar bo‘ladi. Ko‘rsatish mumkinki, $\{\varphi_k\}$, $k \in \mathbb{Z}_+$ sistema $L_2[0, \pi]$ fazoda to‘la ortonormal sistema bo‘ladi.

a). $\lambda_k = \pi^2(k + 0,5)^2$, $k \in \mathbb{Z}$, $x_k(t) = C \sin \pi(k + 0,5)\, t$.

b) $\lambda_k = 4k^2 - 1$, $k \in \mathbb{N}$, $x_k(t) = C \sin 2\, k\, t$.

c) $\lambda_k = (k + 0,5)^2 - 1$, $k \in \mathbb{Z}_+$, $x_k(t) = C \sin(k + 0,5)\, t$.

d) $\lambda_k = -w_k^2$, bu yerda w_k, $\text{tg}w = \dfrac{2w}{1-w^2}$ tenglamaning ildizlari, $x_k(t) = w_k \cos w_k\, t + \sin w_k\, t$.

e) $\lambda_k = w_k^2 + 1$, bu yerda w_k, $\text{tg}w = -\dfrac{2w}{1+w^2}$ tenglamaning ildizlari, $x_k(t) = w_k \cos w_k\, t + \sin w_k\, t$.

f) $\lambda_k = k^2\pi^2$, $x_k(t) = k\,\pi \cos k\,\pi\, t + \sin k\,\pi\, t$. □

10.4. Agar

a) $f(t) = t, \quad K(t,s) = \begin{cases} t(s-1), & agar \;\; 0 \le t \le s \le 1, \\ s(t-1), & agar \;\; 0 \le s < t \le 1; \end{cases}$

b) $f(t) = \cos \pi\, t\,, \qquad K(t,s) = \begin{cases} (t+1)s\,, & agar \;\; 0 \le t \le s \le 1\,, \\ (s+1)t\,, & agar \;\; 0 \le s < t \le 1 \end{cases}$

bo‘lsa, $L_2\,[0,\ 1]$ kompleks Hilbert fazosida

$$x(t) - \lambda \int_0^1 K(t,s)x(s)ds = f(t) \tag{10.22}$$

integral tenglamaning yechimini toping.

Yechish. Misol a) qismining yechimi. Dastlab biz, bir jinsli

$$x(t) = \lambda \int_0^t s(t-1)\, x(s)ds + \lambda \int_t^1 t(s-1)\, x(s)ds \tag{10.23}$$

tenglamaning xarakteristik sonlari λ_n, ya’ni T operatorning xos qiymatlari $\dfrac{1}{\lambda_n}$ larni topamiz. Integral operatorning yadrosi $K(t,s)$ haqiqiy qiymatli va simmetrik $K(t,s) = K(s,t)$ bo‘lganligi uchun, $K(t,s)$ yordamida (10.4) tenglik bilan aniqlangan T operator o‘z-o‘ziga qo‘shma kompakt operator bo‘ladi. Bu yerdan, λ_n larning haqiqiy ekanligi kelib chiqadi. Xuddi 10.7-misoldagi kabi, (10.23) integral tenglamadan

$$\begin{cases} x''(t) - \lambda\, x(t) = 0 \\ x(0) = x(1) = 0 \end{cases}$$

differensial tenglamaga kelamiz. Bu chegaraviy masalaning xarakteristik sonlari $\lambda_k = -k^2\pi^2$, $k \in \mathbb{N}$ lar, ularga mos xos funksiyalar $\varphi_k(t) = \sqrt{2}\sin k\pi t$ lardir. Demak, T operatorning xos qiymatlari $\dfrac{1}{\lambda_k} = -\dfrac{1}{k^2\pi^2}$, $k \in \mathbb{N}$ sonlar, ularga mos normasi bir bo'lgan xos funksiyalar $\varphi_k(t) = \sqrt{2}\sin k\pi t$, $k \in \mathbb{N}$ lar bo'ladi. Bu sistema $L_2[0,\ 1]$ fazoda to'la ortonormal sistema bo'ladi. Endi (10.22) tenglamani yechishga Hilbert-Shidt teoremasini qo'llaymiz. $f(t) = t$ funksiyaning $\{\varphi_k\}$, $k \in \mathbb{N}$ ortonormal sistemadagi Furye koeffitsiyentlarini topamiz:

$$c_k = (f, \varphi_k) = \int_0^1 t\,\varphi_k(t)dt = \sqrt{2}\int_0^1 t\,\sin k\pi t\,dt = \frac{(-1)^{k-1}\sqrt{2}}{k\pi}.$$

Shunday qilib,

$$f(t) = t = \sum_{k=1}^{\infty} c_k\,\varphi_k(t), \quad x(t) = \sum_{k=1}^{\infty} x_k\ \varphi_k(t)$$

larni (10.22) ga qo'yib, quyidagi tenglamani olamiz:

$$\sum_{k=1}^{\infty} x_k\ \varphi_k - \lambda\sum_{k=1}^{\infty}\frac{x_k}{\lambda_k}\varphi_k = \sum_{k=1}^{\infty} c_k\,\varphi_k.$$

$\{\varphi_k\}_{k=1}^{\infty}$ ning ortonormal sistema ekanligidan foydalansak,

$$x_k\left(1 - \frac{\lambda}{\lambda_k}\right) = c_k,\ \ k \in \mathbb{N} \tag{10.24}$$

tengliklarga kelamiz. Agar $\lambda \in \mathbb{C}\setminus\{\lambda_1,\ \lambda_2, \ldots, \lambda_n, \ldots\}$ bo'lsa, (10.24) sistema yagona $x_k = \dfrac{\lambda_k\,c_k}{\lambda_k - \lambda}$, $k \in \mathbb{N}$ yechimga ega. Bu yerdan (10.22) tenglama yechimi $x(t)$ uchun

$$x(t) = \sum_{k=1}^{\infty} x_k\ \varphi_k(t) = \sqrt{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\,k\pi}{k^2\pi^2 + \lambda}\ \sin k\pi t$$

ifodani olamiz. Agar $\lambda \in \{\lambda_1,\ \lambda_2, \ldots, \lambda_n, \ldots\}$ bo'lsa, u holda (10.24) sistema yechimga ega emas, bu esa o'z navbatida (10.22) tenglamaning yechimga ega emasligini bildiradi.

b) agar $\lambda \neq \pi^2$ va $\lambda \neq -4\,\pi^2 n^2,\ \ n \geq 2$ bo‘lsa, yechim

$$x_k(t) = \frac{\pi^3(\pi\cos\pi\, t + \sin\pi\, t)}{(\pi^2-\lambda)(\pi^2+1)} - \sum_{n=2}^{\infty} \frac{16\sqrt{2}\,\pi^2 n^3(2\pi\, n\cos 2\pi\, n\, t + \sin 2\pi\, n\, t)}{(4n^2-1)(4\pi^2 n^2+\lambda)\sqrt{4\pi^2 n^2+1}}$$

bo‘ladi. Agar $\lambda \in \{\pi^2, -4\,\pi^2 n^2,\ \ n \geq 2\}$ bo‘lsa, yechim mavjud emas. □

10.5. $x(s) - \lambda \int\limits_{-\pi}^{\pi} \cos(s+t)x(t)dt = \sin\, s$ integral tenglamani yeching.

Yechish. $\cos(s+t) = \cos s\,\cos t - \sin s \sin t$ ayniyatdan foydalansak, berilgan integral tenglamani quyidagicha yozish mumkin:

$$x(s) = \lambda\cos s \int\limits_{-\pi}^{\pi} \cos t\, x(t)dt - \lambda\sin s \int\limits_{-\pi}^{\pi} \sin t\, x(t)dt + \sin\, s. \tag{10.25}$$

(10.15) belgilashdan foydalanib, (10.25) ni quyidagicha yozamiz:

$$x(s) = \lambda\alpha_1\cos s - \lambda\alpha_2\,\sin s + \sin s. \tag{10.26}$$

$x(s)$ uchun hosil qilingan (10.26) ni (10.15) tengliklarga qo‘yib,

$$\begin{cases} \alpha_1 = \lambda\,\alpha_1\,\pi \\ \alpha_2 = -\lambda\,\alpha_2\,\pi + \pi \end{cases} \tag{10.27}$$

sistemani olamiz. Bu sistema $\lambda \neq \pm\dfrac{1}{\pi}$ da yagona $\alpha_1 = 0$ va $\alpha_2 = \dfrac{\pi}{1+\lambda\,\pi}$ yechimga ega. α_1 va α_2 larning bu qiymatlarini (10.26) ga qo‘yib, berilgan tenglama yechimi (yagona) uchun

$$x(s) = \frac{-\lambda\,\pi}{1+\lambda\,\pi}\,\sin s + \sin s = \frac{\sin\, s}{1+\lambda\,\pi}$$

ifodani olamiz. Agar $\lambda = -\dfrac{1}{\pi}$ bo‘lsa, (10.27) sistema yechimga ega emas, bundan berilgan tenglama ham yechimga ega emas degan xulosa chiqadi. Agar $\lambda = \dfrac{1}{\pi}$ bo‘lsa, (10.27) sistema cheksiz ko‘p yechimga ega bo‘lib, bunda α_1 – ixtiyoriy son, $\alpha_2 = \dfrac{\pi}{2}$ dir. Ularning bu qiymatlarini (10.26) ga qo‘yib,

$$x(s) = C\cos s - \frac{1}{\pi}\cdot\frac{\pi}{2}\,\sin s + \sin s = C\cos s + \frac{1}{2}\,\sin s$$

yechimni olamiz. Bu yerda C ixtiyoriy o‘zgarmas son. □

10.6. Ushbu

$$x(s) = 1 + s + \int\limits_0^s (s-t)\, x(t)\, dt \qquad (10.28)$$

Volterra tipidagi integral tenglamani ketma-ket yaqinlashishlar usuli yordamida yeching.

Yechish. Boshlang‘ich yaqinlashish sifatida $x_0(s) \equiv 1$ funksiyani olib, keyingi yaqinlashishlarni

$$x_n(s) = 1 + s + \int_0^s (s-t)\, x_{n-1}(t)\, dt, \quad n = 1,\ 2,\ \ldots$$

iteratsion formula yordamida topamiz:

$$x_1(s) = 1 + s + \int_0^s (s-t)\, dt = 1 + s + \frac{s^2}{2!},$$

$$x_2(s) = 1 + s + \int_0^s (s-t)\left(1 + t + \frac{t^2}{2!}\right) dt = 1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \frac{s^4}{4!},$$

$$x_3(s) = 1 + s + \int_0^s (s-t)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}\right) dt =$$

$$= 1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \frac{s^4}{4!} + \frac{s^5}{5!} + \frac{s^6}{6!}.$$

Bu jarayonni n marta takrorlash natijasida quyidagiga ega bo‘lamiz:

$$x_n(s) = 1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \cdots + \frac{s^{2n-1}}{(2n-1)!} + \frac{s^{2n}}{(2n)!}.$$

Bu yerdan ko‘rinib turibdiki, $x_n(s)$ funksiya $\sum\limits_{k=0}^{\infty} \frac{s^k}{k!} = e^s$ qatorning $2n$ – xususiy yig‘indisidan iborat. Shuning uchun

$$x(s) = \lim_{n\to\infty} x_n(s) = e^s.$$

Demak, (10.28) integral tenglama yechimi $x(s) = e^s$ funksiya ekan. □

Uy vazifalari va mavzuni o‘zlashtirish uchun masalalar

10.7. Agar

a) $a = -\dfrac{\pi}{4}\,,\quad b = \dfrac{\pi}{4}\,,\quad K(t,s) = \mathrm{tg} s\,,\quad f(t) = 1\,;$

b) $a = 0\,,\quad b = \dfrac{\pi}{2}\,,\quad K(t,s) = \sin t\ \cos s\,,\quad f(t) = \sin t\,;$

c) $a = 0\,,\quad b = \pi\,,\quad K(t,s) = \sin t\ \cos s\,,\quad f(t) = \sin t\,;$

d) $a = 0\,,\quad b = 1\,,\quad K(t,s) = t + s - 2ts\,,\quad f(t) = t + t^2\,;$

e) $a = -1\,,\quad b = 1\,,\quad K(t,s) = ts - t^2 s^2\,,\quad f(t) = t^2 + t^4\,;$

f) $a = 0\,,\quad b = 2\pi\,,\quad K(t,s) = |\pi - s| \sin t\,,\quad f(t) = t\,;$

g) $a = 0\,,\quad b = \pi\,,\quad K(t,s) = \sin s + s\ \cos t\,,\quad f(t) = 1 - \dfrac{2t}{\pi};$

h) $a = 0\,,\quad b = \pi\,,\quad K(t,s) = \sin(t\ - 2s)\,,\quad f(t) = \cos 2t\,;$

bo‘lsa, $C\,[a,\ b]$ fazoda

$$x(t) - \int_a^b K(t,s)x(s)ds = f(t)$$

tenglama yechimini toping.

10.8. Agar

a) $a = 0\,,\quad b = 2\pi\,,\quad K(t,s) = \sin(t + s)\,;$

b) $a = 0\,,\quad b = \pi\,,\quad K(t,s) = \cos(t + s)\,;$

c) $a = 0\,,\quad b = 1\,,\quad K(t,s) = 2ts - 4t^2\,;$

d) $a = -1\,,\quad b = 1\,,\quad K(t,s) = ts + t^2 s^2$

bo‘lsa, $C\,[a,\ b]$ fazoda

$$x(t) - \lambda \int_a^b K(t,s)x(s)ds = 0$$

tenglamaning noldan farqli yechimlarini toping.

10.9. Agar

a) $a=-1\,, \quad b=1\,, \quad K(t,s)=t\,s\,, \quad f(t)=\alpha\, t^2+\beta\, t+\gamma\,;$

b) $a=0\,, \quad b=\pi\,, \quad K(t,s)=\cos(t+s)\,, \quad f(t)=\alpha\,\sin t+\beta\,;$

c) $a=-1\,, \quad b=1\,, \quad K(t,s)=t^2-2ts\,, \quad f(t)=\alpha\, t^2-\beta\, t\,;$

d) $a=-1\,, \quad b=1\,, \quad K(t,s)=3t+ts-5t^2s^2, \quad f(t)=\alpha\, t$

bo‘lsa, bu tenglama ozod hadiga kiruvchi α, β, γ parametrlarning barcha qiymatlarida $C\,[a,b]$ fazoda

$$x(t)-\lambda\int\limits_a^b K(t,s)x(s)ds=f(t)$$

tenglamaning yechimini toping.

10.10. Ixtiyoriy $\lambda\in\mathbb{C}$ va ixtiyoriy $f\in L_2[0,\ 2\pi]$ uchun

$$x(t)-\lambda\int\limits_0^{2\pi}\sin(t-2s)\,x(s)\,ds=f(t)$$

tenglama yechimga ega ekanligini isbotlang va yechimni toping.

10.11-10.28-misollarda berilgan integral tenglama $\lambda\in\mathbb{C}$ parametrning qanday qiymatlarida yechimga ega, qanday qiymatlarida yechim mavjud emas, qanday qiymatlarida yechim cheksiz ko‘p. Yechim mavjud bo‘lgan hollarda yechimni toping.

10.11. $x(s)-\lambda\int\limits_0^1 s\,(1+t)x(t)dt=s^2.$

10.12. $x(s)-\lambda\int\limits_0^1 (s+s^2t)x(t)dt=s^2+1.$

10.13. $x(s)-\lambda\int\limits_0^1 s\,x(t)dt=\sin 2\pi\, s.$

10.14. $x(s)-\lambda\int\limits_0^1 (t+s\,t)x(t)dt=s^2-1.$

10.15. $x(s)-\lambda\int\limits_0^1 (1+2s)x(t)dt=1-\frac{3}{2}s.$

10.16. $x(s)-\lambda\int\limits_{-1}^1 (t+s+s^2t)x(t)dt=s^2+2s.$

10.17. $x(s)-\lambda\int\limits_0^1 s\,\sin 2\pi t\,x(t)dt=s.$

10.18. $x(s)-\lambda\int\limits_0^1 (t+s\,t+s^2t)x(t)dt=2s^2+s.$

10.19. $x(s)-\lambda\int\limits_{-\pi/4}^{\pi/4} \operatorname{tg} t\cdot x(t)dt=\cos\ s.$

10.20. $x(s)-\lambda\int\limits_{-\pi}^{\pi} \sin s\cdot\cos t\,x(t)dt=\cos\ s.$

10.21. $x(s)-\int\limits_{-1}^1 (s\,t+s^2t^2)\,x(t)dt=1+s^2.$

10.22. $x(s)-\lambda\int\limits_{-1}^1 (s+s^2t)x(t)dt=\sin\pi\,s.$

10.23. $x(s)-\lambda\int\limits_0^{\pi} \sin(s+t)\,x(t)dt=\cos\ s.$

10.24. $x(s)-\lambda\int\limits_{-1}^1 (s+t)\,x(t)dt=\frac{1}{2}+\frac{3}{2}s.$

10.25. $x(s)-\lambda\int\limits_{-1}^1 (1+st+t^2)\,x(t)dt=1+s.$

10.26. $x(s)-\lambda\int\limits_0^1 \arccos t\cdot\ x(t)dt=\frac{1}{\sqrt{1-s^2}}.$

10.27. $x(s)-\lambda\int\limits_0^1 e^{s+t}\,x(t)dt=e^{2s}.$

10.28. $x(s)-\lambda\int\limits_{-1}^1 (1+t+st^2)\,x(t)dt=s^2.$

10.29. $x(s)-\lambda\int\limits_0^1 (1+t+s+st)\,x(t)dt=2s+s^2.$

10.30-10.48-misollarda Volterra yoki Fredholm integral tenglamasi berilgan. Ularni ketma-ket yaqinlashish usuli yordamida yeching. Nolinchi yaqinlashish berilgan. Iteratsiyaning ikkinchi hadi $x_2(s)$ ni toping.

10.30. $x(s) = s + \int\limits_0^s x(t)dt, \qquad x_0(s) = s.$

10.31. $x(s) = 1 + \int\limits_0^s (s-t)x(t)dt, \qquad x_0(s) = 1.$

10.32. $x(s) = 2s^2 + 2 - \int\limits_0^s s\,x(t)dt, \qquad x_0(s) = 2.$

10.33. $x(s) = s + 1 + \int\limits_0^s x(t)dt, \qquad x_0(s) = 2s.$

10.34. $x(s) = s + \int\limits_0^s (s-t)x(t)dt, \qquad x_0(s) = 0.$

10.35. $x(s) = 1 + \int\limits_0^s (t-s)x(t)dt, \qquad x_0(s) = 0.$

10.36. $x(s) = \frac{5}{6}s - \frac{1}{9} + \frac{1}{3}\int\limits_0^1 (s+t)x(t)dt, \qquad x_0(s) = 0.$

10.37. $x(s) = 1 + \int\limits_0^s x(t)dt, \qquad x_0(s) = 0.$

10.38. $x(s) = e^s - \frac{e}{2} + \frac{1}{2} + \frac{1}{2}\int\limits_0^1 x(t)dt, \qquad x_0(s) = 0.$

10.39. $x(s) = \frac{1}{2}\int\limits_0^\pi t\,s\,x(t)dt + \frac{5}{6}s, \qquad x_0(s) = s.$

10.40. $x(s) = \frac{1}{3}\int\limits_0^1 (s+t)x(t)dt + \frac{5}{6}s - \frac{1}{9}, \qquad x_0(s) = 2s.$

10.41. $x(s) = \frac{1}{3}\int\limits_0^1 x(t)dt + s, \qquad x_0(s) = 3s.$

10.42. $x(s) = \frac{1}{2}\int\limits_0^1 tx(t)dt + s + 1, \qquad x_0(s) = 1.$

10.43. $x(s) = \dfrac{1}{\pi}\int\limits_0^{\pi} \cos^2 t \cdot x(t)dt + 1, \qquad x_0(s) = 1.$

10.44. $x(s) = \pi \int\limits_0^{1} (1-s)\sin 2\pi\, t\, x(t)dt + \dfrac{1}{2}(1-s), \qquad x_0(s) = 2.$

10.45. $x(s) = \dfrac{1}{2\pi}\int\limits_0^{\pi} \sin s \cdot t\, x(t)dt + 2\sin s, \qquad x_0(s) = 1.$

10.46. $x(s) = \dfrac{1}{2}\int\limits_0^{1} x(t)dt + \sin \pi\, s, \qquad x_0(s) = 3.$

10.47. $x(s) = s + \int\limits_0^{s} x(t)dt, \qquad x_0(s) = s^2.$

10.48. $x(s) = s + 1 + s^2\int\limits_0^{1} x(t)dt, \qquad x_0(s) = 2s.$

III bobni takrorlash uchun test savollari

1. ℓ_2 ni ℓ_2 ga akslantiruvchi A, B, C operatorlar berilgan:

$Ax = (0, 0, \ldots, 0, x_{1,}, x_{2,}, x_3, \ldots)$,

$Bx = (0, 0, \ldots, 0, x_{n+1,}, x_{n+2,}, x_{n+3}, \ldots)$,

$Cx = (x_{1,}, x_{2,}, x_3, 0, \ldots, 0, \ldots)$,

Kompakt operatorlar keltirilgan javobni toping.

A) AB va BC B) B va C C) A va B D) AC va BC

2. $L_2[-1,\ 1]$ ni $L_2[-1,\ 1]$ ga akslantiruvchi A, B, I operatorlar berilgan:
$(Af)(x) = x\,f(x)$, $(Bf)(x) = \int\limits_{-1}^{1}(1+xy)f(y)dy$, $(If)(x) = f(x)$.
Kompakt operatorlar keltirilgan javobni toping.

A) AB va B B) B va I C) A va B D) A va I

3. $L_2[-1,\ 1]$ ni $L_2[-1,\ 1]$ ga akslantiruvchi

$$(Af)(x) = 3\int_{-1}^{1} x\,y\,f(y)\,dy$$

kompakt operatorning xos qiymatlarini toping.

A) 0, 2 B) 2 C) 0, 1, 2 D) 2, 3

4. $L_2[-1,\ 1]$ fazoni o‘zini-o‘ziga akslantiruvchi

$$(Af)(x) = 3\int_{-1}^{1} x\,y\,f(y)\,dy$$

kompakt operatorning xos funksiyalari ko‘rsatilgan javobni toping.

A) $f_0(x) = 1$, $f_3(x) = x$ B) $f_0(x) = 1 + x$, $f_3(x) = x^2$

C) $f_0(x) = 3 + x$, $f_3(x) = 5x^2$ D) $f_0(x) = 4 + x$, $f_3(x) = x^4$

5. Chekli o‘lchamli $A : \ell_2 \to \ell_2$ operatorni toping.

A) $Ax = (0, 0, \ldots, 0, x_{1,}, x_{2,}, x_3, \ldots)$

B) $Ax = (0, 0, \ldots, 0, x_{n+1,}, x_{n+2,}, x_{n+3}, \ldots)$

C) $Ax = (x_{1,}, x_{2,}, x_3, 0, \ldots, 0, \ldots)$

D) $Ax = (x_{n+1}, x_{n+2}, x_{n+3}, \ldots)$

6. Kompakt $A : \ell_2 \to \ell_2$ operatorni toping.

A) $Ax = (0, 0, \ldots, 0, x_{1,}, x_{2,}, x_3, \ldots)$

B) $Ax = (0, 0, \ldots, 0, x_{n+1,}, x_{n+2,}, x_{n+3}, \ldots)$

C) $Ax = (a_1 x_1, a_2 x_2, a_3 x_3, \ldots, a_n x_n, \ldots)$, $\lim\limits_{n\to\infty} a_n = 0$

D) $Ax = (x_{n+1}, x_{n+2}, x_{n+3}, \ldots)$

7. ℓ_2 fazoda berilgan $Ax = (a_1 x_1, a_2 x_2, a_3 x_3, \ldots, a_n x_n, \ldots)$ operatorning kompaktlik kriteriysini toping.

A) $\sup\limits_{n\geq 1} |a_n| < \infty$

B) shunday $n_0 \in \mathbb{N}$ mavjud bo'lib, barcha $n > n_0$ larda $a_n = 0$ bo'lishi

C) $\lim\limits_{n\to\infty} a_n = 0$

D) $\{a_n\}$ ning nolga yaqinlashuvchi qismiy ketma-ketlikni saqlashi

8. Quyidagi tasdiqlar ichidan to'g'rilarini ajrating.

1) Chekli o'lchamli $A \in L(X, Y)$ operator kompakt bo'ladi.

2) Chiziqli $A : \mathbb{C}^n \to \mathbb{C}^n$ operator kompakt bo'ladi.

3) Birlik $I : X \to X$, $\dim X < \infty$ operator kompakt bo'ladi.

A) 1, 2 B) 2, 3 C) 1, 2, 3 D) 1, 3

9. Quyidagi tasdiqlar ichidan to'g'rilarini ajrating.

1) Kompakt operatorlarning yig'indisi kompakt bo'ladi.

2) Kompakt operatorning songa ko'paytmasi kompakt bo'ladi.

3) Kompakt operatorning chegaralangan operatorga ko'paytmasi kompakt bo'ladi.

A) 1, 2 B) 2, 3 C) 1, 2, 3 D) 1, 3

10. Quyidagi tasdiqlar ichidan to'g'rilarini ajrating.

1) Kompakt operatorga qo'shma operator kompakt bo'ladi.

2) Kompakt operatorga teskari operator kompakt bo'ladi.

3) Birlik $I : X \to X$, $\dim X = \infty$ operator kompakt bo'ladi.

A) 1, 2 B) 2, 3 C) 1 D) 1, 3

11. Quyidagi tasdiqlar ichidan to‘g‘rilarini ajrating.

1) Kompakt operatorning xos qiymatlari oddiy bo‘ladi.

2) O‘z-o‘ziga qo‘shma operatorning xos qiymatlari haqiqiy bo‘ladi.

3) O‘z-o‘ziga qo‘shma kompakt operatorning har xil xos qiymatlariga mos xos vektorlari ortogonal bo‘ladi.

A) 1, 2 B) 2, 3 C) 1, 2, 3 D) 1, 3

12. Quyidagi tasdiqlar ichidan to‘g‘rilarini ajrating.

1) Kompakt operatorning noldan farqli xos qiymatlari chekli karralidir.

2) $A : \ell_2 \to \ell_2$ kompakt operatorning spektri nolni saqlaydi.

3) Birlik $I : C[a, b] \to C[a, b]$ operator kompakt emas.

A) 1, 2 B) 2, 3 C) 1 D) 1, 2, 3

13. $C[-1, 1]$ fazoda chekli o‘lchamli operatorlarni ko‘rsating.

$$(Af)(x) = x\,f(x), \quad (Bf)(x) = \int_{-1}^{1} (x-y) f(y)\,dy, \quad (Cf)(x) = f(0)x^2$$

A) $A,\ B$ B) $A,\ C$ C) $B,\ C$ D) $A,\ B,\ C$

14. ℓ_2 fazoda berilgan $Ax = (a_1x_1, a_2x_2, a_3x_3, \ldots, a_nx_n, \ldots)$ operator uchun quyidagilardan qaysilari invariant qism fazo bo‘ladi.

1) $L_1 = \{x \in \ell_2 : x_1 = x_2 = 0\}$

2) $L_2 = \{x \in \ell_2 : x_3 = x_4 = x_5 = 0\}$

3) $L_3 = \{x \in \ell_2 : x_n = 0, \quad n \geq 6\}$

A) 1, 2 B) 2, 3 C) 1, 2, 3 D) 1, 3

15. $L_2[-\pi, \pi]$ fazoda kompakt operatorni ko‘rsating.

A) $(Af)(x) = xf(x)$ B) $(Af)(x) = \int\limits_{-\pi}^{\pi} \cos(x-y)\,f(y)\,dy$

C) $(Af)(x) = f(x)$ D) $(Af)(x) = \big(x^2+1\big)\,f(x)$

16. $\{A_n\} \subset K(X)$ kompakt operatorlar ketma-ketligi, $A \in L(X)$ bo‘lsin. Quyidagi tasdiqlardan qay, biri to‘g‘ri.

A) Agar $A_n \overset{u}{\Longrightarrow} A$ (tekis) bo'lsa, u holda A kompakt bo'ladi.

B) Agar $A_n \overset{s}{\longrightarrow} A$ (kuchli) bo'lsa, u holda A kompakt bo'ladi.

C) Agar $A_n \overset{w}{\longrightarrow} A$ (kuchsiz) bo'lsa, u holda A kompakt bo'ladi.

D) Agar $A_n \to A$ (nuqtali) bo'lsa, u holda A kompakt bo'ladi.

17. Chekli o'lchamli operator ta'rifini toping.

A) Agar $A \in L(X, Y)$ bo'lib, $\dim ImA < \infty$ bo'lsa

B) Agar A har qanday chegaralangan to'plamni nisbiy kompakt to'plamga akslantirsa

C) Agar A har qanday nisbiy kompakt to'plamni kompakt to'plamga akslantirsa

D) Agar $A : \mathbb{C}^n \to \mathbb{C}^n$ bo'lib, $\dim Im\, A < n$ bo'lsa

18. Kompakt operator ta'rifini toping.

A) Agar $A \in L(X, Y)$ bo'lib, $\dim ImA < \infty$ bo'lsa

B) Agar $A \in L(X, Y)$ har qanday chegaralangan to'plamni nisbiy kompakt to'plamga akslantirsa

C) Agar $A \in L(X, Y)$ har qanday nisbiy kompakt to'plamni kompakt to'plamga akslantirsa

D) Agar $A : \mathbb{C}^n \to \mathbb{C}^n$ bo'lib, $\dim Im\, A < n$ bo'lsa

19. Quyidagilar ichidan kompakt operator ta'riflarini ajrating.

1) Agar $A \in L(X, Y)$ har qanday chegaralangan to'plamni nisbiy kompakt to'plamga akslantirsa

2) Agar $A \in L(X, Y)$ operator X dagi birlik sharni nisbiy kompakt to'plamga akslantirsa

3) Agar $A \in L(H)$ operator H dagi ixtiyoriy kuchsiz yaqinlashuvchi ketma-ketlikni kuchli yaqinlashuvchi ketma-ketlikka akslantirsa.

A) 1, 2, 3 B) 2, 3 C) 1, 3 D) 1, 2

20. u ga nisbatan Fredholmning I tur integral tenglamasini toping.

A) $u(x) = \int_a^b K(x,y)u(y)dy + f(x)$ B) $f(x) = \int_a^b K(x,y)u(y)dy$

C) $u(x) = \int_a^x K(x,y)u(y)dy + f(x)$ D) $f(x) = \int_a^x K(x,y)u(y)dy$

21. u ga nisbatan Fredholmning II tur integral tenglamasini toping.

A) $u(x) = \int_a^b K(x,y)u(y)dy + f(x)$ B) $f(x) = \int_a^b K(x,y)u(y)dy$

C) $u(x) = \int_a^x K(x,y)u(y)dy + f(x)$ D) $f(x) = \int_a^x K(x,y)u(y)dy$

22. u ga nisbatan Volterraning I tur integral tenglamasini toping.

A) $u(x) = \int_a^b K(x,y)u(y)dy + f(x)$ B) $f(x) = \int_a^b K(x,y)u(y)dy$

C) $u(x) = \int_a^x K(x,y)u(y)dy + f(x)$ D) $f(x) = \int_a^x K(x,y)u(y)dy$

23. u ga nisbatan Volterraning II tur integral tenglamasini toping.

A) $u(x) = \int_a^b K(x,y)u(y)dy + f(x)$ B) $f(x) = \int_a^b K(x,y)u(y)dy$

C) $u(x) = \int_a^x K(x,y)u(y)dy + f(x)$ D) $f(x) = \int_a^x K(x,y)u(y)dy$

24. $T^* : L_2[-\pi,\ \pi] \to L_2[-\pi,\ \pi]$ kompakt operator bo'lib, 1 uning oddiy xos qiymati bo'lsin. 1 xos qiymatga mos keluvchi xos funksiya esa $\cos 2x$ bo'lsin. $u = Tu + f$ tenglama yechimga ega bo'ladigan f ni toping:

A) $\cos x$ B) $\cos 2x$ C) $1 - \cos 2x$ D) $\cos^2 x$

25. $T^* : L_2[-\pi,\ \pi] \to L_2[-\pi,\ \pi]$ kompakt operator bo'lib, 1 uning ikki karrali xos qiymati bo'lsin. 1 xos qiymatga mos keluvchi xos funksiyalar esa $\cos x$ va $\sin x$ lar bo'lsin. $u = Tu + f$ tenglama yechimga ega bo'ladigan f ni toping:

A) $\cos x$ B) $\cos 2x$ C) $\cos x + \sin x$ D) $\cos x - \sin x$

26. $T^* : L_2[-\pi,\ \pi] \to L_2[-\pi,\ \pi]$ kompakt operator bo'lib, 1 uning oddiy xos qiymati bo'lsin. 1 xos qiymatga mos keluvchi xos funksiya esa $\cos 2x$

bo‘lsin. $u = Tu + f$ tenglama yechimga ega bo‘lmaydigan f ni toping:

A) $\cos x$ B) $\cos 2x$ C) 1 D) $\sin x$

27. $T^* : L_2[-\pi, \pi] \to L_2[-\pi, \pi]$ kompakt operator bo‘lib, 1 uning ikki karrali xos qiymati bo‘lsin. 1 xos qiymatga mos keluvchi xos funksiyalar esa $\cos x$ va $\sin x$ lar bo‘lsin. $u = Tu + f$ tenglama yechimga ega bo‘lmaydigan f ni toping:

A) $\cos x + \sin x$ B) $\cos 2x$ C) 1 D) $\sin 2x$

28. $L_2[-\pi, \pi]$ fazoda $u(x) = 1 + \int\limits_{-\pi}^{\pi} \cos x \sin y u(y) dy$ integral tenglama yechimini toping.

A) $1 + \cos x$ B) $1 + \sin x$ C) 1 D) $1 + \pi \cos x$

29. $L_2[-\pi, \pi]$ fazoda $u(x) = \sin x + \int\limits_{-\pi}^{\pi} \cos x \sin y u(y) dy$ integral tenglama yechimini toping.

A) $\sin x + \cos x$ B) $1 + \sin x$ C) 1 D) $\sin x + \pi \cos x$

30. $L_2[-\pi, \pi]$ fazoda $u(x) = \cos x + \int\limits_{-\pi}^{\pi} \cos x \sin y u(y) dy$ integral tenglama yechimini toping.

A) $\sin x + \cos x$ B) $1 + \sin x$ C) $\cos x$ D) $\sin x + \pi \cos x$

31. $L_2[-\pi, \pi]$ fazoda $(Au)(x) = u(x) - \frac{1}{\pi} \int\limits_{-\pi}^{\pi} \cos x \cos y u(y) dy$ chiziqli operator yadrosining o‘lchamini toping.

A) $\dim Ker A = 0$ B) $\dim Ker A = 1$

C) $\dim Ker A = 2$ D) $\dim Ker A = \infty$

32. $L_2[-\pi, \pi]$ fazoda $(Au)(x) = u(x) - \frac{1}{\pi} \int\limits_{-\pi}^{\pi} \cos(x - y) u(y) dy$ chiziqli operator yadrosining o‘lchamini toping.

A) $\dim Ker A = 0$ B) $\dim Ker A = 1$

C) $\dim Ker A = 2$ D) $\dim Ker A = \infty$

33. $L_2[-\pi,\ \pi]$ fazoda $(Au)(x) = u(x) - \frac{1}{\pi}\int\limits_{-\pi}^{\pi}[\frac{1}{2} + \cos(x-y)]\,u(y)dy$ chiziqli operator yadrosining o‘lchamini toping.

A) $\dim KerA = 1$ B) $\dim KerA = 2$

C) $\dim KerA = 3$ D) $\dim KerA = \infty$

34. $L_2[-\pi,\ \pi]$ fazoda $u(x) = f(x) + \frac{1}{\pi}\int\limits_{-\pi}^{\pi}[\frac{1}{2} + \cos(x-y)]\,u(y)\,dy$ chiziqli integral tenglamaga mos bir jinsli tenglamaning chiziqli bog‘lanmagan yechimlari sonini toping.

A) 1 B) 2 C) 3 D) ∞

35. $L_2[-\pi,\ \pi]$ fazoda $u(x) = f(x) + \frac{1}{\pi}\int\limits_{-\pi}^{\pi}\cos(x-y)\,u(y)\,dy$ chiziqli integral tenglamaga mos bir jinsli tenglamaning chiziqli bog‘lanmagan yechimlari sonini toping.

A) 1 B) 2 C) 3 D) ∞

36. $L_2[-\pi,\ \pi]$ fazoda $u(x) = f(x) + \frac{1}{\pi}\int\limits_{-\pi}^{\pi}\cos x\ \cos y\ u(y)\,dy$ chiziqli integral tenglamaga mos bir jinsli tenglamaning chiziqli bog‘lanmagan yechimlari sonini toping.

A) 1 B) 2 C) 3 D) ∞

37. $L_2[-\pi,\ \pi]$ fazoda $(Au)(x) = \int\limits_{-\pi}^{\pi}\cos(x-y)\,u(y)\,dy,\ \ (Bu)(x) = \int\limits_{-\pi}^{\pi}(\alpha\cos x\cos y - \beta\sin x\sin y)\,u(y)\,dy$ integral operatorlar berilgan. $A^* = B$ tenglik o‘rinli bo‘ladigan $\alpha \in \mathbb{R}$ va $\beta \in \mathbb{R}$ parametrlarning qiymatlarini toping.

A) $\alpha = 1,\ \beta = 1$ B) $\alpha = 1,\ \beta = -1$

C) $\alpha = -1,\ \beta = 1$ D) bunday qiymatlar yo‘q

38. T – $L_2[a,\ b]$ fazodagi kompakt operator, 1 soni uning xos qiymati bo‘lsin. $u = f + Tu$ (1) tenglama uchun quyidagi tasdiqlardan qaysilari to‘g‘ri?

1) (1) tenglama ba'zi $f \in L_2[a,\ b]$ larda yechimga ega emas.

2) (1) tenglama yechimga ega bo'lishi uchun f funksiya $u = T^*u$ tenglamaning barcha yechimlariga ortogonal bo'lishi zarur va yetarli.

3) (1) tenglama yechimga ega bo'lishi uchun $\dim KerT = \dim KerT^*$ bo'lishi zarur va yetarli.

A) 1, 2, 3 $\quad$ $B)$ 2, 3 $\quad$ $C)$ 3 $\quad$ D) 1, 2

39. $L_2[-\pi,\ \pi]$ fazoda ajralgan yadroli integral tenglamalarni ko'rsating.

1) $u(x) = \int\limits_{-\pi}^{\pi} \cos(x-y)\, u(y)\, dy$

2) $u(x) = \int\limits_{-\pi}^{\pi} (\alpha \cos x\ \cos y - \beta \sin x \sin y)\, u(y)\, dy$

3) $u(x) = \int\limits_{-\pi}^{\pi} \ln(1+|x-y|)\, u(y)\, dy.$

A) 1, 2, 3 $\quad$ B) 2, 3 $\quad$ C) 3 $\quad$ D) 1, 2

40. $L_2[-1,\ 1]$ fazoda ajralgan yadroli integral tenglamalarni ko'rsating.

1) $u(x) = \int\limits_{-1}^{1} (x-y)^3 u(y)\, dy$

2) $u(x) = \int\limits_{-1}^{1} (1+xy)^2 u(y)\, dy$

3) $u(x) = \int\limits_{-1}^{1} \ln(1+|x-y|)\, u(y)\, dy.$

A) 1, 2, 3 $\quad$ B) 2, 3 $\quad$ C) 3 $\quad$ D) 1, 2

III bob uchun javoblar va ko‘rsatmalar

9-§. Kompakt operatorlar

5 - 9 - misollarda $\dim ImA = 2$ ekanligini ko‘rsating va 5.1-teoremadan foydalaning.

10. $\dim ImA = 4$ ekanligini ko‘rsating va 5.1-teoremadan foydalaning. 5.1-misoldan foydalaning.

11, 12-misollarda Artsela teoremasidan foydalaning.

13 - misolda $\dim ImA = 2$ ekanligini ko‘rsating va 5.1-teoremadan foydalaning.

14. Ixtiyoriy $x \in L_2[0, \pi]$ uchun

$$(Ax)(s) = s\int\limits_0^\pi \cos t\, x(t)\, dt + \cos\, s\int\limits_0^\pi t\, x(t)\, dt = \alpha\, s + \beta\, \cos s \qquad (9.1j)$$

tenglik bajariladi. Bu yerda

$$\alpha = \int\limits_0^\pi \cos t\, x(t)\, dt, \quad \beta = \int\limits_0^\pi t\, x(t)\, dt.$$

(9.1j) tenglikdan $\dim ImA \leq 2$ ni olamiz. Shunday x_1, x_2 ni topish mumkinki, $\alpha(x_1) = \beta(x_2) = 0$, $\alpha(x_2) = \beta(x_1) = 1$ bo‘ladi, ya’ni A ikki o‘lchamli operator. 9.1-teoremaga ko‘ra u kompakt operator bo‘ladi.

15- misolda 10.1-teoremadan foydalaning.

16, 17-misollarda 9.1-misoldan foydalaning.

18 - 22- misollarda 9.1-misoldan foydalaning.

24-misolda $\dim ImA = 6$ ekanligini ko‘rsating va 9.1-teoremadan foydalaning.

25. A ni kompakt operator deb faraz qilaylik. A ga teskari operator mavjud va chegaralangan:

$$(A^{-1}y)(t) = \frac{y(t)}{t+1}, \qquad D(A^{-1}) = C[0, 1].$$

9.5-teoremaga ko'ra, cheksiz o'lchamli fazoda kompakt operatorning chegaralangan teskarisi mavjud emas. Demak, A kompakt operator emas.

26 - 28 - misollarda 9.25-misoldan va 9.5-teoremadan foydalaning.

29, 30-misollarda 9.4-misoldan foydalaning.

31. Ko'rsatamizki $\lambda = 1$ soni A operator uchun cheksiz karrali xos qiymat bo'ladi. Bu esa 9.6-teorema bilan birgalikda A operatorning kompakt emasligini isbotlaydi. Endi $\lambda = 1$ soni A operatorning cheksiz karrali xos qiymati ekanligini ko'rsatamiz. $C[-1,\ 1]$ fazoni juft funksiyalardan iborat $C^+[-1,\ 1]$ va toq funksiyalardan tashkil topgan $C^-[-1,\ 1]$ qism fazolarning yig'indisiga yoyish mumkin, ya'ni

$$C[-1,\ 1] = C^+[-1,\ 1] \oplus C^-[-1,\ 1].$$

Bu fazolar A operator uchun invariant qism fazolar bo'ladi va quyidagi tengliklar o'rinli:

$$A\,x^+ = x^+,\ x \in C^+[-1,\ 1], \quad A\,x^- = 0,\ x \in C^-[-1,\ 1].$$

Bu yerdan kelib chiqadiki $Ker(A - I) = C^+[-1,\ 1]$ va

$$\dim Ker(A - I) = \dim C^+[-1,\ 1] = \infty.$$

Ya'ni $\lambda = 1$ soni A operatorning cheksiz karrali xos qiymati ekan.

32-34. 9.25-misoldan va 9.5-teoremadan foydalaning.

35-43 -misollarda 9.1-misoldan foydalaning.

44. $\{Ae_n\}$ ketma-ketlikdan yaqinlashuvchi qismiy ketma-ketlik ajratish mumkin emasligini ko'rsating.

45. Hech qanday shartda ham bu operator kompakt bo'lmaydi.

47. a) yo'q. b) yo'q. c) ha.

49. b) va c).

53. $\sigma(A) = \{0\}$.

53. $\sigma(A) = \{0,\ \frac{2}{3},\ -\frac{2}{5}\}.$

57. Singulyar sonlar $v(n),\ \ n \in \mathbb{N}.$

10-§. Integral tenglamalar

7. a) $x(t) = 1,$ b) $x(t) = 2\sin t,$ c) $x(t) = \sin t,$

d) $x(t) = t^2 - t + \frac{1}{6},$ e) $x(t) = t^4 + \frac{25}{49}t^2,$ f) $x(t) = t = \pi^3 \sin\, t,$

g) $x(t) = 1 - \frac{2\,t}{\pi} - \frac{\pi^2 \cos\, t}{18},$ h) $x(t) = \cos 2\,t + \frac{3\pi}{10}\sin\, t,$

8. a) $\lambda = \frac{1}{\pi}$ da yechim $x(t) = \alpha\, \sin\, t + \alpha\, \cos\, t$ ga, $\lambda = -\frac{1}{\pi}$ da yechim $x(t) = \alpha\, \sin\, t - \alpha\, \cos\, t$ bo'ladi.

b) $\lambda = \frac{2}{\pi}$ da yechim $x(t) = \alpha\, \cos\, t$ ga, $\lambda = -\frac{2}{\pi}$ da yechim $x(t) = \alpha\, \sin\, t$ ga teng bo'ladi.

c) $\lambda = -3$ da yechim $x(t) = \alpha\, t - 2\,\alpha\, t^2$ ga teng bo'ladi.

d) $\lambda = \frac{3}{2}$ da yechim $x(t) = \alpha\, t$ ga, $\lambda = \frac{5}{2}$ da yechim $x(t) = \alpha\, t^2$ ga teng bo'ladi.

9. a) Agar $\lambda \neq \frac{3}{2}$ bo'lsa, yechim yagona va u $x(t) = \alpha\, t^2 + \frac{3\beta}{3 - 2\lambda}\, t + \gamma$ ko'rinishda bo'ladi. Agar $\lambda = \frac{3}{2}$ va $\beta \neq 0$ bo'lsa, yechim mavjud emas. Agar $\lambda = \frac{3}{2}$ va $\beta = 0$ bo'lsa, yechim cheksiz ko'p va u $x(t) = \alpha\, t^2 + C\, t + \gamma$ ko'rinishda bo'ladi.

b) Agar $\lambda \neq \pm\frac{2}{\pi}$ bo'lsa, yechim yagona va u $x(t) = \beta + \frac{2\alpha - 4\,\beta\,\lambda}{2 + \lambda\,\pi}\sin\, t$ ko'rinishda bo'ladi. Agar $\lambda = -\frac{2}{\pi}$ va $\frac{\alpha\,\pi}{2} + 2\beta \neq 0$ bo'lsa, yechim mavjud emas. Agar $\lambda = -\frac{2}{\pi}$ va $\frac{\alpha\,\pi}{2} + 2\beta = 0$ bo'lsa, yechim cheksiz ko'p va u $x(t) = \beta + C \sin\, t$ ko'rinishda bo'ladi. Agar $\lambda = \frac{2}{\pi}$ bo'lsa, yechim cheksiz ko'p va u $x(t) = \beta + \frac{\alpha\,\pi - 4\,\beta}{2\,\pi}\sin\, t + C\, \cos t$ ko'rinishda bo'ladi.

c) Agar $\lambda \neq -\frac{3}{4},\ \ \lambda \neq \frac{3}{2}$ bo'lsa, yechim yagona va u $x(t) = \frac{3\alpha}{3 - 2\lambda}\, t^2 - \frac{3\beta}{3 + 4\lambda}\, t$ ko'rinishda bo'ladi. Agar $\lambda = -\frac{3}{4}$ va $\beta \neq 0$ yoki $\lambda = \frac{3}{2}$ va $\alpha \neq 0$

bo‘lsa, yechim mavjud emas. Agar $\lambda = -\dfrac{3}{4}$ va $\beta = 0$ bo‘lsa, yechim cheksiz ko‘p va u $x(t) = C\,t + \dfrac{2\,\alpha}{3}\,t^2$ ko‘rinishda bo‘ladi. Agar $\lambda = \dfrac{3}{2}$ va $\alpha = 0$ bo‘lsa, yechim cheksiz ko‘p va u $x(t) = -\dfrac{19\,\beta}{27}\,t + C\,t^2$ ko‘rinishda bo‘ladi.

d) Agar $\lambda \neq -\dfrac{1}{2}, \quad \lambda \neq \dfrac{3}{2}$ bo‘lsa, yechim yagona va u $x(t) = \dfrac{3\,\alpha\,t}{3-2\lambda}$ ko‘rinishda bo‘ladi. Agar $\lambda = \dfrac{3}{2}$ va $\alpha \neq 0$ bo‘lsa, yechim mavjud emas. Agar $\lambda = \dfrac{3}{2}$ va $\alpha = 0$ bo‘lsa, yechim cheksiz ko‘p va u $x(t) = C\,t$ ko‘rinishda bo‘ladi. Agar $\lambda = -\dfrac{1}{2}$ bo‘lsa, yechim cheksiz ko‘p va u $x(t) = -\dfrac{3}{4}(\alpha - C)\,t + C\,t^2$ ko‘rinishda bo‘ladi.

10. $x(t) = \lambda \int\limits_0^{2\pi} \sin(t-2s)\, f(s)\, ds + f(t).$

11. Parametr λ ning $\dfrac{6}{5}$ dan farqli barcha qiymatlarida tenglama yagona yechimga ega va u $x(s) = s^2 + \dfrac{7\lambda s}{2(6-5\lambda)}$ ko‘rinishga ega.

12. Parametr $\lambda \in \mathbb{C}$ ning $\pm\dfrac{3}{2}$ dan farqli barcha qiymatlarida tenglama yagona yechimga ega va u $x(s) = s^2 + 1 + \dfrac{24\lambda s}{9-4\lambda^2} + \dfrac{16\lambda^2 s^2}{9-4\lambda^2}$ ko‘rinishga ega.

13. Parametr $\lambda \in \mathbb{C}$ ning barcha qiymatlarida tenglama yechimga ega. $\lambda \neq 2$ da yechim yagona $x(s) = \sin 2\pi s$, $\lambda = 2$ da yechim cheksiz ko‘p bo‘lib, uning ko‘rinishi $x(s) = \sin 2\pi s + as, \quad a \in \mathbb{C}$ - ixtiyoriy son.

14. Parametr $\lambda \in \mathbb{C}$ ning barcha qiymatlarida tenglama yechimga ega. $\lambda \neq 3/2$ bo‘lsa, $x(s) = s^2 - 1$ tenglamaning yagona yechimi bo‘ladi. Agar $\lambda = 3/2$ bo‘lsa, tenglama yechimi cheksiz ko‘p bo‘lib, ular $x(s) = s^2 - 1 + \alpha\,(1+s)$, $\alpha \in \mathbb{C}$ ko‘rinishga ega.

15. Parametr $\lambda \in \mathbb{C}$ ning barcha $\lambda \neq 1/2$ qiymatlarida tenglama $x(s) = 1 - \dfrac{3}{2}\,s + \dfrac{\lambda\,(1+2s)}{4\,(1-2\lambda)}$ ko‘rinishdagi yagona yechimga ega. Agar $\lambda = 1/2$ bo‘lsa, tenglama yechimga ega emas.

16. Agar $\lambda = \pm 3/4$ bo‘lsa, tenglama yechimga ega emas. Agar $\lambda \neq \pm 3/4$ bo‘lsa, tenglama yagona $x(s) = s^2 + 2s + \lambda\,\alpha\,s + \lambda\beta(1+s^2)$, bu yerda

$\alpha = \dfrac{2}{3} + \dfrac{8\lambda}{3} + \dfrac{12}{9-16\lambda^2} + \dfrac{4\lambda}{9-16\lambda^2}, \ \beta = \dfrac{12}{9-16\lambda^2} + \dfrac{4\lambda}{9-16\lambda^2}.$

17. Parametr $\lambda \in \mathbb{C}$ ning barcha qiymatlarida tenglama yagona $x(s) = s$ yechimga ega.

18. Parametr $\lambda \in \mathbb{C}$ ning barcha $\lambda \neq 3/2$ qiymatlarida tenglama $x(s) = 2\,s^2 + s + \dfrac{2\lambda}{3-2\lambda}(1 + s + s^2)$ ko'rinishdagi yagona yechimga ega. Agar $\lambda = 3/2$ bo'lsa, tenglama yechimga ega emas.

19. Parametr $\lambda \in \mathbb{C}$ ning barcha qiymatlarida tenglama yagona $x(s) = \cos s$ yechimga ega.

20. Parametr $\lambda \in \mathbb{C}$ ning barcha qiymatlarida tenglama yagona $x(s) = \cos s$ yechimga ega.

21. Agar $\lambda \notin \left\{\dfrac{3}{2}, \dfrac{5}{3}\right\}$ bo'lsa, tenglama yagona $x(s) = 1 + s^2 + \dfrac{16\lambda\, s^2}{3(5-3\lambda)}$ yechimga ega. Agar $\lambda = \dfrac{5}{3}$ bo'lsa, tenglama yechimga ega emas. Agar $\lambda = \dfrac{3}{2}$ bo'lsa, tenglama cheksiz ko'p $x(s) = 1 + 17 \cdot s^2 + \alpha \cdot s\,,\ \forall \alpha \in \mathbb{C}$ yechimlarga ega.

22. Agar $\lambda \neq \pm\dfrac{3}{2}$ bo'lsa, tenglama yagona $x(s) = \sin \pi s$ yechimga ega. Agar $\lambda = \pm\dfrac{3}{2}$ bo'lsa, tenglama cheksiz ko'p $x(s) = \sin \pi\, s + \alpha\, s + \beta\, s^2\,,\ \forall \alpha, \beta \in \mathbb{C}$ yechimlarga ega.

23. Agar $\lambda \neq \pm\dfrac{2}{\pi}$ bo'lsa, tenglama yagona $x(s) = \cos s + \dfrac{2\pi\lambda}{4-\pi^2\lambda^2}\sin s + \dfrac{\pi^2\lambda^2}{4-\pi^2\lambda^2}\cos s$ yechimga ega. Agar $\lambda = \pm\dfrac{2}{\pi}$ bo'lsa, tenglama yechimga ega emas.

24. Agar $\lambda \neq \pm\dfrac{\sqrt{3}}{2}$ bo'lsa, tenglama yagona $x(s) = \dfrac{1}{2} + \dfrac{\lambda\,(3+2\lambda)}{3-4\lambda^2} + \dfrac{2s\left(3+\lambda-2\lambda^2\right)}{2\,(3-4\lambda^2)}$ yechimga ega. Agar $\lambda = \pm\dfrac{\sqrt{3}}{2}$ bo'lsa, tenglama yechimga ega emas.

25. Agar $\lambda \neq \dfrac{3}{2},\ \ \lambda \neq \dfrac{3}{8}$ bo'lsa, tenglama yagona $x(s) = \dfrac{3}{3-8\lambda} + \dfrac{3s}{3-2\lambda}$ yechimga ega. Agar $\lambda = \dfrac{3}{2},\ \ \lambda = \dfrac{3}{8}$ bo'lsa, tenglama yechimga ega emas.

26. Agar $\lambda \neq 1$ bo'lsa, tenglama yagona $x(s) = \dfrac{1}{\sqrt{1-s^2}} + \dfrac{\pi\lambda}{2(1-\lambda)}$ yechimga ega. Agar $\lambda = \dfrac{3}{2}$, $\quad \lambda = \dfrac{3}{8}$ bo'lsa, tenglama yechimga ega emas.

27. Agar $\lambda \neq \dfrac{2}{e^2-1}$ bo'lsa, tenglama yagona $x(s) = e^{2s} + \dfrac{2\lambda\left(e^2-1\right)e^s}{3\left(2-\lambda e^2+\lambda\right)}$ yechimga ega. Agar $\lambda = \dfrac{2}{e^2-1}$ bo'lsa, tenglama yechimga ega emas.

28. Agar $\lambda \neq \dfrac{-9 \pm 3\sqrt{13}}{4}$ bo'lsa, tenglama yagona $x(s) = \dfrac{6\lambda}{9-18\lambda-4\lambda^2} + \dfrac{9-14\lambda-4\lambda^2}{9-18\lambda-4\lambda^2} s$ yechimga ega. Agar $\lambda = \dfrac{-9 \pm 3\sqrt{13}}{4}$ bo'lsa, tenglama yechimga ega emas.

29. Agar $\lambda \neq \dfrac{3}{8}$ bo'lsa, tenglama yagona $x(s) = s + s^2 + \dfrac{6\lambda(1+s)}{3-8\lambda}$ yechimga ega. Agar $\lambda = \dfrac{3}{8}$ bo'lsa, tenglama yechimga ega emas.

30. $x_2(s) = s + \dfrac{s^2}{2} + \dfrac{s^3}{6}; \;\; x(s) = e^s - 1.$

31. $x_2(s) = 1 + \dfrac{s^2}{2} + \dfrac{s^4}{24}; \;\; x(s) = \operatorname{ch} s.$

32. $x_2(s) = 2; \;\; x(s) = 2.$

33. $x_2(s) = 1 + 2s + \dfrac{s^2}{2} + \dfrac{s^3}{3}; \;\; x(s) = 2e^s - 1.$

34. $x_2(s) = s + \dfrac{s^3}{6}; \;\; x(s) = \operatorname{sh} s.$

35. $x_2(s) = 1 - \dfrac{s^2}{2}; \;\; x(s) = \cos s.$

36. $x_2(s) = \dfrac{101}{108}s - \dfrac{1}{27}; \;\; x(s) = \dfrac{461}{474}s - \dfrac{65}{948}.$

37. $x_2(s) = 1 + s; \;\; x(s) = e^s.$

38. $x_2(s) = e^s + \dfrac{1-e}{4}; \;\; x(s) = e^s.$

39. $x_2(s) = x(s) = s.$

40. $x_2(s) = \dfrac{107}{108}s; \;\; x(s) = \dfrac{461}{474}s - \dfrac{65}{948}.$

41. $x_2(s) = s + \dfrac{1}{3}; \;\; x(s) = s + \dfrac{1}{4}.$

42. $x_2(s) = s + 1\dfrac{23}{48}; \;\; x(s) = s + 1\dfrac{5}{9}.$

43. $x_2(s) = 1\dfrac{3}{4}; \;\; x(s) = 2.$

44. $x_2(s) = \frac{3}{4}(1-s); \quad x(s) = 1 - s.$

45. $x_2(s) = \frac{\pi + 24}{8}\sin s; \quad x(s) = 4\sin s.$

46. $x_2(s) = \frac{1}{\pi} + \frac{3}{4} + \sin \pi s; \quad x(s) = \frac{2}{\pi} + \sin \pi s.$

47. $x_2(s) = s + \frac{s^2}{2} + \frac{s^4}{12}; \quad x(s) = e^s - 1.$

48. $x_2(s) = 1 + s + \frac{11}{6}s^2; \quad x(s) = 1 + s + \frac{9}{4}s^2.$

III bobda keltirilgan test javoblari

1-D 2-A 3-A 4-A 5-C 6-C 7-C 8-C 9-C 10-C 11-B 12-D 13-C 14-C 15-B 16-A 17-A 18-B 19-A 20-B 21-A 22-D 23-C 24-A 25-B 26-B 27-A 28-C 29-D 30-C 31-B 32-C 33-C 34-C 35-B 36-A 37-B 38-D 39-D 40-D.

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