18-MAVZU: FUNKSIONAL QATOR. YAQINLASHISH SOHASI. DARAJALI QATOR. YAQINLASHISH RADIUSI.

REJA.

1. Funksional qatorlar.
2. Darajali qator.
3. Yaqinlashish radiusi.

1.Funksional qatorlar

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + u_3(x) + \ldots + u_n(x) + \ldots.. \qquad (1)$$

ko'rinishdagi ifoda funksional qator deb ataladi. Uning har bir hadi x ga bog'liq funksiyadir. x ga har xil sonli qiymatlarni berib turli tuman sonli qatorlarni hosil qilish mumkin. Ularning ayrimlari yaqinlashuvchi, ayrimlari esa uzoqlashuvchi bo'ladi.

Ta'rif. Funksional qatorni yaqinlashuvchi qatorga aylantiradigan x larning sonli qiymatlar to'plami uning yaqinlashish sohasi deyiladi.

Tabiiyki, yaqinlashish sohasida funksional qatorning yig'indisi x ga bog'liq bo'lgan birorta funksiyadan iborat bo'ladi. Shuning uchun funksional qatorning yig'indisi $S(x)$ orqali belgilanadi.

THEOREM 9.10 THE INTEGRAL TEST

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\, dx$$

either both converge or both diverge.

Apply the Integral Test to the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$.

Solution The function $f(x) = x/(x^2 + 1)$ is positive and continuous for $x \geq 1$. To determine whether f is decreasing, find the derivative.

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{-x^2 + 1}{(x^2 + 1)^2}$$

So, $f'(x) < 0$ for $x > 1$ and it follows that f satisfies the conditions for the Integral Test. You can integrate to obtain

$$\begin{aligned}
\int_1^{\infty} \frac{x}{x^2 + 1}\, dx &= \frac{1}{2} \int_1^{\infty} \frac{2x}{x^2 + 1}\, dx \\
&= \frac{1}{2} \lim_{b \to \infty} \int_1^{b} \frac{2x}{x^2 + 1}\, dx \\
&= \frac{1}{2} \lim_{b \to \infty} \Big[\ln(x^2 + 1) \Big]_1^{b} \\
&= \frac{1}{2} \lim_{b \to \infty} \left[\ln(b^2 + 1) - \ln 2 \right] \\
&= \infty.
\end{aligned}$$

So, the series *diverges*.

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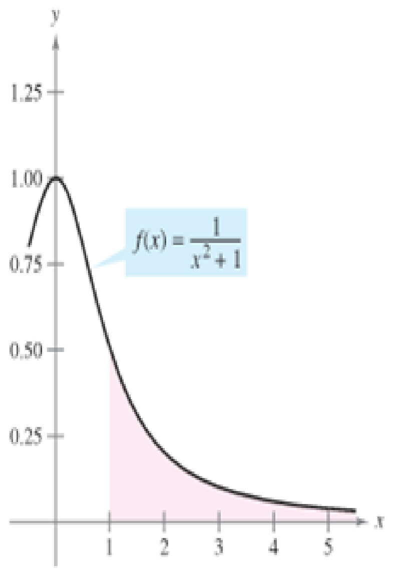


Because the improper integral converges, the infinite series also converges.
Figure 9.9

EXAMPLE 2 Using the Integral Test

Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Solution Because $f(x) = 1/(x^2+1)$ satisfies the conditions for the Integral Test (check this), you can integrate to obtain

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2+1}\,dx$$

$$= \lim_{b\to\infty} \Big[\arctan x\Big]_1^b$$

$$= \lim_{b\to\infty} (\arctan b - \arctan 1)$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

So, the series *converges* (see Figure 9.9). ■

3.Canuto, C., Tabacco, A. Mathematical Analysis II, P.17.

Masalan, $\sum_{n=1}^{\infty} \ln^n x$ qatorning yaqinlashish sohasi topilsin.

Yechish. Berilgan qator maxraji $q=\ln x$ ga teng bo'lgan cheksiz geometrik progressiyani ifodalaydi. Geometrik progressiya $|q|<1$ shart bajarilgandagina yaqinlashgani uchun, berilgan qator $|\ln x|<1$ ya'ni $-1<\ln x<1$ tengsizlik bajarilganda absolyut yaqinlashadi, demak, $e^{-1}<x<e$ tengsizlik berilgan qatorning yaqinlashish sohasini ifodalaydi. Shunday qilib (e^{-1},e) oraliqda berilgan qatorning yig'indisini

$$S(x)=\frac{\ln x}{1-\ln x}$$

formula yordamida hisoblaymiz.

(1) qatorning birinchi n ta hadining yig'indisini $S_n(x)$ deb belgilaylik. Agar bu qator yaqinlashsa va uning yig'indisi $S(x)$ ga teng bo'lsa, u holda

$$S(x)=S_n(x)+r_n(x)$$

tenglikni yozishimiz mumkin, bu yerda

$$r_n(x)=u_{n+1}(x)+u_{n+2}(x)+u_{n+3}(x)+\dots$$

kattalik (1) qatorning qoldig'i deyiladi. Qatorning yaqinlashish sohasida $\lim_{n\to\infty} s_n(x)=S(x)$ munosabat o'rinli, shuning uchun

$$\lim_{n\to\infty} r_n(x)=\lim_{n\to\infty}\left[S(x)-S_n(x)\right]=0,$$

ya'ni yaqinlashuvchi qatorning qoldig'i $r_n(x)$, $n\to\infty$ da nolga intiladi.

THEOREM 9.11 CONVERGENCE OF p-SERIES

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

1. converges if $p > 1$, and
2. diverges if $0 < p \leq 1$.

3.Canuto, C., Tabacco, A. Mathematical Analysis II, P.17.

1. Kuchaytirilgan qatorlar.

Ta'rif

$$u_1(x)+u_2(x)+u_3(x)+\ldots+u_n(x) \qquad (1)$$

funksional qator berilgan bo'lsin. Agar shunday bir musbat hadli yaqinlashuvchi sonli qator

$$\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n + \ldots \qquad (2)$$

mavjud bo'lib,

$$|u_1(x)| \leq \alpha_1, |u_2(x) \leq \alpha_2|, \ldots |u_n(x)| \leq \alpha_n \qquad (3)$$

shart bajarilsa, (1) funksional qatorni o'zining aniqlanish sohasida kuchaytirilgan qator deb ataladi.

Misol uchun:

$$\frac{\cos x}{1}+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\ldots+\frac{\cos nx}{n^2}+\ldots$$

qator $(-\infty,+\infty)$ oraliqda kuchaytirilgan qatordir. Haqiqatdan ham x ning har qanday qiymatlari uchun.

$\left|\frac{\cos nx}{n^2}\right|\leq\frac{1}{n^2}$ $(n=1,2,3,\ldots)$ munosabat bajariladi va bu qator yaqinlashuvchidir, chunki $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots$ umumlashgan garmonik qator yaqinlashuvchi qator hisoblanadi.

Ta'rifdan ko'rinadiki, ma'lum bir sohada berilgan kuchaytirilgan qator o'sha sohaning har bir nuqtasida absolyut yaqinlashadi.

Undan tashqari kuchaytirilgan qator quyidagi muhim xossaga ega.

Veyershtrass teoremasi. $[a, b]$ kesmada

$$u_1(x)+u_2(x)+u_3(x)+\ldots+u_n(x)+\ldots$$

kuchaytirilgan qator berilgan bo'lib, $S(x)$ uning yig'indisi, $S_n(x)$ esa uning birinchi n ta hadining yig'indisi bo'lsin. U holda istalgan kichik musbat son $\varepsilon>0$ uchun shunday musbat son N topiladiki, hamma $n \geq N$ lar uchun $|S(x)-S_n(x)|<\varepsilon$ tengsizlik bajariladi.

Ta'rif. Veyershtrass teoremasiga bo'ysunadigan har qanday qator $[a, b]$ kesmada tekis yaqinlashuvchi qator deb ataladi.

Veyershtrass teoremasidan ko'rinadiki kuchaytirilgan qator tekis yaqinlashuvchi qatordir.

2.Darajali qatorlar.

Ta'rif. Hadlari darajali funksiyalardan iborat bo'lgan

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots = \sum_{n=0}^{\infty} a_nx^n \qquad (1)$$

ko'rinishdagi qator darajali qator deb ataladi.

Abel teoremasi. 1) Agar darajali qator $x=x_0\neq0$ nuqtada yaqinlashsa, u holda bu qator $-|x_0|<x<|x_0|$ oraliqda absolyut yaqinlashadi; 2) Agar darajali qator $x=x_0$ nuqtada uzoqlashsa, u holda bu qator $-|x_0|>x$ va $x>|x_0|$ oraliqlarda uzoqlashadi;

Isboti. 1) Teoremaning shartiga ko'ra

$$a_0+a_1x_0+a_2x_0^2+\ldots+a_nx_0^n+\ldots \qquad (2)$$

qator yaqinlashadi, demak $n \to \infty$ da $a_n x_0^n \to 0$, bu degani shunday bir musbat M soni mavjud bo'ldiki, qatorning hamma hadi absolyut qiymati bo'yicha M dan kichik bo'ladi. (1) qatorni

$$a_0 + a_1 x_0 \left(\frac{x}{x_0}\right) + a_2 x_0^2 \left(\frac{x}{x_0}\right)^2 + \ldots + a_n x_0^n \left(\frac{x}{x_0}\right)^n + \ldots \qquad (3)$$

ko'rinishda yozib olamiz va

$$|a_0| + |a_1 x_0| \left|\frac{x}{x_0}\right| + |a_2 x_0^2| \left|\frac{x}{x_0}\right|^2 + \ldots + |a_n x_0^n| \left|\frac{x}{x_0}\right|^n + \ldots \qquad (4)$$

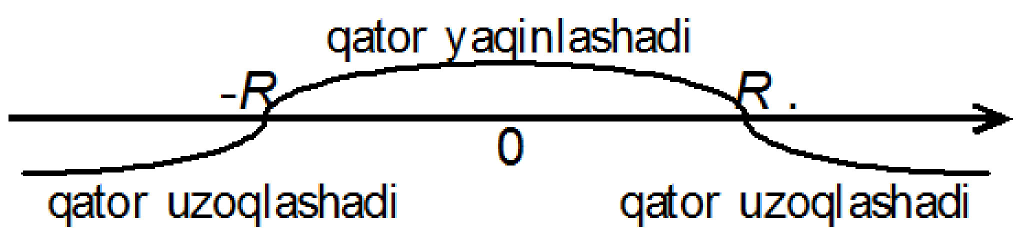
qatorni ko'raylik. Bu qatorning hadlari

$$M + M\left|\frac{x}{x_0}\right| + M\left|\frac{x}{x_0}\right|^2 + \ldots + M\left|\frac{x}{x_0}\right|^n + \ldots \qquad (5)$$

qatorning mos hadidan kichik. $|x| < |x_0|$ tengsizlik bajarilganda (5) qator maxraji $q = \left|\frac{x}{x_0}\right| < 1$ ga teng bo'lgan cheksiz kamayuvchi geometrik progressiyani tashkil etadi, demak, yaqinlashadi. Shunday qilib, (5) qator yaqinlashgani uchun (4) qator ham yaqinlashadi, natijada (3) qator yoki (1) qator absolyut yaqinlashadi.

2) Endi teoremaning ikkinchi qismini ham isbot qilish unchalik qiyin emas: faraz qilaylik x_0^1 nuqtada (1) qator uzoqlashsin. U holda $|x|>|x_0|$ tengsizlikni qanoatlantiruvchi har qanday x nuqtada ham qator uzoqlashadi. Demak, $-|x_0|>x$ va $x>|x_0|$ oraliqlarda (1) qator uzoqlashadi. Shunday qilib, teorema to'la isbot qilindi.

Ta'rif. Darajali qatorning yaqinlashish sohasi markazi koordinat boshida yotadigan intervaldan iboratdir. Darajali qatorning yaqinlashish intervali deb shunday $-R$ dan $+R$ gacha bo'lgan intervalga aytiladiki, bu intervalning ichida yotadigan har qanday x nuqtada qator absolyut yaqinlashadi, intervalning tashqarisida yotadigan istalgan x nuqtada esa uzoqlashadi.



R soni darajali qatorning yaqinlashish radiusi deb aytiladi. Intervalning oxirlarida (ya'ni $x=-R$ va $x=R$ nuqtalarida) berilgan qatorning yaqinlashishi va uzoqlashishi haqidagi savol har bir qator uchun alohida yechiladi.

3.Yaqinlashish radiusi.

(1) darajali qatorni

$$|a_0|+|a_1||x|+|a_2||x|^2+|a_3||x|^3+\ldots+|a_n||x|^n+\ldots \qquad (6)$$

ko'rinishda yozib olamiz. Bu qator musbat hadli qator bo'lgani uchun uning yaqinlashishini Dalamber alomatiga ko'ra aniqlaymiz. Faraz qilaylik, quyidagi limit mavjud bo'lsin:

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right|=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||x|=L|x|$$

Unda agar $L|x|<1$ bo'lsa, ya'ni $|x|<1/L$ yoki $-1/L<x<1/L$ intervalda qator absolyut yaqinlashadi.

Agar $L|x|>1$ bo'lsa, ya'ni $|x|>1/L$ yoki $-1/L>x$ va $x>1/L$ intervallarda qator uzoqlashadi. Yaqinlashish radiusi

$$R=\frac{1}{L}=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

formulaga ko'ra topiladi. Shunga o'xshab R ni Koshi alomatini qo'llab ham topish mumkin: $R=\dfrac{1}{\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}}$;

Misol. $\frac{x}{2}+\frac{2^2x^2}{2^2}+\frac{3^2x^3}{2^3}+\frac{4^2x^4}{2^4}+\ldots+\frac{n^2x^n}{2^n}+\ldots$ darajali qatorning yaqinlashish intervali topilsin.

Yechish. Bu yerda $a_n=\frac{n^2}{2^n}$, $a_{n+1}=\frac{(n+1)^2}{2^{n+1}}$, demak,

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\frac{n^2 2^{n+1}}{2^n(n+1)^2}=2$$

Javob. Berilgan darajali qatorning yaqinlashish intervali $-2<x<2$ tengsizlikdan iborat. Intervalning chegaralarida qator uzoqlashadi.