7-MAVZU : DIFFERENSIAL HISOBNING ASOSIY TEOREMALARI.

REJA

1. Differensial hisobning asosiy teoremalari.
2. Lopital qoidasi.
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1. Differensial hisobning asosiy teoremalari.

Ushbu mavzuda biz differensial hisobning asosiy teoremalarini keltiramiz. Bu teremalar funksiyalarni tekshirishda muhim rol uynaydi.

1-teorema. (Ferma teoremasi (*1601-1665, fransuz matematigi*)).
$f(x)$ funksiya biror X oraliqda aniqlangan va Xning biror ichki c nuqtasida eng katta (eng kichik) qiymatga ega bo'lsa va shu nuqtada chekli $f'(c)$ hosilasi mavjud bo'lsa, u holda $f'(c)=0$ bo'lishi zarurdir.

2-teorema. (Roll teoremasi (*1652-1719, fransuz matematigi*)).
$f(x)$ funksiya $[a,b]$ da aniqlangan, uzluksiz va $(a;b)$ da differensiallanuvchi bo'lib, $f(a)=f(b)$ bo'lsa, $(a;b)$ da uning hosilasi $f'(x)$ nolga teng bo'ladigan kamida bitta c nuqta mavjud bo'ladi (ya'ni $f'(c)=0$, $a<c<b$).

Roll teoremasida quyidagi uchta asosiy shart berilgan:

1) $f(x)$ funksiya $[a,b]$ da uzluksiz;
2) $f(x)$ funksiya hech bo'lmaganda $(a;b)$ da differensiallanuvchi;
3) $[a,b]$ kesma uchlarida $f(a)=f(b)$. U holda $\exists c\in(a;b)$, $f'(c)=0$ bo'ladi.

3-teorema. (Lagranj teoremasi (*1736-1813, fransuz matematigi*)).

$f(x)$ funksiya $[a,b]$ da aniqlangan, uzluksiz va $(a;b)$ da differensiallanuvchi bo'lsa, $(a;b)$ da kamida bitta shunday c nuqta mavjud bo'ladiki, ushbu tengsizlik o'rinli bo'ladi:

$$f(b)-f(a)=(b-a)f'(c),\ a<c<b \qquad (1)$$

va bu formula Lagranj formulasi deyiladi.

Lagranj formulasini quyidagi ko'rinishda yozish mumkin:

$$f(x+\Delta x)-f(x_0)=\Delta x\cdot f'(x_0+\theta\Delta x),\ 0<\theta<1 \qquad (2)$$

Logranj formulasining qulaylik tomoni shundadi, u differemsiallanuvchi funksiyaning chekli orttirmasi uchun taqribiy emas, aniq ifoda beradi. Lekin bu formulaning bitta noqulay joyi shundaki, bu aniq formulada c no'malim bo'ladi. Lekin bu keyinchalik unchalik ahamiyatga ega emas.

4-teorema. (Koshi teoremasi).

$f(x)$ va $\varphi(x)$ funksiyalar $[a,b]$ da aniqlangan, uzluksiz va $(a;b)$ da differensiallanuvchi bo'lib, $(a;b)$ da $\varphi'(x)\neq 0$ bo'lsa, $(a;b)$ da kamida bitta shunday c nuqta mavjud bo'ladiki, ushbu tenglik o'rinli bo'ladi:

$$\frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}=\frac{f'(c)}{\varphi'(c)},\ a<c<b. \qquad (3)$$

2.Lopital qoidasi.

Tegishli funksiyalarning hosilalari mavjud bo‘lganda $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$, 1^{∞}, 0^{0}, ∞^{0} ko‘rinishdagi aniqmasliklarni ochish masalasi engillashadi. Odatda hosilalardan foydalanib, aniqmasliklarni ochish Lopital qoidalari deb ataladi. Biz quyida Lopital qoidalarining bayoni bilan shug‘ullanamiz.

1. $\frac{0}{0}$ **ko‘rinishdagi aniqmaslik.** Ma’lumki, $x\to0$ da $f(x)\to0$ va $g(x)\to0$ bo‘lsa, $\frac{f(x)}{g(x)}$ nisbat $\frac{0}{0}$ ko‘rinishdagi aniqmaslikni ifodalaydi. Ko‘pincha $x\to a$ da $\frac{f(x)}{g(x)}$ nisbatning limitini topishga qaraganda $\frac{f'(x)}{g'(x)}$ nisbatning limitini topish oson bo‘ladi. Bu nisbatlar limitlarining teng bo‘lish sharti quyidagi teoremada ifodalangan.

1-teorema. Agar

1) $f(x)$ va $g(x)$ funksiyalar $(a-\delta;a)\cup(a;a+\delta)$, bu yerda $\delta>0$, to‘plamda uzluksiz, differensiallanuvchi va shu to‘plamdan olingan ixtiyoriy x uchun $g(x)\neq 0$, $g'(x)\neq 0$;

2) $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} g(x)=0$;

3) hosilalar nisbatining limiti (chekli yoki cheksiz)

$$\lim_{x\to a}\frac{f'(x)}{g'(x)}=A$$

mavjud bo‘lsa, u holda funksiyalar nisbatining limiti $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ mavjud va

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)} \qquad (2.1)$$

tenglik o‘rinli bo‘ladi.

Isbot. Har ikkala funksiyani $x=a$ nuqtada $f(a)=0$, $g(a)=0$ deb aniqlasak, natijada ikkinchi shartga ko‘ra $\lim_{x\to a} f(x)=0=f(a)$, $\lim_{x\to a} g(x)=0=g(a)$ tengliklar o‘rinli bo‘lib, $f(x)$ va $g(x)$ funksiyalar $x=a$ nuqtada uzluksiz bo‘ladi.

Avval $x>a$ holni qaraymiz. Berilgan $f(x)$ va $g(x)$ funksiyalar $[a;x]$, bu yerda $x<a+\delta$, kesmada Koshi teoremasining shartlarini qanoatlantiradi. Shuning uchun a bilan x orasida shunday c nuqta topiladiki, ushbu $\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f'(c)}{g'(c)}$ tenglik o‘rinli bo‘ladi. $f(a)=g(a)=0$ ekanligini e’tiborga olsak, so‘ngi tenglikdan

$$\frac{f(x)}{g(x)}=\frac{f'(c)}{g'(c)} \qquad (2.2)$$

bo‘lishi kelib chiqadi. Ravshanki, $a<c<x$ bo‘lganligi sababli, $x\to a$ bo‘lganda $c\to a$ bo‘ladi. Teoremaning 3-sharti va (2.2) tenglikdan $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=A$ kelib chiqadi.

Shunga o‘xshash, $x<a$ holni ham qaraladi. Teorema isbot bo‘ldi.

Misol. Ushbu $\lim\limits_{x\to 2}\frac{\ln(x^2-3)}{x^2+3x-10}$ limitni hisoblang.

Yechish. Bu holda $f(x)=\ln(x^2-3)$, $g(x)=x^2+3x-10$ bo‘lib, ular uchun 1-teoremaning barcha shartlari bajariladi.

Haqiqatan ham,

1) $\lim\limits_{x\to 2} f(x)=\lim\limits_{x\to 2}\ln(x^2-3)=\ln 1=0$, $\lim\limits_{x\to 2} g(x)=\lim\limits_{x\to 2}(x^2+3x-10)=0$;

2) $f'(x)=\frac{2x}{x^2-3}$, $g'(x)=2x+3$, $x\neq\pm\sqrt{3}$;

3) $\lim\limits_{x\to 2}\frac{f'(x)}{g'(x)}=\lim\limits_{x\to 2}\frac{2x}{(x^2-3)(2x+3)}=0$ bo‘ladi.

Demak, 1-teoremaga binoan $\lim\limits_{x\to 2}\frac{\ln\left(x^2-3\right)}{x^2+3x-10}=0$.

2-Teorema. Agar $[c;+\infty)$ nurda aniqlangan $f(x)$ va $g(x)$ funksiyalar berilgan bo‘lib,

1) $(c;+\infty)$ da chekli $f'(x)$ va $g'(x)$ hosilalar mavjud va $g'(x)\neq 0$,

2) $\lim\limits_{x\to+\infty} f(x)=0,\quad \lim\limits_{x\to+\infty} g(x)=0$;

3) hosilalar nisbatining limiti $\lim\limits_{x\to+\infty}\dfrac{f'(x)}{g'(x)}$ (chekli yoki cheksiz) mavjud bo‘lsa, u holda funksiyalar nisbatining limiti $\lim\limits_{x\to+\infty}\dfrac{f(x)}{g(x)}$ mavjud va

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=\lim_{x\to+\infty}\frac{f'(x)}{g'(x)} \qquad (2.3)$$

tenglik o‘rinli bo‘ladi.

2. $\frac{\infty}{\infty}$ **ko'rinishdagi aniqmaslik**. Agar $x \to a$ da $f(x) \to \infty$, $g(x) \to \infty$ bo'lsa, $\frac{f(x)}{g(x)}$ nisbat $\frac{\infty}{\infty}$ ko'rinishidagi aniqmaslikni ifodalaydi. Endi bunday aniqmaslikni ochishda ham $f(x)$ va $g(x)$ funksiyalarning hosilalaridan foydalanish mumkinligini ko'rsatadigan teoremani keltiramiz.

3-teorema. Agar

1) $f(x)$ va $g(x)$ funksiyalar $(a;\infty)$ nurda differensiallanuvchi, hamda $g'(x) \neq 0$,

2) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$,

3) $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ mavjud bo'lsa,

u holda $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ mavjud va $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ bo'ladi.

Recall that the forms $0/0$ and ∞/∞ are called *indeterminate* because they do not guarantee that a limit exists, nor do they indicate what the limit is, if one does exist.

Larson Edvards. /Calculus/ 2010. P.569.

THEOREM 8.4 L'HÔPITAL'S RULE

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies if the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

Larson Edvards. /Calculus/ 2010. P.570.

EXAMPLE 2 Indeterminate Form ∞/∞

Evaluate $\lim_{x\to\infty} \frac{\ln x}{x}$.

Solution Because direct substitution results in the indeterminate form ∞/∞, you can apply L'Hôpital's Rule to obtain

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}[\ln x]}{\frac{d}{dx}[x]}$$ Apply L'Hôpital's Rule.

$$= \lim_{x\to\infty} \frac{1}{x}$$ Differentiate numerator and denominator.

$$= 0.$$ Evaluate the limit. ■

Larson Edvards. /Calculus/ 2010. P.571.

3. Boshqa ko‘rinishdagi aniqmasliklar. Ma’lumki, $\lim_{x\to a} f(x)=0$, $\lim_{x\to a} f(x)=\infty$, bo‘lganda *f(x)·g(x)* ifoda $0\cdot\infty$ ko‘rinishidagi aniqmaslik bo‘lib, uning quyidagi

$$f(x)\cdot g(x)=\frac{f(x)}{\frac{1}{g(x)}}=\frac{g(x)}{\frac{1}{f(x)}}$$

kabi yozish orqali $\frac{0}{0}$ yoki $\frac{\infty}{\infty}$ ko‘rinishidagi aniqmaslikka keltirish mumkin. Shuningdek, $\lim_{x\to a} f(x)=+\infty$, $\lim_{x\to a} g(x)=+\infty$, bo‘lganda *f(x)-g(x)* ifoda ∞-∞ ko‘rinishidagi aniqmaslik bo‘lib, uni ham quyidacha shakl almashtirib

$$f(x)-g(x)=\frac{\frac{1}{g(x)}-\frac{1}{f(x)}}{\frac{1}{f(x)}\cdot\frac{1}{g(x)}}$$

$\frac{0}{0}$ ko‘rinishdagi aniqmaslikka keltirish mumkin.

Ma’lumki, $x\to a$ da *f(x)* funksiya 1, 0 va ∞ ga, *g(x)* funksiya esa mos ravshda ∞, 0 va 0 intilganda $(f(x))^{g(x)}$ darajali-ko‘rsatkichli ifoda 1^{∞}, 0^0, ∞^0 ko‘rinishidagi aniqmasliklar edi. Bu ko‘rinishdagi aniqmasliklarni ochish uchun avval $y=(f(x))^{g(x)}$ ni logarifmlaymiz: $\ln y= g(x)\cdot\ln(f(x))$. Bunda $x\to a$ da $g(x)\ln(f(x))$ ifoda $0\cdot\infty$ ko‘rinishdagi aniqmaslikni ifodalaydi.

Shunday qilib, funksiya hosilalari yordamida $0\cdot\infty$, $\infty-\infty$, 1^{∞}, 0^0, ∞^0, ko‘rinishdagi aniqmasliklarni ochiшda, ularni $\frac{0}{0}$ yoki $\frac{\infty}{\infty}$ ko‘rinishidagi aniqmaslikka keltirib, so‘ng yuqoridagi teoremalar qo‘llaniladi.

In addition to the forms $0/0$ and ∞/∞, there are other indeterminate forms such as $0 \cdot \infty$, 1^{∞}, ∞^0, 0^0, and $\infty - \infty$. For example, consider the following four limits that lead to the indeterminate form $0 \cdot \infty$.

$$\underbrace{\lim_{x\to 0} (x)\left(\frac{1}{x}\right)}_{\text{Limit is 1.}}, \quad \underbrace{\lim_{x\to 0} (x)\left(\frac{2}{x}\right)}_{\text{Limit is 2.}}, \quad \underbrace{\lim_{x\to \infty} (x)\left(\frac{1}{e^x}\right)}_{\text{Limit is 0.}}, \quad \underbrace{\lim_{x\to \infty} (e^x)\left(\frac{1}{x}\right)}_{\text{Limit is } \infty.}$$

Because each limit is different, it is clear that the form $0 \cdot \infty$ is indeterminate in the sense that it does not determine the value (or even the existence) of the limit. The following examples indicate methods for evaluating these forms. Basically, you attempt to convert each of these forms to $0/0$ or ∞/∞ so that L'Hôpital's Rule can be applied.

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The forms $0/0$, ∞/∞, $\infty - \infty$, $0 \cdot \infty$, 0^0, 1^∞, and ∞^0 have been identified as *indeterminate*. There are similar forms that you should recognize as "determinate."

$\infty + \infty \to \infty$	Limit is positive infinity.
$-\infty - \infty \to -\infty$	Limit is negative infinity.
$0^\infty \to 0$	Limit is zero.
$0^{-\infty} \to \infty$	Limit is positive infinity.

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Eslatma. Agar *f(x)* va *g(x)* funksiyalarning *f'(x)* va *g'(x)* hosilalari ham *f(x)* va *g(x)* lar singari yuqorida keltirilgan teoremalarning barcha shartlarini qanoatlantirsa, u holda $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}=\lim\limits_{x\to a}\frac{f''(x)}{g''(x)}$ tengliklar o'rinli bo'ladi, ya'ni bu holda Lopital qoidasini takror qo'llanish mumkin bo'ladi.

Misol. Ushbu $\lim\limits_{x\to 0}\left(\frac{tgx}{x}\right)^{\frac{1}{x^2}}$ limitni hisoblang.

Yechish. Ravshanki, $x\to 0$ da $\left(\frac{tgx}{x}\right)^{\frac{1}{x^2}}$ ifoda 1^∞ ko'rinishdagi aniqmaslik bo'ladi.

Uni logarifmlab, $\frac{0}{0}$ aniqmaslikni ochishga keltiramiz:

Demak, $\lim\limits_{x\to 0}\left(\frac{tgx}{x}\right)^{\frac{1}{x^2}}=e^{\frac{1}{3}}=\sqrt[3]{e}$.

2. Teylor formulasi.

Teylor formulasi matematik analizning eng muhim formulalaridan biri bo‘lib, ko‘plab nazariy tatbiqlarga ega. U taqribiy hisobning negizini tashkil qiladi.

Teylor ko‘phadi. Peano ko‘rinishdagi qoldiq hadli Teylor formulasi. Ma’lumki, funksiyaning qiymatlarini hisoblash ma’nosida ko‘phadlar eng sodda funksiyalar hisoblanadi. Shu sababli funksiyaning x_0 nuqtadagi qiymatini hisoblash uchun uni shu nuqta atrofida ko‘phad bilan almashtirish muammosi paydo bo‘ladi.

Nuqtada differensiallanuvchi funksiya ta’rifiga ko‘ra agar $y=f(x)$ funksiya x_0 nuqtada differensiallanuvchi bo‘lsa, u holda uning shu nuqtadagi orttirmasini $\Delta f(x_0)=f'(x_0)\Delta x+o(\Delta x)$, ya’ni $f(x)=f(x_0)+f'(x_0)(x-x_0)+o(x-x_0)$ ko‘rinishda yozish mumkin.

Boshqacha aytganda x_0 nuqtada differensiallanuvchi $y=f(x)$ funksiya uchun birinchi darajali

$$P_1(x)=f(x_0)+b_1(x-x_0) \qquad (3.1)$$

ko‘phad mavjud bo‘lib, $x\to x_0$ da $f(x)=P_1(x)+o(x-x_0)$ bo‘ladi. Shuningdek, bu ko‘phad $P_1(x_0)=f(x_0)$, $P_1'(x_0)=b=f'(x_0)$ shartlarni ham qanoatlantiradi.

Endi umumiyroq masalani qaraylik. Agar $x=x_0$ nuqtaning biror atrofida aniqlangan $y=f(x)$ funksiya shu nuqtada $f'(x)$, $f''(x)$, ..., $f^{(n)}(x)$ hosilalarga ega bo'lsa, u holda

$$f(x)=P_n(x)+o(x-x_0) \qquad (3.2)$$

shartni qanoatlantiradigan darajasi n dan katta bo'lmagan $P_n(x)$ ko'phad mavjudmi?

Bunday ko'phadni

$$P_n(x)=b_0+b_1(x-x_0)+b_2(x-x_0)^2+ \ldots +b_n(x-x_0)^n, \qquad (3.3)$$

ko'rinishda izlaymiz. Noma'lum bo'lgan b_0, b_1, b_2, ..., b_n koeffitsientlarni topishda

$$P_n(x_0)=f(x_0),\ P_n'(x_0)=f'(x_0),\ P_n''(x_0)=f''(x_0),\ \ldots,\ P_n^{(n)}(x_0)=f^{(n)}(x_0) \qquad (3.4)$$

shartlardan foydalanamiz. Avval $P_n(x)$ ko'phadning hosilalarini topamiz:

$P_n'(x)=b_1+2b_2(x-x_0)+3b_3(x-x_0)^2+ \ldots +nb_n(x-x_0)^{n-1}$,

$P_n''(x)=2\cdot 1b_2+3\cdot 2b_3(x-x_0)+ \ldots +n\cdot(n-1)b_n(x-x_0)^{n-2}$,

$P_n'''(x)=3\cdot 2\cdot 1b_3+ \ldots +n\cdot(n-1)\cdot(n-2)b_n(x-x_0)^{n-3}$,

. . . . ,

$P_n^{(n)}(x)=n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 2\cdot 1b_n$.

Yuqorida olingan tengliklar va (3.3) tenglikning har ikkala tomoniga x o'rniga x_0 ni qo'yib barcha b_0, b_1, b_2, ..., b_n koeffitsientlar qiymatlarini topamiz:

$P_n(x_0)=f(x_0)=b_0$,

$P_n'(x_0)=f'(x_0)=b_1$,

$P_n''(x_0)=f''(x_0)=2\cdot 1b_2=2!b_2$,

. . . .

$P_n^{(n)}(x_0)=f^{(n)}(x_0)=n\cdot(n-1)\cdot\ldots\cdot 2\cdot 1b_n=n!b_n$

Bulardan $b_0=f(x_0)$, $b_1=f'(x_0)$, $b_2=\frac{1}{2!}f''(x_0)$, . . ., $b_n=\frac{1}{n!}f^{(n)}(x_0)$ hosil qilamiz.

Topilgan natijalarni (3.3) qo‘yamiz va

$$P_n(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\ldots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n, \qquad (3.5)$$

ko‘rinishda ko‘phadni hosil qilamiz. Bu ko‘phad Teylor ko‘phadi deb ataladi.

DEFINITIONS OF *n*TH TAYLOR POLYNOMIAL AND *n*TH MACLAURIN POLYNOMIAL

If f has n derivatives at c, then the polynomial

$$P_n(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdot\cdot\cdot+\frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the ***n*th Taylor polynomial for *f* at *c*.** If $c=0$, then

$$P_n(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdot\cdot\cdot+\frac{f^{(n)}(0)}{n!}x^n$$

is also called the ***n*th Maclaurin polynomial for *f*.**

Larson Edvards. /Calculus/ 2010. P.652.

Teylor ko‘phadi (3.2) shartni qanoatlantirishini isbotlaymiz. Funksiya va Teylor ko‘phadi ayirmasini $R_n(x)$ orqali belgilaymiz: $R_n(x)=f(x)-P_n(x)$. (3.4) shartlardan $R_n(x_0)=R_n'(x_0)=\ldots=R_n^{(n)}(x_0)=0$ bo‘lishi kelib chiqadi.

Endi $R_n(x)=o((x-x_0)^n)$, ya’ni $\lim\limits_{x\to x_0}\dfrac{R_n(x)}{(x-x_0)^n}=0$ ekanligini ko‘rsatamiz. Agar $x\to x_0$ bo‘lsa, $\lim\limits_{x\to x_0}\dfrac{R_n(x)}{(x-x_0)^n}$ ifodaning 0/0 tipidagi aniqmaslik ekanligini ko‘rish qiyin emas. Unga Lopital qoidasini n marta tatbiq qilamiz. U holda

$$\lim_{x\to x_0}\frac{R_n(x)}{(x-x_0)^n}=\lim_{x\to x_0}\frac{R_n'(x)}{n(x-x_0)^{n-1}}=\ldots=\lim_{x\to x_0}\frac{R_n^{(n-1)}(x)}{n!(x-x_0)}=$$

$=\lim\limits_{x\to x_0}\dfrac{R_n^{(n)}(x)}{n!}=\dfrac{R_n^{(n)}(x_0)}{n!}=0$, demak $x\to x_0$ da $R_n(x)=o((x-x_0)^n)$ o‘rinli ekan.

Shunday qilib, quyidagi teorema isbotlandi:

Teorema. Agar $y=f(x)$ funksiya x_0 nuqtaning biror atrofida n marta differensiallanuvchi bo‘lsa, u holda $x\to x_0$ da quyidagi formula

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\ldots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+o((x-x_0)^n) \quad (3.6)$$

o‘rinli bo‘ladi, bu yerda $R_n(x)=o((x-x_0)^n)$ Peano ko‘rinishidagi qoldiq had.

Agar (3.6) formulada $x_0=0$ deb olsak, Teylor formulasining xususiy holi hosil bo‘ladi:

$$f(x)=f(0)+f'(0)x+\frac{1}{2!}f''(0)x^2+\ldots+\frac{1}{n!}f^{(n)}(0)x^n+o(x^n). \quad (3.7)$$

Bu formula Makloren formulasi deb ataladi.

Teylor formulasining Lagranj ko'rinishdagi qoldiq hadi. Teylor formulasi $R_n(x)$ qoldiq hadi yozilishining turli ko'rinishlari mavjud. Biz uning Lagranj ko'rinishi bilan tanishamiz.

Qaralayotgan $f(x)$ funksiya x_0 nuqta atrofida $n+1$ –tartibli hosilaga ega bo'lsin deb talab qilamiz va yangi $g(x)=(x-x_0)^{n+1}$ funksiyani kiritamiz. Ravshanki,

$g(x_0)=g'(x_0)=\ldots=g^{(n)}(x_0)=0;\ \ g^{(n+1)}(x_0)=(n+1)!\neq 0.$

Ushbu $R_n(x)=f(x)-P_n(x)$ va $g(x)=(x-x_0)^{n+1}$ funksiyalarga Koshi teoremasini tatbiq qilamiz. Bunda $R_n(x_0)=R_n'(x_0)=\ldots=R_n^{(n)}(x_0)=0$ e'tiborga olib, quyidagini topamiz:

$$\frac{R_n(x)}{g(x)}=\frac{R_n(x)-R_n(x_0)}{g(x)-g(x_0)}=\frac{R_n'(c_1)}{g(c_1)}=\frac{R_n'(c_1)-R_n'(x_0)}{g'(c_1)-g'(x_0)}=\frac{R_n''(c_2)}{g''(c_2)}=\ldots=$$

$$=\frac{R_n^{(n)}(c_n)}{g^{(n)}(c_n)}=\frac{R_n^{(n)}(x)-R_n^{(n)}(x_0)}{g^{(n)}(x)-g^{(n)}(x_0)}=\frac{R_n^{(n+1)}(\xi)}{g^{(n+1)}(\xi)},$$

bu yerda $c_1\in(x_0;x);\ c_2\in(x_0;c_1);\ \ldots;\ c_n\in(x_0;c_{n-1});\ \xi\in(x_0;c_n)\subset(x_0;x)$.

Shunday qilib, biz $\dfrac{R_n(x)}{g(x)}=\dfrac{R_n^{(n+1)}(\xi)}{g^{(n+1)}(\xi)}$ ekanligini ko'rsatdik, bu yerda $\xi\in(x_0;x)$. Endi $g(x)=(x-x_0)^{n+1}$, $g^{(n+1)}(\xi)=(n+1)!$, $R_n^{(n+1)}(\xi)=f^{(n+1)}(\xi)$ ekanligini e'tiborga olsak quyidagi formulaga ega bo'lamiz:

$$R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1},\ \xi\in(x_0;x). \qquad (3.8)$$

Bu (3.8) formulani Teylor formulasining Lagranj ko'rinishidagi qoldiq hadi deb ataladi.

Lagranj ko‘rinishdagi qoldiq hadni

$$R_n(x)=\frac{f^{(n+1)}(x_0+\theta(x-x_0))}{(n+1)!}(x-x_0)^{n+1} \qquad (3.9)$$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

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ko‘rinishda ham yozish mumkin, bu yerda θ birdan kichik bo‘lgan musbat son, ya’ni $0<\theta<1$.

Shunday qilib, $f(x)$ funksiyaning Lagranj ko‘rinishidagi qoldiq hadli Teylor formulasi kuyidagi shakldа yoziladi:

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\ldots$$

$$+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1},$$ bu yerda $\xi\in(x_0;x)$.

Agar $x_0=0$ bo‘lsa, u holda $\xi=x_0+\theta(x-x_0)=\theta x$, bu yerda $0<\theta<1$, bo‘lishi ravshan, shu sababli Lagranj ko‘rinishidagi qoldiq hadli Makloren formulasi

$$f(x)=f(0)+f'(0)x+\frac{1}{2!}f''(0)x^2+\ldots+\frac{1}{n!}f^{(n)}(0)x^n+\frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1} \qquad (3.10)$$

shaklida yoziladi.

Teylor formulasining Koshi ko'rinishidagi qoldiq hadi. Teylor formulasi qoldiq hadining boshqa ko'rinishlariga misol tariqasida Koshi ko'rinishidagi qoldiq hadni keltirish mumkin. Buning uchun

$$\varphi(t) = f(x) - f(t) - f'(t)(x-t) - \ldots - \frac{f^{(n)}(t)}{n!}(x-t)^n$$

yordamchi funksiyani tuzib olamiz va $[x_0;x]$ segmentda uzluksiz, $(x_0;x)$ intervalda esa noldan farqli chekli hosilaga ega bo'lgan biror $\psi(t)$ funksiyani olib, bu funksiyalarga Koshi teoremasini qo'llasak,

$$R_n(x) = \frac{\psi(x) - \psi(x_0)}{\psi'(c)} \cdot \frac{f^{(n+1)}(c)}{n!}(x-c)^n, \quad c \in (x_0; x) \qquad (3.11)$$

ko'rinishdagi qoldiq hadni chiqarish mumkin.

Agar (3.11) formulada $\psi(t)$ funksiya sifatida $\psi(t) = x - t$ funksiya olinsa, natijada Koshi shaklidagi qoldiq hadni hosil qilamiz:

$$R_n(x) = \frac{f^{(n+1)}(c)}{n!}(1-\theta)^n (x-x_0)^{n+1}, \quad c = x_0 + \theta(x-x_0), \quad 0 < \theta < 1$$