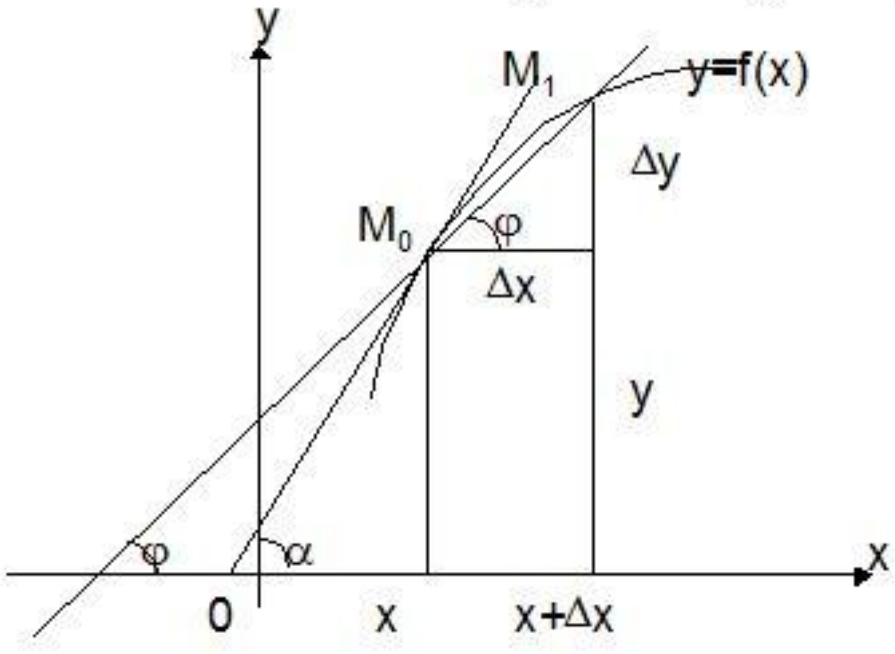
5-MAVZU : FUNKSIYA HOSILANING GEOMETRIK VA FIZIK MA’NOSI

1. Hosilaning geometrik va mexanik ma'nosi.

Bizga berilgan u=f(x) funksiya x nuqta va uning atrofida aniqlangan bo'lsin. Argument x ning biror qiymatida u=f(x) funksiya aniq qiymatga ega bo'ladi, biz uni $M_0(x, u)$ deb belgilaylik. Argumentga Δx orttirma beramiz va natija funksiyaning $u+\Delta u=f(x+\Delta x)$ orttirilgan qiymati to'g'ri keladi. Bu nuqtani $M_1(x+\Delta x,\ u+\Delta u)$ deb belgilaymiz va M_0 kesuvchi o'tkazib uning OX o'qining musbat yo'nalishi bilan tashkil etgan burchagini φ bilan belgilaymiz.



Endi $\frac{\Delta y}{\Delta x}$ nisbatni qaraymiz. Rasmdan ko'rinadiki, $\frac{\Delta y}{\Delta x} = tg\varphi$ (1) ga

Agar $\Delta x \to 0$ ga, u holda M_1 nuqta egri chiziq bo'yicha harakatlanib, M_0 nuqtaga yaqinlasha boradi. M_0M_1 kesuvchi ham $\Delta x \to 0$ da o'z holatini o'zgartira boradi, xususan φ burchak ham o'zgaradi va natijada φ burchak α burchakka intiladi. M_0M_1 kesuvchi esa M_0 nuqtadan o'tuvchi urinma holatiga intiladi. Urinmaning burchak koeffitsienti quyidagicha topiladi

$$tg\alpha = \lim_{\Delta x \to 0} tg\varphi = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x) \qquad (2)$$

Demak, $f'(x) = tg\alpha$, ya'ni, argument x ning berilgan qiymatida $f'(x)$ hosilaning qiymati f(x) funksiyaning grafigiga uning $M_0(x, u)$ nuqtasidagi urinmaning OX o'qining musbat yo'nalishi bilan hosil qilgan burchak tangensiga teng.

1. Geometrik ma'nosi.

Faraz qilaylik bizga $y = f(x)$ funksiya grafiga va unga tegishli bo'lgan $P_0(x_0, f(x_0))$ nuqta berilgan bo'lsin.

$f'(x_0)$ - f funksiyaning grafigiga $P_0(x_0, f(x_0))$ nuqtada o'tkazilgan urinmaning burchak koeffisientiga teng. Bundan foydalanib biz urinma tenglamasini keltirib chiqaramiz. Faraz qilaylik urinma tenglamasi

$$y = kx + l$$

ko'rinishida bo'lsin. Bu yerda $k = f'(x_0)$

$P_0(x_0, f(x_0))$ nuqta bu to'g'ri chiziqqa tegishli ekanidan $f(x_0) = f'(x_0)x_0 + l$

$$l = f(x_0) - f'(x_0)x_0$$

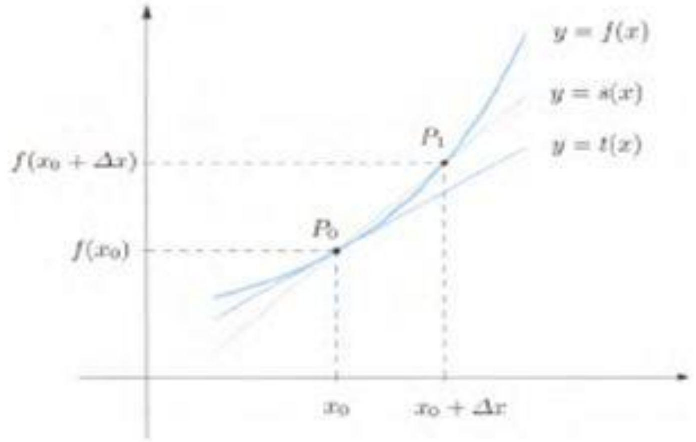
Bundan

$$y = t(x) = f(x_0) + f'(x_0)(x - x_0), \quad x \in R$$

2. Fizik ma'nosi

$$v(t_0) = s'(t_0) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \qquad (**)$$

(**) formula $s = s(t)$ qonun bo'yicha harakatlanayotgan M jismning t_0 vaqtdagi oniy tezligini ifodalaydi.



From the geometric point of view $f'(x_0)$ is the slope of the **tangent line** at $P_0 = \big(x_0, f(x_0)\big)$ to the graph of f: such line t is obtained as the limiting position of the secant s at P_0 and $P = \big(x, f(x)\big)$, when P approaches P_0. From (6.1) and the previous definition we have

$$y = t(x) = f(x_0) + f'(x_0)(x - x_0), \qquad x \in \mathbb{R}.$$

In the physical example given above, the derivative $v(t_0) = s'(t_0) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$ is the instantaneous *velocity* of the particle M at time t_0.

Bibliography:

Claudio Canuto, Anita Tabacco " Mathematical analysis I" pp 168-169

3. Hosilaning geometrik va fizik ma'nolari.

Hosilaning fizik ma'nosi. Hosila tushunchasiga olib keladigan ikkinchi masalada harakat qonuni $s=s(t)$ funksiya bilan tavsiflanadigan to'g'ri chiziq bo'ylab harakatlanayotgan moddiy nuqtaning t vaqt momentidagi oniy tezligi $v_{oniy} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$ ekanligini ko'rgan edik. Bundan hosilaning fizik (mexanik) ma'nosi kelib chiqadi.

$s=s(t)$ funksiya bilan tavsiflanadigan to'g'ri chiziqli harakatda t vaqt momentidagi harakat tezligining son qiymati hosilaga teng: $v_{oniy} = s'(t)$.

Hosilaning mexanik ma'nosini qisqacha quyidagicha ham aytish mumkin: yo'ldan vaqt bo'yicha olingan hosila tezlikka teng.

Hosila tushunchasi nafaqat to'g'ri chiziqli harakatning oniy tezligini, balki boshqa jarayonlarning ham oniy tezligini aniqlashga imkon beradi. Masalan, faraz qilaylik $y=Q(T)$ jismni T tempyeraturaga qadar qizdirish uchun uzatilayotgan issiqlik miqdorining o'zgarishini tavsiflovchi funksiya bo'lsin. U holda jismning issiqlik sig'imi issiqlik miqdoridan tempyeratura bo'yicha olingan hosilaga teng bo'ladi:

$$C=\frac{dQ}{dT} = \lim_{\Delta T \to 0} \frac{\Delta Q}{\Delta T}.$$

Umuman olganda, hosilani $f(x)$ funksiya bilan tavsiflanadigan jarayon oniy tezligining *matematik modeli* deb aytish mumkin.

4. Hosila hisoblash qoidalari

Quyida keltirilgan teoremalar isbotida hosila topish algoritmidan, limitga ega bo'lgan funksiyalar ustida arifmetik amallar haqidagi teoremalardan foydalanamiz. Shuningdek $\Delta u=u(x+\Delta x)-u(x)$ va $\Delta v=v(x+\Delta x)-v(x)$ ekanligini hisobga olgan holda, $u(x+\Delta x)=u(x)+\Delta u$, $v(x+\Delta x)=v(x)+\Delta v$ tengliklardan foydalanamiz.
$u(x)$ va $v(x)$ funksiyalar (a,b) intervalda aniqlangan bo'lsin.

Yig'indining hosilasi.

1-teorema. Agar $u(x)$ va $v(x)$ funksiyalarning $x\in(a,b)$ nuqtada hosilalari mavjud bo'lsa, u holda $f(x)=u(x)+v(x)$ funksiyaning ham x nuqtada hosilasi mavjud va

$$f'(x)=u'(x)+v'(x) \qquad (4.1)$$

tenglik o'rinli bo'ladi.

Isboti. 1^0. $f(x)=u(x)+v(x)$.
2^0. $f(x+\Delta x)= u(x+\Delta x)+ v(x+\Delta x)= u(x)+\Delta u+ v(x)+\Delta v$.
3^0. $\Delta y= f(x+\Delta x)- f(x)= \Delta u+\Delta v$.
4^0. $\dfrac{\Delta y}{\Delta x}=\dfrac{\Delta u+\Delta v}{\Delta x}=\dfrac{\Delta u}{\Delta x}+\dfrac{\Delta v}{\Delta x}$.
5^0. $\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{\Delta u+\Delta v}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{\Delta u}{\Delta x}+\lim\limits_{\Delta x\to 0}\dfrac{\Delta v}{\Delta x}=u'(x)+v'(x)$.

Shunday qilib, (4.1) tenglik o'rinli ekan. Isbot tugadi.

Misol. $(x^2+1/x)'=(x^2)'+(1/x)'=2x-1/x^2$.

Matematik induksiya metodidan foydalanib, quyidagi natijani isbotlash mumkin:

Natija. Agar $u_1(x)$, $u_2(x)$, ... ,$u_n(x)$ funksiyalarning x nuqtada hosilalari mavjud bo'lsa, u holda $f(x)= u_1(x)+ u_2(x+ ...+u_n(x)$ funksiyaning ham x nuqtada hosilasi mavjud va quyidagi formula o'rinli bo'ladi:
$f'(x)=(u_1(x)+ u_2(x+ ...+u_n(x))'= u'_1(x)+ u'_2(x+ ...+u'_n(x)$.

Ko‘paytmaning hosilasi.

2-teorema. Agar $u(x)$ va $v(x)$ funksiyalar $x\in(a,b)$ nuqtada hosilaga ega bo‘lsa, u holda ularning $f(x)=u(x)\cdot v(x)$ ko‘paytmasi ham $x\in(a,b)$ nuqtada hosilaga ega va

$$f'(x)=u'(x)v(x)+u(x)v'(x) \qquad (4.2)$$

tenglik o‘rinli bo‘ladi.

Isboti. 1^0. $f(x)=u(x)\cdot v(x)$.

2^0. $f(x+\Delta x)=u(x+\Delta x)\cdot v(x+\Delta x)=(u(x)+\Delta u)\cdot(v(x)+\Delta v)=$
$=u(x)v(x)+\Delta u v(x)+\Delta v u(x)+\Delta u\Delta v.$

3^0. $\Delta y=f(x+\Delta x)-f(x)=\Delta u v(x)+\Delta v u(x)+\Delta u\Delta v.$

4^0. $\dfrac{\Delta y}{\Delta x}=\dfrac{\Delta u v(x)+\Delta v u(x)+\Delta u\Delta x}{\Delta x}=\dfrac{\Delta u}{\Delta x}v(x)+\dfrac{\Delta v}{\Delta x}u(x)+\dfrac{\Delta u}{\Delta x}\Delta v.$

5^0. $\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}=\left(\lim\limits_{\Delta x\to 0}\dfrac{\Delta u}{\Delta x}\right)\cdot v(x)+\left(\lim\limits_{\Delta x\to 0}\dfrac{\Delta v}{\Delta x}\right)\cdot u(x)+\lim\limits_{\Delta x\to 0}\dfrac{\Delta u}{\Delta x}\cdot\lim\limits_{\Delta x\to 0}\Delta v=$
$=u'(x)\cdot v(x)+u(x)\cdot v'(x)++u'(x)\cdot\lim\limits_{\Delta x\to 0}\Delta v.$

Bunda $v(x)$ funksiyaning uzluksizligini e’tiborga olsak $\lim\limits_{\Delta x\to 0}\Delta v=0$ va natijada (4.2) formulaga ega bo‘lamiz.

1-natija. Quyidagi $(Cu(x))'=C\cdot u'(x)$ formula o'rinli.

Isboti. Ikkinchi teoremaga ko'ra $(Cu(x))'=C'\cdot u(x)+C\cdot u'(x)$. Ammo $C'=0$, demak $(Cu(x))'=C\cdot u'(x)$.

Misollar. 1. $(6x^2)'=6(x^2)'=6\cdot 2x=12x$.
2. $(x^4)'=((x^2)(x^2))'=(x^2)'(x^2)+(x^2)(x^2)'=2x(x^2)+(x^2)\cdot 2x=4x^3$.
3. $(0,25x4-3x2)'=(0,25x^4)'+(3x^2)'=0,25\cdot 4x^3+3\cdot 2x= x^3+6x$.

2-natija. Agar $u_1(x)$, $u_2(x)$, ... ,$u_n(x)$ funksiyalar x nuqtada hosilaga ega bo'lsa, u holda ularning ko'paytmasi $f(x)= u_1(x)\cdot u_2(x)\cdot \ldots\cdot u_n(x)$ ham x nuqtada hosilaga ega va quyidagi formula o'rinli bo'ladi:

$$f'(x)= (u_1(x)\cdot u_2(x)\cdot \ldots\cdot u_n(x))'= u'_1(x)\cdot u_2(x)\cdot \ldots\cdot u_n(x)+ u_1(x)\cdot u'_2(x)\cdot \ldots\cdot u_n(x)+\ldots+ u_1(x)\cdot u_2(x)\cdot \ldots\cdot u'_n(x).$$

Bo'linmaning hosilasi.

3-teorema. Agar $u(x)$ va $v(x)$ funksiyalar $x\in(a,b)$ nuqtada hosilaga ega, $v(x)\neq 0$ bo'lsa, u holda ularning $f(x)=u(x)/v(x)$ bo'linmasi $x\in(a,b)$ nuqtada hosilaga ega va

$$f'(x)=\frac{u'(x)v(x)-u(x)v'(x)}{v^2(x)} \qquad (4.3)$$

formula o'rinli bo'ladi.

Isboti. 1^0. $f(x)=\dfrac{u(x)}{v(x)}$.

2^0. $f(x+\Delta x)=\dfrac{u(x+\Delta x)}{v(x+\Delta x)}=\dfrac{u(x)+\Delta u}{v(x)+\Delta v}$.

3^0. $\Delta y=f(x+\Delta x)-f(x)=\dfrac{u(x)+\Delta u}{v(x)+\Delta v}-\dfrac{u(x)}{v(x)}=\dfrac{\Delta u\cdot v(x)-\Delta v\cdot u(x)}{(v(x)+\Delta v)v(x)}$

4^0. $\dfrac{\Delta y}{\Delta x}=\dfrac{\Delta u\cdot v(x)-\Delta v\cdot u(x)}{(v(x)+\Delta v)v(x)\Delta x}=\left(\dfrac{\Delta u}{\Delta x}v(x)-u(x)\dfrac{\Delta v}{\Delta x}\right)\cdot\dfrac{1}{v^2(x)+v(x)\Delta v}$

5^0. $\Delta x\to 0$ da limitga o'tamiz, limitga ega funksiyalarning xossalari va 2-teorema isbotidagi kabi $\lim\limits_{\Delta x\to 0}\Delta v=0$ tenglikdan foydalansak

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left(\frac{\Delta u}{\Delta x}v(x)-u(x)\frac{\Delta v}{\Delta x}\right)\cdot\frac{1}{v^2(x)+v(x)\Delta v}=\frac{u'(x)v(x)-u(x)v'(x)}{v^2(x)}$$

natijaga yerishamiz, ya'ni (4.3) formula o'rinli ekan.

Misol. Ushbu $f(x)=\frac{3x+7}{5x-4}$ funksiyaning hosilasini toping.

Yechish. $\left(\frac{3x+7}{5x-4}\right)' = \frac{(3x+7)'\cdot(5x-4)-(3x+7)\cdot(5x-4)'}{(5x-4)^2} = \frac{3(5x-4)-5(3x+7)}{(5x-4)^2} = -\frac{47}{(5x-4)^2}$

Shunday qilib biz ushbu paragrafda hosilani hisoblashning quyidagi qoidalarini keltirib chiqardik:

1. Ikkita, umuman chekli sondagi funksiyalar yig'indisining hosilasi hosilalar yig'indisiga teng.
2. O'zgarmas ko'paytuvchini hosila belgisi oldiga chiqarish mumkin.
3. Ikkita $u(x)$ va $v(x)$ funksiyalar ko'paytmasining hosilasi $u'v+uv'$ ga teng.
4. Ikkita $u(x)$ va $v(x)$ funksiyalar bo'linmasining hosilasi $(u'v-uv')/v^2$ ga teng.

1- va 2-teorema natijalaridan foydalangan holda quyidagi qoidaning ham o'rinli ekanligini ko'rish qiyin emas:

5. Chekli sondagi differensiallanuvchi funksiyalar chiziqli kombinatsiyasining hosilasi hosilalarning aynan shunday chiziqli kombinatsiyasiga teng, ya'ni agar $f(x)=c_1u_1(x)+ c_2u_2(x)+...+ c_nu_n(x)$ bo'lsa, u holda $f'(x)=c_1u'_1(x)+ c_2u'_2(x)+...+ c_nu'_n(x)$.

Bu qoidaning isbotini o'quvchilarga havola qilamiz.

Eslatma. Yuqoridagi teoremalar funksiyalar yig'indisi, ko'paytmasi, bo'linmasining hosilaga ega bo'lishining yetarli shartlarini ifodalaydi. Demak, ikki funksiya yig'indisi, ayirmasi, ko'paytmasi va nisbatidan iborat bo'lgan funksiyaning hosilaga ega bo'lishidan bu funksiyalarning har biri hosilaga ega bo'lishi har doim kelib chiqavyermaydi. Masalan, $u(x)=|x|$, $v(x)=|x|$ deb, ularning ko'paytmasini tuzsak, $y=x^2$ ko'rinishdagi funksiya hosil bo'ladi. Bu funksiyaning $\forall x\in(-\infty;+\infty)$ nuqtada, xususan, $x=0$ nuqtada hosilasi mavjud. Ammo, ma'lumki $y=|x|$ funksiyaning $x=0$ nuqtada hosilasi mavjud emas.

Asosiy trigonometrik jadvallar

$$\mathrm{D}\, x^{\alpha} = \alpha x^{\alpha-1} \qquad (\forall \alpha \in \mathbb{R})$$

$$\mathrm{D}\, \sin x = \cos x$$

$$\mathrm{D}\, \cos x = -\sin x$$

$$\mathrm{D}\, \tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}$$

$$\mathrm{D}\, \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\mathrm{D}\, \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\mathrm{D}\, \arctan x = \frac{1}{1+x^2}$$

$$\mathrm{D}\, a^x = (\log a)\, a^x \qquad \text{in particular,} \quad \mathrm{D}\, \mathrm{e}^x = \mathrm{e}^x$$

$$\mathrm{D}\, \log_a |x| = \frac{1}{(\log a)\, x} \qquad \text{in particular,} \quad \mathrm{D}\, \log |x| = \frac{1}{x}$$

Claudio Canuto, Anita Tabacco " Mathematical analysis I" pp 175