O'ZBEKISTON RESPUBLIKASI
OLIY VA O'RTA MAXSUS TA'LIM VAZIRLIGI

MIRZO ULUG'BEK NOMIDAGI
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OLIY MATEMATIKA

Toshkent - 2004

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(Biologiya–tuproqshunoslik va kimyo fakulteti bakalavr talabalari uchun)

Toshkent – 2004

Mazkur o'quv qo'llanma Biologiya—tuproqshunoslik va kimyo fakulteti talabalari uchun mo'ljallangan bo'lib, unda oliy algebra, analitik geometriya, matematik—analiz va differentsial tenglamalar kurslarining muhim tushunchalari bayon qilingan.

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MUNDARIJA

SO'Z BOSHI

Hozirgi vaqtda matematik metodlar fan, texnika va xalq xo'jaligining ko'p sohalarida turli—tuman masalalarni hal qilishda keng qo'llanilmoqda. Ayniqsa, xalq xo'jaligining barcha sohalarida kompyutrlarning yalpi qo'llanilishi munosabati bilan matematik usullarning ahamiyati yanada ortdi.

No matematik mutaxassisliklarda matematikani o'qitishdan maqsad — talabaning mantiqiy fikrlashini o'stirish va tatbiqiy masalalarni matematik usullar bilan tekshirish malakalarini hosil qilishdan iboratdir.

Zamon talabi o'qitilayotgan fanning mos mutaxassisliklar bo'yicha amaliy tatbiqlarini hisobga olishni talab etmoqda.

Ushbu o'quv ·qo'llanma biologiya—tuproqshunoslik va ximiya mutaxassisliklari talabalari uchun muljallangan o'quv reja va o'tiladigan o'quv dasturidan kelib chiqqan holda yozilgan. O`quv qo'llanmada oliy algebra, analitik geometriya, matematik—analiz va differentsial tenglamalar bo'limining elementlari, hamda mustaqil yechish uchun misol va masalalar keltirilgan.

Bundan tashqari ba'zi biologiya va ximiyada uchraydigan ayrim masalalarning matematik modellari keltirilgan.

I–BOB.
DASTLABKI TUSHUNCHALAR.

1–§. To'plamlar va ular ustida amallar.

1⁰. Matematikada yozuvni qisqartirib ifodalash, shuningdek iboralarni ixcham aytish maqsadida maxsus belgilar (mantiqiy belgilar) dan foydalaniladi:

1) $\Rightarrow$ belgi «agar ... bo'lsa, u holda ... bo'ladi» iborasi o'rnida;

2) $\forall$ belgi, «har qanday», «ixtiyoriy», «barchasi uchun» so'zlari va so'z birikmalari o'rnida;

3) $\exists$ belgi, «mavjudki», «topiladiki» so'zlari o'rnida;

4) $\Leftrightarrow$ belgi, ikki ekvivalent tasdiqni ifodalashda;

5) := belgi, «ta'rifga ko'ra teng» so'z birikmalari o'rnida;

6) $\in$ belgi, «tegishli», $\notin$ yoki $\overline{\in}$ belgilari esa «tegishli emas» so'zlari o'rnida ishlatiladi.

To'plam matematikaning boshlang'ich tushunchalaridan bo'lib, uni ob'ektlarning ma'lum belgilar bo'yicha birlashmasi (majmuasi) sifatida tushuniladi.

Bir qancha ob'ektlarni birgalikda yangi ob'ekt deb qarash mumkin. Bu yangi ob'ektga **to'plam** deyladi.

Bitta ham elementga ega bo'lmagan to'plam bo'sh to'plam deyiladi va $\varnothing$ kabi belgilanadi.

Odatda, to'plamlar bosh harflar, uni tashkil etuvchilari (elementlari) kichik harflar bilan belgilanadi.

To'plamning P xususiyatli x elementlaridan tashkil topgan to'plam quyidagicha $\{x \mid P\}$ belgilanadi.

2⁰. Ikki A va B to'plamlar berilgan bo'lsin. A to'plam B ning qismi ($A \subset B$), A va B to'plamlar tengligi ($A = B$), A va B to'plamlar yig'indisi ($A \cup B$), A va B to'plamlar ko'paytmasi ($A \cap B$), A to'plamdan B to'plamning ayirmasi ($A \setminus B$) quyidagicha ta'riflanadi:

$$A \subset B: \quad a \in A \Rightarrow a \in B;$$
$$A = B: \quad A \subset B, \; B \subset A;$$
$$A \cup B := \{x \mid x \in A \text{ ёки } x \in B\};$$
$$A \cap B := \{x \mid x \in A, \quad x \in B\};$$
$$A \setminus B := \{x \mid x \in A, \quad x \overline{\in} B\}.$$

1−misol. $A = \{x \in N \mid -3 < x \leq 5\}$ va $B = \{x \in N \mid x^2 + 2x - 3 = 0\}$ to'plamlar uchun $A \cup B,\ A \setminus B,\ B \setminus A,\ A \cap B$ lar topilsin.

◀ $A = \{x \in N \mid -3 < x \leq 5\} = \{1, 2, 3, 4, 5\}$ va $B = \{x \in N \mid x^2 + 2x - 3 = 0\} = \{1\}$ ekanligidan quyidagilarni hosil qilamiz:

$$A \cup B = \{1, 2, 3, 4, 5\},$$
$$A \setminus B = \{2, 3, 4, 5\},$$
$$B \setminus A = \varnothing,$$
$$A \cap B = \{1\}.$$

2 − misol. Agar ikki A va B to'plamlari uchun

$$A \setminus B = B \setminus A$$

bo'lsa, A va B lar haqida nima deyish mumkin?

◀ Ravshanki,

$$\left.\begin{array}{l} x \in A \setminus B \Rightarrow x \in A,\ x \,\overline{\in}\, B, \\ x \in B \setminus A \Rightarrow x \in B,\ x \,\overline{\in}\, A. \end{array}\right\} \Rightarrow x \,\overline{\in}\, A \cap B.$$

Demak, bu to'plamlar kesishmaydi ▶

3 − misol. $A = \{x \mid x^2 - 8x + 15 = 0\}$ to'plamning barcha qism to'plamlarini yozing.

◀ $A = \{3;5\}$ ekanligini ko'rish qiyin emas. Bu to'plamning barcha qism to'plamlari quyidagi to'plamlardan iborat bo'ladi:

$$\varnothing,\ \{3\},\ \{5\},\ \{3;5\}$$ ▶

4 − misol. Ushbu

1) $\{1;2\} \in \{\{1\}, \{2\}, \{1;3\}, \{1;2;3\}\}$

2) $\{1;2\} \subset \{\{1\}, \{2\}, \{1;3\}, \{1;2;3\}\}$

munosabatlar to'g'rimi?

◀ Agar $A = \{1;2\}$ va $B = \{\{1\}, \{2\}, \{1;3\}, \{1;2;3\}\}$ deb belgilasak, B to'plam elementlari orasida A to'plamga teng bo'lgan element bo'lmaganligi sababli $A \,\overline{\in}\, B \Rightarrow 1)$ − munosabat to'g'ri emas. $A \subset B$ esa bajariladi, chunki $1 \in B$ va $2 \in B$ Demak, 2) − munosabat to'g'ri. ▶

5 – misol. A, B, S to'plamlari uchun ushbu

$$(A\setminus B)\cap C=(A\cap C)\setminus(B\cap C)$$

tenglikning o'rinli bo'lishi ko'rsatilsin.

◄ Aytaylik, $x\in(A\setminus B)\cap C$ bo'lsin. Yuqorida keltirilgan to'plamlar ustida amallar ta'riflaridan foydalanib topamiz:

$$x\in(A\setminus B)\cap C\Rightarrow x\in A\setminus B,\ x\in C\Rightarrow$$

$$\Rightarrow x\in A,\ x\notin B,\ x\in C\Rightarrow x\in A\cap C,\ x\notin B\cap C\Rightarrow$$
$$\Rightarrow x\in(A\cap C)\setminus(B\cap C).$$

Demak,

$$(A\setminus B)\cap C\subset(A\cap C)\setminus(B\cap C). \qquad (1)$$

Endi $x\in(A\cap C)\setminus(B\cap C)$ bo'lsin. Unda

$$x\in A\cap C,\ x\notin B\cap C\Rightarrow x\in A,\ x\in C,$$
$$x\notin B\Rightarrow x\in A|B,\ x\in C\Rightarrow x\in(A\setminus B)\cap C.$$

Demak,

$$(A\cap C)\setminus(B\cap C)\subset(A\setminus B)\cap C. \qquad (2)$$

(1) va (2) munosabatlardan

$$(A\setminus B)\cap C=\subset(A\cap C)\setminus(B\cap C)$$

bo'lishi kelib chiqadi. ►

6 – misol. A, B, S to'plamlari uchun ushbu

$$(A\setminus B)\setminus C=(A\setminus C)\setminus(B\setminus C)$$

tenglikning bajarilishi ko'rsatilsin.

◄ Faraz qilaylik, $x\in(A\setminus B)\setminus C$ bo'lsin. Unda

$$x\in A\setminus B,\ x\notin C\Rightarrow x\in A,\ x\notin B,$$
$$x\notin C\Rightarrow x\in A\setminus C,\ x\notin B\setminus C\Rightarrow x\in(A\setminus C)\setminus(B\setminus C),$$

Demak,

$$(A\setminus B)\setminus C\subset(A\setminus C)\setminus(B\setminus C). \qquad (3)$$

Endi $x \in (A \setminus C) \setminus (B \setminus C)$ bo'lsin. Unda

$$x \in A \setminus C,\ x \overline{\in} B \setminus C \Rightarrow x \in A,\ x \overline{\in} C,$$
$$x \in A \setminus B. \Rightarrow x \in A \setminus Bx \overline{\in} C. \Rightarrow x \in (A \setminus B) \setminus C,$$

Demak,
$$(A \setminus C)(B \setminus C) \subset (A \setminus B) \setminus C. \ (4)$$

(3) va (4) munosabatlardan

$$(A \setminus B) \setminus C = (A \setminus C) \setminus (B \setminus C)$$

bo'lishi kelib chiqadi. ▸

Mustaqil yechish uchun misol va masalalar.

Berilgan A va B to'plamlar uchun

$$A \cup B,\ A \setminus B,\ B \setminus A,\ A \cap B$$

to'plamlar topilsin.

1. $A = \{x \in N \mid -2 < x \leq 3\},\ B = \{x \mid x^2 - x - 2 = 0\}$

J: $A \cup B = \{-1; 1; 3\},\ A \setminus B = \{1; 3\},\ B \setminus A = \{-1\},\ A \cap B = \{2\}$

2. $A = \{x \in N \mid \sin x = 0,\ x \in [-\pi;\ \pi]\},\ B = \left\{x \in N \mid \cos\frac{\pi x}{2} = 0,\ x \in [0;\ 2\pi]\right\}.$

J: $A \cup B = \{-1; 2; 3; 5\},\ A \setminus B = \{2\},\ B \setminus A = \{5\},\ A \cap B = \{1; 3\}.$

3. $A = \{1; 2; 4\},\ B = \{2;\ 3\}.$

j: $A \cup B = \{1;2;3;4\}, \quad A \setminus B = \{1;4\}, \quad B \setminus A\{3\}, \quad A \cap B = \{2\}.$

4. $A = \{x \mid x^2 - 7x + 6 = 0\}, \quad B = \{1; 6\}.$

J: $A \cup B = \{1;6\}, \quad A \setminus B = \phi, \quad B \setminus A = \phi, \quad A \cap B = \{1;6\}.$

5. Agar $A = \{x \in N \mid 2 < x \leq 6\},\ B = \{x \in N \mid 1 < x < 4\}$ va $C = \{x \in N \mid x^2 - 4 = 0\}$ bo'lsa,
1) $B \cup C$, 2) $A \cap B \cap C$, 3) $A \cup B \cup C$, 4) $(A \cap B) \cup (B \cup C)$ to'plamlar topilsin.

J: 1) $\{2;3\}$, 2) ϕ, 3) $\{2; 3; 4; 5; 6\}$, 4) $\{2; 3\}$.

6. $A=\{x\in N \mid 1<x\le 4\}$, $B=\{x \mid x^2-9=0\}$ bo'lsa, $A\cup B$ to'plamning barcha qism to'plamlari topilsin.

J: ϕ, $\{-3\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{-3; 2\}$, $\{-3; 3\}$, $\{-3; 4\}$, $\{2; 3\}$, $\{2; 4\}$, $\{3; 4\}$, $\{-3; 2; 3\}$, $\{-3; 2; 4\}$, $\{-3; 3, 4\}$, $\{2; 3; 4\}$, $\{-3; 2; 3\}$.

7. Agar $A=\{x\in N \mid 0\le x<3\}$, $B=\{x\in N \mid -1<z<4\}$ va $C=\{x\in N \mid x^2-9=0\}$ bo'lsa, 1) $B\cup C$, 2) $A\cap B\cap C$, 3) $A\cup B\cup C$, 4) $(A\cap B)\cup(B\cup C)$ to'plamalr topilsin.

J: 1)$\{2;3\}$,2)ϕ,3)$\{1;2;3\}$,4)$\{2;3\}$.

8. $A=\{x\in N \mid x^2-16=0\}$ $B=\{x\in N \mid -4\le x\le 2\}$ bo'lsa, $A\cup B$ to'plamning barcha qism to'plamlari topilsin.

J: $\phi,\{1\},\{2\},\{4\},\{1;2\},\{1;4\},\{2;4\},\{1;2;4\}$.

9. A to'plam 2 ga bo'linuvchi barcha natural sonlar to'plami, B esa 3 ga bo'linuvchi barcha natural sonlar to'plami bo'lsa, $A\cap B$ to'plam qanday bo'ladi?

J: 6 ga bo'linuvchi barcha natural sonlar to'plami.

10. Agar $B\subset A$ bo'lib, ushbu

$$A\cap X=B$$

tenglik bajarilsa, unda X to'plam qanday bo'ladi?

J: X = B

11. Agar $X=A\setminus(B\cup C)$, $Y=(A\setminus B)\cup(A\setminus C)$ bo'lsa, unda X va Y to'plamlar qanday munosabatda bo'ladi?

J: $X\subset Y$

A, B va S to'plamlar uchun quyidagi munosabatlarning bajarilishi ko'rsatilsin.

12. $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$.

13. $A\setminus(B\cup C)=(A\setminus B)\setminus C$.

14. Agar

$$X=A\cup(B\setminus C),\ Y=(A\cup B)\setminus(A\cup C)$$

bo'lsa, X va Y to'plamlar qanday munosabatda bo'ladi?

J: $X\supset Y$

15. A to'plam 3 ga bo'linuvchi barcha natural sonlar to'plami, B esa 5 ga bo'linuvchi barcha natural sonlar to'plami bo'lsa, $A\cap B$ to'plam qanday bo'ladi?

J: 15 ga bo'linuvchi barcha natural sonlar to'plami

16. Ushbu

$$(A\setminus B)\cup B=A$$

tenglik faqat $B\subset A$ bo'lgandagina bajarilishi ko'rsatilsin.
17. Ixtiyoriy A, B va S to'plamlari uchun ushbu

$$(A\cup B)\setminus B\subset(A\setminus B)\cup C$$

munosabatning bajarilishi ko'rsatilsin.

Ixtiyoriy A, B va S to'plamlar uchun quyidagi munosabatlarning bajarilishi ko'rsatilsin.
18. $A\cap(A\cup B)=A.$
19. $A\setminus(A\setminus B)=A\cap B.$
20. $(A\setminus B)\cup(B\setminus A)\cup(A\cap B)=A\cup B.$
21. $(A\setminus B)\cup(B\setminus A)=(A\cup B)\setminus(A\cap B).$
22. $A\cap(B\cup C)=(A\cap B)\cup(A\cup C).$
23. Agar $X=(A\cap B)\setminus C,\quad Y=(A\setminus C)\cap(B\setminus C)$ bo'lsa, X va Y to'plamlar qanday munosabatda bo'ladi?

J: $X=Y$.
24. Agar A to'plam 4 ta elementdan iborat bo'lsa, uning qismiy to'plamlaridan tashkil topgan to'plamning elementlari soni 16 ta bo'lishi isbotlansin.
25. Agar A to'plamning elementlari soni m ta, B niki n ta bo'lib, p va q to'plamlarning elementlari soni mos ravishda r va q ta bo'lsa

$$p=m+n-q$$

bo'lishi isbotlansin.

2–§. Ҳақиқий sonlar.

1. Ҳақиқий sonlar. Sanash jarayonida ishlatiladigan 1, 2, 3, ... sonlar **natural sonlar** deyilib, natural sonlar to'plami N bilan belgilanadi:

$$N=\{1,2,3,\ldots,n,\ldots\}$$

Barcha natural sonlar, nol soni va barcha manfiy ishora bilan olingan natural sonlardan tashkil topgan to'plam **butun sonlar** to'plami Z ni hosil qiladi:

$$Z = \{\ldots,-3,-2,-1,0,1,2,\ldots\}.$$

$$n! = 1 \bullet 2 \bullet 3 \bullet \ldots \bullet (n-1) \bullet n.$$

Bu "n faktorial" deb o'qiladi.
$0! = 1$, deb qabul qilingan.
C_n^k binomial koeffitsentni quyidagi ko'rinishda aniqlaymiz:

$$C_n^k = \frac{n!}{k!(n-k)!}, \qquad n,k \in N, \qquad k \le n.$$

Barcha $n \in N$ lar uchun $C_n^0 = 1$ deb hisoblaymiz.

Quyidagi xossalarni keltiramiz,

1) $C_n^k = C_n^{n-k}$,

2) $C_{n+1}^{k+1} = C_n^k + C_n^{k+1}$.

Bu xossalardan binomial koeffitsent natural son ekanligi kelib chiqadi.

C_n^k dan foydalanib Nyuton binomi formulasini keltiramiz:

$$(x+a)^n = x^n + C_n^1 x^{n-1} a + C_n^2 x^{n-2} a^2 + \ldots + C_n^{n-1} x a^{n-1} + a^n = \sum_{k=0}^{n} C_n^k x^{n-k} a^k, n \in N.$$

Nyuton binomi formulasi xususiy hollarda quyidagi ko'rinishlarda bo'ladi:

$$(x+a)^2 = x^2 + 2xa + a^2,$$
$$(x+a)^3 = x^3 + 3x^2 a + 3xa^2 + a^3,$$
$$(x+a)^4 = x^4 + 4x^3 a + 6x^2 a^2 + 4xa^3 + a^4.$$

Ma'lumki, qisqarmaydigan $\frac{p}{n} (p \in Z, n \in N)$ kasr ko'rinishida ifodalanadigan son **ratsional son** deyiladi va barcha ratsional sonlar to'plami Q deb belgilanadi:

$$Q = \left\{ r = \frac{p}{n} \middle| p \in Z, n \in N, (p,n) = 1 \right\}.$$

Agar $\frac{p}{n}$ kasrning maxraji 10^{K*} ko'rinishda bo'lsa, uni **o'nli kasr** deyiladi. O'nli kasrlar uch xil bo'ladi:

1) chekli o'nli kasrlar (1,5; 0,17; −0,01);
2) cheksiz davriy o'nli kasrlar (0,888..=0,(8); 0,11777..=0,11(7));
3) cheksiz davriy bo'lmagan o'nli kasrlar (3,14..; 2,71..).

Har qanday ratsional son chekli o'nli kasr yoki cheksiz davriy o'nli kasr ko'rinishida ifodalanadi va aksincha.

Cheksiz davriy bo'lmagan o'nli kasr **irratsional son** deyiladi.

Ratsional hamda irratsional sonlar umumiy nom bilan haqiqiy son deb ataladi va barcha haqiqiy sonlar to'plami R deb belgilanadi.

2. Sonlar o'qi. Haqiqiy sonlar to'plami R bilan to'gri chiziq nuqtalari to'plami o'zaro bir qiymatli moslikda bo'ladi. Shu sababli haqiqiy sonlar to'plami sonlar **o'qi yoki** haqiqiy o'q, haqiqiy sonning o'zi esa **sonlar o'qining nuqtasi** ham deyiladi.

$x \in R$ son uchun

$$|x| = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$$

miqdor x ning **absolyut qiymati** deyiladi.

$x \in R$ sonning o'zidan katta bo'lmagan butun qismi $[x]$ kabi yoziladi, $\{x\} = x - [x]$ esa shu sonning kasr qismi bo'ladi.

Aytaylik,, $a \in R$ va $b \in R$ bo'lib, $a < b$ bo'lsin. Unda

$[a,b] = \{x \in R | a \le x \le b\}$ kesma (segment,)

$(a,b) = \{x \in R | a < x < b\}$ interval,

$[a,b) = \{x \in R | a \le x < b\}$ yarim interval,

$(a,b] = \{x \in R | a < x \le b\}$ yarim interval deyilib, a va b lar ularning chetki nuqtalari deyiladi. Shuningdek,

$[a,+\infty) = \{x \in R | x \ge a\}$,

$(-\infty,a) = \{x \in R | x < a\}$,

$(-\infty,+\infty) = R$

deb qaraymiz.

3. To'plamning chegaralari. Biror E $(E \subset R)$ to'plam berilgan bo'lsin.

Arap

$$\exists b_0 \in E, \forall x \in E \Rightarrow x \le b_0$$

bo'lsa, b_0 soni E to'plamning **eng katta elementi** deyiladi va

$$b_0 = \max E = \max\{x | x \in E\}$$

kabi belgilanadi.

Agar

$$\exists a_0 \in E, \forall x \in E \Rightarrow x \ge a_0$$

bo'lsa a_0 soni E to'plamning **eng kichik elementi** deyiladi va

$$a_0 = \min E = \min\{x | x \in E\}$$

kabi belgilanadi.

Agar

$$\exists M \in R \quad \forall x \in E \Rightarrow x \le M \qquad (\exists m \in R \quad \forall x \in E \Rightarrow m \le x)$$

bo'lsa, E to'plam yuqoridan (quyidan) chegaralangan, M(m) soni esa E to'plamning **yuqori (quyi) chegarasi** deyiladi.

Agar $a*$ (a_*) soni

1) E to'plamning yuqori (quyi) chegarasi bo'lib,

2) E to'plamning ixtiyoriy yuqori (quyи) chegarasi d uchun $a* \le d$ $(a_* \ge d)$ bo'lsa,

a^* soni E to'plamning **aniq yuqori (aniq quyi) chegarasi** deyiladi va

$$a^* = \sup E \quad (a_* = \inf E)$$

kabi belgilanadi.

1–teorema. *a^* soni E to'plamning aniq yuqori chegarasi bo'lishi uchun ushbu*

1) a^* soni E to'plamning yuqori chegarasi,

2) $\forall d < a^* \exists x \in E \Rightarrow x > d$ shartlarning bajarilishi zarur va etarli.

a_* soni E to'plamning aniq quyi chegarasi bo'lishi uchun ushbu

1') a_* soni E to'plamning quyi chegarasi,

2') $\forall d > a_*$ $\exists x \in E \Rightarrow x < d$ shartlarning bajarilishi zarur va etarli.

2–теорема. *Faraz qilaylik, $E \subset R$ bo'lib, $E \neq \varnothing$ bo'lsin. Agar E to'plam yuqoridan chegaralangan bo'lsa, u aniq yoqori chegaraga ega. Agar E quyidan chegaralangan bo'lsa, u aniq quyi chegaraga ega.*

Agar E yuqoridan chegaralanmagan bo'lsa $\sup E = +\infty$, quyidan chegaralanmagan bo'lsa $\inf E = -\infty$ deb qaraladi.

1–misol. Ushbu

$$\alpha = \sqrt{3}, \quad \beta = \log_7 6$$

sonlarning irratsional ekanligi ko'rsatilsin.

Faraz qilaylik, α ratsional son bo'lsin. Unda $\sqrt{3} = \frac{p}{q}$ bo'lib, p va q lar o'zaro tubdir: $(p,q) = 1$.

Ravshanki, $p^2 = 3q^2$ bo'ladi. Demak, p^2 son 3 ga bo'linadi. Haqiqatdan ham, agar r son 3 ga bo'linmaydigan bo'lsa, unda $p = 3k+1$ yoki $p = 3k+2$ (k∈Z) bo'ladi.

Agar $p = 3k+1$ bo'lsa, $p^2 = 9k^2 + 6k + 1$, yani $p^2 - 3(3k^2 + 2k) = 1$ bo'ladi. Ayni paytda p^2 son 3 ga bo'linganligi sababli, $p^2 - 3(3k^2 + 2k)$ ham 3 ga bo'linishi lozim. Biroq, bu ayirma 1 ga teng bo'lishi ziddiyatga olib keladi.

Xuddi shunga o'xshash $p = 3k+2$ bo'lganda ham ziddiyat kelib chiqadi. Demak, p soni 3 ga bo'linmaydi : $p = 3m$.

Shuni etiborga olib, $q^2 = 3m^2$ bo'lishini va yuqoridagi mulohazaga ko'ra q ning ham 3 ga bo'linishini topamiz: $q = 3n$.

Demak, p va q sonlarning har biri 3 ga bo'linadi. Bu esa p va q larning o'zaro tub ekanligiga ziddir. Ziddiyat, $\sqrt{3}=\frac{p}{q}$ deb olinishi oqibatida sodir bo'ldi. Demak, $\sqrt{3}$ ni $\frac{p}{q}$ ko'rinishida ifodalab bo'lmaydi. $\sqrt{3}-$ irratsional son .

$\beta=\log_7 6$ sonini ham ratsional, yani $\log_7 6=\frac{p}{q}$ deb faraz qilamiz. Unda $7^{\frac{p}{q}}=6,\Rightarrow 7^p=6^q$ bo'ladi. Bu tenglik hech qachon o'rinli bo'lmaydi, chunki 6^q $-$juft, 7^p $-$toq sondir.

Demak, $\log_7 6$ ni $\frac{p}{q}$ ko'rinishda ifodalab bo'lmaydi, u irratsional son.

2–misol. Agar $r-$ratsional son va $\alpha-$irratsional son bo'lsa, $\alpha+r$ irratsional son bo'lishi isbotlansin.

$\alpha+r=\beta$ deb belgilab, $\beta-$ratsional son bo'lsin deb faraz qilamiz. $\Rightarrow$ $\alpha=\beta-r$ $-$ratsional son. Ziddiyat hosil bo'ldi. Demak, $\alpha+r=\beta$ $-$irratsional son.

3–misol. Agar α va β lar irratsional sonlar bo'lsa, unda

1) $\alpha+\beta$,

2) α^β lar ratsional son bo'lishi mumkinmi?

$\alpha+\beta$ ratsional son bo'lishi mumkin. Masalan, $\alpha=\sqrt{3}$ va $\beta=2-\sqrt{3}$ lar irratsional, lekin $\alpha+\beta=2$ $-$ ratsional.

$\alpha+\beta$ α^β ham ratsional son bo'lishi mumkin.

Masalan, $\alpha=\sqrt{3}$ va $\beta=2\log_3 2$ larning irratsional ekanini ko'rish qiyin emas. Lekin $\alpha^\beta=\sqrt{3}^{2\log_3 2}=2$ $-$ratsional son.

4–misol. $\alpha=\sqrt{3}+\sqrt{5}$ sonining irratsional son ekanligi isbotlansin.

Teskarisini faraz qilamiz, yani $\sqrt{3}+\sqrt{5}=r$ ratsional son bo'lsin. Unda $\sqrt{3}+\sqrt{5}=r$ $\Rightarrow, 3+2\sqrt{3}\bullet\sqrt{5}+5=r^2$ $\sqrt{15}=r^2-\frac{8}{2}$. Bu tenglikning o'ng tomonida ratsional son turibdi, chapdagi $\sqrt{15}$ ning irratsional ekanini ko'rish qiyin emas. Ziddiyat $\alpha=\sqrt{3}+\sqrt{5}$ ning irratsional son ekanini isbotlaydi.

5–misol. Ushbu

$$\left|\frac{x}{x+1}\right|>\frac{x}{x+1}$$

tengsizlik yechilsin.

$|x|>x$ tengsizlik $x<0$ bo'lgandagina bajarilgani uchun berilgan tengsizlikdan

$$\frac{x}{x+1}<0$$

ekanligini topamiz. Intervallar usulidan foydalansak –

$$-1<x<0$$

ekanligini ko'rish qiyin emas.

6–misol. $E\subset R, F\subset R$ to'plamlar yuqoridan chegaralangan va

$$E+F=\{x+y|x\in E, y\in F\}$$

bo'lsin. Unda

$$\sup(E+F)=\sup E+\sup F$$

bo'lishi isbotlansin.

Aytaylik, $\sup E=a$ va $\sup F=b$ bo'lsin. Unda 1–teoremaga ko'ra

$$\sup E \Leftrightarrow \begin{cases} 1)\forall x\in E, \quad x\le a \\ 2)\forall d_1<a\exists x_0\in E\Rightarrow x_0>d_1, \end{cases}$$

$$\sup F \Leftrightarrow \begin{cases} 1)\forall y\in F, \quad y\le b \\ 2)\forall d_2<b\exists y_0\in E\Rightarrow y_0>d_2, \end{cases}$$

bo'ladi. Ravshanki,

$$\forall x\in F, x\le a; \forall y\in F, y\le b\Rightarrow x+y\le a+b$$
$$\forall d_1<a\exists x_0\in E\Rightarrow x_0>d_1, \forall d_2<b\exists y_0\in F\Rightarrow y_0>d_2$$

bo'lishidan

$\forall d<a+b$ olib $\exists$ $x_1+y_1\in E+F$: $x_1+y_1>d$ bo'lsin.

Yana 1– teoremaga ko'ra

$$\sup(E+F)=a+b$$

bo'ladi. Demak,

$$\sup(E+F)=\sup E+\sup F.$$

Mustaqil yechish uchun misol va masalalar.

1. Ushbu $\alpha=\sqrt{5}$ va $\beta=\lg 2$ sonlarining irratsional ekanligi isbotlansin.

2. Ushbu $\alpha=\sqrt{2}$ va $\beta=\log_2 3$ sonlarining irratsional ekanligi isbotlansin.

3. Ushbu 1) $2^x=11$, 2) $x^2-7x+1=0$ tenglamani qanoatlantiruvchi ratsional son mavjud emasligi ko'rsatilsin.

4. Agar a, b va $\sqrt{a}+\sqrt{b}$ ratsional sonlar bulsa, $\sqrt{a}$ va $\sqrt{b}$ sonlar to'g'risida nima deyish mumkin?

5. Agar α va β – irratsional sonlar, r – ratsional son bo'lsa, ushbu 1) $\alpha+r$, 2) $\sqrt{r}$, 3) α•r, 4) $\sqrt{\alpha+\sqrt{\beta}}$, 5) $\sqrt{\alpha+\sqrt{r}}$ sonlarning qaysilari ratsional bo'lishi mumkin?

J: $\sqrt{r}$, $\sqrt{\alpha+\sqrt{\beta}}$.

6. Ushbu 1) $x^3=7$, $x^2+3x+1=0$ tenglamani qanoatlantiruvchi ratsional son mavjud emasligi isbotlansin.

7. $\sqrt{2}$ va $\sqrt{3}$ sonlar orasida joylashgan ratsional son topilsin.

J: bunday sonlar cheksiz ko'p, masalan $\frac{3}{2}$.

8. Agar α va β – irratsional sonlar, r – ratsional bo'lsa, ushbu 1) $\alpha+\beta$, 2) $\sqrt{\alpha}$, 3) α• β, 4) $\sqrt{\alpha}+r$ va 5) $\sqrt{\alpha}+\sqrt{r}$ sonlarning qaysilari ratsional bo'lishi mumkin?

J: $\alpha+\beta$, α• β, $\sqrt{\alpha+\sqrt{r}}$.

9. $\alpha=\sqrt{2}+\sqrt{3}$ sonining irratsional ekanligi isbotlansin.

10. Agar α va β – irratsional sonlar bo'lib, $\alpha+\beta$ – ratsional bo'lsa, unda $\alpha-\beta$ va $\alpha+2\beta$ larning irratsional son bo'lishi ko'rsatilsin.

Quyidagi sonlar irratsional ekanligi isbotlansin.

11. $\alpha = \frac{1}{\sqrt{2}+\sqrt{5}}$.

12. $\sqrt{5}-\sqrt{2}$.

Tengsizliklar yechilsin.

13. $|x+3|>2$ J: $x \in (-\infty;-5) \cup (-1;+\infty)$.

14. $|x-4|+|x+4| \le 10$. J: $-5 \le x \le 5$.

15. $|x| < x+1$. J: $x < -\frac{1}{2}$.

16. $|x-2| < 3$. J: $-1 < x < 5$.

17. $|x+2|+|x-2| \ge 12$ J: $x \in (-\infty;-6] \cup [6;+\infty)$

18. $|x^2-2x| > x^2-|2x|$ J: $x < 2 \quad (x \ne 0)$

19. Agar $a > \frac{1}{2}$, bo'lsa, $\frac{1}{a}$ sonning butun qismi topilsin.

J: 1, agar $\frac{1}{2} < a \le 1$ bo'lsa, 0 agar $a > 1$, bo'lsa.

20. Ushbu $E = \left\{\frac{n}{n+1} \middle| n \ge 1\right\}$ to'plamning chegaralanganligi ko'rsatilsin.

21. Ushbu $A = \{r \in Q \mid r^2 < 2\}$ to'plamning aniq yuqori va quyi chegaralari topilsin.

J: J: $SupA = \sqrt{2}$, $\inf A = -\sqrt{2}$

22. Ushbu $\sup\left\{\frac{1}{\sqrt{n}} \middle| n \in N\right\} = 1$ tenglik isbotlansin.

23. Aytaylik, $E \subset R$ va $-E = \{x \mid -x \in E\}$ bo'lsin. Unda $\sup(-E) = -\inf E$, $\inf(-E) = -\sup E$ bo'lishi isbotlansin.

24. Ushbu $E = \left\{\frac{n^2}{n^2+4} \middle| n \in N\right\}$ to'plam uchun $\sup E = 1, \inf E = \frac{1}{5}$ bo'lishi ko'rsatilsin.

3–§. Matematik induktsiya usuli.

Ma'lumki, fikr yuritish asosan ikki – deduktiv va induktiv formalarda olib boriladi. *Deduktsiya* – fikrlashning umumiy tasdiqlaridan xususiy tasdiqlarga o'tish, *induktsiya* esa fikrlashning xususiy tasdiqlaridan umumiy tasdiqlarga o'tish formasidir.

Aytaylik, $A(n)$ biror tasdiqni (fikr, mulohazani) ifodalasin $(n \in N)$. Matematik induktsiya usulida:

1) $n = n_0$ bo'lganda $A(n)$ tasdiqning to'g'riligi (mulohazaning rostligi) tekshiriladi;
2) $n = k$ bo'lganda $(k > n_0)$ $A(n)$ tasdiqni to'g'ri deb faraz qilib, $n = k+1$ uchun $A(n)$ ning to'g'riligi isbotlanadi $\left(A(k) \Rightarrow A(k+1)\right)$;
3) yuqoridagi 1) va 2) bandlar ijobiy hal bo'lgan taqdirda $A(n)$ tasdiq barcha $n \geq n_0$ lar uchun to'g'ri deb xulosa chiqariladi.

1–misol. Dastlabki n ta toq natural sonlar yig'indisi uchun

$$S_n = 1+3+5+...+(2n-1) = n^2 \qquad (1)$$

bo'lishi isbotlansin.

▸ Bu ayniyatni matematik induktsiya usuli yordamida isbotlaymiz.

1) $n = 1$ bo'lganda $S_1 = 1 = 1^2$ bo'lib, A(1) to'g'ri.
2) $n = k$ da A(k) to'g'ri bo'lsin:

$$S_k = 1+3+5+...+(2k-1) = k^2.$$

Unda

$$S_{k+1} = 1+3+5+...+(2k-1)+(2k+1) = S_k + 2k+1 =$$
$$k^2 + 2k + 1 = (k+1)^2$$

bo'ladi. Demak,

$$A(k) \Rightarrow A(k+1).$$

Matematik induktsiya usuliga binoan (1) tenglik ixtiyoriy natural n soni uchun to'g'ri bo'ladi ▸

2–misol. $\forall n \in N$ uchun $n(n^2+5)$ ning 6 ga bo'linishi ko'rsatilsin.

◂ 1) $n=1$ bo'lganda $1\cdot(1^2+5)=6$ bo'lib, A(1) tasdiq to'g'ri.

2) $n=k$ da A(k) to'g'ri bo'lsin, ya'ni $k\cdot(k^2+5)6$ ga bo'linsin.

Unda

$$(k+1)\cdot\left[(k+1)^2+5\right]=(k+1)\cdot(k^2+5+2k+1)=$$

$$=k\cdot(k^2+5)+2k^2+k+k^2+5+2k+1=k(k^2+5)+3k(k+1)+6$$

bo'lib, $k(k+1)$ har doim juft son bo'lganligidan $3k(k+1)$ soni 6 ga bo'linadi. Undan

$$A(k)\Rightarrow A(k+1)$$

ni topamiz. Demak, $\forall n \in N$ da $n(n^2+5)$ son 6 ga bo'linadi ▸

3–misol. $\forall n \in N$ va $\forall x > -1$ uchun ushbu

$$(1+x)^n \geq 1+nx \text{ (Bernulli tengsizligi) } (2)$$

tengsizlik isbotlansin.

◂ 1) $n=1$ bo'lganda $(1+x)^1=1+1\cdot x$ bo'lib, A(1) to'g'ri.

2) $n=k$ da A(k) to'g'ri bo'lsin:

$$(1+x)^k \geq 1+kx \qquad (3)$$

Ravshanki, $x>-1 \Rightarrow 1+x>0$. (3) tengsizlikning har ikki tomonini $1+x$ ga ko'paytirib topamiz:

$$(1+x)^{k+1} \geq (1+kx)\cdot(1+x)=1+kx+x+kx^2 \geq 1+(k+1)x$$

Demak,

$$A(k)\Rightarrow A(k+1)$$

Binobarin, $\forall n \in N$ va $\forall x > -1$ uchun

$$(1+x)^n \geq 1+nx$$

bo'ladi ▸

4–misol. $\forall n \in N$ uchun Ushbu

$$\cos\alpha\cdot\cos 2\alpha\cdot\cos 4\alpha\cdot\ldots\cdot\cos 2^n\alpha=\frac{\sin 2^{n+1}\alpha}{2^{n+1}\cdot\sin\alpha},\alpha\neq 0 \quad (4)$$

tenglikning o'rinli bo'lishi ko'rsatilsin.

◂ 1) $n=1$ da (4) tenglik o'rinli:

$$\cos\alpha\cdot\cos 2\alpha=\frac{1}{4\sin\alpha}\cdot 4\sin\alpha\cdot\cos\alpha\cdot\cos 2\alpha=$$

$$=\frac{1}{4\sin\alpha}\cdot 2\sin 2\alpha\cdot\cos 2\alpha=\frac{\sin 4\alpha}{4\sin\alpha}=\frac{\sin 2^2\alpha}{2^2\cdot\sin\alpha}$$

2) $n=k$ da A(k) to'g'ri, ya'ni

$$\cos\alpha\cdot\cos 2\alpha\cdot\cos 4\alpha\cdot\ldots\cdot\cos 2^k\alpha=\frac{\sin 2^{k+1}\alpha}{2^{k+1}\cdot\sin\alpha}$$

deb, $n=k+1$ da $A(k+1)$ ning to'g'ri bo'lishini ko'rsatamiz:

$$\cos\alpha\cdot\cos 2\alpha\cdot\cos 4\alpha\cdot\ldots\cdot\cos 2^k\alpha\cdot\cos 2^{k+1}\alpha=\frac{\sin 2^{k+1}\alpha}{2^{k+1}\cdot\sin\alpha}\cdot\cos 2^{k+1}\alpha=$$

$$=\frac{2\cdot\sin 2^{k+1}\alpha\cdot\cos 2^{k+1}\alpha}{2\cdot 2^{k+1}\cdot\sin\alpha}=\frac{\sin(2\cdot 2^{k+1}\alpha)}{2^{k+2}\cdot\sin\alpha}=\frac{\sin 2^{k+2}\alpha}{2^{k+2}\cdot\sin\alpha}$$

Demak, (4) tenglik $\forall n\in N$ uchun o'rinli ▸

5–misol. Ixtiyoriy natural $n>1$ uchun

$$(2n)!<2^{2n}\cdot(n!)^2 \quad (5)$$

tengsizlik isbotlansin.

◂ 1) $n=2$ da $(2\cdot 2)!=4!=24$ va $2^{2\cdot 2}\cdot(2!)^2=16\cdot 4=64$ bo'lib, A(1) to'g'ri.

2) $n=k$ da A(k) to'g'ri bo'lsin, ya'ni $(2k)!<2^{2k}\cdot(k!)^2$ tengsizlik bajarilsin. Unda

$$(2k+2)!=(2k)!\cdot(2k+1)\cdot(2k+2)<2^{2k}\cdot(k!)^2\cdot(2k+1)\cdot(2k+2)=$$

$$=2^{2k+2}\cdot[(k+1)!]^2\cdot\frac{2^{2k}\cdot(k!)^2\cdot(2k+1)\cdot(2k+2)}{2^{2k+2}\cdot[(k+1)!]^2}=$$

$$=2^{2k+2}\cdot[(k+1)!]^2\cdot\frac{(2k+1)(2k+2)}{2^2\cdot(k+1)^2}=2^{2k+2}\cdot[(k+1)!]^2\cdot\frac{2k^2+3k+1}{2k^2+4k+2}<$$

$$<2^{2k+1}\cdot[(k+1)!]^2$$

bo'ladi. Demak, (5) tengsizlik ixtiyoriy natural $n>1$ uchun o'rinli. ▸

Mustaqil yechish uchun misol va masalalar.

Matematik induktsiya usulini qo'llab, $\forall n\in N$ uchun quyidagi munosabatlarning to'g'riligi ko'rsatilsin.

1. $1+2+3+...+n=\frac{n(n+1)}{2}$.

2. $1^3+2^3+3^3+...+n^3=(1+2+3+...+n)^2$.

Ko'rsatma: Avval 1-misoldan foydalanib, isbotlanishi kerak bo'lgan tenglikni ushbu

$$1^3+2^3+3^3+...+n^3=\left[\frac{n(n+1)}{2}\right]^2$$

tenglik ko'rinishida yozib oling.

3. $1^2+3^2+5^2+...+(2n-1)^2=\frac{n(4n^2-1)}{3}$.

4. $\left(1-\frac{1}{4}\right)\cdot\left(1-\frac{1}{9}\right)\cdot...\cdot\left[1-\frac{1}{(n+1)^2}\right]=\frac{n+2}{2n+2}$.

5. $1^3+3^3+5^3+...+(2n-1)^3=n^2\cdot(2n^2-1)$.

6. $arctg\frac{1}{2}+arctg\frac{1}{8}+...+arctg\frac{1}{2n^2}=arctg\frac{n}{n+1}$.

Ko'rsatma:

$$arctg\,\alpha + arctg\,\beta = arctg\,\frac{\alpha+\beta}{1-\alpha\beta}$$

tenglikdan foydalaning.

7. $1^2+2^2+3^2+\ldots+n^2=\dfrac{n(n+1)(2n+1)}{6}$.

8. $1+3+6+10+\ldots+\dfrac{(n-1)n}{2}+\dfrac{n(n+1)}{2}=\dfrac{1}{6}n(n+1)(n+2)$.

9. $1+2+2^2+\ldots+2^{n-1}=2^n-1$.

10. $1\cdot2+2\cdot3+3\cdot4+\ldots+(n-1)n=\dfrac{(n-1)\cdot n\cdot(n+1)}{3}$.

11. $\dfrac{0}{1!}+\dfrac{1}{2!}+\dfrac{2}{3!}+\ldots+\dfrac{n-1}{n!}=1-\dfrac{1}{n!}$.

12. $1\cdot1!+2\cdot2!+3\cdot3!+\ldots+n\cdot n!=(n+1)!-1$.

13. $2^2+6^2+\ldots+(4n-2)^2=\dfrac{4n\cdot(2n-1)\cdot(2n+2)}{3}$.

14. $\cos\dfrac{\alpha}{2}\cdot\cos\dfrac{\alpha}{4}\cdot\cos\dfrac{\alpha}{8}\cdot\ldots\cdot\cos\dfrac{\alpha}{2^n}=\dfrac{\sin\alpha}{2^n\cdot\sin\dfrac{\alpha}{2^n}}$.

15−20 misollarda ko'rsatilgan ifodalarni $\forall n\in N$ uchun qav ichida ko'rsatilgan songa bo'linishi isbotlansin.

15. n^5-n (5 ga).

16. $6^{2n-2}+3^{n+1}+3^{n-1}$ (11 ga). n^3+11n (6 ga).

17. $11^{n+1}+12^{2n-1}$ (133 ga).

18. $7\cdot4^{2n}+56$ (3 ga).

19. $2^{n+5}\cdot3^{4n}+5^{3n+1}$ (37 ga).

20. Ixtiyoriy $a\in R, b\in R$ va $\forall n\in N$ lar uchun ushbu

$$(a+b)^n=\sum_{k=0}^{n}C_n^k a^{n-k}b^k=a^n+n\cdot a^{n-1}b+ \\ +\frac{n\cdot(n-1)}{1\cdot2}a^{n-2}b^2+\ldots+na\cdot b^{n-1}+b^n \qquad (6)$$

tenglik (Nyuton binomi) isbotlansin. Bu yerda

$$C_n^k:=\frac{n!}{k!\cdot(n-k)!},\quad 0!=1.$$

J. Ko'rsatma: Matematik induktsiya usulidan foydalanamiz.

1) A(1) to'g'ri. 2) A(k) to'g'ri, ya'ni $(a+b)^k = \sum_{m=0}^{k} C_k^m a^{k-m} b^k$ tenglik bajariladi deb faraz qilamiz. Unda

$$(a+b)^{k+1} = (a+b)\cdot(a+b)^{k+1} = (a+b)\cdot\sum_{m=0}^{k} C_k^m a^{k-m} b^k = \sum_{m=0}^{k} C_k^m a^{k+1-m} b^m + \sum_{m=0}^{k} C_k^m a^{k-m} b^{m+1} =$$

$$= \sum_{m=0}^{k} C_k^m a^{k+1-m} b^m + \sum_{m=0}^{k+1} C_k^{m-1} a^{k+1-m} b^m = a^{k+1} + \sum_{m=1}^{k} (C_k^m + C_k^{m-1}) a^{k+1-m} b^m + b^{k+1}.$$

Agar $C_k^m + C_k^{m-1} = C_{k+1}^m$ va $C_{k+1}^0 = C_{k+1}^{k+1} = 1$ ekanligini e'tiborga olsak, oxirgi tenglikdan

$$(a+b)^{k+1} = a^{k+1} + \sum_{m=1}^{k} C_{k+1}^m a^{k+1-m} b^m + b^{k+1} = \sum_{m=1}^{k+1} C_{k+1}^m a^{k+1-m} b^m$$

tenglikni hosil qilamiz. Demak, $A(k) \Rightarrow A(k+1) \Rightarrow \forall n \in N$ uchun Nyuton binomi o'rinli.

21. $\forall n \in N$ uchun ushbu

$$2 \le \left(1 + \frac{1}{n}\right)^n < 3$$

tengsizlik isbotlansin.

Ko'rsatma: Nyuton binomidan foydalaning.

22. Ixtiyoriy natural $n > 1$ uchun ushbu

$$\frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} < 1$$

tengsizlik isbotlansin.

23. Agar $a > b > 0$ bo'lsa, $\forall n \in N$ uchun ushbu

$$a^n > b^n$$

tengsizlik isbotlansin.

24. Ixtiyoriy natural $n \in N$ uchun

$$\frac{1}{2}\cdot\frac{3}{4}\cdot...\cdot\frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$$

tengsizlik isbotlansin.

25. Agar $x_1, x_2, ..., x_n$ lar ixtiyoriy musbat sonlar bo'lib,

$$x_1 \cdot x_2 \cdot ... \cdot x_n = 1 \quad (7)$$

munosabat bajarilsa, u holda

$$x_1 + x_2 + ... + x_n \geq n \qquad (8)$$

tengsizlikning bajarilishi isbotlansin.

J. Ko'rsatma: 1) n=1 bo'lganda, (7) shartga ko'ra $x_1=1$ bo'lib, A(1) o'rinli.

2) $n=k$, bo'lganda A(k) o'rinli deb faraz qilamiz va n=k+1 da A(k+1) ning o'rinli bo'lishini ko'rsatamiz.

Aytaylik, $x_1, x_2, ..., x_k, x_{k+1}$ lar $x_1 \cdot x_2 \cdots x_k \cdot x_{k+1} = 1$ munosabatni qanoatlantiruvchi ixtiyoriy musbat sonlar bo'lsin.

(7) munosabat bajarilayotganda quyidagi ikki hol bo'lishi mumkin: yoki (7) munosabatdagi barcha sonlar 1 ga teng bo'lib, A (k+1) bajariladi; yoki bu sonlar ichida kamida bitta birdan farqli son bor. Hech bo'lmaganda yana bitta birdan farqli son bo'lib, agar ularning biri 1 dan kichik bo'lsa, ikkinchisi albatta 1 dan katta bo'ladi. Umumiylikka ziyon keltirmagan holda $x_k>1$ va $x_{k+1}<1$ deb faraz qilish mumkin.

Endi ushbu k ta $x_1, x_2, ..., x_{k-1}, (x_k \cdot x_{k+1})$ sonni ko'ramiz. Induktiv farazga ko'ra $x_1 + x_2 + ... + x_{k-1} + x_k \cdot x_{k+1} \geq k$ tengsizlik o'rinli, chunki bu **k** ta sonning ko'paytmasi 1 ga teng. Bu tengsizlikning ikkala tomoniga $x_k + x_{k+1}$ ni qo'shib , $x_k \cdot x_{k+1}$ ni o'ng tomonga o'tkazamiz:

$$x_1 + x_2 + ... + x_{k-1} + x_k + x_{k+1} \geq k - x_k \cdot x_{k+1} + x_k + x_{k+1} = k + 1 + x_k(1 - x_{k+1}) + x_{k+1} - 1 =$$
$$= k + 1 + x_k \cdot (1 - x_{k+1}) - (1 - x_{k+1}) = k + 1 + (1 - x_{k+1}) \cdot (x_k - 1) \geq k + 1$$

Demak, $A(k) \Rightarrow A(k+1)$. Shunday qilib, (8)−tengsizlik $\forall n \in N$ uchun isbotlanadi.

Isbotdan shu narsa ko'rinib turibdiki, (8)−munosabat faqat $x_1 = x_2 = ... = x_n = 1$ bo'lgandagina tenglikka aylanadi.

26. Agar $a_1, a_2, ..., a_n$ lar ixtiyoriy musbat sonlar bo'lsa, ushbu

$$\frac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdot ... \cdot a_n} \qquad (9)$$

tengsizlikning bajarilishi isbotlansin.

Ko'rsatma.

$$x_k = \frac{a_k}{\sqrt[n]{a_1 \cdot a_2 \cdot ... \cdot a_n}} \quad (k = 1, 2, ..., n)$$

deb olib, 25 – misoldan foydalaning.

Matematik induktsiya usulidan foydalanib quyidagi munosabatlarning o'rinli ekanligi isbotlansin

27. $n! < \left(\frac{n+1}{2}\right)^n \quad (n > 1)$.

Ko'rsatma: $a_k = k \quad (k = \overline{1,n})$ deb olib, 26 – misoldan foydalaning.

28. $\sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}$ $(n \in N)$ bu yerda $x_k > 0,\ k = \overline{1,n}$.

Ko'rsatma: (9) tengsizlikni $\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}$ sonlar uchun qo'llang.

29. $\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right) \cdot \left(\sum_{i=1}^{n} y_i^2\right)$ – Koshi – Bunyakovskiy tengsizligi. Bu yerda $x_i, y_i (i = \overline{1,n})$ lar ixtiyoriy haqiqiy sonlar.

J. **Ko'rsatma:** $\forall t \in R$ uchun $\sum_{i=1}^{n}(x_i t + y_i)^2 \geq 0$ tengsizlik o'rinli bo'lishidan foydalaning.

30. $(n!)^2 < \left[\frac{(n+1)(2n+1)}{6}\right]^n \quad (n > 1)$.

Ko'rsatma: $a_k = k^2$ deb olamiz, 26 – misoldan foydalaning.

31. $n^{n+1} > (n+1)^n \qquad (n \geq 3)$.

32. $(2n)! > \frac{4n}{n+1} \cdot (n!)^2 \qquad (n > 1)$.

II–BOB.
OLIY ALGEBRA ELEMENTLARI.

1–§. 2–va 3–tartibli determinantlar, ularning xossalari. Chiziqli tenglamalar sistemasini Kramer usuli bilan yechish.

Aytaylik, a_1, a_2, b_1, b_2 sonlar berilgan bo'lsin.

Ushbu $a_1b_2 - a_2b_1$ songa **ikkinchi tartibli determinant** deb ataladi va u,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

kabi yoziladi.

a_1, a_2, b_1, b_2 sonlarga determinantning elementlari deyiladi.

Misol. $$\begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} = 2\cdot(-4) - 5\cdot 3 = -23.$$

Xossalari.

1⁰. Agar determinantning mos qatorini mos ustuni bilan almashtirilsa, uning qiymati o'zgarmaydi, ya'ni

$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ – tenglik o'rinli.

2⁰. Agar determinantning yo'llari (yoki ustunlari) almashtirilsa, unda determinant ishorasini o'zgartiradi:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = -\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix}, \qquad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = -\begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix}.$$

3⁰. $\begin{vmatrix} ka_1 & b_1 \\ ka_2 & b_2 \end{vmatrix} = k\cdot\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$

4^0. Birorta yo'lning (yoki ustunning) elementlari nollardan iborat bo'lsa, unda determinantning qiymati nolga teng bo'ladi.

5^0. Agar ikkita yo'ldagi (yoki ustundagi) elementlar bir-biriga teng bo'lsa, unda determinant nolga teng bo'ladi.

Masalan,

$$\begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} = a_1a_2 - a_1a_2 = 0$$

6^0. Agar birorta yo'ldagi (yoki ustundagi) elementlar o'zgarmas songa ko'paytirilib boshqasiga qo'shilsa, unda determinantning qiymati o'zgarmaydi.

Masalan,

$$\begin{vmatrix} a_1 \pm ka_2 & a_2 \\ b_1 \pm kb_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Endi 3-tartibli determinantning ta'rifini beramiz.

Aytaylik, $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$ sonlar berilgan bo'lsin. Ushbu

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}. \quad (1)$$

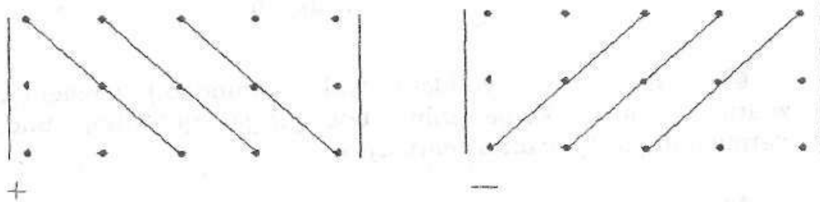
songa **<u>3-tartibli determinant</u>** deyiladi.

Δ ning qiymatini hisoblashning (1)-formulasiga **<u>uchburchak usuli</u>** deyiladi. Uni eslab qolishda quyidagi sxema yordam beradi:

+ −

Δ ning qiymatini ***Sarryus usuli*** bilan ham hisoblash mumkin;

$$\begin{matrix} a_{11}a_{12}a_{13}a_{11}a_{12} \\ a_{21}a_{22}a_{23}a_{21}a_{22} \\ a_{31}a_{32}a_{33}a_{31}a_{32} \end{matrix}$$



Uchinchi tartibli determinantning qiymatini determinantning tartibini pasaytirib, minorlar yordamida ham hisoblash mumkin. Masalan, birinchi yo'lning elementlari bo'yicha minorlarga yoyish yordamida quyidagicha hisoblanadi:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \quad (2)$$

$a_{ij}(i,j=1,2,3)$ elementlarga ko'paytirilgan ikkinchi tartibli determinantlar bu elementlarning **algebraik to'ldiruvchisi** deyiladi va A_{ij} kabi belgilanadi.

Agar $a_{ij}(i,j=1,2,3)$ element turgan satr hamda ustunning nomerlari yig'indisi juft bo'lsa, u holda $A_{ij}=M_{ij}$ bo'ladi. Agar yig'ndi toq bo'lsa, u holda $A_{ij}=-M_{ij}$ bo'ladi, ya'ni

$$A_{ij}=(-1)^{i+j}M_{ij}$$

Uchinchi tartibli determinantlar ham $\mathbf{1^0}$–$\mathbf{6^0}$ xossalarga ega. n – tartibli determinantlar ham yuqoridagi kabi aniqlanadi, faqat uning qiymati minorlar yordamida hisoblanadi.

Determinantlarning qo'llanish doirasi juda keng. Xususan, ular yordamida chiziqli tenglamalar sistemasi osongina yechiladi.

Aytaylik, ushbu

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases} \qquad (3)$$

uchta noma'lum chiziqli tenglamalar sistemasi berilgan bo'lib, uni yechish talab qilinsin. Uning uchun ushbu determinantlarni hisoblaymiz:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$ — asosiy determinant.

$$\Delta_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix}, \quad \Delta_z = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}$$

$\Delta_x, \Delta_y, \Delta_z$ — yordamchi determinantlar.

Teorema (Kramer). *Agar* $\Delta \neq 0$ *bo'lsa, unda (3)–tenglamalar sistemasi yagona yechimga ega bo'lib, uning yechimi ushbu*

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}, \quad z = \frac{\Delta_z}{\Delta} \qquad (4)$$

formulalar yordamida topiladi.

(4) — formulaga **Kramer formulasi** deyiladi va tenglamalar sistemasini yechishning bu usuliga esa **Kramer usuli** deb ataladi.

Izoh. *Agar* $\Delta = 0$ *bo'lib,* $\Delta_x, \Delta_y, \Delta_z$ *lardan birortasi* $\neq 0$ *bo'lsa, unda (3)–sistema yechimga ega emas. Agar* $\Delta = \Delta_x = \Delta_y = \Delta_z = 0$ *bo'lsa, (3)–sistema cheksiz ko'p yechimga ega bo'ladi.*

n ta noma'lumli chiziqli tenglamalar sistemasi uchun ham Kramer teoremasi o'rinli.

Nazorat savollari.

1) 2– tartibli determinant deb nimaga aytiladi? Misollar keltiring.
2) 3– tartibli determinant deb nimaga aytiladi? Misollar keltiring.
3) Determinant xossalari. Misollar.
4) Chiziqli tenglamalar sistemasini yechishning Kramer usuli. Misollar.

2–§. Matritsalar va ular ustida amallar. Gauss usuli.

Ushbu

$$\begin{pmatrix} a_{11} & a_{12} & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & a_{2m} \\ \circ & \circ & \circ & \circ \\ a_{n1} & a_{n2} & \cdot & a_{nm} \end{pmatrix} = A$$

jadvalga $(n \times m)$ **o'lchovli matritsa** deb ataladi. $n = m$ bo'lsa unga **kvadrat matritsa** deb ataladi.

Endi,

$$B = \begin{pmatrix} в_{11} & в_{12} & \cdot & в_{1m} \\ в_{21} & в_{22} & \cdot & в_{2m} \\ \circ & \circ & \circ & \circ \\ в_{n1} & в_{n2} & \cdot & в_{nm} \end{pmatrix}$$

matritsa berilgan bo'lsin.

Agar $a_{ij} = в_{ij}$ $(i = \overline{1,n}, j = \overline{1,m})$ bo'lsa, unda $A = B$ deyiladi.

$$A + B := \begin{pmatrix} a_{11} + b_{11} & \cdot & a_{1m} + b_{1m} \\ \circ & \circ & \circ \\ a_{n1} + b_{n1} & \cdot & a_{nm} + b_{nm} \end{pmatrix},$$

$$\lambda \cdot A := \begin{pmatrix} \lambda a_{11} & \cdot & \lambda a_{1m} \\ \circ & \circ & \circ \\ \lambda a_{n1} & \cdot & \lambda a_{nm} \end{pmatrix}, \qquad 0 := \begin{pmatrix} 0 & \cdot & 0 \\ 0 & \cdot & 0 \end{pmatrix}$$

deb qabul qilamiz.

Xossalari:

1^0. $A+B=B+A$
2^0. $(A+B)+C=A+(B+C)$
3^0. $A\lambda=\lambda A$

Endi $n=m=2$ bo'lganda, matritsani ko'paytirishni o'rganamiz.

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ va } B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$A\cdot B:=\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix} \qquad (5)$$

(5) − tenglikdan ko'rinib turibdiki, ∀A va B matritsalar berilganda, A·B aniqlanishi uchun A matritsaning ustunlar soni B matritsaning yo'llar soniga teng bo'lishi kerak.

Misollar.

1) $$\begin{pmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 2 \end{pmatrix}=\begin{pmatrix} 2\cdot1+1\cdot2+0\cdot2 & 2\cdot2+1\cdot1+0\cdot2 \\ 3\cdot1+1\cdot2+1\cdot2 & 3\cdot2+1\cdot1+1\cdot2 \end{pmatrix}=\begin{pmatrix} 4 & 5 \\ 7 & 9 \end{pmatrix}.$$

2) $$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} a_{11}x_1+a_{12}x_2 \\ a_{21}x_1+a_{22}x_2 \end{pmatrix}.$$

4^0. $A\cdot(B\cdot C)=(A\cdot B)\cdot C$
5^0. $(A+B)\cdot C=A\cdot C+B\cdot C$
6^0. $A\cdot E=E\cdot A=A$, bu yerda $E=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ − birlik matritsa.

A − kvadrat matritsa bo'lib, uning determinanti $|A|\neq 0$ bo'lsin.

Unda A^{-1} deb belgilanadigan shunday matritsa topiladiki, $A\cdot A^{-1}=A^{-1}A=E$ tenglik bajariladi, hamda A^{-1}ga teskari matritsa deb ataladi.

Agar, $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ bo'lsa,

$$A^{-1}=\begin{pmatrix} \frac{A_{11}}{|A|} & \frac{A_{21}}{|A|} & \frac{A_{31}}{|A|} \\ \frac{A_{12}}{|A|} & \frac{A_{22}}{|A|} & \frac{A_{32}}{|A|} \\ \frac{A_{13}}{|A|} & \frac{A_{23}}{|A|} & \frac{A_{33}}{|A|} \end{pmatrix} \quad (6)$$

formula yordamida topiladi. Bu yerda $A_{ii}-a_{ii}$ elementining algebraik to'ldiruvchisi;

Masalan, $A_{11}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, $A_{12}=-\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \ldots$

Matritsalar yordamida chiziqli tenglamalar sistemasi osongina yechiladi.

Aytaylik,

$$\begin{cases} a_{11}x+a_{12}y+a_{13}z=c_1 \\ a_{21}x+a_{22}y+a_{23}z=c_2 \\ a_{31}x+a_{32}y+a_{33}z=c_3 \end{cases} \quad (7)$$

(7) – tenglamalar sistemasi berilgan bo'lsin. Agar

$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ va $C=\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ deb belgilasak,

(7) ni $A\cdot X=C$ (8) matritsa ko'rinishida yozish mumkin.

$$X=A^{-1}\cdot C \quad (9)$$

1–misol. $\begin{cases} x_1+2x_2=10 \\ 3x_1+2x_2+x_3=23 \\ x_2+2x_3=13 \end{cases}$ tenglamalar sistemasi matritsalar usuli bilan yechilsin.

$$A=\begin{pmatrix}1&2&0\\3&2&1\\0&1&2\end{pmatrix},\ X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix},\ C=\begin{pmatrix}10\\23\\13\end{pmatrix}\Rightarrow A^{-1}=\begin{pmatrix}-\frac{1}{3}&\frac{4}{9}&-\frac{2}{9}\\\frac{2}{3}&-\frac{2}{9}&\frac{1}{9}\\-\frac{1}{3}&\frac{1}{9}&\frac{4}{9}\end{pmatrix}\Rightarrow$$

$$\Rightarrow X=A^{-1}\cdot C=\begin{pmatrix}4\\3\\5\end{pmatrix}\Rightarrow x_1=4,x_2=3,x_3=5.$$

(7) tenglamalar sistemasini noma'lumlarni ketma−ket yo'qotish usuli, ya'ni Gauss usuli bilan ham yechish mumkin.

Uni misolda tushuntiramiz.

2−misol. $$\begin{cases}x_1+0{,}5x_2-0{,}5x_3=0{,}5\\3x_1+2x_2-2x_3=1\\x_1-x_2+2x_3=5\end{cases}$$

Avval 1−qatorni 3 ga ko'paytirib, keyin ikkinchi qatordan ayiramiz, so'ng 1−qatorni uchinchi qatordan ayirib qo'yidagi sistemani hosil qilamiz (bunda biz determinantlarning 6^0−xossasidan foydalanamiz):

$$\left.\begin{matrix}x_1+0{,}5x_2-0{,}5x_3=0{,}5\\0{,}5x_2-0{,}5x_3=-0{,}5\\-1{,}5x_2+2{,}5x_3=45\end{matrix}\right\},\Rightarrow\left.\begin{matrix}x_1+0{,}5x_2-0{,}5x_3=0{,}5\\x_2-x_3=-1\\-1{,}5x_2+2{,}5x_3=4{,}5\end{matrix}\right\},=\left.\begin{matrix}x_1+0{,}5x_2-0{,}5x_3=0{,}5\\x_2-x_3=-1\\x_3=3\end{matrix}\right\}\Rightarrow$$

$$\Rightarrow x_3=3\Rightarrow x_2=-1+x_3=2\Rightarrow x_1=1.$$

Shunday qilib, $x_1=1,x_2=2,x_3=3$.

Nazorat savollari.

1.Matritsa ta'rifi. Misollar.
2.Matritsalarni qo'shish va ko'paytirish. Misollar.
3.Teskari matritsa. Misollar.
4.Tenglamalar sistemasini yechishning matritsalar usuli. Misollar.
5.Gauss usuli. Misollar.

3-§. Chiziqli (vektor) fazo va chiziqli almashtirishlar.

Vektor tushunchasi, ular ustida amallar, ularning skalyar ko'paytmasi maktab kursidan ma'lum bo'lganligi sababli biz ularga to'xtalib o'tirmaymiz.

n ta $x_1, x_2, ..., x_n$ tartiblangan sonlar to'plamini **n o'lchovli x vektor** deb ataymiz va x vektorni $x = \{x_1, x_2, ..., x_n\}$ ko'rinishda yozamiz. Agar $A(a_1, a_2, ..., a_n)$ va $B(b_1, b_2, ..., b_n)$ nuqtalarning koordinatalari uchun $x_1 = b_1 - a_1$, $x_2 = b_2 - a_2, ..., x_n = b_n - a_n$ munosabatlar o'rinli bo'lsa A nuqta x vektorning boshi, B nuqta x vektorning oxiri ekanligini bildiradi. Vektorlar quyidagicha ham belgilanadi x, $\bar{x}$ va $\overline{AB}$.

x vektorni tashkil qiluvchi $x_1, x_2, ..., x_n$ sonlar uning **komponentlari (koordinatalari)** deyiladi.

Barcha komponentlari nolga teng bo'lgan vektor **nol vektor** deyiladi.

Vektorlarni qo'shish va vektorlarni haqiqiy songa ko'paytirish vektorlar ustida **chiziqli amallar bajarish** deyiladi.

Ushbu

$$x = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + ... + \lambda_n e_n$$

(bu yerda $e_1, e_2, e_3, ..., e_n$ — turli vektorlar, $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ — sonlar) ko'rinishda tasvirlangan x vektor $e_1, e_2, e_3, ..., e_n$ vektorlarning **chiziqli kombinatsiyasidan** iborat deyiladi.

Agar hammasi bir vaqtda nolga teng bo'lmagan $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ $(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + ... + \lambda_m^2 > 0)$ shunday sonlar mavjud bo'lib,

$$\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + ... + \lambda_n e_n = 0 \qquad (*)$$

tenglik o'rinli bo'lsa, $e_1, e_2, e_3, ..., e_n$ vektorlar **chiziqli bog'liq vektorlar** deyiladi. Agar bunday $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ — sonlar mavjud bo'lmasa, ya'ni (*) tenglik faqat $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$ bo'lgandagina bajarilsa, unda $e_1, e_2, e_3, ..., e_n$ vektorlar **chiziqli erkli vektorlar** deyiladi.

Ikkita vektor o'zaro kollinear bo'lganda ular *chiziqli bog'liq* bo'ladilar. Agar uchta vektor o'zaro komplanar bo'lsa, ular *chiziqli bog'liqli* bo'ladilar.

V - fazoning istalgan vektori chiziqli ifodalanadigan chiziqli erkli vektorlar to'plami shu fazoning **bazisi** deyiladi. V - fazoning bazisini tashkil etuvchi vektorlar **bazis vektorlar** deyiladi.

V - fazoning chiziqli erkli vektorlarining eng katta soni shu fazoning **o'lchovi** deyiladi.

Shunga muvofiq ravishda, to'g'ri chiziqni **bir o'lchovli V_1 fazo** deyiladi, birorta to'g'ri chiziqqa joylashgan vektorlar to'plamining bazisi bitta vektordan iborat. Tekislikni **ikki o'lchovli V_2 fazo** deyiladi, birorta tekislikda joylashgan vektorlar to'plamining bazislari ikkita vektordan iborat. Odatdagi fazoni **uch o'lchovli V_3 fazo** deyiladi, uning bazislari uchta nokomplanar vektordan iborat.

Quyidagi teorema o'rinli: *V_3 fazoning istalgan x vektori shu fazoning uchta chiziqli erkli e_1, e_2, e_3 vektorlarining chiziqli kombinatsiyasi yordamida bir qiymatli aniqlanadi, ya'ni*

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3 .$$

Bu ifoda x vektorning e_1, e_2, e_3 bazisi bo'yicha *yoyilmasi* deyiladi. Vektorning bazis bo'yicha yoyilmasidagi x_1, x_2, x_3 koeffitsientlar vektorning shu bazisdagi *koordinatalari* deyiladi.

O'zaro perpendikulyar birlik vektorlardan iborat bo'lgan bazis **ortonormallangan bazis** deyiladi va quyidagicha yoziladi.

$$(e_i, e_j) = \begin{cases} 1, & агар \ i = j \ бўлса \\ 0, & агар \ i \neq j \ бўлса \end{cases}$$

$(i, j = 1, 2, ..., n)$.

V_3 fazoda biror qoida bo'yicha har bir vektor x ga shu fazoning y vektori mos qo'yilgan bo'lsin, ya'ni vektor argument x ning $y = A(x)$ vektor qiymatli funktsiyasi berilgan bo'lsa, u holda bu fazoda **vektorlarni almashtirish** berilgan deyiladi.

Agar

$$A(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 A(x_1) + \lambda_2 A(x_2) \quad (1)$$

shart bajarilsa A almashtirish **chiziqli almashtirish** deyiladi, bu yerda x_1 va x_2 — qaralayotgan fazoning ixtiyoriy vektorlari, λ_1 va λ_2 esa istalgan sonlar.

V_3 fazoda biror e_1, e_2, e_3 bazis tanlab olamiz va bazisning har bir vektori uchun A almashtirishni qo'llaymiz:

$$\begin{aligned} A(e_1) &= a_{11}e_1 + a_{21}e_2 + a_{31}e_3 \\ A(e_2) &= a_{12}e_1 + a_{22}e_2 + a_{32}e_3 \\ A(e_3) &= a_{13}e_1 + a_{23}e_2 + a_{33}e_3 \end{aligned} \quad (2)$$

Ixtiyoriy $x \in V_3$ vektorni olamiz va uni $x = x_1e_1 + x_2e_2 + x_3e_3$ bazis bo'yicha yoyamiz. U holda

$$y = A(x) = A(x_1e_1 + x_2e_2 + x_3e_3) = x_1A(e_1) + x_2A(e_2) + x_3A(e_3)$$

(2) formuladan foydalanib va o'xshash hadlarni ixchamlab quyidagini hosil qilamiz

$$\begin{aligned} y &= x_1(a_{11}e_1 + a_{21}e_2 + a_{31}e_3) + x_2(a_{12}e_1 + a_{32}e_2 + a_{32}e_3) + x_3(a_{13}e_1 + a_{23}e_2 + a_{33}e_3) = \\ &= (a_{11}x_1 + a_{12}x_2 + a_{13}x_3)e_1 + (a_{21}x_1 + a_{22}x_2 + a_{23}x_3)e_2 + (a_{31}x_1 + a_{32}x_2 + a_{33}x_3)e_3 \end{aligned}$$

y vektorning e_1, e_2, e_3 bazisdagi koordinatalarini y_1, y_2, y_3 orqali belgilab, V_3 fazodagi har qanday vektorning koordinatalarini chiziqli almashtirishini aniqlovchi formulani hosil qilamiz.

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ y_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned} \quad (3)$$

(3)ni matritsa shaklida yozamiz:

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

ya'ni

$$Y = AX$$

Vektorlarning har bir $y = Ax$ chiziqli almashtirishga shu vektorlar koordinatalarining $Y = AX$ chiziqli almashtirishi mos keladi va aksincha.

Misol. Ikkita chiziqli

$$\begin{cases} x_1' = 2x_1 - x_2 + 5x_3 \\ x_2' = x_1 + 4x_2 - x_3 \\ x_3' = 3x_1 - 5x_2 + 2x_3 \end{cases} \quad \text{va} \quad \begin{cases} x_1'' = x_1' + 4x_2' + 3x_3' \\ x_2'' = 5x_1' - x_2' - x_3' \\ x_3'' = 3x_1' + 6x_2' + 7x_3' \end{cases}$$

almashtirishlar berilgan. x_1'', x_2'', x_3'' ni x_1, x_2, x_3 orqali ifodalovchi almashtirishni toping.

Yechilishi: Koordinatalarning birinchi chiziqli almashtirish matritsasini A bilan ikkinchisini B bilan belgilaymiz.

$$A = \begin{pmatrix} 2 & -1 & 5 \\ 1 & 4 & -1 \\ 3 & -5 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 & 3 \\ 5 & -1 & -1 \\ 3 & 6 & 7 \end{pmatrix}$$

x, x', x'' vektorlarning koordinatalaridan olingan vektor ustunlarini mos ravishda X, X', X'' bilan belgilaymiz, ya'ni

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad X' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}, \quad X'' = \begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix}.$$

U holda koordinatalarning berilgan chiziqli almashtirishlarini matritsa shaklida yozamiz:

$X' = AX$, $X'' = BX'$ ularga mos vektorlarning chiziqli almashtirishlari esa $x' = Ax$, $x'' = Bx'$ ko'rinishda yoziladi.

Hosil qilingan formulalar quyidagini ifodalaydi: A chiziqli almashtirish x vektorni x' vektorga o'tkazadi, hosil qilingan x'

vektor B almashtirish bilan x'' vektorga o'tkaziladi. x vektorni y vektorga o'tkazuvchi chiziqli almashtirish bunday yoziladi:

$$x'' = B(Ax)$$

koordinatalarning unga mos chiziqli almashtirishi esa quyidagicha yoziladi:

$$X'' = BAX \quad (4)$$

B va A matritsalarni ko'paytiramiz:

$$BA = \begin{pmatrix} 1 & 4 & 3 \\ 5 & -1 & -1 \\ 3 & 6 & 7 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 & 5 \\ 1 & 4 & -1 \\ 3 & -5 & 2 \end{pmatrix} = \begin{pmatrix} 15 & 0 & 7 \\ 6 & -4 & 24 \\ 33 & -14 & 23 \end{pmatrix}$$

Olingan BA matritsani (4) tenglikka qo'yamiz:

$$\begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} = \begin{pmatrix} 15 & 0 & 7 \\ 6 & -4 & 24 \\ 33 & -14 & 23 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

bundan

$$x_1'' = 15x_1 + 0 \cdot x_2 + 7x_3$$
$$x_2'' = 6x_1 + 4x_2 + 24x_3$$
$$x_3'' = 33x_1 - 14x_2 + 23x_3$$

shuni topish talab qilingan edi.

Nazorat savollari.

1) Chiziqli bog'liqli va chiziqli erkli vektorlar.
2) Bazis tshunchasi. Misollar.
3) Chiziqli almashtirishlar.

Mustaqil yechish uchun misollar.

Chiziqli tenglamalar sistemasini uchta usul:

1) Kramer usuli.
2) Matritsa usuli .
3) Gauss usuli bilan yeching.

1. $$\begin{cases} 3x_1 + 4x_2 + 2x_3 = 8 \\ 2x_1 - 4x_2 - 3x_3 = -1 \\ x_1 + 5x_2 + x_3 = 0 \end{cases}$$

2. $$\begin{cases} 5x_1 + 8x_2 - x_3 = 7 \\ 2x_1 - 3x_2 + 2x_3 = 9 \\ x_1 + 2x_2 + 3x_3 = 1 \end{cases}$$

3. $$\begin{cases} 3x_1 + x_2 + x_3 = 5 \\ x_1 - 4x_2 - 2x_3 = -3 \\ -3x_1 + 5x_2 + 6x_3 = 7 \end{cases}$$

4. $$\begin{cases} 2x_1 - x_2 + 5x_3 = 4 \\ 5x_1 + 2x_2 + 13x_3 = -23 \\ 3x_1 - x_3 + 5 = 0 \end{cases}$$

5. $$\begin{cases} x_1 + x_2 - x_3 = -2 \\ 4x_1 - 3x_2 + x_3 = 1 \\ 2x_1 + x_2 - 5 = 1 \end{cases}$$

6. $$\begin{cases} 7x_1 - 5x_2 = 31 \\ 4x_1 - 11x_3 = -43 \\ 2x_1 + 3x_2 + 4x_3 = -20 \end{cases}$$

7. $$\begin{cases} x_1 + 2x_2 + x_3 = 4 \\ 3x_1 - 5x_2 + 3x_3 = 1 \\ 2x_1 + 7x_2 - x_3 = 8 \end{cases}$$

8. $$\begin{cases} x_1 + x_2 - x_3 = 1 \\ 8x_1 + 3x_2 - 6x_3 = 2 \\ -4x_1 - x_2 + 3x_3 = -3 \end{cases}$$

9. $$\begin{cases} x_1 - 2x_2 + 3x_3 = 6 \\ 2x_1 + 3x_2 - 4x_3 = 20 \\ 3x_1 - 2x_2 - 5x_3 = 6 \end{cases}$$

10. $$\begin{cases} 4x_1 - 3x_2 + 2x_3 = 9 \\ 2x_1 + 5x_2 - 3x_3 = 14 \\ 5x_1 + 6x_2 - 2x_3 = 18 \end{cases}$$

11. $$\begin{cases} 3x_1 + 2x_2 + x_3 = 5 \\ 2x_1 + 3x_2 + x_3 = 1 \\ 2x_1 + x_2 + 3x_3 = 11 \end{cases}$$

12. $$\begin{cases} x_1 - 2x_2 + 3x_3 = 6 \\ 2x_1 + 3x_2 - 4x_3 = 20 \\ 3x_1 - 2x_2 - 5x_3 = 6 \end{cases}$$

13. $$\begin{cases} 4x_1 - 3x_2 + 2x_3 = 9 \\ 2x_1 + 5x_2 - 3x_3 = 4 \\ 5x_1 + 6x_2 - 2x_3 = 18 \end{cases}$$

14. $$\begin{cases} x_1 + x_2 + 2x_3 = -1 \\ 2x_1 - x_2 + 2x_3 = -4 \\ 4x_1 + x_2 + 4x_3 = -2 \end{cases}$$

15. $$\begin{cases} 2x_1 - x_2 - x_3 = 4 \\ 3x_1 + 4x_2 - 2x_3 = 1 \\ 3x_1 - 2x_2 + 4x_3 = 11 \end{cases}$$

16. $$\begin{cases} 3x_1 + 4x_2 + 2x_3 = 8 \\ 2x_1 - x_2 - 3x_3 = -4 \\ x_1 + 5x_2 + x_3 = 0 \end{cases}$$

17. $$\begin{cases} x_1 + x_2 - x_3 = 1 \\ 8x_1 + 3x_2 - 6x_3 = 2 \\ 4x_1 + x_2 - 3x_3 = 3 \end{cases}$$

18. $$\begin{cases} x_1 - 4x_2 - 2x_3 = -3 \\ 3x_1 + x_2 + x_3 = 5 \\ 3x_1 - 5x_2 - 6x_3 = -9 \end{cases}$$

19. $\begin{cases} 7x_1 - 5x_2 = 31 \\ 4x_1 - 11x_3 = -43 \\ 2x_1 + 3x_2 + 4x_3 = -20 \end{cases}$ 20. $\begin{cases} x_1 + 2x_2 + 4x_3 = 31 \\ 5x_1 + x_2 + 2x_3 = 20 \\ 3x_1 - x_2 + x_3 = 9 \end{cases}$

Ikkita chiziqli almashtirish berilgan x_1'', x_2'', x_3'' ni x_1, x_2, x_3 orqali ifodalovchi almashtirishni toping.

1. $\begin{cases} x_1' = 5x_1 - x_2 + 3x_3 \\ x_2' = x_1 - 2x_2 \\ x_3' = 7x_2 - x_3 \end{cases}$ va $\begin{cases} x_1'' = 2x_1' + x_3' \\ x_2'' = x_2' - 5x_3' \\ x_3'' = 2x_1' \end{cases}$

2. $\begin{cases} x_1' = x_1 + 2x_2 + 2x_3 \\ x_2' = -3x_2 + x_3 \\ x_3' = 2x_1 + 3x_3 \end{cases}$ va $\begin{cases} x_1'' = 3x_1' + x_2' \\ x_2'' = x_1' - 2x_2' - x_3' \\ x_3'' = 3x_2' + 2x_3' \end{cases}$

3. $\begin{cases} x_1' = x_1 - 3x_2 + 4x_3 \\ x_2' = 2x_1 + 2x_2 - 5x_3 \\ x_3' = -3x_1 + 5x_2 + x_3 \end{cases}$ va $\begin{cases} x_1'' = 4x_1' + 5x_2' - 3x_3' \\ x_2'' = x_1' - x_2' - x_3' \\ x_3'' = 7x_1' + 4x_3' \end{cases}$

4. $\begin{cases} x_1' = 4x_1 + 3x_2 + 5x_3 \\ x_2' = 6x_1 + 7x_2 + x_3 \\ x_3' = 9x_1 + x_2 + 8x_3 \end{cases}$ va $\begin{cases} x_1'' = -x_1' + 5x_2' - 3x_3' \\ x_2'' = x_1' - x_2' - x_3' \\ x_3'' = 7x_1' + 4x_3' \end{cases}$

5. $\begin{cases} x_1' = -x_1 - x_2 - x_3 \\ x_2' = -x_1 + 4x_2 + 7x_3 \\ x_3' = 8x_1 + x_2 - x_3 \end{cases}$ va $\begin{cases} x_1'' = 9x_1' + 3x_2' + 5x_3' \\ x_2'' = 2x_1' + 3x_3' \\ x_3'' = x_2' - x_3' \end{cases}$

6. $\begin{cases} x_1' = 4x_1 + 3x_2 + 2x_3 \\ x_2' = -2x_1 + x_2 - x_3 \\ x_3' = 3x_1 + x_2 + x_3 \end{cases}$ va $\begin{cases} x_1'' = x_1' - x_2' - x_3' \\ x_2'' = 3x_1' + x_2' + 2x_3' \\ x_3'' = x_1' + 2x_2' + 2x_3' \end{cases}$

7. $\begin{cases} x_1' = 3x_1 + 5x_3 \\ x_2' = x_1 + x_2 + x_3 \\ x_3' = 3x_2 - 6x_3 \end{cases}$ va $\begin{cases} x_1'' = 2x_1' - x_2' - 5x_3' \\ x_2'' = 7x_1' + x_2' + 4x_3' \\ x_3'' = 6x_1' + 4x_2' - 7x_3' \end{cases}$

8. $\begin{cases} x_1' = 2x_2 \\ x_2' = -2x_1 + 3x_2 + 2x_3 \\ x_3' = 4x_1 - x_2 + 5x_3 \end{cases}$ va $\begin{cases} x_1'' = -3x_1' + x_3' \\ x_2'' = 2x_2' + x_3 \\ x_3'' = -x_2' + 3x_3' \end{cases}$

9. $\begin{cases} x_1' = 7x_1 + 4x_3 \\ x_2' = 4x_2 - 9x_3 \\ x_3' = 3x_1 + x_2 \end{cases}$ va $\begin{cases} x_1'' = x_2' - 6x_3' \\ x_2'' = 3x_1' + 7x_3' \\ x_3'' = x_1' + x_2' - x_3' \end{cases}$

10. $\begin{cases} x_1' = 7x_1 - 4x_3 \\ x_2' = 4x_2 - 9x_3 \\ x_3' = 3x_1 + x_2 \end{cases}$ va $\begin{cases} x_1'' = x_2' + 6x_3' \\ x_2'' = 3x_1' - 7x_3' \\ x_3'' = x_1' - x_2' + x_3' \end{cases}$

11. $\begin{cases} x_1' = 4x_1 + 3x_2 + 5x_3 \\ x_2' = 6x_1 + 7x_2 + x_3 \\ x_3' = 9x_1 + x_2 + 8x_3 \end{cases}$ va $\begin{cases} x_1'' = -x_1' + 3x_2' - 2x_3' \\ x_2'' = -4x_1' + x_2 + 2x_3' \\ x_3'' = 3x_1' - 4x_2' + 5x_3' \end{cases}$

12. $\begin{cases} x_1' = x_1 - x_2 - x_3 \\ x_2' = -x_1 + 4x_2 + 7x_3 \\ x_3' = 8x_1 + x_2 - x_3 \end{cases}$ va $\begin{cases} x_1'' = 9x_1' + 3x_2' + 5x_3' \\ x_2'' = 2x_1' + 3x_3' \\ x_3'' = x_2' - x_3 \end{cases}$

13. $\begin{cases} x_1' = 7x_1 + 4x_3 \\ x_2' = 4x_2 - 9x_3 \\ x_3' = 3x_1 + x_2 \end{cases}$ va $\begin{cases} x_1'' = x_2' - 6x_3 \\ x_2'' = 3x_1' + 7x_3' \\ x_3'' = x_1' + x_2' - x_3' \end{cases}$

14. $\begin{cases} x_1' = 2x_2 \\ x_2' = -2x_1 + 3x_2 + 2x_3 \\ x_3' = 4x_1 - x_2 + 5x_3 \end{cases}$ va $\begin{cases} x_1'' = -3x_1' + x_3' \\ x_2'' = 2x_2' + x_3' \\ x_3'' = -x_2' + 3x_3' \end{cases}$

15. $\begin{cases} x_1' = 3x_1 - x_2 + 5x_3 \\ x_2' = x_1 + 2x_2 + 4x_3 \\ x_3' = 3x_1 + 2x_2 - x_3 \end{cases}$ va $\begin{cases} x_1'' = 4x_1' + 3x_2' + x_3' \\ x_2'' = 3x_1' + x_2' + 2x_3' \\ x_3'' = x_1' - 2x_2' + x_3' \end{cases}$

16. $\begin{cases} x_1' = 4x_1 + 3x_2 - x_3 \\ x_2' = -2x_1 + x_2 - x_3 \\ x_3' = 3x_1 + x_2 + x_3 \end{cases}$ va $\begin{cases} x_1'' = x_1' - 2x_2' - x_3' \\ x_2'' = 3x_1' + x_2' + 2x_3' \\ x_3'' = x_1' + 2x_2' + 2x_3 \end{cases}$

17. $\begin{cases} x_1' = 4x_1 + 3x_2 + 8x_3 \\ x_2' = 6x_1 + 9x_2 + x_3 \\ x_3' = 2x_1 + x_2 + 8x_3 \end{cases}$ va $\begin{cases} x_1'' = -x_1' + 8x_2' - 2x_3' \\ x_2'' = -4x_1' + 3x_2' + 2x_3' \\ x_3'' = 3x_1' - 8x_2' + 5x_3' \end{cases}$

18. $\begin{cases} x_1' = 5_1 - 3x_2 + 4x_3 \\ x_2' = 2x_1 + x_2 - 5x_3 \\ x_3' = -3x_1 + 5x_2 + x_3 \end{cases}$ va $\begin{cases} x_1'' = 4x_1' + 5x_2' - 3x_3' \\ x_2'' = x_1' - x_2' - x_3' \\ x_3'' = 7x_1' + 4x_3' \end{cases}$

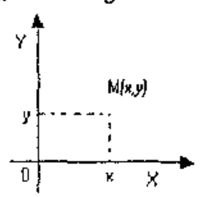
19. $\begin{cases} x_1' = 3x_1 + 0 + 5x_3 \\ x_2' = x_1 + x_2 + x_3 \\ x_3' = 3x_2 - 6x_3 \end{cases}$ va $\begin{cases} x_1'' = 2x_1' - x_2' - 5x_3' \\ x_2'' = 7x_1' + x_2' + 4x_3' \\ x_3'' = 6x_1' + 4x_2' - 7x_3' \end{cases}$

20. $\begin{cases} x_1' = x_1 + 2x_2 + 2x_3 \\ x_2' = -3x_2 + x_3 \\ x_3' = 2x_1 + 3x_3 \end{cases}$ va $\begin{cases} x_1'' = 3x_1' + x_2' \\ x_2'' = x_1' - 2x_2' - x_3' \\ x_3'' = 3x_2' + 2x_3' \end{cases}$

III–BOB.
ANALITIK GEOMETRIYA ELEMENTLARI.

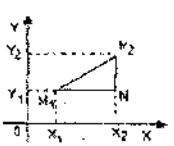
1–§. Dekart koordinatalar sisitemasi. Ikki nuqta orasidagi masofa. Kesmani berilgan nisbatda bo'lish.

Tekislikda O nuqta orqali o'zaro perpendikulyar ikkita x va y to'g'ri chiziqlarni o'tkazamiz. x o'qi (u odatda gorizontal bo'ladi) **abstsissalar o'qi** deyiladi, y o'qi esa (u vertikal holatda bo'ladi) **ordinatalar o'qi** deyiladi. Kesishish nuqtasi O koordinatalar boshi deb ataladi. O nuqta o'qlarning har birini ikkita yarim o'qqa ajratadi. Ulardan birini *musbat yarim o'q* deb, uni strelka bilan belgilaymiz, ikkinchisini *manfiy yarim o'q* deyiladi. Bu o'qlarda uzunlik birligi tanlangan bo'lsa, birgalikda XOU – **Dekart koordinatalar sistemasi** berilgan deyiladi.



1 – chizma.

$M_1(x_1,u_2)$, $M_2(x_2,u_2)$ nuqtalar berilgan bo'lib, $d = M_1M_2 - ?$



2 – chizma.

Pifagor teoremasiga ko'ra,

$$d = M_1M_2 = \sqrt{(M_1N)^2 + (M_2N)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad (*)$$

bo'ladi.

(*) − <u>ikki nuqta orasidagi masofani hisoblash formulasi</u> deyiladi.

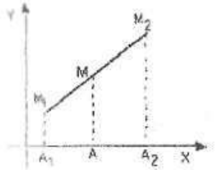
Endi $M_1(x_1,u_2)$, $M_2(x_2,u_2)$ nuqtalar berilgan bo'lib, M_1,M_2 nuqtalarda shunday $M(x,u)$ nuqtani topish kerakki,

$$\frac{|M_1M|}{|M_2M|}=\lambda$$

shart bajarilsin.

Elementar matematikadan ma'lumki,

$$\frac{|M_1M|}{|M_2M|}=\frac{|A_1A|}{|AA_2|}\Rightarrow\lambda=\frac{x-x_1}{x_2-x}\Rightarrow \qquad x=\frac{x_1+\lambda x_2}{1+\lambda}, \quad y=\frac{y_1+\lambda y_2}{1+\lambda}$$



3−chizma.

Demak,

$$\begin{cases} x=\dfrac{x_1+\lambda x_2}{1+\lambda} \\ y=\dfrac{y_1+\lambda y_2}{1+\lambda} \end{cases} \qquad (1)$$

Agar (1) da, $\lambda=1$ desak,

$$\begin{cases} x=\dfrac{x_1+x_2}{2} \\ y=\dfrac{y_1+y_2}{2} \end{cases} \qquad (2)$$

<u>kesma o'rtasining koordinatalarini hisoblash formulasini</u> hosil qilamiz.

Misol. Uchlari $A(-5;3)$, $B(2;-4)$ nuqtalarda bo'lgan AB kesma berilgan. $C(x;y)$ nuqta kesmani $\frac{1}{4}$ nisbatda bo'ladi. $C(x;y)$ nuqta koordinatalari bilan AB kesma uzunligini toping.

Yechilishi. Quyidagilarni yozib olamiz:

$$\frac{AC}{CB}=\lambda=\frac{1}{4}, \qquad x_1=-5, \quad y_1=3, \qquad x_2=2, \quad y_2=-4.$$

(1) formuladan foydalanib $C(x;y)$ nuqtaning koordinatalarini topamiz.

$$x=\frac{-5+\frac{1}{4}\cdot 2}{1+\frac{1}{4}}=-3{,}6; \qquad y=\frac{3+\frac{1}{4}\cdot(-4)}{1+\frac{1}{4}}=1{,}6. \qquad C(-3{,}6;1{,}6)$$

AB kesma uzunligini (*) formuladan foydalanib topamiz.

$$AB=d=\sqrt{(2+5)^2+(-4-3)^2}=\sqrt{49+49}=\sqrt{98}=7\sqrt{2}.$$

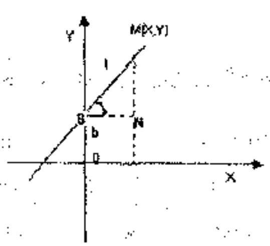
Nazorat savollari.

1. Tekislikda Dekart koordinatalar sistemasini kiriting. Nuqtalar belgilab koordinatalarini aniqlang.
2. Ikki nuqta orasidagi masofa. Misollar.
3. Kesmani berilgan nisbatda bo'lish. Misollar.
4. Kesmaning o'rtasini topish. Misollar.

2–§. To'g'ri chiziq va uning tenglamalari.

1.To'g'ri chiziqning burchak koeffitsientli tenglamasi.

Aytaylik, l to'g'ri chiziq OU o'qiga parallel bo'lmasin va OX o'qi bilan φ burchakni tashkil qilib, OU o'qidan b birlik kesma kessin.



4 – chizma.

ΔMBN dan $\Rightarrow tg\varphi = \frac{MN}{BN} = \frac{y-в}{x} \Rightarrow y = tg\varphi \cdot x + в \Rightarrow tg\varphi = k$ desak,

$y = kx + в$ (1) to'g'ri chiziqning burchak koeffitsientli tenglamasi.

$y = в$ OX o'qiga parallel to'g'ri chiziq.

$x = a$ OU o'qiga parallel to'g'ri chiziq.

2. To'g'ri chiziqning umumiy tenglamasi.

Teorema. Tekislikda dekart koordinatalar sistemasida berilgan ixtiyoriy to'g'ri chiziq ushbu

$$Ax + By + C = 0 \quad (2)$$

tenglama yordamida aniqlanadi va aksincha.

Bu yerda A, B, C lar o'zgarmas koeffitsentlar bo'lib A va B lardan hech bo'lmasa biri noldan farqli deb qaraladi.

Shuning uchun ham (2) tenglamaga **to'g'ri chiziqning umumiy tenglamasi** deb ataladi.

A, B, C o'zgarmas koeffitsentlar turli qiymatlarga teng bo'lganda turli to'g'ri chiziqlar hosil bo'ladi. Demak, to'g'ri chiziqning tekislikdagi vaziyati shu A, B, C sonlar bilan to'liq aniqlanadi.

3. Berilgan nuqtalardan o'tuvchi to'g'ri chiziq tenglamasi.

Aytaylik, M(x_1, u_1) nuqta berilgan bo'lib, shu nuqtadan o'tuvchi to'g'ri chiziq tenglamasini topish talab qilinsin.

$y = kx + b$ ko'rinishda qidiramiz. To'g'ri chiziq M(x_1, u_1) nuqtadan o'tganligi uchun $y_1 = kx_1 + b_1$ bo'ladi.

$$\Rightarrow y - y_1 = k(x - x_1) \quad (3)$$

izlanayotgan tenglama bo'ladi.

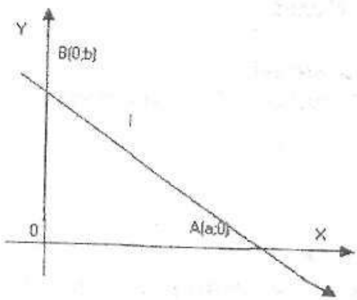
Agar M(x_1, u_1) va M_2(x_2, u_2) nuqtalar berilgan bo'lsa, bu nuqtalardan o'tuvchi *to'g'ri chiziq tenglamasi* ushbu

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \quad (4)$$

tenglikdan topiladi.

4. To'g'ri chiziqning kesmalardagi tenglamasi.

Aytaylik, l to'g'ri chiziq OX va OY o'qlarni mos ravishda a va b birlik kesmalarda kessin, ya'ni A (a,0) va B (0,b) nuqtalardan o'tsin.



5 – chizma.

(4) formuladan quyidagini yozib olamiz:

$$\frac{y-0}{b-0}=\frac{x-a}{0-a}$$

Y holda uning tenglamasi

$$\frac{x}{a}+\frac{y}{в}=1 \quad (5)$$

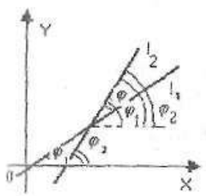
bo'lishini topish qiyin emas.

(3) – tenglamaga to'g'ri chiziqning **kesmalardagi tenglamasi** deb ataladi.

5. Ikkita to'g'ri chiziq orasidagi burchak.

Faraz qilaylik, $l_1: y=k_1x+в_1$ va $l_2: y=k_2x+в_2$ to'g'ri chiziqlar berilgan bo'lsin.

$$\Rightarrow tg\varphi_1=k_1, \quad tg\varphi_2=k_2 \qquad \varphi=\left(l_1\hat{,}l_2\right)=\varphi_2-\varphi_1 \qquad (0\le\varphi\le\pi)$$



6 – chizma.

$$tg\varphi = tg(\varphi_2 - \varphi_1) = \frac{tg\varphi_2 - tg\varphi_1}{1 + tg\varphi_2 tg\varphi_1} = \frac{k_2 - k_1}{1 + k_1 k_2} \Rightarrow tg\varphi = \frac{k_2 - k_1}{1 + k_1 k_2} \quad (6)$$

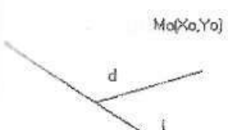
(6) **ikkita to'g'ri chiziqning orasidagi burchakni hisoblash** formulasi.

Agar ikkita to'g'ri chiziq orasidagi burchak $\varphi = 0$ bo'lsa, ravshanki, bu *to'g'ri chiziqlar o'zaro parallel* bo'ladi yoki *ustma–ust* tushadi. Bu holda $k_1 = k_2$ bo'lishi kelib chiqadi.

Agar ikkita to'g'ri chiziq orasidagi burchak $\varphi = \frac{\pi}{2}$ bo'lsa, unda *to'g'ri chiziqlar perpendikulyar* bo'ladi, ya'ni $k_1 k_2 = -1$.

Berilgan nuqtadan to'g'ri chiziqqacha bo'lgan masofani quyidagi formuladan foydalanib topiladi.

$$l: Ax + By + C = 0$$



7 – chizma.

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}} \quad (7)$$

Masala. Uchlarining koordinatalari A $(1; -1)$, B $(\frac{1}{3}, \frac{1}{3})$, C$(0,0)$ bo'lgan ABS uchburchak uchun quyidagilarni aniqlang:

1. AB tomonining uzunligini hisoblang;
2. Tomonlarining tenglamasini tuzing;
3. S uchidan o'tkazilgan balandlikning tenglamasini tuzing;
4. B uchidan AS tomongacha bo'lgan masofani hisoblang;
5. Ichki A burchak bissektrisasining tenglamasini toping.

Yechilishi.

1.

$$d=\sqrt{(x_2 - x_1)^2+(y_2-y_1)^2}=\sqrt{\left(\frac{1}{3}-1\right)^2+\left(\frac{1}{3}+1\right)^2}=\frac{2\sqrt{5}}{3}$$

2.

$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$ dan foydalanib quyidagilarni topamiz:

AB : $2x+y-1=0$

AC: $x+y=0$

BC: $-x+y=0$

3. S(0,0) nuqtadan $2x+y-1=0$ to'g'ri chiziqqa perpendikulyar bo'lgan to'g'ri chiziq tenglamasini yozamiz.

AB to'g'ri chiziqning burchak koeffitsenti $k_1=-2$ ($y=-2x+1$dan) ga teng. Perpendikulyarlik shartidan $k_2=\frac{1}{2}$ ekanini topamiz.

$\Rightarrow y-y_C=k(x-x_C)$ dan qidirayotgan tenglamamiz $y=\frac{1}{2}x$ ekanini topamiz.

4.

$$d=\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}=\frac{\left|1\left(\frac{1}{3}\right)+1\left(\frac{1}{3}\right)+0\right|}{\sqrt{2}}=\frac{\sqrt{2}}{3}.$$

5. S nuqtaning koordinatalarini AB tomon tenglamasiga qo'ysak

$$2\cdot 0+0-1=-1<0$$

B nuqtaning koordinatalarini AC tomon tenglamasiga qo'ysak

$$\frac{1}{3}+\frac{1}{3}=\frac{2}{3}>0,$$

Demak, ABS uchburchakning ichki A burchak bissektrisasi AB to'g'ri chiziqdan tekislikning manfiy qismidan o'tadi va AC to'g'ri chiziqdan tekislikning musbat qismidan o'tadi ya'ni bissektrisa nuqtalari uchun $2x+y-1<0, \quad x+y>0$. Shuning uchun ichki A burchak bissektrisasining tenglamasi quyidagicha:

$$\frac{y+2x-1}{\sqrt{5}}=\pm\frac{y+x}{\sqrt{2}}.$$

Nazorat savollari.

1.To'g'ri chiziqning turli tenglamalari va ular orasidagi bog'lanish.
2. To'g'ri chiziqning burchak koeffitsienti.
3. Berilgan nuqtalardan o'tuvchi to'g'ri chiziq tenglamasi.
4. Ikki to'g'ri chiziq orasidagi burchak.
5. Nuqtadan to'g'ri chiziqqacha bo'lgan masofa.

3−§. Ikkinchi tartibli chiziqlar.

Ushbu

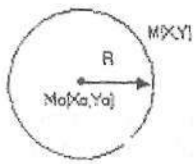
$$Ax^2+Bxy+Cy^2+Dx+Ey+C=0 \qquad (1)$$

$(1)-$tenglama yordamida aniqlanadigan chiziqqa **ikkinchi tartibli chiziq** deb ataladi. Ular jumlasiga matematikada muhim rol o'ynaydigan *aylana, ellips, giperbola va parabolalar* kiradi.

1. Aylana. Markaz deb ataladigan nuqtadan bir xil uzoqlikda joylashgan nuqtalarning geometrik o'rniga **aylana** deyiladi.

$M_0M=R \quad \forall M(x,y)$ uchun aylananing *kanonik tenglamasi* quyidagicha:

$$(x-x_0)^2+(y-y_0)^2=R^2 \quad (2)$$

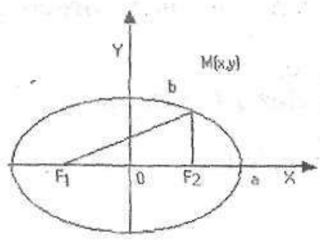


8 – chizma.

2. Ellips. Ixtiyoriy nuqtasidan fokus deb ataladigan ikkita qo'zg'almas F_1 va F_2 nuqtalargacha bo'lgan masofalar yig'indisi o'zgarmas $2a$ ga teng bo'lgan tekislik nuqtalarining geometrik o'rniga **ellips** deb ataladi.

Ellips tenglamasini keltirib chiqaramiz. Ellipsdan $\forall M(x,y)$ nuqta olamiz. Aytaylik, $F_1(-c;0)$ va $F_2(c;0)$ bo'lsin. Shartga ko'ra

$$MF_1+MF_2=2a$$



9 – chizma.

MF_1 va MF_2 larning o'rniga qo'ysak:

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a,$$

tenglamani hosil qilamiz. Tenglikni chap tomonidagi qo'shiluvchilarning birini o'ng tomonga o'tkazib kvadratga ko'taramiz. Hosil bo'lgan ifodani soddalashtirib, yana kvadratga ko'taramiz. Natijada,

$$(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2),$$

tenglik hosil bo'ladi.

$$2a > 2c \Rightarrow a^2 - c^2 > 0$$

ekanligidan

$$в^2 = a^2 - c^2$$

deb belgilay olamiz

$$в^2x^2 + a^2y^2 = a^2в^2.$$

Bundan,

$$\frac{x^2}{a^2} + \frac{y^2}{в^2} = 1 \quad (3)$$

tenglik kelib chiqadi.

(3) – tenglik ellipsning kanonik tenglamasini ifodalaydi.

$\varepsilon = \frac{c}{a}$ – ellipsning ekstsentrisiteti deyiladi, $0 \le \varepsilon < 1$.

Ekstsentrisitet ellipsning siqilish darajasini aniqlaydi. Ekstsentrisitet ta'rifidan va $b^2 = a^2 - c^2$ tenglikdan quyidagi kelib chiqadi:

$$\varepsilon^2 = \frac{c^2}{a^2} = \frac{a^2 - b^2}{a^2} = 1 - \left(\frac{b}{a}\right)^2 \quad \Rightarrow \frac{b}{a} = \sqrt{1 - \varepsilon^2}.$$

Bundan kelib chiqadiki, ε qancha katta bo'lsa $\frac{b}{a}$ nisbat shuncha kichik bo'ladi va ellips shuncha cho'zilgan bo'ladi.

Agar $\varepsilon = 0$ bo'lsa *ellips aylanaga* aylanadi.

$r_1 = a - ex$ va $r_2 = a + ex$ **ellipsning fokal radiuslari** deyiladi.

Yarim o'qlar a va b, fokuslar orasidagi masofa 2s ga teng.

1–misol. Agar yarim o'qlar yig'indisi 16 ga va fokuslar orasidagi masofa 8 ga teng bo'lsa, ellips tenglamasini yozing.

Yechilishi. $a + b = 16;$ $2c = 8 \Rightarrow c = 4$ ma'lumki $a^2 - b^2 = c^2$,

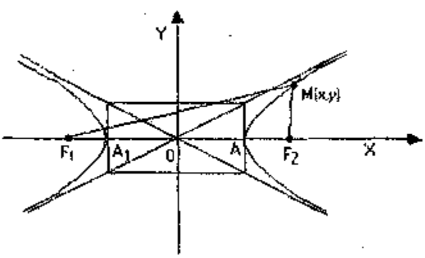
$$16(a - b) = 16 \Rightarrow a - b = 1.$$

$$\begin{cases} a + b = 16 \\ a - b = 1 \end{cases}$$

tenglamalar sistemasini yechamiz. $a = 8{,}5, \quad b = 7{,}5$.. Qidirilayotgan *ellips tenglamasi* quyidagicha yoziladi:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{x^2}{\frac{17^2}{4}}+\frac{y^2}{\frac{15^2}{4}}=1 \Rightarrow \frac{4x^2}{17^2}+\frac{4y^2}{15^2}=1.$$

3. Giperbola. Ixtiyoriy nuqtasidan fokuslar deb ataladigan ikkita qo'zg'almas F_1 va F_2 nuqtalargacha bo'lgan masofalar ayirmasining absolyut qiymati o'zgarmas $2a$ ga teng bo'lgan tekislik nuqtalarining geometrik o'rniga **giperbola** deyiladi.



10 – chizma.

Giperbola ta'rifidan foydalanib, so'ngra ellips kanonik tenglamasini keltirib chiqarishga o'xshash amallar bajarib,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad (4)$$

giperbolaning *kanonik tenglamasini* keltirib chiqarish mumkin, bu yerda $b^2 = c^2 - a^2$.

$y = \pm\frac{b}{a}x$ – giperbolaning asimptotalari.

$\varepsilon = \frac{c}{a}$ – giperbolaning ekstsentrisiteti, $\varepsilon > 1$.

$r_1 = ex - a$ va $r_2 = ex + a$ **giperbolaning fokal radiuslari** deyiladi.

Agar giperbolada $a = b$ bo'lsa, hosil bo'lgan

$$x^2 - y^2 = a^2$$

giperbolaga **teng tomonli giperbola** deb ataladi.

2−misol. Berilgan $9x^2-25y^2=225$ giperbolada a va b lar uzunligini, fokuslar koordinatalarini va ekstsentrisitetini toping.

Yechilishi:

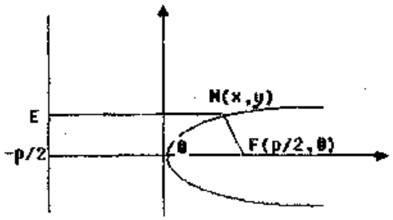
$$\frac{9x^2}{225}-\frac{25y^2}{225}=1\Rightarrow\frac{x^2}{25}-\frac{y^2}{9}=1\Rightarrow a=5,\quad b=3$$

ekanligi kelib chiqadi. Bundan foydalanib c ni topamiz:

$$c=\sqrt{5^2+3^2}=\sqrt{34}\Rightarrow\varepsilon=\frac{\sqrt{34}}{5}.$$

Demak, $F_2(\sqrt{34};0)$, $F_1(-\sqrt{34};0)$.

4. Parabola. Fokus deb ataladigan qo'zg'almas F nuqtadan va direktrisa deb ataladigan l to'g'ri chiziqdan teng uzoqlikda joylashgan tekislik nuqtalarining geometrik o'rniga **parabola** deyiladi.



11 – chizma.

Parabolaning ta'rifiga ko'ra uning ixtiyoriy M nuqtasidan F nuqtagacha va M nuqtadan l to'g'ri chiziqqacha masofalar teng ekan. F nuqtadan l to'g'ri chiziqqacha masofani p, x o'qini l to'g'ri chiziqqa perpendikulyar hamda F nuqtadan o'tadigan, y o'qni esa b to'g'ri chiziq va F nuqtaning o'rtasidan o'tadigan qilib olsak

$$\sqrt{\left(x+\frac{p}{2}\right)^2}=\sqrt{\left(x-\frac{p}{2}\right)^2+y^2}$$

tenglik hosil bo'ladi. Tenglikni ikkala tarafini kvadratga ko'tarib, ixchamlashtirganimizdan so'ng,

$$y^2 = 2px \qquad (5)$$

parabolaning kanonik tenglamasi hosil bo'ladi.

$F\left(\frac{p}{2};0\right)$ parabola fokusi;

$x = -\frac{p}{2}$ parabola direktrisasi;

$r = x + \frac{p}{2}$ parabolaning fokal radiusi deyiladi.

3–misol. $y^2 = 6x$ paraboladagi fokal radiusi 4,5 ga teng nuqtalarni toping.

Yechilishi. Berilgan parabola Ox o'qiga nisbatan simmetrik:

$$2p = 6 \Rightarrow \frac{p}{2} = 1{,}5 \qquad r = x + \frac{p}{2} \text{ dan } \quad 4{,}5 = x + 1{,}5; \quad x = 3.$$

$$y^2 = 6 \cdot 3 = 18 \Rightarrow y_{1,2} = \pm\sqrt{18} = \pm 3\sqrt{2}.$$

Demak, qidirilayotgan nuqtalar quyidagicha:

$$A(3;3\sqrt{2}), B(3;-3\sqrt{2})$$

Nazorat savollari.

1. Aylana. Misollar.
2. Ellips va uning elementlari.
3. Giperbola va uning elementlari.
4. Parabola va uning elementlari.

4–§. Ikkinchi tartibli egri chiziqlarning umumiy tenglamasi.

Biz yuqorida ikkinchi tartibli egri chiziqlardan aylana, ellips, giperbola, parabolalarni keltirdik va ularning sodda xossalarini o'rgandik.

Endi biz ikkinchi tartibli egri chiziqlarning umumiy tenglamasini keltiramiz:

$$a_{11}x^2+2a_{12}xy+a_{22}y^2+2a_{13}x+2a_{23}y+a_{33}=0 \quad (1)$$

1) Koordinata o'qlarini parallel ko'chirish;

2) Koordinata o'qlarini ma'lum α burchakka burish natijasida.

Ikkinchi tartibli egri chiziqlarning umumiy tenglamasini *kanonik ko'rinishga* keltirish mumkin.

Dekart koordinatalar sistemasining biridan ikkinchisiga o'tishda ikkinchi tartibli chiziqning tenglamasidagi koeffitsientlar umuman aytganda, o'zgaradi. Biroq chiziq tenglamasi koeffitsientlarining funktsiyalari bo'lgan ba'zi kattaliklar mavjud bo'lib, ularning qiymatlari bir dekart koordinatalar sistemasidan ikkinchisiga o'tganda o'zgarmaydi. Bunday kattaliklar **invariantlar** deyiladi.

Ushbu

$$I_1=a_{11}+a_{22},\quad I_2=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},\quad I_3=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

(bu yerda $a_{21}=a_{12},\ a_{31}=a_{13},\ a_{32}=a_{23}$) kattaliklar ikkinchi tartibli chiziq tenglamasining dekart koordinatalar sistemasi almashtirishlariga nisbatan invariantdir.

Barcha ikkinchi tartibli chiziqlar quyidagi uchta tipga ajraladi:

agar $I_2>0$ bo'lsa, chiziq elliptik tipda;

agar $I_2<0$ bo'lsa, chiziq giperbolik tipda;

agar $I_2=0$ bo'lsa, chiziq parabolik tipda bo'ladi.

Agar ikkinchi tartibli chiziqning tenglamasi koordinatalar sistemasini almashtirish yo'li bilan

$$\lambda_1 x^2 + \lambda_2 y^2 + m = 0 \quad (2)$$

ko'rinishga keltirilgan bo'lsa, chiziqning invariantlarini yozamiz:

$$I_1 = a_{11} + a_{22} = \lambda_1 + \lambda_2 \quad (3)$$

$$I_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} = \lambda_1 \lambda_2 \quad (4)$$

$$I_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} = \lambda_1 \lambda_2 m \quad (5)$$

(3) va (4) tengliklardan Viet teoremasiga muvofiq λ_1 va λ_2 lar

$$\lambda^2 - I_1 \lambda + I_2 = 0 \quad (6)$$

kvadrat tenglamaning ildizlari ekanligi kelib chiqadi.
(4) va (5) tengliklardan quyidagiga ega bo'lamiz:

$$m = \frac{I_3}{I_2} \quad (7)$$

Agar (1) tenglama

$$\lambda_1 x^2 + 2ky = 0 \quad (8)$$

ko'rinishga keltirilsa, u holda quyidagi ifodalar uning invariantlari bo'ladi:

$$I_1 = \lambda_1, \quad I_2 = 0, \quad I_3 = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & k \\ 0 & k & 0 \end{vmatrix} = -\lambda_1 k^2,$$

bundan

$$k = \pm\sqrt{-\frac{I_3}{I_1}} \quad (9)$$

Agar (1) tenglama

$$\lambda_1 x^2 + n = 0 \quad (10)$$

ko'rinishga keltirilsa, tenglama bir juft to'g'ri chiziqni (ustma-ust tushgan haqiqiy yoki mavhum parallel to'g'ri chiziqlarni) aniqlaydi. Bu holda invariantlar quyidagi ko'rinishda bo'ladi:

$$I_1 = \lambda_1, \quad I_2 = 0, \quad I_3 = 0.$$

1-misol. $x^2 - 8xy + 7y^2 - 18 = 0$ ikkinchi tartibli egri chiziq tenglamasini kanonik ko'rinishga keltiring.

Yechilishi:

$$I_1 = 8, \quad I_2 = \begin{vmatrix} 1 & -4 \\ -4 & 7 \end{vmatrix} = -9, \quad I_3 = \begin{vmatrix} 1 & -4 & 0 \\ -4 & 7 & 0 \\ 0 & 0 & 18 \end{vmatrix} = 162$$

$I_2 < 0$ bo'lgani uchun qaralayotgan tenglama giperbolik tipdagi chiziqni aniqlaydi.

$$\lambda^2 - 8\lambda - 9 = 0, \quad \lambda_1 = 9, \quad \lambda_2 = -1.$$

(7) formuladan foydalanib, topamiz.

$$m = \frac{I_3}{I_2} = -18$$

va giperbola tenglamasi quyidagicha bo'ladi:

$$9x^2 - y^2 - 18 = 0 \text{ yoki } \frac{x^2}{2} - \frac{y^2}{18} = 1$$

2-misol. $8x^2 + 4\sqrt{2}xy + y^2 + 6x - 12\sqrt{2}y = 0$ ikkinchi tartibli egri chiziq tenglamasini kanonik ko'rinishga keltiring.

Yechilishi:

$$I_1 = 9, \quad I_2 = 0, \qquad I_3 = \begin{vmatrix} 8 & 2\sqrt{2} & 3 \\ 2\sqrt{2} & 1 & -6\sqrt{2} \\ 3 & -6\sqrt{2} & 0 \end{vmatrix} = -729$$

$$k = \pm\sqrt{-\frac{I_3}{I_1}} = \pm\sqrt{\frac{729}{9}} = \pm 9$$

$I_2 = 0$ bo'lgani uchun qaralayotgan chiziq parabolik tipdagi chiziq bo'lib, uning tenglamasi quyidagicha:

$$9x^2 - 18y = 0 \quad \text{yoki} \quad 9x^2 + 18y = 0$$

bundan

$$x^2 = 2y \quad \text{yoki} \quad x^2 = -2y.$$

3-misol. $x^2 - 4xy + 4y^2 + 2x - 4y - 3 = 0$ ikkinchi tartibli egri chiziq tenglamasini kanonik ko'rinishga keltiring.

Yechilishi:

$$I_1 = 5, \quad I_2 = 0, \qquad I_3 = \begin{vmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & -3 \end{vmatrix} = 0$$

Bu holda $k = \pm\sqrt{-\frac{I_3}{I_1}} = 0$, demak, tenglamaning chap qismi chiziqli ko'paytuvchilarning ko'paytmasidan iborat.

$$x^2 - 2(2y-1)x + 4y^2 - 4y - 3 = 0$$

bundan

$$x_{1,2} = 2y - 1 \pm \sqrt{(2y-1)^2 - 4y^2 + 4y + 3},$$

$$x_1 = 2y + 1, \quad x_2 = 2y - 3,$$

yoki

$$(x - 2y - 1)(x - 2y + 3) = 0.$$

Shunday qilib, berilgan tenglama bilan aniqlangan chiziq bir juft parallel to'g'ri chiziqqa ajraladi:

$$x - 2y - 1 = 0 \quad \text{va} \quad x - 2y + 3 = 0$$

(1) tenglama koordinatalar sistemasini tanlash yo'li bilan, shu sistemada qaralayotgan quyidagi kanonik ko'rinishlardan bittasiga keltiriladi.

1) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (ellips)

2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$ (mavhum ellips)

3) $a^2x^2 + c^2y^2 = 0$ (ikki mavhum kesishuvchi chiziqlar)

4) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (giperbola)
5) $a^2x^2-c^2y^2=0$ (ikki kesishuvchi chiziqlar)
6) $y^2=2px$ (parabola)
7) $y^2-a^2=0$ (ikki parallel chiziqlar)
8) $y^2+a^2=0$ (ikki parallel mavhum chiziqlar)
9) $y^2=0$ (ikki o'zaro ustma – ust tushuvchi chiziqlar)

Nazorat savollari.

1) Ikkinchi tartibli egri chiziqlarning umumiy tenglamasi.
2) Invariantlik xossasi.
3) Ikkinchi tartibli chiziqlarning tiplari.
4) Ikkinchi tartibli egri chiziqlarni kanonik kurinishga keltirish. Misollar.

Mustaqil yechish uchun misollar va masalalar.

1–masala. ABC uchburchak uchlarining koordinatalari berilgan, quyidagilar talab qilinadi:

1. AB tomonning uzunligini hisoblash;
2. AB kesma o'rtalarining koordinatalarini topish:
3. AB tomonning tenglamasini tuzish;
4. C uchidan o'tkazilgan balandlikning tenglamasini tuzish;
5. B uchidan AC tomongacha bo'lgan masofani hisoblash;
6. Ichki A burchak bissektrisasining tenglamasini topish;

1.1.	A(–6;–4),	B(–10;–1),	C(6;1).
1.2.	A(–2;–4),	B(2;–8),	C(10;2).
1.3.	A(12;0),	B(1;8),	C(0;5).
1.4.	A(–10;–2),	B($\sqrt{2}$; $-\sqrt{7}$)	C($2\sqrt{2}$;0).
1.5.	A(–2;–6),	B(–6;–3),	C(10;–1).
1.6.	A(4;3),	B(7;6),	C(2;11).
1.7.	A(8;2),	B(14;10),	C(–4;7).
1.8.	A(2;–1),	B(4;2),	C(5;1).
1.9.	A(2;–4),	B(–2;–1),	C(14;1).

1.10. $A(1;-5)$,	$B(2;7)$,	$C(-4;11)$.
1.11. $A(2;-1)$,	$B(8;7)$,	$C(-10;4)$.
1.12. $A(4;11)$,	$B(-1;-1)$,	$C(5;7)$.
1.13. $A(5;-3)$,	$B(1;0)$,	$C(17;2)$.
1.14. $A(3;8)$,	$B(10;2)$,	$C(2;7)$.
1.15. $A(14;-6)$,	$B(20;2)$,	$C(2;-1)$.
1.16. $A(7;2)$,	$B(1;9)$,	$C(-8;-4)$.
1.17. $A(3;4)$,	$B(-1;7)$,	$C(15;9)$.
1.18. $A(5;-7)$,	$B(-4;-2)$,	$C(15;-1)$.
1.19. $A(1;-2)$,	$B(7;6)$,	$C(-11;3)$.
1.20. $A(-3;-3)$,	$B(-1;3)$,	$C(11;-1)$.

2-masala.

2.1. Agar parallelogrammning diagonallari $(-1;0)$ nuqtada kesishishi ma'lum bo'lsa, uning $x+y-1=0$ va $y+1=0$ tomonlarining kesishish nuqtasidan o'tmaydigan diagonalining tenglamasini toping.

2.2. $2x+u+11=0$ to'g'ri chiziqda berilgan ikki: $A(1;1)$ va $B(3;0)$ nuqtadan baravar uzoqlashgan nuqtani toping.

2.3. $(2;-4)$ nuqtaga $4x+3y+1=0$ tenglamaga nisbatan simmetrik bo'lgan nuqtaning koordinatalarini toping:

2.4. Uchlari $A(-1;1)$, $B(2;-1)$, $C(4;0)$ bo'lgan uchburchakka tashqi chizilgan aylana markazining koordinatalarini hisoblang;

2.5. $(2;6)$ nuqtadan o'tuvchi va koordinata o'qlari bilan ikkinchi chorakda joylashib 3 kv. birlik yuzaga ega bo'lgan uchburchak tashkil etuvchi to'g'ri chiziqning tenglamasini tuzing.

2.6. $A(-1;2)$ nuqtadan o'tuvchi to'g'ri chiziqlar tenglamasini shunday tuzingki, uning $x+2y+1=0$ va $x+2y-3=0$ parallel to'g'ri chiziqlar orasida joylashgan kesmasining bir uchi $x-y-6=0$ to'g'ri chiziqda yotsin.

2.7. Uchburchak ikkita tomonining tenglamalari berilgan: $4x-5y+9=0$ va $x+4y-3=0$. Agar bu uchburchakning meridianalari $(3;1)$ nuqtada kesishishi ma'lum bo'lsa, uchburchak tomonining tenglamasini toping.

2.8. Romb ikkita tomonining tenglamasi: $2x-y+4=0$ va $2x-y+10=0$, hamda diagonallaridan birining tenglamasi $x+y+2=0$ ma'lum bo'lsa, romb uchlarining koordinatalarini toping.

2.9. Agar $A(-5;5)$ va $B(3;1)$ – uchburchakning ikkita uchi; $M(2;5)$ esa uning balandliklarining kesishgan nuqtasi bo'lsa, uchburchak tomonlarining tenglamasini tuzing.

2.10. $x+3u-7=0$ kvadrat tomonlaridan birining tenglamasi va $C(0;-1)$ – bu kvadratning qolgan uchta tomonlarining kesishgan nuqtasi. Qolgan tomonlarning tenglamasini tuzing.

3–masala.

3.1. Elipsning ekstsentrisiteti 0,8 ga, uning nuqtalaridan birining fokal radiuslari 2 va 3 ga teng, ellipsning katta o'qi abtsissalar o'qi bilan, uning markazi esa koordinatalar boshi bilan mos keladi deb olib, shu ellipsning tenglamasini tuzing.

3.2. $9x^2-16y^2=144$ giperbolada shunday nuqtalarni topingki bu nuqtalar bilan giperbolaning chap fokusi orasidagi masofa ularning o'ng fokusigacha bo'lgan masofasidan ikki marta kichik bo'lsin.

3.3. Agar $y^2=2rx$ parabolaning $y=x$ to'g'ri chiziq bilan $x^2+y^2-6x=0$ aylananing kesishish nuqtalaridan o'tishi ma'lum bo'lsa, shu parabolaning parametrini va uning direktrisasi tenglamasini toping.

3.4. $\frac{x^2}{441}+\frac{y^2}{216}=1$ ellipsda shunday nuqtalarni topingki, ularda fokal radiuslar o'zaro perpendikulyar bo'lsin.

3.5. Giperbolaning fokuslari $F_1(\sqrt{7},0)$ va $F_2(-\sqrt{7},0)$ nuqtalarda joylashgan. Giperbola $A(2;0)$ nuqtadan o'tadi. Uning asimptotalarining tenglamasini va ular orasidagi burchakni toping.

3.6. $A(2;2)$ nuqtadan va abtsissalar o'qidan teng masofada joylashgan nuqtalar geometrik o'rnining tenglamasini tuzing.

3.7. Har biri $A(3;0)$ nuqtadan ordinata o'qlariga qaraganda ikki marta uzoqroq masofada joylashgan nuqtalar geometrik o'rnining tenglamasini tuzing.

3.8. Koordinata boshigacha bo'lgan masofalarning $3x+16=0$ to'g'ri chiziqqacha bo'lgan masofalariga nisbati 0,6 ga teng bo'lgan nuqtalar geometrik o'rnining tenglamasini tuzing.

3.9. Har biri $A(1;0)$ nuqtaga $B(-2;0)$ nuqtaga qaraganda ikki marta yaqin bo'lgan nuqtalar geometrik o'rnining tenglamasini tuzing.

3.10. Abtsissalar o'qiga urinuvchi va A(0;3)nuqtadan o'tuvchi aylana markazlari geometrik o'rinlarining tenglamasini tuzing.

3.11. Ellipsning ekstsentrisiteti 0,4 ga uning nuqtalaridan birining fokal radiuslari 4 va 6 ga teng ellipsning katta o'qi abstsissalar o'qi bilan, uning markazi esa koordinatalar boshi bilan mos keladi, deb olib, shu ellipsning tenglamasini tuzing.

3.12. $36x^2-64y^2=2304$ giperbolada shunday nuqtalarni topingki, bu nuqtalar bilan giperbolaning chap fokusi orasidagi masofa ularning o'ng fokusigacha bo'lgan masofalaridan ikki marta kichik bo'lsin.

3.13. $9x^2-4y^2=36$ giperbola asimptotalari va $9x-2y-24=0$ to'g'ri chiziqlardan hosil bo'lgan uchburchak yuzi topilsin.

3.14. Asimptotasi $y=\pm\frac{3}{4}x$ va (2;1) nuqtadan o'tgan giperbola tenglamasini yozing.

3.15. Fokus $4x-3y-4=0$ to'g'ri chiziq va OX o'qi bilan kesishish nuqtasida yotgan parabola tenglamasi tuzilsin.

3.16. $4x+3y+10=0$ to'g'ri chiziqdan 2 birlik o'zoqlikda yotgan $y^2 = 32x$ parabolaga tegishli nuqta topilsin.

3.17. $\frac{x^2}{16}-\frac{y^2}{9}=1$ giperbolaning $x^2+y^2=91$ aylana bilan kesishish nuqtasida yotgan fokal radius topilsin.

3.18. M(9;8) nuqtadan o'tgan, $y=\pm(\frac{2\sqrt{2}}{3})x$ asimptotaga ega bo'lgan giperbola tenglamasi tuzilsin.

3.19. $M(\sqrt{3};\sqrt{2})$ nuqtadan o'tuvchi, ekstsentrisiteti giperbolaning $\sqrt{2}$ ga teng bo'lgan giperbola tenglamasi tuzilsin.

3.20. $\frac{x^2}{16}-\frac{y^2}{9}=1$ parabolaning o'ng tarmog'idan shunday nuqta topingki uning o'ng fokusigacha bo'lgan masofa chap fokusigacha bo'lgan masofadan 2 marta kichik bo'lsin.

4–masala. Berilgan parabola va giperbola tenglamalarini kanonik ko'rinishga keltiring.

4.1. a) $y=3x^2-12x+11;$ b) $y=\frac{4x+1}{x-1}.$

4.2. a) $y=-2x^2+8x-9;$ b) $y=\frac{2x+3}{2x-1}.$

4.3. a) $y=\frac{1}{3}x^2+2x+5;$ b) $y=\frac{-4x+5}{6x-3}.$

4.4. a) $y=3x^2-18x+25;$ b) $y=\frac{4x+5}{2x+1}.$

4.5. a) $y=-2x^2-4x;$ b) $y=\frac{3x-3}{2x+5}.$

4.6. a) $y=-2x^2+12x+7;$ b) $y=\frac{3x+1}{2x-5}.$

4.7. a) $y=5x^2+10x+2;$ b) $y=\frac{8x+2}{2x+3}.$

4.8. a) $y=-x^2+2x+3;$ b) $y=\frac{2x+5}{2x-7}.$

4.9. a) $y=3x^2+6x+5;$ b) $y=\frac{3x-7}{x-2}.$

4.10. a) $y=-x^2+2x-2;$ b) $y=\frac{-2x+5}{x-4}.$

5–masala. Ikkinchi tartibli egri chiziq tenglamasini kanonik ko'rinishiga keltiring.

5.1 $5x^2+4xy+2p^2=18$

5.2 $4x^2+2\sqrt{6}xy+3y^2=24$

5.3 $6x^2+2\sqrt{5}xy+2y^2=21$

5.4 $5x^2+4\sqrt{2}xy+3y^2=14$

5.5 $7x^2+2\sqrt{18}xy+4y^2=15$

5.6 $3x^2+2\sqrt{14}xy+8y^2=10$

5.7 $7x^2+2\sqrt{6}xy+2y^2=24$

5.8 $9x^2+4\sqrt{2}xy+2y^2=20$

5.9 $6x^2+2\sqrt{10}xy+3y^2=16$

5.10 $4x^2+4\sqrt{3}xy+5y^2=40$

5.11 $5x^2+4xy+8y^2-32x-56y+80=0$

5.12 $5x^2+12xy-22x-12y-19=0$

5.13 $x^2-4xy+4y^2+4x-3y-7=0$

5.14 $x^2-5xy+4y^2+x+2y-2=0$

5.15 $4x^2-12xy+9y^2-2x+3y-2=0$

5.16 $9x^2-4xy+6y^2+16x-8y-2=0$

5.17 $8x^2+6xy-26x-12y+11=0$

5.18 $x^2-2xy+y^2-10x-6y+25=0$

5.19 $2x^2-5xy-12y^2-x+26y-10=0$

5.20. $4x^2-4xy+y^2-6x-3y-4=0$

IV BOB.
FUNKTSIYA TUSHUNCHASI VA GRAFIGI.

1–§. Funktsiya tushunchasi. Misollar. Murakkab funktsiya. Funktsiyaning aniqlanish sohasi va qiymatlar sohasi.

1. Funktsiya tushunchasi. *Aytaylik,* $X \subset R$ *va* $Y \subset R$ *to'plamlar berilgan bo'lib, x o'zgaruvchi X da, y o'zgaruvchi Y to'plamda o'zgarsin* $x \in X$, $u \in Y$.

Agar har bir $x \in X$ songa biror f qoidaga ko'ra bitta $y \in Y$ son mos qo'yilsa, X to'plamda funktsiya berilgan deyiladi. Uni

$$f : X \to Y \text{ yoki } y = f(x)$$

kabi belgilanadi.

X to'plam funktsiyaning **aniqlanish to'plami (sohasi)**, Y esa funktsiyaning **o'zgarish to'plami (sohasi)**, **x–erkli o'zgaruvchi** yoki **funktsiya argumenti**, **u–erksiz o'zgaruvchi** yoki **x o'zgaruvchining funktsiyasi** deyiladi.

Ushbu $Y_f = \{y | \quad y = f(x), x \in X\}$ **to'plam funktsiyaning qiymatlari to'plami** deyiladi.

Odatda, $y = f(x)$ funktsiyaning aniqlanish sohasi D(f), qiymatlari to'plami esa $E(f)$ kabi belgilanadi.

Agar $f(x)$ va $g(x)$ funktsiyalar berilgan bo'lib,
1) $D(f) = D(g)$,
2) $\forall x \in D(f)$ uchun $f(x) = g(x)$ shartlar bajarilsa, unda $f(x)$ va $g(x)$ funktsiyalar D to'plamda teng deyiladi.

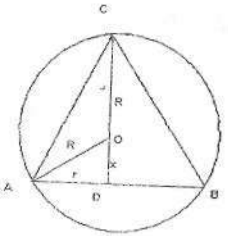
Masalan, $f(x) = \sqrt{x^2}$, $g(x) = x$ va $\varphi(x) = |x|$ bo'lsin. Bu funktsiyalar uchun $D(f) = D(g) = D(\varphi) = R$, lekin f funktsiya g bilan $A = [0, +\infty)$ to'plamdagina ustma–ust tushadi, φ bilan esa butun R da teng.

Aytaylik, $f : X \to Y$, $g : Y \to Z$ funktsiyalari berilgan bo'lsin. Ushbu $z = \varphi(x) = g(f(x))$, $x \in X$ munosabat bilan aniqlangan. $\varphi : X \to Z$ funktsiya **murakkab funktsiya** yoki f va g funktsiyalarning **superpozettsiyasi** deyiladi.

2. Misollar.

1–misol. Radiusi R ga teng bo'lgan sharga ichki chizilgan konusning o'q kesimining hajmi uning uchidagi burchak funktsiyasi ko'rinishida ifodalansin.

Yechilishi. R radiusli shar olib, unga ichki konus chizamiz. Konusning o'q kesimini olamiz (1 – chizma).



1 – chizma.

Aytaylik, $\angle ACB = \alpha$, $AD = r$, $CD = H$ bo'lsin.
Ma'lumki,

$$V = \frac{1}{3}\pi r^2 \cdot H$$

formula o'rinli. Masalani yechish uchun r va H larni R va α lar yordamida ifodalashimiz kerak.

$OD = x$ desak, bundan kelib chiqadiki $H = R + x$.

$$\Delta AOC \Rightarrow AO = OC = R \Rightarrow \angle ACO = \angle CAO = \frac{\alpha}{2}.$$

$\Rightarrow \angle AOC = \pi - \left(\frac{\alpha}{2} + \frac{\alpha}{2}\right) = \pi - \alpha. \Rightarrow \angle AOD = \alpha.$ Unda ΔAOD dan qo'yidagilarni topamiz:

$$x = R\cos\alpha, \quad r = R\sin\alpha \Rightarrow H = R + x = R\cdot(1+\cos\alpha).$$

Demak,

$$V = \frac{1}{3}\pi R^3 \sin^2\alpha \cdot (1+\cos\alpha).$$

2–misol. Parashyutchi a sek. davomida erkin tushib, keyin parashyutni ochdi va b sek. davomida o'zgarmas v_0

tezlik bilan tushdi. Parashyutchining t vaqt davomida bosib o'tgan yo'li $S(t)$ topilsin.

Yechilishi. Bu masalada har xil oraliqda funktsiya ham turli analitik ifodalar yordamida beriladi. Haqiqatan ham, agar $0 \le t \le a$ bo'lsa, maktab fizika kursidan ma'lumki,

$$S(t)=\frac{g\cdot t^2}{2}$$

bo'ladi. Bu yerda g – erkin tushish tezlanishi.

Agar $a \le t \le a+b$ bo'lsa,

$$S(t)=\frac{g\cdot a^2}{2}+v_0\cdot(t-a)$$

ekanini ko'rish qiyin emas. Demak,

$$S(t)=\begin{cases}\dfrac{g\cdot t^2}{2}, & 0\le t\le a\\[2ex] \dfrac{g\cdot a^2}{2}+v_0\cdot(t-a), & a<t\le a+b\end{cases}$$

ekan.

3. Funktsiyaning aniqlanish sohasi va qiymatlar to'plamini topish.

3–misol. Agar $f=x^2$ va $\varphi=2^x$ bo'lsa, $f[f(x)]$, $\varphi[\varphi(x)]$, $f[\varphi(x)]$, $\varphi[f(x)]$ murakkab funktsiyalar topilsin.

Yechilishi:

$$f[f(x)]=[f(x)]^2=(x^2)^2=x^4,$$

$$\varphi[\varphi(x)]=2^{\varphi(x)}=2^{2^x}$$

$$f[\varphi(x)]=[\varphi(x)]^2=(2^x)^2=2^{2x}$$

$$\varphi[f(x)]=2^{f(x)}=2^{x^2}$$

bo'ladi.

4-misol. Ushbu $y=\frac{1}{x^2-5x+6}$ funktsiyaning aniqlanish sohasi topilsin.

Yechilishi:

$$D(f)=R\setminus\{x\in R|x^2-5x+6=0\}=R\setminus\{2;3\}=(-\infty;2)\cup(2;3)\cup(3;+\infty)$$

5-misol. Agar $f(x)=\frac{1}{1-x}$ bo'lsa ,

1) $\varphi(x)=f(f(x))$,

2) $\varphi(x)=f(f(f(x)))$ funktsiyalarning aniqlanish sohasi topilsin.

Yechilishi:

1) $\varphi(x)=f(f(x))=\frac{1}{1-f(x)}=\frac{1}{1-\left(\frac{1}{1-x}\right)}=x-\frac{1}{x}$ bo'lib, uning aniqlanish sohasi

$$D(\varphi)=R\setminus\{0;1\}=(-\infty;0)\cup(0;1)\cup(1;+\infty)$$

bo'ladi.

2) $\varphi(x)=f(f(f(x)))=\frac{1}{1-f(f(x))}=\frac{1}{1-\left(\frac{1}{1-x}\right)}=x$ bo'lib, bu funktsiyaning ham aniqlanish sohasi $D(\varphi)=(-\infty;0)\cup(0;1)\cup(1;+\infty)$ bo'ladi.

6-misol. Agar $f(\frac{x}{x+1})=x^2$ bo'lsa, f(x) topilsin.

Yechilishi: Berilgan munosabatda $\frac{x}{x+1}=t$ deb belgilaymiz. Unda

$\frac{x}{x+1}=t\Rightarrow x=tx+t\Rightarrow x=\frac{t}{1-t}$ bo'lib $f(t)=\left(\frac{t}{1-t}\right)^2$, ya'ni $f(x)=\left(\frac{x}{1-x}\right)^2$ bo'ladi.

7-misol. Ushbu funktsiyalar aynan tengmi?

1) $f(x)=\lg x^2$ bilan $\varphi(x)=2\lg|x|$,

2) $f(x)=\lg x^2$ bilan $\varphi(x)=2\lg x$,

3) $f(x)=\left(\frac{2x^2}{x^2}\right)$ bilan $\varphi(x)=2$

Yechilishi:

1) $f(x)$ va $\varphi(x)$ funktsiyalarning aniqlanish sohasi bir xil: $D(f)=D(\varphi)=R\setminus\{0\}$ bo'lib, $\forall x\in D(f)$da $f(x)=\lg x^2=2\lg|x|=\varphi(x)$ bo'ladi.

2) $\varphi(x)=2\lg x$ funktsiyaning aniqlanish sohasi esa $D(\varphi)=(0;+\infty)$ bo'ladi. Demak, $D(f)\neq D(\varphi)$. $f(x)$ va $\varphi(x)$ funktsiyalar aynan teng emas.

3) $f(x)=\left(\frac{2x^2}{x^2}\right)$ funktsiyaning aniqlanish sohasi $D(f)=(-\infty;0)\cup(0;+\infty)$ bo'lib, $\varphi(x)=2$ funktsiyaning aniqlanish soxasi $D(f)=(-\infty;+\infty)$ bo'ladi. Bu to'plamlar bir–biriga teng emas. Binobarin, $f(x)$ va $\varphi(x)$ funktsiyalar aynan teng emas.

8–misol. Ushbu $f(x)=\frac{1+\sin x}{\sin x}$ funktsiyaning aniqlanish sohasi D(f) va qiymatlari to'plami Ye(f)lar topilsin.

Yechilishi:

Ravshanki, $D(f)=\{x\in R|\sin x\neq 0\}=\{x\in R|x\neq \pi k, k\in Z\}$ bo'ladi. Ye(f) ni topish uchun $f(x)=\frac{1+\sin x}{\sin x}=1+\frac{1}{\sin x}$ deb olib, $-1\leq \sin x\leq 1$ foydalanamiz. Demak, $\frac{1}{\sin x}\in(-\infty;-1)\cup[1;+\infty)\Rightarrow E(f)=(-\infty;0)\cup[2;+\infty)$

Mustaqil yechish uchun misol va masalalar.

1. Radiusi R ga teng bo'lgan doiraga ichki chizilgan to'g'ri to'rtburchakning yuzi uning asosining funktsiyasi ko'rinishida ifodalansin.

J: $S(x)=x\cdot\sqrt{4R^2-x^2},\quad 0<x<2R.$

2. Radiusi R ga teng bo'lgan sharga ichki chizilgan konusning hajmi konus balandligining funktsiyasi ko'rinishida ifodalansin.

J: $V=\frac{\pi H^2\cdot(2R-H)}{3},\quad 0<H<2R.$

3. Radiusi R ga teng bo'lgan sharga ichki chizilgan konusning hajmi konus asosi radiusining funktsiyasi ko'rinishida ifodalansin.

J: $V = \frac{\pi r^2 \cdot \left(R + \sqrt{R^2 - r^2}\right)}{3}, \quad 0 < r < R.$

Modul belgisini ishlatmay qo'yidagi funktsiya analitik ko'rinishda yozilsin.

4. a) $y = |x^2 - 5x + 6|$, b) $y = |\lg x|$.

J: a) $y = \begin{cases} x^2 - 5x + 6, & -\infty < x \le 2, \\ -x^2 + 5x - 6, & 2 < x \le 3 \\ x^2 - 5x + 6, & 3 < x < +\infty \end{cases}$ b) $y = \begin{cases} -\lg x, & 0 < x \le 1, \\ \lg x, & 1 < x < +\infty \end{cases}$

5. a) $y = \frac{1}{2}(x + |x|)$, b) $y = \frac{1}{2}(x - |x|)$,

J: a) $y = \begin{cases} x, & x \ge 0, \\ 0, & x < 0. \end{cases}$ b) $y = \begin{cases} 0, & x \ge 0, \\ x, & x < 0. \end{cases}$

Quyidagi funktsiyalarning aniqlanish sohalarini toping.

6. a) $f(x) = \frac{3x - 1}{x^2 - 3x + 2}$, b) $f(x) = \arccos \frac{2x}{1 + x^2}$

J: a) $D(f) = (-\infty;1) \cup (1;2) \cup (2;+\infty,)$, b) $D(f) = (-\infty;+\infty)$.

7. a) $f(x) = \lg(3\sin^2 x - 4)$, b) $f(x) = \log_2 |4 - x^2|$.

J: a) $D(f) = \varnothing$, b) $D(f) = (-\infty;2) \cup (-2;2) \cup (2;+\infty)$.

8. Agar f(x) funktsiyaning aniqlanish sohasi $[-1; 0]$ kesma bo'lsa, quyidagi funktsiyalarning aniqlanish sohasi topilsin.

a) $f(-x^2)$, b) $f(\cos x)$.

J: a) $[-1 ;1]$, b) $\left(2k + \frac{1}{2}\right)\pi \le x \le \left(2k + \frac{3}{2}\right)\pi, k \in Z.$

9. Qo'yidagi funktsiyalarning qiymatlar to'plami topilsin.

a) $f(x) = x + \frac{1}{x}(x \in (0;+\infty))$, b) $f(x) = 2^{x^2 + 4x - 5}$

J: a) $E(f) = [2^{-9};+\infty)$ b) $E(f) = [2;+\infty)$,

2–§. Funktsiya xossalarini tekshirish. Teskari funktsiya.

1⁰. Agar $\forall x \in X \subset R$ uchun $-x \in X$ bo'lsa, X to'plam O nuqtaga nisbatan **simmetrik to'plam** deyiladi. Bunday to'plamda $f(x)$ funktsiya aniqlangan bo'lsin.

Agar $\forall x \in X \subset R$ *uchun* $f(-x) = f(x)$ *bo'lsa, juft;* $f(-x) = -f(x)$ *bo'lsa, toq funktsiya deyiladi.*

Endi davriy funktsiya tushunchasini keltiramiz. Aytaylik, $f(x)$ funktsiya $X \subset R$ to'plamda berilgan bo'lsin.

Agar, $\exists T \neq 0 (T \in R)$, $\forall x \in X$ uchun $x - T \in X$, $x + T \in X$, $f(x+T) = f(x)$ shartlar bajarilsa, $f(x)$ **davriy funktsiya** deyiladi, T esa **funktsiyaning davri** deyiladi.

Agar $\forall x_1, x_2 \in X$ *uchun* $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ $(x_1 < x_2 \Rightarrow f(x_1) < f(x_2))$ *bo'lsa,* $f(x)$ *funktsiya X to'plamda o'suvchi (qatiy o'suvchi),* $(x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2))(x_1 < x_2 \Rightarrow f(x_1) > f(x_2))$ *bo'lsa,* $f(x)$ *funktsiya X to'plamda kamayuvchi (qatiy kamayuvchi) deyiladi.*

O'suvchi va kamayuvchi funktsiyalar umumiy nom bilan **monoton funktsiyalar** deyiladi. $y = f(x)$ funktsiya $X \subset R$ to'plamda berilgan bo'lib, uning qiymatlari to'plami $E(f)$ bo'lsin. Agar har bir $y \in E(f)$ ga $y = f(x)$ munosabatni qanoatlantiruvchi bitta $x \in X$ son mos qo'yilsa, unda yuzaga kelgan funktsiya $y = f(x)$ga **teskari funktsiya** deyiladi va u $x = f^{-1}(y)$ kabi yoziladi.

2⁰. Ma'lumki, asosiy elementar funktsiyalarga darajali funktsiya, ko'rsatkichli funktsiya, logarifmik funktsiya, trigonometrik va teskari trigonometrik funktsiyalar kiritiladi. Asosiy elementar funktsiyalarning aniqlanish sohasi, qiymatlar to'plami va ularning xossalari maktab matematika kursidan ma'lum.

Asosiy elementar funktsiyalar ustida arifmetik amallar bajarish yordamida hosil qilinib, bitta formula yordamida beriladigan funktsiyalarga **elementar funktsiyalar** deyiladi.

Masalan: 1) $f(x) = x^2 - \cos x$ 2) $f(x) = \lg\left(x + \sqrt{1+x^2}\right)$ 3) $f(x) = (x-1)^2 \cdot \sin^2 x$ funktsiyalar elementar funktsiyalarga misol bo'ladi.

Elementar funktsiyalarning superpozitsiyasi ko'rinishida ifodalab bo'lmaydigan funktsiyaga **elementar bo'lmagan**

funktsiyalar deyiladi. Masalan, bir nechta formula yordamida aniqlangan ushbu

$$y=\begin{cases} x^3, & x \le 0 \\ x+2, & x>0 \end{cases}$$

funktsiya elementar bo'lmagan funktsiyaga misol bo'ladi.

3⁰. 1–misol. Ushbu $a)\ f(x)=\frac{a^x+a^{-x}}{2}$, $b)\ f(x)=\lg\frac{1+x}{1-x}$ funktsiyalar juft yoki toqlikka tekshirilsin.

$a)\ f(-x)=\frac{a^{-x}+a^{-(-x)}}{2}=\frac{a^{-x}+a^x}{2}=f(x)$. Demak, $f(x)=\frac{a^x+a^{-x}}{2}$ funktsiya juft.

$b)\ f(-x)=\lg\frac{1+(-x)}{1-(-x)}=\lg\frac{1-x}{1+x}=\lg\left(\frac{1+x}{1-x}\right)^{-1}=-\lg\frac{1+x}{1-x}$.

Demak, $f(x)=\lg\frac{1+x}{1-x}$ funktsiya toq.

2–misol. Simmetrik to'plamda aniqlangan har qanday funktsiya juft va toq funktsiyalar yig'indisi ko'rinishida ifodalanishi isbotlansin.

Aytaylik, $f(x)$ funktsiya berilgan bo'lsin. Agar

$$f(x)=\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}=\varphi(x)+\psi(x)$$

deb olsak, $\varphi(x)=\frac{f(x)+f(-x)}{2}$ funktsiya juft, $\psi(x)=\frac{f(x)-f(-x)}{2}$ funktsiyaning esa toq bo'lishini ko'rish qiyin emas.

3–misol. Ushbu, $f(x)=A\sin(\alpha x+\beta)$ funktsiyaning davri topilsin, bunda A, α, β – o'zgarmas sonlar.

Aytaylik, $f(x)$ funktsiyaning davri T bo'lsin $(T\neq 0)$. Unda $f(x+T)=f(x)$, yani $A\sin[\alpha(x+T)+\beta]\equiv A\sin(\alpha x+\beta)$ bo'ladi. Keyingi tenglikdan topamiz:

$$\sin\frac{\alpha T}{2}\cdot\cos\left(\alpha x+\beta+\frac{\alpha T}{2}\right)\equiv 0$$

bu tenglikdan $\sin\frac{\alpha T}{2}=0$ bo'lishi kelib chikadi. Demak, $\frac{\alpha T}{2}=\pi n$, yani $T=\frac{2\pi n}{\alpha}$ bo'ladi, bunda $n\in Z$. Berilgan funktsiyaning eng kichik musbat davri $T_0=\frac{2\pi}{\alpha}$ bo'ladi.

4–misol. Ushbu $f(x)=\frac{x}{1+x^2}$ funktsiya monotonlikka tekshirilsin.

Ravshanki, bu funktsiyaning aniqlanish sohasi $D(f)=(-\infty;+\infty)$ bo'ladi.

$\forall x_1, x_2\in R,\quad x_1<x_2$ bo'lsin. Unda

$$f(x_2)-f(x_1)=\frac{x_2}{1+x_2^2}-\frac{x_1}{1+x_1^2}=\frac{(x_2-x_1)(1-x_1\bullet x_2)}{(1+x_1^2)(1+x_2^2)}$$

bo'ladi. Bu ayirmaning ishorasi $1-x_1x_2$ ning ishorasiga bog'liq bo'ladi.

Agar $x_1,x_2\in(-1\ ;\ 1)$ bo'lsa, $1-x_1\bullet x_2>0$, $x_1x_2\in(-\infty;-1)\cup(1;+\infty)$ bo'lsa $1-x_1x_2<0$ bo'ladi. Binobarin, $\forall x_1x_2\in(-1;1)$ da

$$x_1<x_2\Rightarrow f(x_2)-f(x_1)>0\Rightarrow f(x_1)<f(x_2)$$

bo'lib, berilgan funktsiya $(-1\ ;\ 1)$ oraliqda **o'suvchi** ↑ bo'ladi. $\forall x_1,x_2\in(-\infty;-1)\cup(1;+\infty)$ da $x_1<x_2\Rightarrow f(x_2)-f(x_1)>0\Rightarrow f(x_1)>f(x_2)$ bo'lib, berilgan funtsiya $(-\infty;-1)\cup(1;+\infty)$ to'plamda **kamayuvchi** ↓ bo'ladi.

5–misol. Ushbu

$$y=\frac{1-x}{1+x}\qquad (x\neq 1)$$

funktsiyaga teskari funktsiya topilsin. Ravshanki,

$$y=\frac{1-x}{1+x}\Rightarrow\quad y+yx=1-x\quad\Rightarrow(1+y)x=1-y\quad\Rightarrow x=\frac{1-y}{1+y}\quad(y\neq-1).$$

Demak, berilgan funktsiyalarga teskari funktsiya $y=\frac{1-x}{1+x}$ $(x\neq-1)$ bo'ladi.

Mustaqil yechish uchun misol va masalalar.

1. Quyidagi funktsiyalar juft yoki toqlikka tekshirilsin.

a) $f(x)=x^2-\cos x$

b) $f(x)=\lg\left(x+\sqrt{1+x^2}\right)$

v) $f(x)=(x-1)^2\cdot\sin^2 x$

J: a) juft; b) toq; v)juft ham, toq ham emas.

Quyidagi funktsiyalar davriylikka tekshirilsin, davriy bo'lsa, eng kichik musbat davri topilsin.

2. a) $f(x)=x^2+x-1$, b) $f(x)=3$.

J: a) Davriy emas; b) $f(x+T)=f(x)$ tenglik $\forall T$ uchun bajariladi, lekin eng kichik musbat davr mavjud emas.

3. a) $f(x)=\sqrt{\sin 3x}$ b) $f(x)=\sin^2 x$

J : a) davriy, $T_0=\frac{2\pi}{3}$; b) davriy, $T_0=\pi$

4. a) $f(x)=\cos x^2$, b) $f(x)=\{x\}=x-[x]$

J: a) davriy emas; b)davriy, $T_0=1$

5. a) $f(x)=\sin 2\pi x$, b) $f(x)=\sin^4 x+\cos^4 x$.

J: a) davriy, $T_0=1$; b)davriy , $T_0=\frac{\pi}{2}$

6. Agar $f(x)$ davriy funktsiya bo'lib, uning davri T (T≠0) bo'lsa, $y=f(ax+b)$ $(a\neq 0)$ davriy funktsiya bo'lib, uning davri $\frac{T}{a}$ bo'lishi isbotlansin.

7. Ushbu $f(x)=x-\sin x$ funktsiya $\left(0;\frac{\pi}{2}\right)$ intervalda o'suvchi bo'lishi isbotlansin.

Ko'rsatma: $\forall$ $x_1x_2\in\left(0;\frac{\pi}{2}\right)$ va $x_1 < x_2$ uchun $\sin x_1<\sin x_2$ tengsizlikning bajarilishidan foydalaning.

8. a) $f(x)=\sin 4x+5\cos 6x$, b) $f(x)=\sin^3 x+\cos^3 x$.

J: a) Davriy, $T_0=\pi$; b) Davriy, $T_0=2\pi$.

9. a) $f(x)=\cos\pi x$; b) $f(x)=\left|\sin\left(\sqrt{2}x+1\right)\right|$.

J: a) Davriy, $T_0=2$; b) Davriy, $T_0=\frac{\pi}{\sqrt{2}}$.

10. Davriy bo'lmagan shunday $f(x)$ va $g(x)$ funktsiyalarga shunday misollar keltirilsinki,

1) $f(x)+g(x)$,

2) $f(x)\bullet g(x)$

funktsiyalar davriy bo'lsin.

11. Shunday davriy $f(x)$ va davriy bo'lmagan $g(x)$ funktsiyalarga misollar keltirilsinki, $f(x)+g(x)$, $f(x)\bullet g(x)$ funktsiyalar davriy bo'lsin.

12. Ushbu $f(x)=\left(x^2-1\right)^2$ funktsiyalarning $(-1\ ;0)$ intervalda o'suvchi, $(0;1)$ intervalda esa kamayuvchi ekanligi isbotlansin.

Quyidagi funktsiyalar monotonlikka tekshirilsin.

13. a) $f(x)=\dfrac{x^2}{x^2+1}$, b) $f(x)=\dfrac{x^2}{x^2-1}$.

J: a) $(-\infty;0)$ da $\downarrow$, $(0;+\infty)$ da $\uparrow$; b) $(-\infty;-1)$ va $(-1;0)$ da $\uparrow$, $(0;1)$ va $(1;+\infty)$ da $\downarrow$.

14. a) $f(x)=\dfrac{x-1}{|x|+1}$; b) $f(x)=1-\dfrac{|x-1|}{1+|x|}$.

J: a) $(-\infty;0)$ da o'zgarmas, $(0;+\infty)$ da $\uparrow$; b) $(-\infty;1)$ da $\uparrow$, $(1;+\infty)$ da$\downarrow$.

15. a) $f(x)=\sin^4 x+\cos^4 x$, $x\in[0\ ;\pi]$,

b) $f(x)=\dfrac{2-\sin x}{2+\sin x}$, $x\in[0\ ;2\pi]$.

J: a) $\left[0;\dfrac{\pi}{4}\right]$ va $\left[\dfrac{\pi}{2};\dfrac{3\pi}{4}\right]$ da $\downarrow$, $\left[\dfrac{\pi}{4};\dfrac{\pi}{2}\right]$ va $\left[\dfrac{3\pi}{4};\pi\right]$ da $\uparrow$.

b) $\left[0;\dfrac{\pi}{2}\right]$ va $\left[\dfrac{3\pi}{2};2\pi\right]$ da $\downarrow$, $\left[\dfrac{\pi}{2};\dfrac{3\pi}{2}\right]$ da $\uparrow$.

16. a) $f(x)=[x]$, b) $f(x)=|x|-x$

J: a) o'suvchi, b) $x\leq0$ da kamayuvchi, $x>0$ da o'zgarmas 0 ga teng.

17. a $f(x)=x^4+6x^2+1$), b) $f(x)=\dfrac{1}{x^2+4x+5}$.

J: a) $x>0$ da o'suvchi, $x<0$ da kamayuvchi. b)$x<-2$ da o'suvchi, $x>-2$ da kamayuvchi.

18. Quyidagi funktsiyalarning o'zaro teskari funktsiyalar ekanligi isbotlansin:

a) $f(x)=\dfrac{x}{x+1}$, $g(x)=\dfrac{x}{1-x}$

b) $f(x)=1-\sqrt[3]{x}$, $g(x)=(1-x)^3$

v) $f(x)=\sqrt[3]{x^5}$, $g(x)=\sqrt[5]{x^3}$

19. Aytaylik, $y=f(x)$ va $y=g(x)$ funktsiyalar $M\subset R$ to'plamda aniqlangan bo'lib, $y=f(x)>0$ kamayuvchi va $y=g(x)<0$ esa o'suvchi bo'lsin. Ushbu funktsiyalar M to'plamda monotonlikka tekshirilsin.

a) $f(x)+g(x)$; b) $f(x)+4g(x)$; v) $y=f^2(x)$; g) $y=g^2(x)$.

J: a) o'suvchi, b) kamayuvchi, v) kamayuvchi, g) kamayuvchi.

Quyidagi funktsiyalarga teskari bulgan funktsiyalar topilsin.

20. a) $f(x)=2x-x^2$, $x\geq 1$: b) $f(x)=2x-x^2$, $x\leq 1$:

v) $f(x)=\dfrac{2x}{1+x^2}$, $(|x|\leq 1)$

J: a) $f=1+\sqrt{1-x}$, $x\leq 1$; b) $f=1-\sqrt{1-x}$, $x\leq 1$;

v) $f(x)=1+\dfrac{\sqrt{1-x^2}}{x}$, $0<x\leq 1$.

21. a) $y=\lg(x-1)$, b) $y=\sqrt[3]{1-x^3}$.

J: a) $y=10^x+1$; b) $y=\sqrt[3]{1-x^3}$.

22. $y=\dfrac{2x}{1-x^2}$, $x<-1$.

J: $y=\dfrac{1+\sqrt{1-x^2}}{x}$, $-1\leq x<0$

23. $y=x|x|+2x$, $x\in R$; b) $y=\sin x$ $x\in\left[\dfrac{5\pi}{2},\dfrac{7\pi}{2}\right]$.

J: a) $y=\begin{cases}1-\sqrt{1-x} & x<0\\ -1+\sqrt{1+x} & x\geq 0\end{cases}$ b) $y=3\pi-arcSinx$, $x\in[-1;1]$.

3–§. Funktsiyalarning grafiklari.

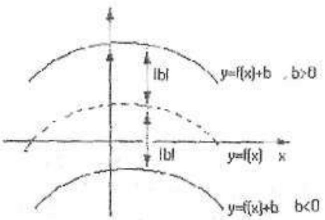
1⁰. Aytaylik, $y=f(x)$ funktsiya $X\subset R$ to'plamda berilgan bo'lsin. $x_0\in X$ da $f(x_0)=y_0$ deylik. Ravshanki, (x_0,y_0) juftlik tekislikda nuqtani tasvirlaydi.

Tekislikning nuqtalaridan iborat ushbu $\{(x,y)|x\in X, y=f(x)\}$ to'plam $y=f(x)$ **funktsiyaning grafigi** deyiladi.

1) Juft funktsiyaning grafigi ordinatalar o'qiga nisbatan, toq funktsiyaning grafigi esa koordinata boshiga nisbatan simmetrik bo'ladi. Shu sababli juft va toq funktsiyalarning grafigini faqat argumentning musbat qiymatlari uchun bilish yetarli.
2) Davriy funktsiya uchun esa uning grafigini bir davr oralig'ida bilish kifoya.

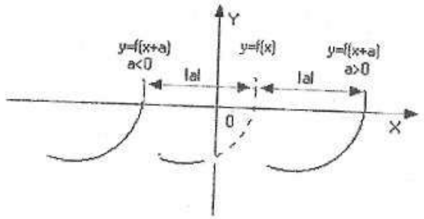
2⁰. Faraz qilaylik, $y=f(x)$ funktsiya grafigining tekislikdagi tasviri ma'lum bo'lsin. Unga ko'ra quyidagi $y=f(x)+b$, $y=f(x+a)$, $y=-f(x)$, $y=f(-x)$, $y=-f(-x)$ $y=|f(x)|$, $y=f(|x|)$, $y=af(x)$, $y=f(ax)$ (bunda a,b – o'zgarmas sonlar) funktsiyalarning grafiklarini yasash usullarini keltiramiz:

1) $y=f(x)+b$ **funktsiyaning grafigi.** Bu funktsiyaning grafigi $y=f(x)$ funktsiyaning grafigini ordinatalar o'qi bo'yicha $|b|$ masofaga parallel ko'chirish bilan yasaladi. Bunda grafik $b>0$ bo'lganda yuqoriga, $b<0$ bo'lganda parallel ko'chiriladi.



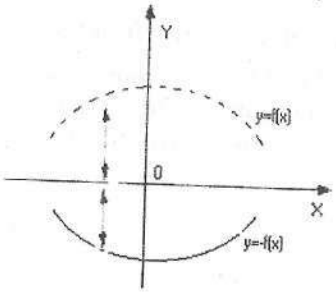
2 – chizma.

2) $y=f(x+a)$ **funktsiyaning grafigi.** Bu funktsiyaning grafigi $y=f(x)$ funktsiyaning grafigini abstsissalar o'qi bo'yicha $|a|$ masofaga parallel ko'chirish bilan yasaladi. Bunda grafik $a>0$ bo'lganda chapga, $a<0$ bo'lganda o'ngga parallel ko'chiriladi.



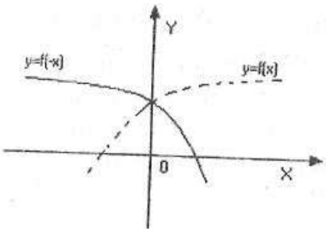
3 — chizma.

3) $\underline{y=-f(x)}$ **funktsiyaning grafigi.** Bu funktsiyaning grafigi $y=f(x)$ funktsiyaning grafigini abstsissalar o'qiga nisbatan simmetrik ko'chirish bilan yasaladi.



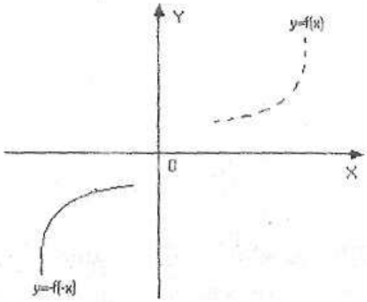
4 — chizma.

4) $\underline{y=f(-x)}$ **funktsiyaning grafigi.** Bu funktsiyaning grafigi $y=f(x)$ funktsiya grafigini ordinatalar o'qiga nisbatan simmetrik ko'chirish bilan yasaladi.



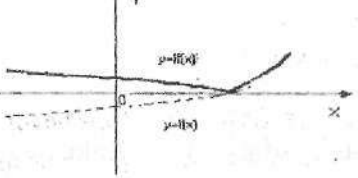
5 — chizma.

5) $\underline{y=-f(-x)}$ **funktsiyaning grafigi.** Bu funktsiyaning grafigi $y=f(x)$ funktsiya grafigini koordinata boshiga nisbatan simmetrik ko'chirish bilan yasaladi.



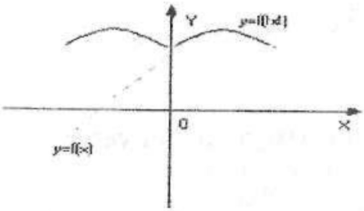
6 – chizma.

6) $y = |f(x)|$ **funktsiyaning grafigi.** Bu funktsiyaning grafigi $y = f(x)$ funktsiya grafigining OX o'qidan yuqorida joylashgan qismini qoldirish, OX o'qidan pastda joylashgan qismini esa shu o'qqa simmetrik ravishda yuqoriga ko'tarish bilan yasaladi.



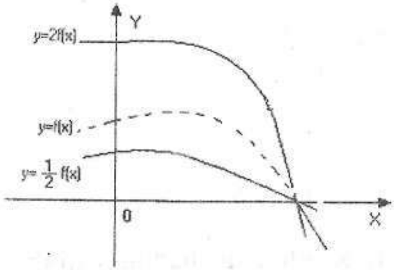
7 – chizma.

7) $y = f(|x|)$ **funktsiyaning grafigi.** Bu funktsiyaning grafigi $y = f(x)$ funktsiya grafigining OU o'qidan o'ng tomonda joylashgan qismini qoldirish, OU o'qidan chap tomonda joylashgan qismini tashlab, uning o'rniga OU ning o'ng tomondagi qismini shu o'qqa simmetrik ravishda chapga o'tkazish bilan yasaladi.



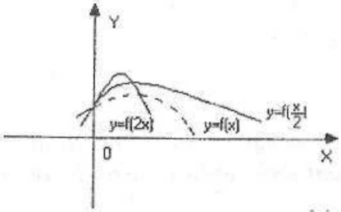
8 – chizma.

8) $y = af(x)$ **funktsiyaning grafigi.** Bu funktsiyaning grafigi $y = f(x)$ funktsiyaning grafigining har bir ordinatasini a ga ko'paytirish bilan yasaladi.



9 – chizma.

9) $y = f(ax)$ **funktsiyaning grafigi.** Bu funktsiyaning grafigi $y = f(x)$ funktsiya grafigining har bir abstsissani a ga bo'lish bilan yasaladi.



10 – chizma.

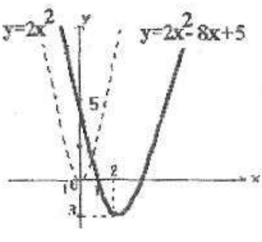
3⁰. Ko'pincha $y = f(x)$ va $y = \varphi(x)$ funktsiyalarning grafiklarini bilgan holda bu funktsiyalarning mos ordinatalarining (ikkalasining ordinatalari bitta x nuqtada hisoblanadi) qo'shish, ayirish, ko'paytirish va bo'lish bilan

$f(x)+\varphi(x)$, $f(x)-\varphi(x)$, $f(x)\bullet\varphi(x)$ $\dfrac{f(x)}{\varphi(x)}$ funktsiyalarning grafigini yasash mumkin bo'ladi.

1–misol. Ushbu $y=2x^2-8x+5$ funktsiya grafigi yasalsin.

Kvadrat uchhadning ko'rinishini o'zgartiramiz:

$y=2x^2-8x+5=2\left(x^2-4x+\dfrac{5}{2}\right)=2(x-2)^2-3$. Demak, berilgan funktsiyalarning grafigini 1) va 2)–qoidalarga asosan $y=2x^2$ parabolaning grafigini 2 birlik o'ngga surish va 3 birlik pastga siljitish yordamida hosil qilinadi (11–chizma).



11–chizma

Izoh. Shu yo'l bilan ixtiyoriy kvadrat uchhadning grafigini yasash mumkin.

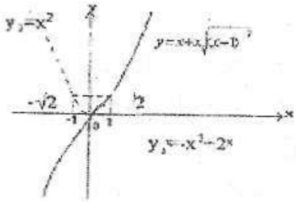
2–misol. Ushbu $y=x+x\sqrt{(x-1)^2}$ funktsiyaning grafigi yasalsin.

Agar x>1 bo'lsa $y=x+x(x-1)=x^2$, x≤1 bo'lsa

$$y=x+x(1-x)=-x^2+2x$$

bo'ladi.

Demak, berilgan funktsiyaning grafigi $(-\infty\ ;\ 1)$ oraliqda $y_1=-x^2+2x$ hamda $[1;+\infty)$ da $y=x^2$ parabolalardan tashkil topgan chiziqni ifodalaydi (12–chizma).

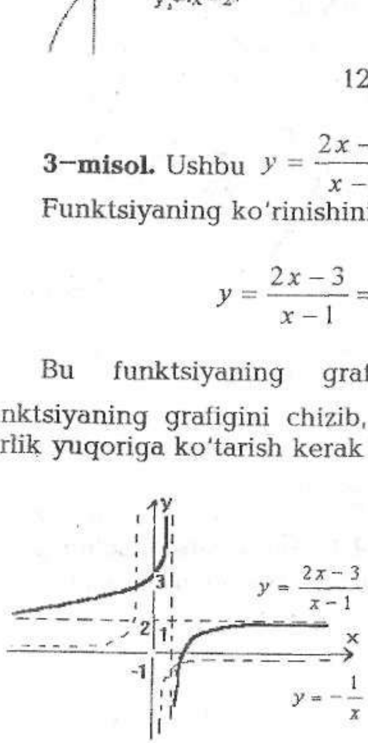


12 – chizma.

3–misol. Ushbu $y = \dfrac{2x-3}{x-1}$ funktsiya grafigi yasalsin.
Funktsiyaning ko'rinishini quyidagicha o'zgartiramiz:

$$y = \frac{2x-3}{x-1} = \frac{2(x-1)-1}{x-1} = 2 - \frac{1}{x-1}.$$

Bu funktsiyaning grafigini yasash uchun $y = -\dfrac{1}{x}$ funktsiyaning grafigini chizib, uni 1 birlik o'ngga surish va 2 birlik yuqoriga ko'tarish kerak (13 – chizma).



13 – chizma

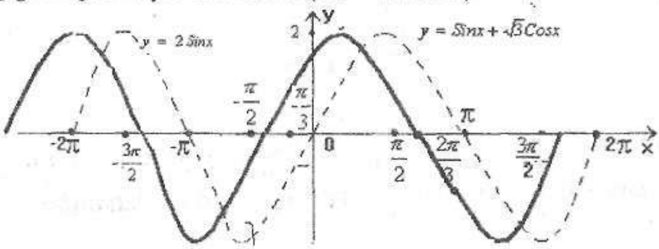
Izoh: Shu yo'l bilan ixtiyoriy kasr – chiziqli $y = \dfrac{ax+в}{cx+d}$ funktsiyaning grafigini yasash mumkin.

4–misol. Ushbu $y = Sinx + \sqrt{3}Cosx$ funktsiyaning grafigi yasalsin.

Quyidagi amallarni bajaramiz:

$$y = Sinx + \sqrt{3}Cosx = 2(\frac{1}{2}Sinx + \frac{\sqrt{3}}{2}Cosx) = 2(SinxCos\frac{\pi}{3} + CosxSin\frac{\pi}{3}) = 2Sin(x + \frac{\pi}{3})$$

Shunday qilib, berilgan funktsiyaning grafigini chizish uchun $y = 2Sinx$ funktsiyaning grafigini chizib, uni $\frac{\pi}{3}$ birlik chapga siljitish yetarli ekan (14—chizma)

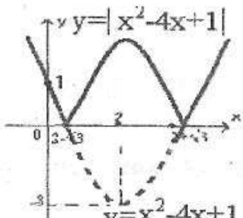


14—chizma

Izoh: Shu yo'l bilan $y = aSinx + bCosx$ ko'rinishidagi ixtiyoriy funktsiyaning grafigini yasash mumkin.

5—misol. Ushbu $y = |x^2 - 4x + 1|$ funktsiyaning grafigi yasalsin.

Bu funktsiyaning grafigini yasash uchun avval $y_1 = x^2 - 4x + 1 = (x-2)^2 - 3$ funktsiyaning grafigini yasab, so'ng $y = |y_1|$ funktsiyaning grafigini yasashda 6—xossadan foydalanamiz (15—chizma).

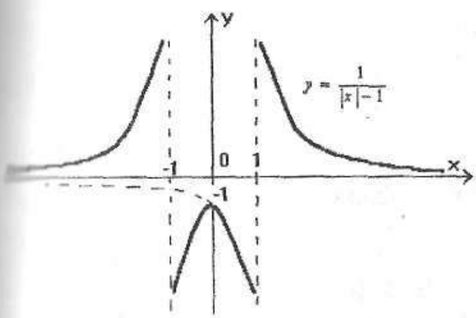


15—chizma.

6—misol. Ushbu $y = \frac{1}{|x| - 1}$ funktsiyaning grafigi yasalsin.

Bu funktsiya juft bo'lganligi sababli uning grafigini OY o'qidan o'ng tomonda joylashgan qismini chizish yetarli bo'ladi (chap tomonga OY o'qiga nisbatan simmetrik ko'chiriladi).

Demak, x≥0 da $y=\frac{1}{x-1}$ bo'lib, uning grafigini o'ng yarim tekislikda chizamiz va bu grafik yordamida berilgan funktsiyaning grafigini hosil qilamiz. (16 – chizma).



16 – chizma.

7–misol. Ushbu

$$y=\arcsin(\sin x)$$

funktsiyaning grafigi yasalsin.

Bu funktsiya 2π davrli funktsiya bo'ladi, chunki

$$\arcsin(\sin(x+2\pi))=\arcsin(\sin x)$$

Arksinus ta'rifiga binoan, birinchidan

$$-\frac{\pi}{2}\le y\le\frac{\pi}{2}$$

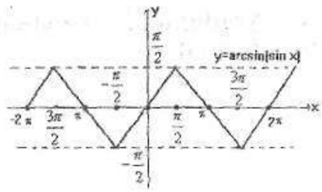
ikkinchidan esa $\sin y=\sin x$ bo'ladi.

Demak,

$$-\frac{\pi}{2}\le x\le\frac{\pi}{2} \quad \text{da} \quad y=x,$$

$$\frac{\pi}{2}<x\le\frac{3\pi}{2} \quad \text{da} \quad y=\pi-x$$

bo'ladi (17 – chizma).



17 – chizma.

8–misol. Ushbu

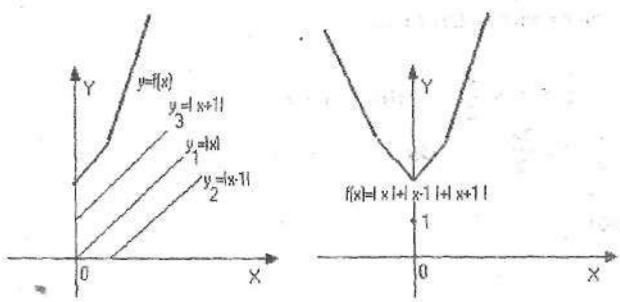
$$f(x) = |x| + |x-1| + |x+1|$$

funktsiyaning grafigi yasalsin.

Bu funktsiya $R = (-\infty;+\infty)$ deb aniqlangan juft funktsiyalardir. Haqiqatdan ham,

$$f(-x) = |-x| + |-x-1| + |-x+1| = |x| + |(-1)(x+1)| + |-(x-1)| = |x| + |x+1| + |x-1| = f(x)$$

Demak, berilgan funktsiya grafigining OY o'qidan o'ng tomonda joylashgan qismini topish yetarli bo'ladi (chap tomonga OY o'qiga nisbatan simmetrik ko'chiriladi). Ravshanki, $y = f(x)$ funktsiya $y_1 = |x|$, $y_2 = |x-1|$, $y_3 = |x+1|$, funktsiyalar yig'indisidan iborat. Bu funktsiyalar grafiklarini chizib (y_2 va y_3 funktsiyalarning grafiklarini chizishda $y_1 = |x|$, funktsiyaning grafigi va 2 – qoidadan foydalaniladi), so'ng har bir x ga mos ordinatalarini qo'shib, berilgan funktsiya grafigi topiladi (18 – chizma).



18-chizma

Mustaqil yechish uchun misol va masalalar.

Quyidagi funktsiyalarning grafiklari yasalsin.

1. $y = 3x^2 - 6x - 17$

2. $y = -2x^2 - 4x + 4$

3. $y = \dfrac{2x+5}{x-2}$

4. $y = \dfrac{x^2-1}{x^2+1}$

5. $y = \sqrt{\lg Sinx}$

6. $y = Sin^2 x$

7. $y = 0{,}5(|x+1| + |x-1|)$

8. $y = arctg(tgx)$

9. $y = \dfrac{1}{x^2+4x+5}$

10. $y = Cos(\lg x)$

11. $y = Sign(Sinx)$

12. $y = \max\left\{x^3; \dfrac{1}{x}\right\}$

13. $y = \dfrac{2x-3}{3-x}$

4. $y = 2^{1-x^2}$

5. $y = 2^{arctgx}$

6. $y = \log_2 \log_2 x$

7. $y = \dfrac{Cos2x}{Sinx}$

3. $y = \arcsin(3x-1)$

9. $y = x^2 + \dfrac{1}{x}$

0. $y = \dfrac{x^2}{|x|-1}$

1. $y = x \cdot Sign(Sinx)$

2. $y = \min\{\log_2 x, \log_x 2\}$

4-§. Parametrik ko'rinishda va qutb koordinatalar sistemasida berilgan funktsiyalar.

1⁰. Aytaylik, $T \subset R$ to'plamda

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \quad t \in T \end{cases} \qquad (1)$$

funktsiyalar berilgan bo'lsin. Faraz qilaylik, X va Y to'plamlar, mos ravishda, $T \subset R$ to'plamda aniqlangan $x = \varphi(t)$ va $y = \psi(t)$ funktsiyalarning qiymatlar to'plami bo'lsin. Agar ixtiyoriy $t \in T$ olinganda ham unga mos $x = \varphi(t) \in X$ uchun yagona $y = \psi(t) \in Y$ mos kelsa, unda (1) sistema parametrik ko'rinishda berilgan funktsiyani aniqlaydi deyiladi.

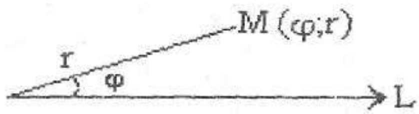
Masalan,

$$\begin{cases} x = t \\ y = t^2 \end{cases} \qquad t \in R$$

funktsiyalar oshkor ko'rinishda berilgan $y = x^2$ funktsiyaning parametrik ko'rinishini ifodalaydi.

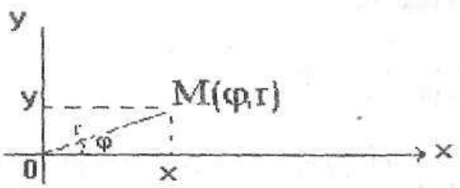
2⁰. Amaliyotda dekart koordinatalar sistemasidan keyin eng ko'p ishlatiladigan sistema – qutb koordinatalar sistemasi hisoblanadi.

Qutb koordinatalar sistemasida tekislikdagi ixtiyoriy M nuqta, O qutb nuqtagacha bo'lgan masofa $|OM| = r$ va OM vektorning OL qutb o'qi bilan hosil qilgan burchagi φ yordamida aniqlanadi (19 – chizma).



19 – chizma

Agar dekart koordinatalar sistemasi markazini qutb nuqta, Ox o'qining musbat qismini qutb o'qi bilan ustma – ust tushirsak, unda qutb va dekart koordinatalar sistemalari orasidagi bog'lanishlar quyidagi tengliklar yordamida ifodalanadi (20 – chizma):



20 – chizma

$$\begin{cases} x = rCos\varphi \\ y = rSih\varphi \end{cases} \quad \text{va} \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ tg\,\varphi = \dfrac{y}{x} \end{cases} \qquad (2)$$

Izoh. Qutb koordinatalar sistemasidagi ixtiyoriy (φ;0) juftlikka Dekart koordinatalar sistemasidagi O (0;0) nuqtani mos qo'yamiz.

3°. **1–misol.** A(0 ; –1) va V(1,6 ; –0,2) nuqtalarning qaysilari Ushbu

$$l=\begin{cases} x = S\text{int}+1 \\ y = Cost-1 \end{cases}$$ chiziqda yotishi aniqlansin.

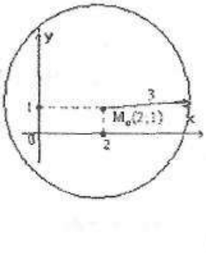
Yechilishi:

Avval A(0 ; –1) nuqtani tekshiramiz. $\Rightarrow x=0,\quad y=-1,\quad x=0$ bo'lganda $\sin t+1=0$ bo'lib, bu tenglik $t=\frac{3\pi}{2}$ da bajariladi. $\Rightarrow y\left(\frac{3\pi}{2}\right)=Cos\frac{3\pi}{2}-1=-1.$ Demak (0 ; – 1) $\in l$.

B(1,6; – 0,2) uchun $x=1,6$ va $y=-0,2$.
$\Rightarrow \sin t+1=1,6 \Rightarrow \sin t=0,6. \Rightarrow \cos t=\sqrt{1-\sin^2 t}=0,8. \Rightarrow y=\cos t-1=$
$=0,8-1=-0,2 \Rightarrow$ B(1,6 ; – 0,2) $\in l$.

2–misol. Chiziqning ushbu $\begin{cases} x = 2-3Cost \\ y = 1+3S\text{int} \end{cases}$ parametrik tenglamasidan t parametr yo'qotilib, uning Dekart koordinatalar sistemasidagi tenglamasi topilsin va grafigi chizilsin.

Yechilishi: Berilgan sistemadan $\begin{cases} x-2 = -3Cost \\ y-1 = 3S\text{int} \end{cases}$ ekanligi va bu yerdan $(x-2)^2+(y-1)^2=9\left(Con^2t+Sin^2t\right)=9$ bo'lishini topamiz. Demak, berilgan chiziq markazi (2;1) nuqtada, radiusi 3 ga teng bo'lgan ushbu $(x-1)^2+(u-1)^2=9$ aylanadan iborat ekan. (21 – chizma).



21 – chizma

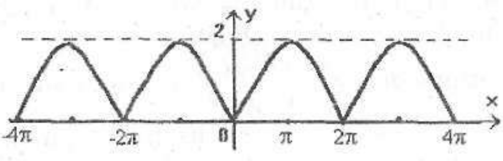
3–misol. Ushbu $\begin{cases} x = t - S\,int \\ y = 1 - Cost \end{cases}$ (3)

tsikloidaning grafigi yasalsin.

Yechilishi: (3) – sistema yordamida aniqlanadigan funktsiyani $y = f(x)$ oshkor ko'rinishda ifodalab bo'lmaydi. Bu funktsiya x ning barcha qiymatlarida aniqlangan $(x \in (-\infty; +\infty))$ va chegaralangan $y \in [0\ ;\ 2]$ hamda t parametr bo'yicha davriy bo'lib, davri 2π ga teng. Xarakterli nuqtalar uchun quyidagi jadvalni tuzamiz.

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
x	0	$\frac{\pi}{4}-\frac{\sqrt{2}}{2}$	$\frac{\pi}{2}-1$	$\frac{3\pi}{4}-\frac{\sqrt{2}}{2}$	π	$\frac{5\pi}{4}+\frac{\sqrt{2}}{2}$	$\frac{3\pi}{2}+1$	$\frac{7\pi}{4}+\frac{\sqrt{2}}{2}$	2π
y	0	$1-\frac{\sqrt{2}}{2}$	1	$1+\frac{\sqrt{2}}{2}$	2	$1+\frac{\sqrt{2}}{2}$	1	$1-\frac{\sqrt{2}}{2}$	0

Bu jadvaldan foydalanib tsikloidaning grafigini chizamiz. (22 – chizma)



22 – chizma.

4–misol. Ushbu A(2√3;2), V(√2; –√2) nuqtalarning qutb koordinatalari va S($\frac{\pi}{2}$; 10), D($\frac{5\pi}{4}$; 2) nuqtalarning Dekart koordinatalari topilsin.

Yechilishi: Masalani yechish uchun Dekart va qutb koordinatalar sistemalarini bog'lovchi (2) tengliklardan foydalanamiz:

$$A\ (2\sqrt{3};2) \Rightarrow x = 2\sqrt{3}\ ,\ y = 2. \Rightarrow r = \sqrt{x^2+y^2} = \sqrt{12+4} = \sqrt{16} = 4,$$

$$tg\varphi = \frac{y}{x} = \frac{1}{\sqrt{3}} \Rightarrow \varphi = \frac{\pi}{6} \Rightarrow A\left(\frac{\pi}{6};4\right)\ B(\sqrt{2}; -\sqrt{2})$$

$$\Rightarrow x=\sqrt{2}, \quad y=-\sqrt{2} \Rightarrow \quad r=\sqrt{x^2+y^2}=\sqrt{2+2}=2, \ tg\ \varphi = \frac{y}{x} = -1 .$$

B nuqta IV – chorakda yotgani uchun $\varphi = 2\pi - arctg1 = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$ bo'ladi. $\Rightarrow$ $V\left(\frac{7\pi}{4};2\right)$. $S(\frac{\pi}{2};10) \Rightarrow \varphi=\frac{\pi}{2}$ $, r=10 \Rightarrow C(0;10)$.

$$D\left(\frac{5\pi}{4};10\right) => \varphi=\frac{5\pi}{4}; r=2 => x=rCos\varphi=2Cos\frac{5\pi}{4}=-2\frac{\sqrt{2}}{2}=-\sqrt{2};$$

$$y=rSin\varphi=2Sin\frac{5\pi}{4}=\sqrt{2} => D\left(-\sqrt{2};\sqrt{2}\right)$$

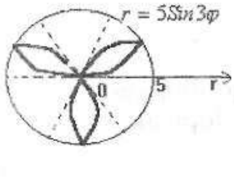
5–misol. Ushbu

$r=5Sin3\varphi$ "uch yaproqli gul" grafigi yasalsin.

Yechilishi: Avval berilgan funktsiyaning aniqlanish sohasini topamiz $: D(r)=\{\varphi | Sin3\varphi \geq 0\} = \left[0;\frac{\pi}{3}\right] \cup \left[\frac{2\pi}{3};\pi\right] \cup \left[\frac{4\pi}{3};\frac{5\pi}{3}\right]$. Qiymatlar to'plami Ye(r)= [0;5] funktsiya davriy bo'lib, uning asosiy davri $T_o=\frac{2\pi}{3}$ bo'ladi. Demak, funktsiya grafigini $\left[0;\frac{2\pi}{3}\right]$ kesmada, xususan $\left[0;\frac{\pi}{3}\right]$ kesmada, chizish kifoya. Buning uchun quyidagi jadvalni tuzamiz:

φ	0	$\frac{\pi}{18}$	$\frac{\pi}{9}$	$\frac{\pi}{6}$	$\frac{2\pi}{9}$	$\frac{5\pi}{18}$	$\frac{\pi}{3}$
r	0	2.5	$\frac{5\sqrt{3}}{2}$	5	$\frac{5\sqrt{3}}{3}$	2.5	0

Jadvaldan va funktsiyaning davriyligidan foydalanib $r=5Sin3\varphi$ funktsiyaning grafigini chizamiz (23 – chizma).



23 – chizma.

Mustaqil yechish uchun misol va masalalar.

1.A(0 ;0) va B(3 ;3) nuqtalarning qaysilari ush

$$l:\begin{cases} x=t^2-1 \\ y=t^3-t \end{cases}$$

chiziqqa tegishli ekanligi aniqlansin.

J: A tegishli, B tegishli emas.

2. $A\left(\frac{3}{2};\sqrt{3}\right)$ va B(1;2) nuqtalarning qaysilari ushbu $l:\begin{cases} x=2Cost-Cos2t \\ y=2S\,int-Sin2t \end{cases}$, chiziqqa tegishli ekanligi aniqlansin.

J : B tegishli, A tegishli emas.

Berilgan A va B nuqtalarning qaysilari tenglamalari parametrik ko'rinishda berilgan chiziqqa tegishli ekanligi aniqlansin.

3. $x=t^2$, $y=t^3$; $A(1;1)$, $B(4;-8)$. J: A va B.

4. $x=1+Cost$; $y=S\,int$; $A(0;0)$ $B\left(\frac{1}{2};\frac{1}{2}\right)$.

J: A tegishli, B tegishli emas

5 . $x=2^t S\,int$; $y=2^t Cost$; $A(2;2)$, $B(0;2\pi)$.

J: A ham, B ham tegishli emas.

6.Ushbu

$\begin{cases} x=Cost \\ y=Cos^2t \end{cases}$, sistema qanday chiziqni aniqlashi topilsin.

J: $y=x^2$ parabolaning A(−1;1) va B(1;1) nuqtalarini tutashtiruvchi AOB yoyini aniqlaydi.

7.Chiziqning ushbu

$$\begin{cases} x=3Cost \\ y=2S\,int \end{cases}$$

parametrik tenglamasidan t parametr yo'qotilib, uning dekart koordinatalar sistemasidagi tenglamasi topilsin va grafigi chizilsin.

Ko'rsatma: Avval birinchi tenglamani 3 ga, ikkinchi tenglamani 2 ga bo'lib, so'ngra t parametrni yo'qoting.

J: $\frac{x^2}{9}+\frac{y^2}{4}=1$ — ellips.

Berilgan chiziqning parametrik tenglamasidan t parametr yo'qotilib, uning Dekart koordinatalar sistemasidagi tenglamasi topilsin va grafigi chizilsin.

8. $x=t-1,\ y=t^2-2t+2$. J: $y=x^2+1$.

9. $x=|\ln t|,\ y=1+t^3$. J: $3x=|\ln(y-1)|$.

10. $x=(t+1)^2,\ y=(t-1)^2$. J: $8(x+y)=(x-y)^2+16$.

11 . $x=aSect,\ y=вtgt$. J: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ — giperbola.

12. Ushbu

$$\begin{cases} x=Cos^3t \\ y=Sin^3t \end{cases}$$

astroidaning grafigi chizilsin.

13. Ushbu Dekart koordinatalar sistemasida berilgan A(2;0), B(1;1), C(−2;0) va D $(-\sqrt{2};-\sqrt{2})$ nuqtalarning qutb koordinatalari topilsin.

J: $A(0;2)$, $B\left(\frac{\pi}{4};\sqrt{2}\right)$, $C(\pi;2)$, $D\left(\frac{5\pi}{4};2\right)$

14. Qutb koordinatalar sistemasida berilgan ushbu $A\left(\frac{\pi}{3};2\right)$, $B\left(\frac{3\pi}{2};3\right)$, $C\left(\frac{3\pi}{4};\sqrt{2}\right)$ va $D\left(\frac{7\pi}{6};4\right)$ nuqtalarning Dekart koordinatalari topilsin

J: $A(1;\sqrt{3})$, $B(0;-3)$, $C(-1;1)$, $D(-2\sqrt{3};-2)$

15. Dekart koordinatalar sistemasida berilgan quyidagi chiziqlarning tenglamalari qutb koordinatalar sistemasida yozilsin:

a) $x\ +y^2=2x$, b) $x=y^2-\frac{1}{4}$ sya

J: a) $r=2Cos\varphi$, b) $r=\frac{1}{4Sin^2\frac{\varphi}{2}}$

16. Quyidagi chiziqlarning grafiklari yasalsin:
a) $r = 2(1 - Cos\varphi)$ – kardioida, b) $r = 3 + 2Sin\varphi$

17. Ushbu $(x^2 + y^2)^2 = 2(x^2 - y^2)$ tenglama bilan berilgan "Bernulli lemniskatasi"ning grafigini qutb koordinatalar sistemasiga o'tish yordamida chizing.

Quyidagi chiziqlarning grafiklari chizilsin.

18. $x = 2 - t$, $y = 1 - t^2$

19. $x = 2Cos^2 t$, $y = 3Sin^2 t$

20. $x = S\text{int}$, $y = Cosec t$

Ko'rsatma: t parametr yo'qotilsa, $y = \frac{1}{x}$ tenglama hosil bo'ladi. Grafik chizilayotganda $|x| \le 1$ va $|y| \ge 1$ bo'lishini e'tiborga olish kerak.

21. Dekart koordinatalar sistemasida berilgan ushbu $A(2\sqrt{3};2)$, $B(0;-1)$, $C(2\sqrt{3};-2)$ va $D(-1;1)$ nuqtalarning qutb koordinatalari topilsin.

J: $A\left(\frac{\pi}{6};4\right)$, $B\left(\frac{3\pi}{2};1\right)$, $C\left(\frac{11\pi}{6};4\right)$, $D\left(\frac{5\pi}{4};\sqrt{2}\right)$.

22. Qutb koordinatalar sistemasida berilgan ushbu $A(\frac{5\pi}{3};2)$, $B(\pi;4)$, $C(\frac{4\pi}{3};2)$ va $D(\frac{5\pi}{4};4)$ nuqtalarning Dekart koordinatalari topilsin.

J: $A(1;-\sqrt{3})$, $B(-4;0)$, $C(-1;-\sqrt{3})$, $D(-2\sqrt{2};-2\sqrt{2})$.

23. Qutb koordinatalar sistemasida berilgan $M(\frac{\pi}{4};3)$ va $N(\frac{3\pi}{4};4)$ nuqtalar orasidagi masofa hisoblansin. **J:** 5.

Ko'rsatma: M va N nuqtalarning Dekart koordinatalarini toping hamda $M(x_1,y_1)$ va $N(x_2,y_2)$ nuqtalar orasidagi masofani hisoblash uchun $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ formuladan foydalaning.

Dekart koordinatalar sistemasida berilgan quyidagi chiziqlarning tenglamalari qutb koordinatalar sistemasida yozilsin.

24. $x+y+1=0$. J: $r=\dfrac{1}{\sqrt{2}Cos\left(\varphi+\dfrac{3\pi}{4}\right)}$.

25. $2xy=x^2-y^2$. J: $\varphi=\dfrac{\pi}{8}+\dfrac{\pi n}{2}$ $(n=1,2,3)$ va $r=0$

Quyidagi chiziqlarning grafiklari yasalsin.

26. $r=\varphi$, $\varphi\in[0;2\pi]$ – Arximed spirali.
27. $r=2(1+Sin\varphi)$ – Kardioda
28. $r=4Sin(\varphi-\dfrac{\pi}{4})$ – Aylana
29. $r=\dfrac{1}{Cos\varphi-Sin\varphi}$ – To'g'ri chiziq
30. $r=Sin^2 2\varphi$ – to'rt yaproqli gul.

V BOB.
LIMIT, UZLUKSIZLIK, HOSILA.

1-§. Ketma-ketlik va uning limiti

Ketma-ketlik tushunchasi maktab kursidan ma'lum.

Aytaylik, $\{x_n\}$: $x_1, x_2, \ldots x_n, \ldots$ ketma-ketlik berilgan bo'lsin.

Agar $\forall n \in N$ uchun $x_n \leq x_{n+1}$ $(x_n \geq x_{n+1})$ bo'lsa, unda $\{x_n\}$ketma- ketlik **o'suvchi (kamayuvchi)** deyiladi.

Agar $\forall n \in N$ uchun $\exists M(m)$ son : $\forall n \in N$ uchun $x_n \leq M$ $(x_n \geq m)$bo'lsa, u holda $\{x_n\}$ ketma-ketlik **yuqoridan (quyidan) chegaralangan** deb ataladi. Agar ketma-ketlik ham yuqoridan, ham quyidan chegaralangan bo'lsa, u holda bunday ketma-ketlik **chegaralangan ketma-ketlik** deyiladi.

Agar ketma-ketlik o'suvchi yoki kamayuvchi bo'lsa, unda unga **monoton ketma-ketlik** deb ataladi.

Aytaylik, $\{x_n\}$ ketma- ketlik va a son berilgan bo'lsin.
Agar $\forall \varepsilon > 0$ uchun $\exists n_0 = n_0(\varepsilon) \in N : \forall n \geq n_0$ uchun $|x_n - a| < \varepsilon$ bo'lsa a son$\{x_n\}$ **ketma-ketlikning limiti** deb ataladi va $\lim\limits_{n\to\infty} x_n = a$ kabi belgilanadi.

Misollar.

1) $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ tenglik isbotlansin.

$\forall \varepsilon > 0$ olamiz, $|x_n - a| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon \Rightarrow n > \frac{1}{\varepsilon} \Rightarrow n_0 = \left[\frac{1}{\varepsilon}\right]$

2) $\lim\limits_{n\to\infty} \frac{n-1}{n} = 1$ tenglik isbotlansin.

$x_n = \frac{n-1}{n}$, $a = 1$ $\forall \varepsilon > 0$ olamiz.

$$|x_n - a| = \left|\frac{n-1}{n} - 1\right| = \frac{1}{n} < \varepsilon \Rightarrow n > \frac{1}{\varepsilon} \Rightarrow n_0 = \left[\frac{1}{\varepsilon}\right].$$

3) $\lim\limits_{n\to\infty}(-1)^n \neq \exists$

Shuni isbotlaylik. Buning uchun teskarisini faraz qilaylik, ya'ni

$\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty}(-1)^n = \exists$ bo'lsin. Unda ta'rifga ko'ra $\forall \varepsilon > 0$ $\exists n_0(\varepsilon) \in N$ $\forall n > n_0$ uchun $|x_n - a| < \varepsilon$ bo'ladi. Aytaylik,

$\varepsilon = \frac{1}{2}$ bo'lsin $|x_{2n} - a| < \varepsilon$ va $|x_{2n-1} - a| < \varepsilon \Rightarrow |x_{2n} - x_{2n-1}| = |x_{2n} - a - (x_{2n-1} - a)| \le |x_{2n} - a| + |x_{2n-1} - a| < \varepsilon + \varepsilon = 2\varepsilon = 1$ bo'ladi. Lekin $|x_{2n} - x_{2n-1}| = |(-1)^{2n} - (-1)^{2n-1}| = 2 > 1$ ziddiyat kelib chiqdi. Demak, $\lim_{n\to\infty}(-1)^n \ne \exists$

Sonli ketma–ketlikning xossalari.

1^0. Sonli ketma–ketlik limitga ega bo'lsa, uning limiti yagona bo'ladi.
2^0. Agar ketma–ketlik limitga ega bo'lsa, ya'ni yaqinlashuvchi bo'lsa, u chegaralangan bo'ladi.

Izoh. Bu xossaning teskarisi o'rinli emas, ya'ni chegaralangan ketma–ketlik yaqinlashuvchi bo'lishi shart emas. Masalan, $\{x_n\} = \{(-1)^n\}$ ketma–ketlik chegaralangan, lekin $\lim_{n\to\infty} x_n \ne \exists$.

3^0. Agar $\{x_n\}$ ketma–ketlik o'suvchi (kamayuvchi) bo'lib, yuqoridan (quyidan) chegaralangan bo'lsa, u holda bunday ketma–ketlik limitga ega bo'ladi.

Bu xossa ketma–ketlik limitini hisoblashda keng qo'llaniladi.

4–misol, $x_n = \left(1 + \frac{1}{n}\right)^n$ ketma–ketlik berilgan bo'lsin. Bu ketma–ketlik monoton o'suvchi va chegaralangan.
Nyuton binomi formulasidan foydalanib topamiz:

$$x_n = \left(1 + \frac{1}{n}\right)^n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \ldots + \frac{n(n-1)(n-2)\ldots 1}{n!} \cdot \frac{1}{n^n}$$

yoki

$$x_n = 2 + \frac{1}{2}\left(1 - \frac{1}{n}\right) + \frac{1}{2 \cdot 3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots + \frac{1}{2 \cdot 3 \cdot 4 \ldots \cdot n}\left(1 - \frac{1}{n}\right)\ldots\left(1 - \frac{n-1}{n}\right)$$

Bu tenglikdan ko'rinadiki:

$$2 < x_n = 2+\frac{1}{2}\left(1-\frac{1}{n}\right)+\frac{1}{2\cdot 3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+...+\frac{1}{2\cdot 3\cdot 4\cdot ...n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)...\left(1-\frac{n-1}{n}\right) <$$

$$< 2+\frac{1}{2}+\frac{1}{3!}+...+\frac{1}{n!} < 2+\frac{1}{2}+\frac{1}{2^2}+...+\frac{1}{2^{n-1}} < 2+\frac{1}{2}+...+\frac{1}{2^{n-1}}+\frac{1}{2^n}+... = 2+\frac{\frac{1}{2}}{1-\frac{1}{2}} = 3$$

Demak, x_n ketma-ketlik chegaralangan $2 < x_n < 3$.

Unda 3^0-xossaga asosan $\lim\limits_{n\to\infty} x_n = \exists$ Bu limit e deb belgilanadi va ushbu

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

tenglikka ikkinchi **ajoyib limit** deb ataladi. e-irratsional son bo'lib, $e \approx 2{,}718281....$ taqribiy tenglik o'rinli.

e soni asos qilib olingan logarifmlarga **natural logarifmlar** deb ataladi va $\ln x$ kabi belgilanadi.

e soni matematik analiz va uning tatbiqida muhim o'rin tutadi.

Misol uchun radioaktiv elementlarning xossalaridan biri uning yemirilishidir. Faraz qilaylik, t=0 da radiyning miqdori m_0 bo'lsin. t vaqt o'tgandan so'ng radiy miqdori x ga teng bo'lsin. U holda radiyning yemirilish qonuni quyidagicha:

$$x = m_0\, e^{-kt}$$

k-bu yerda proportsionallik koeffitsenti.

Nazorat savollari.

1. Sonli ketma-ketliklar. Misollar.
2. Monoton ketma-ketliklar. Misollar.
3. Ketma-ketlik limitining xossalari.
4. e-soni.

2–§. Sonli qatorlar.

1. Sonli qatorlar.

Ketma–ketliklar nazariyasida maxsus ko'rinishdagi ketma–ketliklar, qatorlar muhim ahamiyatga ega.

1–Ta'rif. Aytaylik, $\{a_n\}$– haqiqiy sonlar ketma–ketligi berilgan bo'lsin. U holda

$$a_1 + a_2 + a_3 + ... + a_n + ... = \sum_{n=1}^{\infty} a_n \qquad (1)$$

ifoda **sonli qator** deb ataladi.

a_n $(n = 1,2,3,...)$ son qator hadlari, a_n qatorning **umumiy hadi** deyiladi.

$$S_1 = a_1,$$
$$S_2 = a_1 + a_2,$$
$$S_3 = a_1 + a_2 + a_3,$$
....
$$S_n = a_1 + a_2 + ... + a_n = \sum_{k=1}^{n} a_k \qquad (2)$$

Yig'indilarni tuzamiz. Bu yig'indilar qatorning **qismiy yig'indilari** deyiladi.

2–Ta'rif. Agar $n \to \infty$ da (1) qatorning qismiy yig'indilaridan iborat $\{S_n\}$ ketma–ketlik chekli limitga ega, ya'ni

$$\lim_{n \to \infty} S_n = S$$

bo'lsa, u holda (1)– qator **yaqinlashuvchi** deyiladi. Aks holda, ya'ni limiti cheksiz yoki limit mavjud bo'lmasa (1) qator **uzoqlashuvchi** deyiladi.

S son (1) qatorning yig'indisi deyiladi va quyidagicha yoziladi:

$$S = a_1 + a_2 + a_3 + ... + a_n + ... = \sum_{n=1}^{\infty} a_n .$$

1–misol. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + ... + \frac{1}{n(n+1)} + ...$ qatorni yaqinlashishga tekshiring.

Qatorning qismiy yig'indisi ta'rifga ko'ra:

$$S_1 = a_1 = \frac{1}{2}$$

$$S_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$S_3 = a_1 + a_2 + a_3 = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$$

....

$$S_n = a_1 + a_2 + \ldots + a_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}$$

S_n ni quyidagicha yozib olamiz:

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

$n \to \infty$ da limitga o'tamiz:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$

Demak, berilgan qator yaqinlashuvchi, uning yig'indisi 1 ga teng.

2–misol. $1 + 2 + 3 + \ldots + n + \ldots$ qatorni yaqinlashishga tekshiring.

Ma'lumki, arifmetik progressiyaning dastlabki n ta hadining yig'indisini hisoblash formulasidan foydalanib, berilgan qatorning qismiy yig'indisi quyidagiga teng ekanini topamiz:

$$S_n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

Bundan,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty,$$

demak, qator uzoqlashuvchi.

***1–Teorema** (qator yaqinlashishi uchun Koshi kriteriyasi). (1) qator yaqinlashishi uchun* $\forall \varepsilon > 0$ *son olinganda ham shunday* $n_0 \in N$ *son mavjud bo'lib, barcha* $n \ge n_0$ va $m \ge n_0$ *lar uchun*

$$|S_m - S_n| = |a_{n+1} + a_{n+2} + \ldots + a_m| < \varepsilon \qquad (3)$$

tengsizlikning bajarilishi zarur va yetarli.

Natija 1. Agar sonli qatorning chekli sondagi hadlari o'zgartirilsa, u holda asl qatorning *yaqinlashishidan (uzoqlashishidan)* hosil bo'lgan yangi qator ham *yaqinlashadi (uzoqlashadi).*

Natija 2. (1) qatorning yaqinlashishi uchun

$$\lim_{n\to\infty} a_n = 0$$

tenglikning bajarilishi zarur.

Isbot. (3) shartni $m = n+1$ bo'lganda qarash yetarli:

$$|S_m - S_n| = |a_{n+1}| < \varepsilon .$$

Bu tengsizlikdan quyidagi kelib chiqadi.

$$\lim_{n\to\infty} a_{n+1} = 0 \quad \text{yoki} \quad \lim_{n\to\infty} a_n = 0 .$$

Shunday qatorlar borki $n \to \infty$ da $a_n \to 0$, lekin qator uzoqlashuvchi.

3–misol.

$$1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + ...$$

qatorga **garmonik qator** deyiladi, chunki bu nom quyidagi shartning bajarilishi bilan bog'langan.

$$\frac{1}{a_n} = \frac{1}{2}\left(\frac{1}{a_{n-1}} + \frac{1}{a_{n+1}}\right),$$

bu yerda a_n a_{n-1} va a_{n+1} orasidagi **o'rta garmonik** deyiladi.

2. Musbat hadli qatorlar. Solishtirish teoremalari.

3–Ta'rif. Agarda $\sum_{n=1}^{\infty}|a_n|$ qator yaqinlashsa, $\sum_{n=1}^{\infty}a_n$ qator **absolyut yaqinlashadi** deyiladi.

2–Teorema. Agar qator absolyut yaqinlashsa, u holda qator yaqinlashadi.

Isbot. Quyidagi tengsizlik:

$$|a_{n+1}+a_{n+2}+...+a_m|\le|a_{n+1}|+|a_{n+2}|+...+|a_m|,$$

va Koshi kriteriyasidan teoremaning o'rinli bo'lishi kelib chiqadi.

3–Teorema.

$$a_1+a_2+a_3+...+a_n+...=\sum_{n=1}^{\infty}a_n,$$

manfiy bo'lmagan hadlardan iborat qator yaqinlashuvchi bo'lishi uchun uning qismiy yig'indilaridan iborat ketma–ketlikning yuqoridan chegaralangan bo'lishi zarur va yetarli.

Bu teoremadan foydalanib quyidagi solishtirish teoremasini hosil qilamiz.

4–Teorema. Faraz qilaylik, hadlari manfiy bo'lmagan ikkita qator berilgan bo'lsin:

$$a_1+a_2+a_3+...+a_n+... \qquad (A),$$
$$b_1+b_2+b_3+...+b_n+... \qquad (B).$$

Agar shunday $n_0\in N$ *nomer topilsaki, barcha* $n>n_0$ *lar uchun*

$$a_n\le b_n$$

$a_n\le b_n$ *tengsizlik o'rinli bo'lsa, u holda*

1) (B) qator yaqinlashuvchi bo'lsa, (A) qator ham yaqinlashuvchi bo'ladi.

2) (A) qator uzoqlashuvchi bo'lsa, (B) qator ham uzoqlashuvchi bo'ladi.

4–misol.

$$1+\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{n^2}+... \qquad (4),$$

qatorni solishtirish alomatidan foydalanib yaqinlashishga tekshiramiz.

Ushbu

$$1+\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{(n-1)n}+... \quad (5),$$

qatorni yaqinlashishga tekshirish oson.

$$S_n=1+\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+...+\left(\frac{1}{n-1}-\frac{1}{n}\right)=1+1-\frac{1}{n}=2-\frac{1}{n},$$

$$\lim_{n\to\infty} S_n=\lim_{n\to\infty}\left(2-\frac{1}{n}\right)=2.$$

Demak, qator yaqinlashadi. Bundan va ushbu

$$\frac{1}{n^2}<\frac{1}{(n-1)n}, \qquad n=1,2,...$$

tengsizlikdan solishtirish teoremasiga ko'ra (4) qator yaqinlashadi.

5–Teorema (Absolyut yaqinlashuvchi qator uchun Veyershtras alomati). *Faraz qilaylik, (A) va (B) qatorlar berilgan va (B) qator yaqinlashuvchi ((A) ixtiyoriy qator) bo'lsin.*
Agar shunday $n_0\in N$ *nomer topilsaki, barcha* $n>n_0$ *lar uchun*

$$|a_n|\le b_n$$

$|a_n|\le b_n$ *tengsizlik o'rinli bo'lsa, u holda (A) qator absolyut yaqinlashadi.*

$|a_n|\le b_n$

5–misol. $\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$ qator absolyut yaqinlashadi, chunki $\left|\frac{\sin n}{n^2}\right|\le\frac{1}{n^2}$ $\forall n\in N$ uchun $\sum_{n=1}^{\infty}\frac{1}{n^2}$ qator yaqinlashadi.

3. Yaqinlashish alomatlari.

Koshi alomati. *Agar (A) qatorning umumiy hadi* a_n *uchun biror* $n_0 \in N$ *nomerdan boshlab, barcha* $n \geq n_0$ *lar uchun*

$$\sqrt[n]{a_n} \leq q < 1 \quad \left(\sqrt[n]{a_n} \geq 1\right)$$

o'rinli bo'lsa, u holda $\sum_{n=1}^{\infty} a_n$ *qator yaqinlashuvchi (uzoqlashuvchi) bo'ladi.*

Amaliy masalalarni hal qilishda ko'pincha, Koshi alomatining quyidagi limit ko'rinishidan foydalaniladi.

Agar ushbu

$$\lim_{n \to \infty} \sqrt[n]{a_n} = k$$

limit mavjud bo'lsa, u holda $\sum_{n=1}^{\infty} a_n$ *qator* $k < 1$ *bo'lganda yaqinlashuvchi,* $k > 1$ *bo'lganda esa uzoqlashuvchi bo'ladi.*

6-misol. $\sum_{n=1}^{\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$

qatorni Koshi alomatidan foydalanib yaqinlashishga tekshiring.

$$a_n = \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}, \quad \lim_{n \to \infty} \sqrt[n]{a_n} = \lim_{n \to \infty} \sqrt[n]{\frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}} = \lim_{n \to \infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n}} = \frac{1}{e}$$

bo'ladi. $\frac{1}{e} < 1$ bo'lgani uchun Koshi alomatiga ko'ra berilgan qator yaqinlashuvchi.

Dalamber alomati. *Agar (A) qatorning* a_n *va* a_{n+1} $(n \geq n_0)$ *hadlari uchun*

$$\frac{a_{n+1}}{a_n} = q < 1$$

bo'lsa, $\sum_{n=1}^{\infty} a_n$ *qator yaqinlashuvchi bo'ladi.*

$$\frac{a_{n+1}}{a_n} = q \geq 1$$

bo'lsa, $\sum_{n=1}^{\infty} a_n$ *qator uzoqlashuvchi bo'ladi.*

Dalamber alomatini ham limit ko'rinishida ifodalash mumkin.

Agar ushbu

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q$$

limit mavjud bo'lsa, u holda $\sum_{n=1}^{\infty} a_n$ *qator* $q<1$ *bo'lganda yaqinlashuvchi,* $q>1$ *bo'lganda esa uzoqlashuvchi bo'ladi.*

7–misol.

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

qatorni Dalamber alomatidan foydalanib yaqinlashishga tekshiring.

$$q = \lim_{n\to\infty} \frac{(n+1)!\,n^n}{(n+1)^{n+1} n!} = \lim_{n\to\infty} \frac{(n+1)}{n+1}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}.$$

Shunday qilib, $q<1$ va qator yaqinlashadi.

4–Ta'rif.

$$a_1 - a_2 + a_3 - \ldots + (-1)^{n-1} a_n + \ldots \qquad (6)$$

ko'rinishda bo'lib, $a_n > 0$ bo'lsa, u holda bunday qatorga **hadlarining ishorasi almashinib keluvchi qator** deyiladi.

6–Teorema (Leybnits alomati). *Agar*

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

qator berilgan bo'lib,

1) $\{a_n\}\downarrow$, *ya'ni* $a_n \ge a_{n+1} > 0 \quad (n = 1,2,\ldots)$

2) $\lim_{n\to\infty} a_n = 0$

bo'lsa, u holda (6) qator yaqinlashuvchi bo'ladi.

8–misol.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+(-1)^{n-1}\frac{1}{n}+...$$

qatorni yaqinlashishga tekshiring.

Berilgan qatorning hadlari absolyut qiymat bo'yicha monoton kamayadi:

$$1>\frac{1}{2}>\frac{1}{3}>....,$$

uning umuiy hadi esa $n\to\infty$ da nolga intiladi:

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}(-1)^{n-1}\frac{1}{n}=0.$$

Qator Leybnits alomatiga ko'ra yaqinlashadi.

Nazorat savollari.

1) Sonli qator deb nimaga aytiladi va u qachon yaqinlashadi.
2) Qator yaqinlashishi uchun Koshi kriteriyasi.
3) Musbat hadli qatorlar.
4) Solishtirish teoremalari. Misol.
5) Yaqinlashish alomatlari (Koshi, Dalamber va Leybnits).

3–§. Funktsiya limiti.

Ushbu

$$U_\varepsilon(a)=\{x: x\in R, \quad a-\varepsilon<x<a+\varepsilon\}$$

to'plam $x=a$ nuqtaning **atrofi (ε– atrofi)** deb ataladi.

Faraz qilaylik, $f(x)$ funktsiya biror X oraliqda aniqlangan bo'lib, $x=a$ nuqta biror X oraliqning limit nuqtasi bo'lsin ya'ni a nuqtaning ixtiyoriy $U_\varepsilon(a)$ atrofi olinganda ham $U_\varepsilon(a)\setminus\{a\}$ to'plamda X ning kamida bitta nuqtasi mavjud bo'lsin.

1–ta'rif. Agar $\forall\varepsilon>0$ uchun $\exists\delta(\varepsilon,a)=\delta>0$: $\forall x\in X\cap\{0<|x-a|<\delta\}$, uchun $|f(x)-b|<\varepsilon$ tengsizlik bajarilsa, u holda b soni $f(x)$ funktsiyaning $x\to a$ dagi limiti deb ataladi va $\lim\limits_{x\to a}f(x)=b$ kabi belgilanadi.

1–misol. $\lim\limits_{x\to 1}(2x+3)=5$ ushbu tenglikni ta'rif yordamida isbotlang.

$f(x)=2x+3,\quad b=5,\quad a=1,\quad \forall\varepsilon>0$ olamiz.

$$|f(x)-b|=|2x+3-5|=|2x-2|=2|x-1|<2\delta=\varepsilon\Rightarrow\delta=\frac{\varepsilon}{2}$$

2–ta'rif. Agar $\forall\varepsilon>0$ uchun $\exists E>0$: $\forall|x|>E$ $|f(x)-b|<\varepsilon$ bo'lsa, unda $\lim\limits_{x\to\infty}f(x)=b$ deb ataladi.

2–misol. $\lim\limits_{x\to\infty}\frac{x}{x-1}=1$ ushbu tenglikni ta'rif yordamida isbotlang.

$f(x)=\frac{x}{x-1},\quad b=1,\quad \forall\varepsilon>0$ olamiz,

$$|f(x)-b|=\left|\frac{x}{x-1}-1\right|=\frac{1}{|x-1|}=\left(|x|>E\Rightarrow|x-1|>E-1\right)<\frac{1}{E-1}=\varepsilon\Rightarrow E=1+\frac{1}{\varepsilon}.$$

Bundan kelib chiqadiki $\lim\limits_{x\to\infty}\frac{x}{x-1}=1$.

3–ta'rif. Agar $\lim\limits_{x\to a}f(x)=0$ (∞) bo'lsa, unda $x\to a$ da $f(x)$ funktsiya **cheksiz kichik (katta) funktsiya** deb ataladi.

Masalan, $x\to 1$ da $f(x)=x-1$ cheksiz kichik, $\varphi(x)=\frac{1}{x-1}$ funktsiya esa **cheksiz katta funktsiya** bo'ladi.

Xossalar:

1^0. $\alpha_1(x)$ va $\alpha_2(x)$ funktsiyalar cheksiz kichik bo'lsa, unda $\alpha_1(x)\pm\alpha_2(x)$ ham cheksiz kichik bo'ladi.

2^0. O'zgarmas sonni cheksiz kichikka ko'paytmasi cheksiz kichik bo'ladi.

3^0. Chegaralangan funktsiyaning cheksiz kichik funktsiyaga ko'paytmasi yana cheksiz kichik bo'ladi.
4^0. Chekli sondagi cheksiz kichiklarning ko'paytmasi cheksiz kichik bo'ladi.
5^0. Agar $f(x)$ cheksiz katta bo'lsa, unda $\frac{1}{f(x)}$ cheksiz kichik bo'ladi. $(f(x)\neq 0)$.
6^0. $\alpha(x)$ cheksiz kichik bo'lsa, unda $\frac{1}{\alpha(x)}$ cheksiz katta bo'ladi $(\alpha(x)\neq 0)$.

Endi funktsiyaning limiti haqidagi asosiy teoremalarni keltiramiz.

1–teorema. *$\lim\limits_{x\to a} f(x)=b$ bo'lishi uchun $f(x)=b+\alpha(x)$ bo'lishi zarur va yetarlidir, bu yerda $\alpha(x)$ cheksiz kichik funktsiya.*

Isbot. *Zaruriyligi:* faraz qilaylik $\lim\limits_{x\to a} f(x)=b$ bo'lsin. Bu degani $\forall\varepsilon>0 \quad \exists\delta>0, \quad \forall x \quad 0<|x-a|<\delta \Rightarrow |f(x)-b|<\varepsilon$,
ya'ni $\alpha(x)=f(x)-b$ funktsiya cheksiz kichik miqdor va $f(x)=b+\alpha(x)$.

Yetarliligi: faraz qilaylik $f(x)=b+\alpha(x)$, bu yelda $\alpha(x)$ – cheksiz kichik miqdor. U holda $\forall\varepsilon>0 \quad \exists\delta>0$, $0<|x-a|<\delta$ dan olingan x lar uchun $|\alpha(x)|=|f(x)-b|<\varepsilon$ o'rinli, ya'ni b soni $f(x)$ funktsiyaning $x\to a$ dagi limiti.

2–teorema. *Agar $\forall x\in U_\delta(a)=(a-\delta, \quad a+\delta)\setminus\{a\}$ uchun $f(x)\geq 0$ $(f(x)\leq 0)$ bo'lib, $\lim\limits_{x\to a} f(x)$ mavjud bo'lsa, u holda $\lim\limits_{x\to a} f(x)\geq 0$ $\left(\lim\limits_{x\to a} f(x)\leq 0\right)$ bo'ladi.*

Izoh. Agar 2–teoremada qat'iy tengsizlik qo'yilsa, u o'rinli bo'lmay qolishi mumkin. Masalan, $f(x)=|x|$ va $a=0$ bo'lsin. $\forall x\in U_\delta(0)$ uchun $f(x)>0$ va $\lim\limits_{x\to a} f(x)-\exists$, lekin $\lim\limits_{x\to a} f(x)=0$.

3–teorema. *Agar $f(x)$ va $g(x)$ funktsiyalar $x=a$ nuqtada chekli limitga ega bo'lsa, u holda shu nuqtada*

$$a)\ f(x)\pm g(x),$$
$$b)\ f(x)\cdot g(x),$$
$$c)\ \lim_{x\to a} g(x)\neq 0$$

bo'lganda, $\frac{f(x)}{g(x)}$ funktsiyalar ham chekli limitga ega bo'lib, *ushbu*

$$1.\ \lim_{x\to a}[f(x)\pm g(x)]=\lim_{x\to a}f(x)\pm\lim_{x\to a}g(x),$$

$$2.\ \lim_{x\to a}[f(x)\cdot g(x)]=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x),$$

$$3.\ \lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)},\quad \left(\lim_{x\to a}g(x)\neq 0\right)$$

tengliklar o'rinli bo'ladi.

1–natija. $\lim\limits_{x\to a}[f(x)]^n=\left[\lim\limits_{x\to a}f(x)\right]^n$

2–natija. $\lim\limits_{x\to a}[c\cdot f(x)]=c\cdot\lim\limits_{x\to a}f(x).$

4–teorema. *Agar a nuqtaning biror atrofida $f_1(x)\le f(x)\le f_2(x)$ bo'lib, $\lim\limits_{x\to a}f_1(x)=\lim\limits_{x\to a}f_2(x)=b$ bo'lsa, u holda $\lim\limits_{x\to a}f(x)=b$ bo'ladi.*

Isbot. Teorema shartidan a nuqtaning $0<|x-a|<\delta$ atrofida bir vaqtning o'zida quyidagi tengsizlik bajariladi:

$$|f_1(x)-b|<\varepsilon,\qquad |f_2(x)-b|<\varepsilon,$$

bu yerda ε – ixtiyoriy musbat son. Bundan quyidagini yozib olamiz:

$$b-\varepsilon<f_1(x)<b+\varepsilon,\qquad b-\varepsilon<f_2(x)<b+\varepsilon.$$

Bu tengsizliklardan quyidagini yozib olamiz:

$$b-\varepsilon<f_1(x)\le f(x)\le f_2(x)<b+\varepsilon.$$

Yuqoridagi tengsizliklardan

$$b-\varepsilon<f(x)<b+\varepsilon\quad \text{yoki}\quad |f(x)-b|<\varepsilon$$

kelib chiqadi.

Ushbu, $\lim\limits_{x\to a}\frac{\sin x}{x}=1$ va $\lim\limits_{x\to a}\left(1+\frac{1}{x}\right)^x=e$ limitlarga mos ravishda birinchi va ikkinchi **ajoyib limitlar** deyiladi. Bu limitlar limit hisoblashda, $\frac{0}{0}$ va 1^{∞} ko'rinishdagi aniqmasliklarni ochishda keng qo'llaniladi.

3–misol. $\lim\limits_{x\to0}\frac{\sqrt{4-x}-\sqrt{4+x}}{x}$ limitni toping.

$x\to0$ da limit $\frac{0}{0}$ ko'rinishga keladi. Aniqmaslikdan qutilish uchun kasr suratini qo'shmasiga ko'paytirib bo'lamiz.

$$\lim_{x\to0}\frac{(\sqrt{4-x}-\sqrt{4+x})\left(\sqrt{4-x}+\sqrt{4+x}\right)}{x\left(\sqrt{4-x}+\sqrt{4+x}\right)}=\lim_{x\to0}\frac{4-x-4-x}{x\left(\sqrt{4-x}+\sqrt{4+x}\right)}=$$

$$=-2\lim_{x\to0}\frac{1}{\sqrt{4-x}+\sqrt{4+x}}=-\frac{1}{2}$$

4–misol . $\lim\limits_{x\to0}\frac{\sin^2 x}{x}=\lim\limits_{x\to0}(\frac{\sin x}{x}\cdot\sin x)=\lim\limits_{x\to0}\frac{\sin x}{x}\lim\limits_{x\to0}\sin x=1\cdot0=0.$

5–misol.

$$\lim_{x\to\infty}\left(1-\frac{1}{x^2}\right)^{x}=\left(1^{\infty}\right)=\lim_{x\to\infty}\left[\left(1+\frac{1}{-x^2}\right)^{-x^2}\right]^{\frac{x}{-x^2}}=e^{\lim\limits_{x\to\infty}\left(\frac{-x}{x^2}\right)}=e^{\lim\limits_{x\to\infty}-\frac{1}{x}}=e^{0}=1.$$

6–misol.

$$\lim_{x\to\infty}(1-\sin x)^{\frac{1}{x}}=\left(1^{\infty}\right)=\lim_{x\to0}\left\{\left[1+(-\sin x)\right]^{\frac{1}{\sin x}}\right\}^{-\frac{\sin x}{x^2}}=e^{\lim\limits_{x\to\infty}\left(-\frac{\sin x}{x^2}\right)}=$$

$$=e^{-\lim\limits_{x\to\infty}\frac{\sin x}{x}\cdot x}=e^{-1\cdot0}=e^{0}=1.$$

7–misol.

$$\lim_{x\to\frac{\pi}{2}}(\sin x)^{tgx}=\left(1^{\infty}\right)=\lim_{x\to\frac{\pi}{2}}\left(1-\cos^2 x\right)^{\frac{tgx}{2}}=\lim_{x\to\frac{\pi}{2}}\left[\left(1-\cos^2 x\right)^{\frac{-1}{\cos^2 x}}\right]^{-\frac{\cos^2 x\,tgx}{2}}=$$

$$=e^{-\frac{1}{2}\lim\limits_{x\to\frac{\pi}{2}}\cos x\sin x}=e^{0}=1$$

Nazorat savollari.

1. Funktsiya limiti.
2. Cheksiz katta (kichik) funktsiyalar.
3. Funktsiya limitining xossalari.
4. Birinchi va ikkinchi ajoyib limitlar.

4–§. Funktsiyaning uzluksizligi

Aytaylik, $y=f(x)$ funktsiya x_0 nuqtaning biror atrofida aniqlangan bo'lib, x nuqta shu intervalga (atrofga) tegishli nuqta bo'lsin.

$x-x_0=\Delta x$ deb belgilaymiz $\Rightarrow$

$x=x_0+\Delta x$, Δx – argument orttirmasi.

$\Delta y=f(x_0+\Delta x)-f(x_0)$ – funktsiya orttirmasi.

1–ta'rif: Agar $\lim\limits_{\Delta x\to 0}\Delta y=0$ bo'lsa, unda $y=f(x)$ funktsiya $x=x_0$ nuqtada uzluksiz deyiladi.

2–ta'rif: Agar $\lim\limits_{\Delta x\to 0}f(x)=f(x_0)$ bo'lsa, u holda $y=f(x)$ funktsiya $x=x_0$ nuqtada **uzluksiz** deyiladi.

Misollar: 1) $y=x$, 2)$y=x^3$, 3)$y=\sin x$ funktsiyalar $\forall x_0\in R$ nuqtalarda uzluksiz.

1–teorema: Agar $f(x), g(x)$ funktsiyalar x_0 nuqtada uzluksiz bo'lsa, u holda

$1) f(x)\pm g(x)$, $2) f(x)\cdot g(x)$, $3) f(x)/g(x)$ $(g(x)\neq 0)$,

funktsiyalar ham x_0 nuqtada uzluksiz bo'ladi.

2–teorema: *Agar $f(x)$ funktsiya x_0 nuqtada uzluksiz bo'lib, $f(x_0)\neq 0$ bo'lsa, u holda $f(x)$ funktsiya x_0 nuqtaning biror atrofida bir xil ishorani saqlaydi, $f(x)>0$ $(f(x)<0)$ bo'lsa, $\exists U_\delta(x_0)$ atrof: $\forall x\in U_\delta(x_0)$ uchun $f(x)<0$ $(f(x)<0)$bo'ladi.*

3–ta'rif: Funktsiyaning uzluksizligi buziladigan nuqtalarga uning **uzilish nuqtalari** deb ataladi.

Uzilish nuqtalari uch turga bo'linadi:

1) $\lim\limits_{x\to a}f(x)=b\neq f(a)$ bo'lsin.

Bu holda $\lim\limits_{x\to a+0} f(x) = f(a+0)$ va $\lim\limits_{x\to a-0} f(x) = f(a-0)$ Lar mavjud bo'lib, $f(a+0) = f(a-0) \neq f(a)$ bo'ladi. Bunday nuqta bartaraf qilish mumkin bo'lgan uzilish nuqtasi deb ataladi.

Misol. $f(x) = \begin{cases} x^2, агар & x \neq 0 \quad бўлса \\ 1, агар & x = 0 \quad бўлса \end{cases}$

Funktsiya uchun $x = 0$ nuqta bartaraf qilish mumkin bulgan uzilish nuqtasi bo'ladi, chunki $\lim\limits_{x\to a+0} f(x) = \lim\limits_{x\to a-0} f(x) = 0$ va $f(0) = 1$. Agar $f(0) = 0$ deb qabul qilsak, funktsiya uzluksiz bo'lib qoladi.

2) $\lim\limits_{x\to a+0} f(x)$ – mavjud bo'lmasin.

Bunda quyidagi uchta hol bo'lishi mumkin.

a) $\lim\limits_{x\to a-0} f(x) = f(a-0)$ va $\lim\limits_{x\to a+0} f(x) = f(a+0)$ lar mavjud va $f(a-0) \neq f(a+0)$.

Funktsiyaning bunday nuqtadagi uzilishi birinchi tur uzilish va $|\, f(a-0) - f(a+0)\,|$ ayirmaga funktsiyaning a nuqtadagi sakrashi deyiladi.

b) $x \to a$ da $f(x)$ funktsiyaning o'ng va chap limitlaridan hech bo'lmaganda biri mavjud bo'lmasin. Funktsiyaning a nuqtadagi bo'nday uzilishi ikkinchi tur uzilish deyiladi.

Misol. $f(x) = \begin{cases} \sin\dfrac{1}{x}, агар & x > 0 \\ -x, агар & x \leq 0 \end{cases}$

Funktsiya $x = 0$ nuqtada ikkinchi tur uzilishga ega, chunki

$\lim\limits_{x\to -0} f(x) = \lim\limits_{x\to -0}(-x) = 0 = f(0)$, lekin $\lim\limits_{x\to +0} f(x) = \lim\limits_{x\to 0} \sin\dfrac{1}{x}$ – mavjud emas.

b) $x \to a$ da $f(x)$ funktsiyaning o'ng va chap limitlaridan biri cheksiz yoki o'ng va chap limitlar turli ishorali cheksiz. Funktsiyaning a nuqtadagi bo'nday uzilishi ham ikkinchi tur uzilish deyiladi.

Agar $\lim\limits_{x\to x_0-0} f(x) = f(x_0)$ $\left(\lim\limits_{x\to x_0+0} f(x) = f(x_0)\right)$ bo'lsa, unda $y = f(x)$ funktsiya x_0 nuqtalarda **<u>chapdan (o'ngdan) uzluksiz</u>** deb ataladi.

Agar $f(x)$ funktsiya $[a,b]$ kesmada aniqlangan bo'lib, (a,b) intervalda uzluksiz hamda a nuqtada o'ngdan, b nuqtada esa chapdan uzluksiz bo'lsa, u holda $f(x)$ funktsiya $[a,b]$ kesmada **uzluksiz** deb ataladi va $f(x)\in C[a,b]$ kabi belgilanadi.

Endi kesmada uzluksiz bo'lgan funktsiyalarning xossalarini keltiramiz:

1^0. Agar $f(x)\in C[a,b]$ bo'lib, kesmaning chetki nuqtalarida turli ishoralarni qabul qilsa, $\exists c\in(a,b)$ nuqtalarda $f(c)=0$ bo'ladi.

2^0. $f(x)\in C[a,b]\Rightarrow f(x)$ — chegaralangan, ya'ni $\exists M$ son: $\forall x\in[a,b]$ uchun $|f(x)|\le M$ bo'ladi.

3^0. $f(x)\in C[a,b]\Rightarrow \exists x_1,x_2\in[a,b]$ nuqtalar: $f(x_1)$ va $f(x_2)$ lar mos ravishda $f(x)$ funktsiyaning $[a,b]$ kesmadagi eng katta va eng kichik qiymatlari bo'ladi.

Izoh. Xossalardagi oraliqlarning yopiq kesmadan iborat bo'lishlik sharti muhim shartdir.

Masalan, 2^0 – xossada oraliq $(0,1)$ intervaldan iborat bo'lib, $f(x)=\frac{1}{x}$ bo'lsin. $f(x)=\frac{1}{x}\in C(0,1)$, lekin funktsiya $(0,1)$ oraliqda chegaralangan emas.

Uzluksiz funktsiyaning biologik masalalarda qo'llanilishi.

Ba'zi biologik hodisalar uzluksiz va uzilishga ega bo'lgan funktsiyalar yordamida ifodalanadi.

Shuni ta'kidlash kerakki eshitish, ko'rish, ultra tovushlarni eshitish tebranma harakatlar bilan bog'langan bo'lib, bu harakatlar sinx, cosx kabi trigonometrik funktsiyalar yordamida ifodalanadi.

Aniq tajribalarda populyatsiya biomassasi, populyatsiya miqdori, yo'l, temperatura, vaqt va hokazolar har qanday haqiqiy sonlarni qabul qilmasligi mumkin. Masalan, yo'l kilometr yoki millimetrda, biomassa tonnada yoki milligrammda, vaqt yillar yoki o'ndan bir sekundlarda o'lchanadi. Shartli ravishda kelishilganda keltirilgan funktsiyalarning qiymatlar va aniqlanish sohalari oraliqlar bo'lmasdan, balki juda kichik maxsus to'plamlardir. Juda katta hayvonlarning yoshi yillar bilan, ba'zi mikroorganizmlarning yoshi sekundning o'ndan biri bilan o'lchanadi. Mikroorganizmlar 3,1 yoki 3,2 sekund yashaydi deya olishimiz mumkin, lekin π sekund yashadi deb ayta olmaymiz. Bunday funktsiyalar to'g'risida gapirganda uzluksiz funktsiya haqida gapirib bo'lmaydi. Matematik analiz tushunchalaridan foydalanganda funktsiyaning aniqlanish sohasi ikkita elementdan iborat bo'lsa uni oraliq bilan almashtirish maqsadga muvofiq emas. Biroq funktsiyaning aniqlanish va qiymatlar sohasi cheklita, lekin qandaydir ma'noda yetarlicha katta miqdordagi bir–biriga yaqin joylashgan elementlardan iborat bo'lsa (ya'ni o'lchami juda kichik oraliqlarga bo'lingan bo'lsa), u holda bo'nday funktsiyaning Aniqlanish sohasini shartli ravishda oraliq bilan almashtirsa bo'ladi. Bu keltirgan izoh uzluksiz funktsiyadan foydalanish uchun keltirilgan matematik modeldir. Qaralayotgan funktsiyalarni uzluksiz deb faraz qilamiz va bundan buyon bu haqda eslatmaymiz.

Biologiyada uzluksiz funktsiyaga doir misolni qaraymiz. Hujayralar bo'linishidagi mikroorganizmlar o'sishini o'rganayotganda $f(t)=ae^{kt}$ kabi funktsiyalarga duch kelamiz (bu yerda argument t vaqtni aniqlaydi).

Darajali funktsiya $f(x)=Ax^{\alpha}$ - hayvon og'irligidan asosiy intensiv almashinuvning bog'liqligini ifodalaydi. Bu yerda x – hayvon og'irligi, $f(x)$ – birlik vaqt ichida yutiladigan kislorod miqdori, A va α – tirik mavjudotning turiga qarab aniqlanadigan o'zgarmas parametrlar. Misol uchun qushlarda $A=70$, $\alpha=0{,}74$ ga, baliqlarda $A=0{,}3$ $\alpha=0{,}8$ ga teng.

Uzilishga ega bo'lgan funktsiya uchun misol keltiramiz. Tashqi ta'sirdan qo'zg'aladigan ho'jayralarni qaraylik. Misol uchun nerv hujayralari, mushak hujayralari va boshqalar. t_0 vaqtda hujayra signal qabul qiladi. Biroq ta'sir birmuncha keyinroq bo'ladi, ya'ni $t_1>t_0$. $[t_0;t_1]$ kesma latent davri deyiladi. t_1 vaqtda hujayra oniy qo'zg'aladi va eng yuqori qiymatga erishadi, so'ngra asta sekin kamayib toki boshqa ta'sir bo'lmaguncha nolga yaqinlashib boradi.

Shunday qilib harakat miqdorining funktsiyasi t ga bog'liq bo'lib, latent davrining oxirida uzilishga ega bo'lar ekan.

Nazorat savollari.

1. Funktsiya uzluksizligining ta'rifi.
2. Funktsiya uzilish nuqtalari.
3. Uzluksiz funktsiyalarning xossalari.
4. Uzluksiz funktsiyaning biologik masalalarda qo'llanilishi.

5–§. Hosila. Elementar funktsiyalarning hosilalari.

Tabiatdagi ko'p masalalar hosila tushunchasiga olib keladi. Masalan, harakatdagi nuqtaning tezligi haqidagi masala, egri chiziqqa o'tkazilgan urinma masalasi va boshqalar.

Ximik reaktsiya tezligi haqidagi masala. Aytaylik, $m = m(t)$ funktsiya berilgan bo'lsin, bu yerda $m-$ t vaqtdagi ximiyaviy reaktsiyaga kirishgan modda miqdori. Δt orttirmaga m miqdorning Δm orttirmasi mos keladi. $\frac{\Delta m}{\Delta t}$ nisbat Δt vaqtdagi ximik reaktsiyaning o'rtacha tezligi. Bu nisbatning $\Delta t \to 0$ dagi limiti t vaqtdagi ximik reaktsiya tezligini beradi.

Aytaylik, $y = f(x)$ funktsiya (a,b) oraliqda aniqlangan bo'lib, $x \in (a;b)$ bo'lsin. x nuqtaga $\exists \Delta x \begin{pmatrix} \le \\ \ge \end{pmatrix} 0$ orttirma beraylikki, $x + \Delta x \in (a,b)$ bo'lsin. Unda funktsiya ham $\Delta y = f(x + \Delta x) - f(x)$ orttirma qabul qiladi.

Agar

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

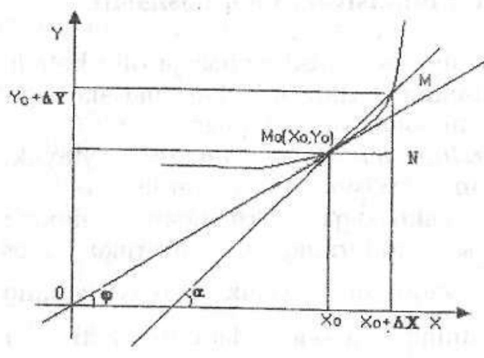
mavjud bo'lib, chekli songa teng bo'lsa, shu songa $y = f(x)$ funktsiyaning x **nuqtadagi hosilasi** deb ataladi va u $f'(x)$, $\frac{df}{dx}$, y' – kabi belgilanadi.

Shunday qilib,

$$f'(x) := \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (1)$$

Misollar. 1) $(C)'=0$, 2) $(x)'=1$, 3) $(x^3-1)'=3x^2$

Hosilaning geometrik ma'nosi.



Quyidagi belgilashlarni kiritamiz:

$$\angle MM_0N=\alpha$$

$$tg\alpha=\frac{MN}{M_0N}=\frac{\Delta y}{\Delta x}$$

$$tg\varphi=\lim_{\Delta x\to 0} tg\alpha=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=f'(x) \quad (2)$$

$y-y_0=f'(x_0)\cdot(x-x_0)$ (3) – urinma tenglamasi.

$y-y_0=-\frac{1}{f'(x_0)}\cdot(x-x_0)$ (4) – normal tenglamasi.

1–teorema: *Agar $y=f(x)$ funktsiya biror x nuqtada differentsiallanuvchi bo'lsa, u holda u shu x nuqtada uzluksiz bo'ladi.*

Izoh. Teoremaning teskarisi har doim ham o'rinli bo'lavermaydi. Masalan, $y=|x|$ funktsiya $x=0$ nuqtada uzluksiz, lekin differentsiallanuvchi emas.

Chunki,

$$f'(0)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{|\Delta x|}{\Delta x}=\begin{cases}1, agap\Delta x>0\\ -1, agap\Delta x<0\end{cases}\Rightarrow f'(0)$$

mavjud emas.

2—teorema. Aytaylik, $u=u(x)$ *va* $v=v(x)$ *funktsiyalar* $x=0$ *nuqtada differentsiallanuvchi bo'lib, chekli* $u'(x)$ *va* $v'(x)$ *hosilalarga ega bo'lsin.*

Unda,

$$1)\ [u\pm v]'=u'\pm v',$$

$$2)\ [u\cdot v]'=u'v\pm uv',$$

$$3)\ \left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}\qquad (v\neq 0)$$

bo'ladi.

Natija: $[C\cdot f(x)]'=C\cdot f'(x)$.

3—teorema. Aytaylik, $y=f(u)$ *bo'lib,* $u=\varphi(x)$ *bo'lsin. Agar* $u=\varphi(x)$ *funktsiya* x *nuqtada* $\varphi'(x)$ *hosilaga,* $y=f(u)$ *funktsiya esa* x *nuqtaga mos* $u=\varphi(x)$ *nuqtada* $\varphi'(x)$ *hosilaga ega bo'lsa, u holda* $y=f[\varphi(x)]=F(x)$ *murakkab funktsiya ham* x *nuqtada hosilaga ega bo'ladi va*

$$y'=F'(x)=f'(u)\cdot\varphi'(x)\ (5)$$

tenglik o'rinli bo'ladi.

(5) – tenglikka **murakkab funktsiyaning hosilasini hisoblash formulasi** deyiladi.

Endi $y=f(x)$ funktsiya va unga teskari bo'lgan $x=f^{-1}(y)=\varphi(y)$ funktsiyalar berilgan bo'lsin.

4—teorema: Agar $y=f(x)$ *va* $x=\varphi(y)$ *funktsiyalar differentsiallanuvchi bo'lsa, unda* $\varphi'(y)=\frac{1}{f'(x)}$ *yoki* $x'_y=\frac{1}{y'_x}$ *(6) formulalar o'rinli bo'ladi.*

2 – 4 – teoremalarni qo'llash natijasida quyidagi hosilalar jadvalini hosil qilamiz.

1) $(C)' = 0,$

2) $(C \cdot u)' = C \cdot u',$

3) $(u + v)' = u' + v',$

4) $(u \cdot v)' = u'v + uv',$

5) $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2},$

6) $(x'_y) = \frac{1}{y'_x},$

7) $(u^\alpha)' = \alpha \cdot u^{\alpha-1} \cdot u',$

8) $(a^u)' = a^u \cdot u' \cdot \ln a, \quad (a > 0)$

9) $(e^u)' = e^u \cdot u',$

10) $(\log_a u)' = \frac{u'}{u}, \quad (a > 0, a \neq 1)$

11) $(\ln u)' = \frac{u'}{u},$

12) $(\sin u)' = u' \cos u,$

13) $(\cos u)' = -u' \cdot \sin u,$

14) $(tgu)' = \frac{u'}{\cos^2 u},$

15) $(ctgu)' = \frac{-u'}{\sin^2 u},$

16) $(\arcsin u)' = \frac{u'}{\sqrt{1 - u^2}},$

17) $(\arccos u)' = -\frac{u'}{\sqrt{1 - u^2}},$

18) $(arctgu)' = \frac{u'}{1 + u^2},$

19) $(arcctgu)' = -\frac{u'}{1 + u^2}.$

Bu jadvalda $u = u(x)$ va $v = v(x)$ lar differentsiallanuvchi funktsiyalar.

Agar funktsiya orttirmasini ushbu,

$$\Delta f(x) = f(x + \Delta x) - f(x) = A(x) \cdot \Delta x + \alpha(x, \Delta x) \cdot \Delta x \quad (7)$$

ko'rinishda ifodalash mumkin bo'lib, bu yerda A – o'zgarmas son va $\lim_{\Delta x \to 0} \alpha(x, \Delta x) = 0$ bo'lsa unda $y = f(x)$ funktsiya x nuqtada **differentsiallanuvchi** deyiladi va $A \cdot \Delta x$ ga funktsiya orttirmasining **chiziqli bosh qismi** deyiladi, hamda $df(x)$ kabi belgilanadi.

(7) tenglikdan, $df(x) = A \cdot \Delta x = f'(x) \cdot \Delta x$ ekanligini ko'rish qiyin emas. Agar bu tenglikda $\Delta x = dx$ desak, u holda

$$df(x) = f'(x)dx \quad (8)$$

tenglikni hosil qilamiz.

(7) va (8) dan

$$\Rightarrow \Delta y = dy + \alpha\cdot\Delta x \Rightarrow \lim_{\Delta x\to 0}\Delta y = \lim_{\Delta x\to 0}(dy + \alpha\cdot\Delta x) = \lim_{\Delta x\to 0} dy \Rightarrow \Delta y \approx dy = f'(x)\cdot\Delta x \Rightarrow$$
$$\Rightarrow f(x+\Delta x) \approx f(x) + f'(x)\cdot\Delta x \qquad (9)$$

(9) — taqribiy hisoblash formulasi.

1–misol. $\sqrt[3]{1,1}$ ni hisoblang.

Funktsiyani $f(x)=\sqrt[3]{x}$ deb, $f'(x)=\dfrac{1}{3\sqrt[3]{x^2}}$, $x=1$, $\Delta x=0,1$ kabi olamiz. (9) formuladan foydalanamiz:

$$\sqrt[3]{1,1} \approx \sqrt[3]{1} + \frac{1}{3\sqrt[3]{1^2}}\cdot 0,1 = 1 + 0,1\cdot\frac{1}{3} = 1,033.$$

Jadvalga ko'ra $\sqrt[3]{1,1}\approx 1,032$ ga teng.

$y=f(x)$ funktsiyaning birinchi tartibli hosilasi $y'=f'(x)$ dan olingan hosilaga (agar u mavjud bo'lsa) $y=f(x)$ funktsiyaning **ikkinchi tartibli hosilasi** deb ataladi va

$$y'',\ f''(x),\ \frac{d^2 f}{dx^2}$$

belgilarining biri yordamida belgilanadi. Funktsiyaning n–tartibli hosilasi quyidagi tenglik bilan aniqlanadi:

$$y^{(n)} := \left[f^{(n-1)}(x)\right]'$$

Funktsiyaning yuqori tartibli differentsiallari ham shu kabi aniqlanadi.

Nazorat savollari.

1. Funktsiya hosilasining ta'rifi.
2. Hosilaning geometrik ma'nosi.
3. Hosila jadvali.
4. Hosila yordamida taqribiy hisoblash.

6–§. Differentsial hisobning asosiy xossalari.

1–teorema. (Ferma): *Agar (a,b) da berilgan $y=f(x)$ funktsiya shu intervalning birorta ichki s nuqtasida eng katta yoki eng kichik qiymatiga erishsa va $f'(c)-\exists$ bo'lsa, unda $f'(c)=0$ bo'ladi.*

Isbot. Aytaylik, $y=f(x)$ funktsiya c nuqtada $(c\in(a,b))$ o'zining eng katta qiymatiga erishsin. c qiymatga yetarlicha kichik Δx orttirma beramiz. U holda $f(c+\Delta x)<f(c)$. Agar $\Delta x<0$ bo'lsa,

$$\frac{\Delta y}{\Delta x}=\frac{f(c+\Delta x)-f(c)}{\Delta x}>0 \quad \text{va} \quad \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=f'(c)\geq 0 \qquad (1).$$

bo'ladi. Agar $\Delta x>0$ bo'lsa $\frac{\Delta y}{\Delta x}<0$ va

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=f'(c)\leq 0 \qquad (2)$$

bo'ladi. (1) va (2) lardan $f'(c)=0$ kelib chiqadi.

Bu teoremaning geometrik ma'nosi funktsiya grafigining $(c;f(c))$ nuqtasiga o'tkazilgan urinma OX o'qiga parallel bo'ladi.

Izoh. Teoremaning barcha shartlari muhim. Misol uchun $0\leq x\leq\frac{\pi}{2}$ segmentda $y=\sin x$ funktsiya $x_0=0$ eng kichik qiymatga erishadi, lekin funktsiyaning hosilasi shu nuqtada birga teng.

2–teorema (Roll): *Aytaylik, $y=f(x)$ funktsiya $[a,b]$ kesmada aniqlangan bo'lib, quyidagi shartlarni qanoatlantirsin:*

1) $f(x)\in C[a,b]$
2) $\forall x\in(a,b)$ *uchun* $f'(x)-\exists$
3) $f(a)=f(b)$

U holda $\exists c\in[a,b]$ nuqta: $f'(c)=0$ bo'ladi.

Izoh. Roll teoremasining shartlari ham muhim. $x=0$ nuqtadan tashqarida $f(x)=|x|$ funktsiya $-1\leq x\leq 1$ da teoremaning barcha shartlarini qanoatlantiradi. $(-1;1)$

oraliqda birorta ham nuqta yo'qki hosilasi 0 ga teng bo'lsa: $(-1;0)$ da $f'(x)=-1$, $(0;1)$, $f'(x)=1$ ga teng. $x=0$ da funktsiya hosilaga ega emas.

3—teorema (Lagranj): *$[a,b]$ da aniqlangan $y=f(x)$ funktsiya quyidagi shartlarni qanoatlantirsin:*

1) $f(x)\in C[a,b]$,

2) $\forall x\in(a,b)$ *uchun* $f'(x)-\exists$.

Unda $\exists c\in[a,b]$ *nuqta topiladiki,*

$$\frac{f(b)-f(a)}{b-a}=f'(c) \qquad (3)$$

bo'ladi.

Agar Lagranj teoremasida $f(a)=f(b)$ bo'lsa, unda $f'(c)=0$ bo'ladi, ya'ni Lagranj teoremasidan Roll teoremasi kelib chiqadi.

Natija: Agar (a,b) da $f'(x)=0$ bo'lsa, unda shu intervalda $f(x)=const$ bo'ladi.

4—teorema (Lopital qoidasi): *Faraz qilamiz* $\lim\limits_{x\to a}f(x)=0$ (∞) *va* $\lim\limits_{x\to a}g(x)=0$ (∞) *bo'lib,* $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}-\exists$ *bo'lsin.*

U holda,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)} \qquad (4)$$

bo'ladi.

Izoh: Boshqa barcha aniqmasliklar $\frac{0}{0}$ yoki $\frac{\infty}{\infty}$ ko'rinishdagi aniqmasliklarga keltirish yordamida yechiladi.

1—misol.

$$\lim_{x\to 0}\frac{2^x-1}{\sin x}=\frac{0}{0}=\lim_{x\to 0}\frac{(2^x-1)'}{(\sin x)'}=\lim_{x\to 0}\frac{2^x\ln 2}{\cos x}=\ln 2.$$

2—misol.

$$\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2}=\frac{0}{0}=\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{\pi-2x}=\lim_{x\to\frac{\pi}{2}}\frac{\sin x}{2}=\frac{1}{2}.$$

3–misol.

$$\lim_{x\to 0}\left(ctgx-\frac{1}{x}\right)=(\infty-\infty)=\lim_{x\to 0}\left(\frac{\cos x}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{(x\cos x-\sin x)'}{(x\sin x)'}=$$

$$=\lim_{x\to 0}\frac{\cos x-x\sin x-\cos x}{\sin x+x\cos x}=-\lim_{x\to 0}\frac{(x\sin x)'}{(\sin x+x\cos x)'}=-\lim_{x\to 0}\frac{\sin x+x\cos x}{\cos x+\cos x-x\sin x}=-\frac{0}{2}=0.$$

Nazorat savollari.

1. Ferma teoremasi.
2. Roll teoremasi.
3. Lagranj teoremasi.
4. Lopital qoidasi.

7–§. Funktsiyani to'liq tekshirish va uning grafigini yasash.

1. Funktsiyaning o'sishi va kamayishi.

Aytaylik, $y=f(x)$ funktsiya (a,b) intervalda berilgan bo'lsin. Agar shu intervaldan olingan ixtiyoriy $x_2>x_1$ uchun $f(x_2)>f(x_1)$ $(f(x_2)<f(x_1))$ bo'lsa, unda $f(x)$ funktsiya (a,b) oraliqda **o'suvchi (kamayuvchi)** deyiladi.

O'suvchi va kamayuvchi funktsiyalarga **monoton funktsiyalar** deb ataladi. Monoton funktsiyalar hayotda ko'p uchraydi. Masalan, o'sayotgan daraxtning bo'yi, yetilayotgan donning og'irligi–vaqtning o'suvchi funktsiyalari; yorug'lik manbaidan ma'lum masofadagi yoritilganlik – masofaning kamayuvchi funktsiyasi bo'ladi.

1–teorema: *Agar $f(x)$ funktsiya (a,b) da differentsiallanuvchi bo'lib, kamayuvchi (o'suvchi) bo'lmasa, u holda (a,b) oraliqda $f'(x)\geq 0$ $(f'(x)\leq 0)$ bo'ladi.*

2–teorema: *Agar $f(x)$ funktsiya (a,b) oraliqda differentsiallanuvchi bo'lib, $f'(x)>0$ $(f'(x)<0)$ shartni qanoatlantirsa, u holda (a,b) oraliqda funktsiya o'suvchi (kamayuvchi) bo'ladi.*

Misollar.

1) $y=e^x$ funktsiya $(-\infty,+\infty)$ da o'suvchi, chunki $\forall x\in R$ uchun $y'=e^x>0$.
2) $y=x^2$ funktsiya uchun $y'=2x\Rightarrow x^2$ funktsiya $[-\infty;0]$ da kamayuvchi va $[0;+\infty)$ da o'suvchi.

2. Funktsiyaning ekstremumlari.

Agar x_0 nuqtaning $\exists\cup_\delta(x_0)$ atrofi; $\forall x\in\cup_\delta(x)$ uchun $f(x)<f(x_0),(f(x)>f(x_0))$ tengsizlik bajarilsa, unda $f(x)$ funktsiya x_0 nuqtada lokal **maksimumga (minimumga) erishadi** deyiladi va $f(x_0)$ qiymatga funktsiyaning $\cup_\delta(x_0)$ atrofdagi **maksimum (minimum) qiymati** deb ataladi, hamda u

$$\max_{\cup_\delta(x)}\{f(x)\},\qquad \min_{\cup_\delta(x)}\{f(x)\}$$

kabi belgilanadi.

2—teorema: (Ekstremum mavjudligining zaruriy sharti).

Agar $f(x)$ funktsiya (a,b) oraliqda differentsiallanuvchi bo'lib, x_0 $(a<x<b)$ nuqtada ekstremumga erishsa, u holda $f'(x_0)=0$, bo'ladi.

Izoh: $f(x)=x^3$ funktsiya uchun $x=0$ nuqtada $f'(x)=0$, lekin bu nuqtada funktsiya ekstremumga erishmaydi, funktsiyaning hosilasi nolga teng bo'lishi yetarli bo'lmay, zaruriy shart ekan.

*3—teorema: (Yetarli shart). Agar $f'(x_0)=0$ bo'lib, $f(x)$ funktsiya x_0 nuqtadan chapdan o'ngga o'tganda ishorasini musbat*dan *(manfiy*dan) *manfiy*ga *(musbat*ga*)* o'zgartirsa, *u holda $f(x)$ funktsiya x_0 nuqtada maksimumga (minimumga) erishadi. Agar x_0 nuqtadan o'tganda ishorasini o'zgartirmasa, u holda $f(x)$ funktsiya x_0 nuqtada ekstremumga erishmaydi.*

Misollar.

1) $f(x)=e^x-x,\quad f^{1}(x)=e^x-1,\quad f^{1}(x)=0\Rightarrow x=0$

x	x<0	X=0	x>0
$f'(x)$	−	0	+
$f(x)$	kamayuvchi	min	o'suvchi

$y_{min}=f(0)=e^0-0=1$

2) $f(x)=x^3-3x+2,\quad f^{1}(x)=3x^2-3,\quad f^{1}(x)=0,\ x=-1,\quad x=1.$

x	$(-\infty;-1)$	-1	$(-1;1)$	1	$(1;+\infty)$
$f'(x)$	+	0	−	0	+
$f(x)$	o'suvchi	max	kamayuvc hi	min0	o'suvchi

$y_{max}=y(-1)=4,\qquad y_{min}=y(1)=0$

4–teorema. *Faraz qilaylik,* $f(x)$ *funktsiya* x_0 *nuqta va uning atrofida uzluksiz hamda* $f'(x),f''(x)\in C\{x_0\}$ *bo'lib,* $f'(x)=0, f''(x)\neq 0$ *bo'lsin. Agar,* $f''(x_0)>0$ $(f''(x_0)<0)$ *bo'lsa, u holda* $y=f(x)$ *funktsiya* x_0 *nuqtada minimumga (maksimumga) erishadi.*

Izohlar:

1) Funktsiya hosilasi mavjud bo'lmagan nuqtada ham ekstremumga erishishi mumkin. Masalan, $f(x)=|x|$ funktsiya uchun $f'(0)$ mavjud emas, lekin funktsiya $x=0$ nuqtada minimumga erishadi.

2) Shuningdek funktsiyaning hosilasi ∞ ga aylanadigan nuqtada ham funktsiya ekstremumga erishishi mumkin. Masalan, $f(x)=\sqrt[3]{x^2}$ funktsiya uchun $f'(0)=\infty$ bo'lsa ham shu nuqtada funktsiya min ga erishadi.

3. Funktsiyaning botiq va qavariqligi. Burilish nuqtasi.

Faraz qilaylik, (a,b) oraliqda differentsiallanuvchi $f(x)$ funktsiya berilgan bo'lsin. Unda bu funktsiya grafigining ixtiyoriy nuqtasidan urinma o'tkazish mumkin. Agar har doim urinma grafikdan pastda (yuqorida) joylashgan bo'lsa, unda funktsiya (a,b) oraliqda **botiq (qavariq)** deb ataladi. Funktsiya grafigi o'z botiqligi yoki qavariqligini o'zgartiradigan nuqtaga esa **egilish nuqtasi** deyiladi.

5–teorema. Agar (a,b) *oraliqda* $f''(x) > 0(< 0)$, *bo'lsa, u holda* $f(x)$ *funktsiya grafigi* (a,b) *oraliqda botiq (qavariq) bo'ladi.*

4. Funktsiyaning asimptotalari.

a) Vertikal asimptota. Agar $\lim\limits_{x\to a} f(x) = \infty$ bo'lsa, $x = a$ to'g'ri chiziq vertikal asimptota bo'ladi.

b) Gorizontal asimptota. Agar $\lim\limits_{x\to\infty} f(x) = b$ bo'lsa, $y = b$ to'g'ri chiziq gorizontal asimptota bo'ladi.

v) Og'ma asimptota. Agar $\lim\limits_{x\to a}[f(x) - (kx + b)] = 0$ bo'lsa, $y = kx + b$ to'g'ri chiziq og'ma asimptota bo'ladi.

To'g'ri chiziqning parametrlari ushbu,

$$k = \lim_{x\to\infty}\frac{f(x)}{x}, \quad b = \lim_{x\to\infty}[f(x) - kx]$$

tengliklar yordamida topiladi.

Endi funktsiya grafigini yasashga o'tish mumkin.

U quyidagi sxema asosida bajariladi:

1) Funktsiyaning aniqlanish sohasini topish.
2) Funktsiyaning juft–toqligini aniqlash.
3) Funktsiyaning davriyligini aniqlash.
4)Funktsiyani uzluksizlikka tekshirish va uzilish nuqtalarini topish.
5) Funktsiya grafigining koordinata o'qlari bilan kesishish nuqtalarini topish.
6) Monotonlik oraliqlarini aniqlash.
7) Ekstremumga tekshirish.
8) Botiq va qavariqlikka tekshirish.
9) Funktsiyaning asimptotalarini topish.
10) Funktsiya grafigini chizish.

Misollar.

$y=\frac{2x-1}{(x-1)^2}$ funktsiyani tekshiring va grafigini chizing.

1. Funktsiya $x=1$ nuqtadan tashqari barcha sonlar o'qida aniqlangan.
2. $f(-x)=\frac{-2x-1}{(-x-1)^2}\neq f(x)$ va $f(-x)\neq f(x)$ demak, funktsiya toq ham emas, juft ham emas.
3. Funktsiya davriy emas.
4. $x=1$ nuqtada II – tur uzilishga ega.

$\lim_{x\to 1\pm 0} f(x)=\lim_{x\to 1\pm 0}\frac{2x-1}{(x-1)^2}=+\infty$ qolgan nuqtalarda funktsiya uzluksiz.

5. $x=1$ funktsiyaning vertikal asimptotasi $y=0$ gorizontal asimptotasi, ya'ni $\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\frac{2x-1}{(x-1)^2}=\lim_{x\to\pm\infty}\frac{\frac{2}{x}-\frac{1}{x_2}}{\left(1-\frac{1}{x}\right)^2}=0$

Og'ma asimptotasini topamiz.

$$k_{1,2}=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\frac{2x-1}{x(x-1)^2}=\lim_{x\to\pm\infty}\frac{2-\frac{1}{x}}{x^2\left(1-\frac{1}{x}\right)^2}=0,\quad \text{u holda } k=0,$$

$b_{1,2}=\lim_{x\to\pm\infty}(f(x)-kx)=\lim_{x\to\pm\infty}\frac{2x-1}{(x-1)^2}=0,$ u holda $b=0$. Bundan kelib chiqadiki, $y=kx+b$ og'ma asimptota yo'q.

6. $y'=-\frac{2x}{(x-1)^2}$ funktsiya aniqlanish sohasini quyidagi oraliqlarga bo'lamiz: $(-\infty;0)$, $(0;1)$, $(1;+\infty)$. $(-\infty;0)$ oraliqlarda funktsiya kamayadi. $(0;1)$ oraliqda esa funktsiya o'sadi. $(1;+\infty)$ oraliqda funktsiya kamayadi.

7. $x=0$ nuqtada funktsiya *aniqlangan* va uzluksiz, $y'(x)$ hosila ishorasini manfiydan musbatga o'zgartiradi, u holda funktsiya, bu nuqtada $y_{min}=y(0)=-1$ erishadi.

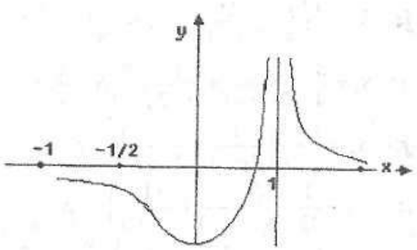
8. Funktsiya botiq va qavariqligini tekshirish uchun ikkinchi tartibli hosilani olamiz. $y''=2\cdot\frac{2x+1}{(x-1)^4}$ funktsiyaning aniqlanish sohasini quyidagi oraliqlarga ajratamiz.

$\left(-\infty,-\frac{1}{2}\right)$, $\left(-\frac{1}{2};1\right)$, $(1;+\infty)$. $\left(-\infty,\frac{1}{2}\right)$ da $f''(-1)=-\frac{1}{8}<0,$ funktsiya qavariq,

$\left(-\frac{1}{2};1\right)$ da $f''(0)=2>0$ funktsiya botiq, $(1;+\infty)$ oraliqda $f''(0)=10>0$ funktsiya botiq. Funktsiyaning ikkinchi tartibli hosilasi $x=-\frac{1}{2}$ dan $x=1$ nuqtaga o'tishdan o'z ishorasini o'zgartiradi, bundan kelib chiqadiki, $f\left(-\frac{1}{2}\right)=-\frac{8}{9}$ nuqta egilish nuqtasi bo'ladi.

9. Agar $x=0$ bo'lsa, u holda $y=-1$ va $y=0$ da $x=1$. Bundan kelib chiqadiki, $(0;-1)$ va $\left(\frac{1}{2};0\right)$ nuqtalar funktsiya grafigining koordinata o'qlari bilan kesishish nuqtalari.

10. Funktsiya grafigi:



Nazorat savollari:

1.Funktsiyaning o'sishi va kamayishi.
2.Funktsiyaning ekstremumlari.
3.Funktsiyaning botiq va qavariqligi. Burilish nuqtasi.
4.Funktsiyaning asimptotalari.

Mustaqil yechish uchun misollar va masalalar.

1–masala. Qatorlarni yaqinlashishga tekshiring va yig'indisini toping.

1.1. $\frac{2}{5}+\frac{2}{25}+...+\frac{2}{5^n}+...$ **j:** $S_n=\frac{1}{2}-\frac{1}{2}\left(\frac{1}{5}\right)^n$, $S=\frac{1}{2}$.

1.2. $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+...+\frac{(-1)^{n-1}}{3^{n-1}}+...$ **j:** $S_n=\frac{3}{4}+\frac{1}{4}\cdot\frac{(-1)^{n-1}}{3^{n-1}}$, $S=\frac{3}{4}$.

1.3. $\left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{3^2+5^2}\right)+...+\left(\frac{1}{3^n}+\frac{1}{5^n}\right)+...$ **j:** $S_n=\frac{3}{4}-\frac{1}{2\cdot3^n}-\frac{1}{4\cdot5^n}$, $S=\frac{3}{4}$.

1.4. $\left(3+\frac{1}{2}\right)+\left(\frac{3}{2}-\frac{1}{6}\right)+\left(\frac{3}{4}+\frac{1}{18}\right)+...+\left(\frac{3}{2^{n-1}}+\frac{(-1)^{n-1}}{2\cdot3^{n-1}}\right)+...$

j: $S_n=\frac{51}{8}-\frac{3}{2^{n-1}}+\frac{(-1)^{n-1}}{8\cdot3^{n-1}}$, $S=\frac{51}{8}$.

1.5. $\sum_{n=1}^{\infty}\frac{1}{16n^2-8n-3}$. **j:** $S_n=\frac{1}{4}\left(1-\frac{1}{4n+1}\right)$, $S=\frac{1}{4}$.

1.6. $\sum_{n=1}^{\infty}\frac{1}{25n^2+5n-6}$. **j:** $S_n=\frac{1}{5}\left(\frac{1}{3}-\frac{1}{5n+3}\right)$, $S=\frac{1}{15}$.

1.7. $\sum_{n=1}^{\infty}\frac{1}{36n^2-24n-5}$. **j:** $S_n=\frac{1}{6}\left(1-\frac{1}{6n+1}\right)$, $S=\frac{1}{6}$.

1.8. $\sum_{n=1}^{\infty}\frac{1}{49n^2+7n-12}$. **j:** $S_n=\frac{1}{7}\left(\frac{1}{4}-\frac{1}{7n+4}\right)$, $S=\frac{1}{28}$.

1.9. $\sum_{n=1}^{\infty}\frac{1}{n^2-1}$. **j:** $S_n=\frac{3}{4}-\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+2}\right)$, $S=\frac{3}{4}$.

1.10. $\sum_{n=1}^{\infty}\frac{1}{4n^2+4n-3}$. **j:** $S_n=\frac{1}{3}-\frac{1}{4}\left(\frac{1}{2n+1}+\frac{1}{2n+3}\right)$, $S=\frac{1}{3}$.

1.11. $\sum_{n=1}^{\infty}\frac{1}{16n^2-8n-15}$. **j:** $S_n=-\frac{1}{12}-\frac{1}{8}\left(\frac{1}{4n-1}+\frac{1}{4n+3}\right)$, $S=-\frac{1}{12}$.

1.12. $\sum_{n=1}^{\infty}\frac{1}{36n^2+12n-35}$. **j:** $S_n=\frac{2}{21}-\frac{1}{12}\left(\frac{1}{6n+1}+\frac{1}{6n+7}\right)$, $S=\frac{2}{21}$.

1.13. $\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$ **j:** $S=\frac{1}{2}$

1.14. $\sum_{n=1}^{\infty}\frac{1}{2^n}$ **j:** $S=2$

1.15. $\sum_{nn=1}^{\infty}\frac{4}{(2n-1)(2n+1)}$ **j:** $S=2$

1.16. $\sum_{n=1}^{\infty}\frac{1}{(3n-2)(3n+1)}$ **j:** $S=\frac{1}{3}$

1.17. $\sum\limits_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}$ **j:** $S=1$

1.18. $\sum\limits_{n=1}^{\infty}\frac{n+1}{n^2(n+2)^2}$ **j:** $S=\frac{3}{8}$.

1.19. $\sum\limits_{n=1}^{\infty}\frac{2}{n(n+1)(n+2)}$ **j:** $S=\frac{1}{2}$.

2–masala. Quyidagi $\sum\limits_{n=1}^{\infty}a_n$ qatorlarni yaqinlashishga yaqinlashish alomatlaridan foydalanib tekshiring.

2.1. $a_n=\frac{3^n}{n^n}$.

2.2. $a_n=\frac{n!}{10^n}$.

2.3. $a_n=\frac{n}{3^n}$.

2.4. $a_n=\frac{3^n}{n!}$.

2.5. $a_n=\frac{n^3}{3^n}$.

2.6. $a_n=\frac{n!a^n}{n^n}$, $a\neq e, a>0$

2.7. $a_n=\frac{1\cdot 3\cdot 5\cdot\ldots(2n-1)}{3^n n!}$.

2.8. $a_n=\frac{(2n)!}{(n!)^2}$.

2.9. $a_n=\frac{100^n}{n!}$.

2.10. $a_n=\frac{(n!)^2}{3^{n^2}}$.

2.11. $a_n=\left(\frac{3}{n}\right)^n$.

2.12. $a_n=\left(\frac{100}{n}\right)^n$.

2.13. $a_n=2^n\left(\frac{n}{n+1}\right)^{n^2}$.

2.14. $a_n=3^{n+1}\left(\frac{n+2}{n+3}\right)^{n^2}$.

2.15. $a_n=\left(\frac{n^2+5}{n^2+6}\right)^{n^3}$.

2.16. $a_n=\left(\frac{2n-1}{2n+1}\right)^{n(n-1)}$.

2.17. $a_n=\left(\frac{n-1}{n+1}\right)^{n^2+4n+5}$.

2.18. $a_n=\left(\frac{n-1}{n+1}\right)^{\sqrt{n^3+3n+2}}$.

2.19. $a_n=3^{-n}\left(\frac{n+1}{n}\right)^{n^2}$.

2.20. $a_n=\left(\frac{6n+1}{5n-3}\right)^{\frac{n}{2}}\left(\frac{5}{6}\right)^{\frac{2n}{3}}$.

3–masala. Quyidagi tengliklar ta'rif yordamida isbotlansin :

3.1. $\lim\limits_{x\to -3}\frac{2x^2+5x-3}{x+3}=-7$.

3.2. $\lim\limits_{x\to -1}\frac{5x^2-4x-1}{x-1}=6$.

3.3. $\lim\limits_{x\to -2}\frac{3x^2+5x-2}{x+2}=-7$.

3.4. $\lim\limits_{x\to 3}\frac{4x^2-14x+6}{x-3}=10$

3.5. $\lim\limits_{x\to-\frac{1}{2}}\dfrac{6x^2+x-1}{x+\frac{1}{2}}=-5.$

3.6. $\lim\limits_{x\to-\frac{1}{2}}\dfrac{6x^2-x-1}{x-\frac{1}{2}}=5.$

3.7. $\lim\limits_{x\to-\frac{1}{3}}\dfrac{9x^2-1}{x+\frac{1}{3}}=-6.$

3.8. $\lim\limits_{x\to2}\dfrac{3x^2-5x-2}{x-2}=7.$

3.9. $\lim\limits_{x\to-\frac{1}{3}}\dfrac{3x^2-2x-1}{x+\frac{1}{3}}=-4.$

3.10. $\lim\limits_{x\to-1}\dfrac{7x^2+8x+1}{x+1}=-6.$

3.11. $\lim\limits_{x\to3}\dfrac{x^2-4x+3}{x-3}=2.$

3.12. $\lim\limits_{x\to\frac{1}{2}}\dfrac{2x^2+3x-2}{x-\frac{1}{2}}=5.$

3.13. $\lim\limits_{x\to-\frac{1}{3}}\dfrac{6x^2-5x+1}{x-\frac{1}{3}}=-1.$

3.14. $\lim\limits_{x\to-\frac{7}{5}}\dfrac{10x^2+9x-7}{x-\frac{7}{5}}=-19.$

3.15. $\lim\limits_{x\to-\frac{7}{2}}\dfrac{2x^2+13x+21}{2x+7}=-\dfrac{1}{2}.$

3.16. $\lim\limits_{x\to-\frac{5}{2}}\dfrac{2x^2-9x+10}{2x-5}=\dfrac{1}{2}.$

3.17. $\lim\limits_{x\to\frac{1}{3}}\dfrac{6x^2+x-1}{x-\frac{1}{3}}=5.$

3.18. $\lim\limits_{x\to-\frac{1}{2}}\dfrac{6x^2+75x-39}{x-\frac{1}{2}}=-81.$

3.19. $\lim\limits_{x\to-11}\dfrac{2x^2-21x-11}{x-11}=23.$

3.20. $\lim\limits_{x\to-5}\dfrac{5x^2-24x-5}{x-5}=26.$

3.21. $\lim\limits_{x\to-7}\dfrac{2x^2+15x+7}{x+7}=-13.$

3.22. $\lim\limits_{x\to-4}\dfrac{2x^2+6x-8}{x+4}=-10.$

3.23. $\lim\limits_{x\to-\frac{1}{3}}\dfrac{6x^2-x-1}{3x+1}=\dfrac{5}{3}.$

3.24. $\lim\limits_{x\to-5}\dfrac{x^2+2x-15}{x+5}=-8.$

3.25. $\lim\limits_{x\to8}\dfrac{3x^2-40x+128}{x-8}=8.$

3.26. $\lim\limits_{x\to-10}\dfrac{5x^2-51x+10}{x-10}=49.$

3.27. $\lim\limits_{x\to\frac{1}{2}}\dfrac{2x^2-5x+2}{x-\frac{1}{2}}=-3.$

3.28. $\lim\limits_{x\to-6}\dfrac{3x^2+17x-6}{x+6}=-19.$

3.29. $\lim\limits_{x\to\frac{1}{3}}\dfrac{15x^2-2x-1}{x-\frac{1}{3}}=8.$

3.30. $\lim\limits_{x\to-\frac{1}{5}}\dfrac{15x^2-2x-1}{x+\frac{1}{5}}=-8.$

4– masala. Fuktsiya limiti hisoblansin.

4.1. $\lim\limits_{x\to4}\dfrac{\sqrt{1+2x}-3}{\sqrt{x}-2}$

4.2. $\lim\limits_{x\to4}\dfrac{\sqrt{1-x}-3}{2+\sqrt[3]{x}}$

4.3. Нт /1-х
2 + ЧхГл

л л ,. 4х + ХЎ-24х + \
4.4. Нт———:———
**> x²-9

4.5. Нт /х-6 + 2
• x' + 8

4.6. Нт У1-2
л/х-4

4.7. Нт /9 + 2х-5
"» ^х -2

4.8. Нт л/1-2х+х²-(1 + х)

4.9. Ит- 3 + 3х + Х² - 2
х + х²

4.10. Нт У27 + х² -421-х
х + 2-\/х

4.11. Н т ^ " 1
ы\ + х-^2х

4.12. Нт 1 + х - VI -х
+ х-Щ-х

4.13. Ит 1ЏЛ-2
12 +х- ы2х

4.14. й п $ г 1
*-' х² - 1

4.15. Ит У9^-3
""Чъ + х-Лх

4.16.1т -6 + 2
.с-» 2 х +2

4.17. Ит
А-ДТх~42х

4.18. Нт 9 + 2х-5
" 8 г 4 х² - 4

МЛ

4.19. Н т , Д 1 ²
«1 11 /—
2,(- + х-у2х
1/2

4.20. Нт- х 1
9 3

4.21. И т - Д Н ⁴ -
-Ы2х

4.22. Нт 4\+х -у/г-х

4.23. щ ^ - ™ ^

4.24. й д Г ^ * ¹ ^ - ²

4.25. И Ь ^ ^ - Р Й *

4.26. Н ш У ⁹ ^ * ⁵

4.27. Ит , ^ ~ ² ,
х ^Щ7х^4)²

4.28. Ит 3-/х -6 + 2
х~*-2 3/„3 х + 8

4.29. Ит /* ~ ²
м 4 ^ х² - 1 6

4.30. Нт Ш-х-бл/ГГх
2 + ^х

5~та$а1а. ҒипкМуашпд НшШ ЫзоЫапзш.

5.1. !ип х² - 1
*->> 1пх

с ~ „ л/х² -х + 1-1
Э.2. ШШ———————
*-»> тх

5.3. $\lim\limits_{x\to\pi}\frac{1+\cos 3x}{\sin^2 7x}$

5.4. $\lim\limits_{x\to\pi/4}\frac{1-\sin 2x}{(\pi-4x)^2}$

5.5. $\lim\limits_{x\to 1}\frac{1+\cos\pi x}{tg^2\pi x}$

5.6. $\lim\limits_{x\to\pi/2}\frac{tg3x}{tgx}$

5.7. $\lim\limits_{x\to\pi}\frac{\sin^2 x-tg^2 x}{(x-\pi)^4}$

5.8. $\lim\limits_{x\to 1}\frac{\sqrt{x^2-x+1}-1}{tg\pi\ x}$

5.9. $\lim\limits_{x\to\pi}\frac{\cos 5x-\cos 3x}{\sin^2 x}$

5.10. $\lim\limits_{x\to 2\pi}\frac{\sin 7x-\sin 3x}{e^{x2}-e^{4\pi 2}}$

5.11. $\lim\limits_{x\to 2}\frac{\sin 7\pi\ x}{\sin 8\pi\ x}$

5.12. $\lim\limits_{x\to 2}\frac{\ln(5-2x)}{\sqrt{10-3x}-2}$

5.13. $\lim\limits_{x\to 2}\frac{\sqrt{x^2-3x+3}-1}{\sin\pi\ x}$

5.14. $\lim\limits_{x\to\pi}\frac{x^2-\pi^2}{\sin\ x}$

5.15. $\lim\limits_{x\to 1}\frac{3^{5x-3}-3^{2x^2}}{tg\pi\ x}$

5.16. $\lim\limits_{x\to 4}\frac{2^x-16}{\sin\pi x}$

5.17. $\lim\limits_{x\to\pi/2}\frac{\ln 2x-\ln\pi}{\sin\frac{5x}{2}\cos x}$

5.18. $\lim\limits_{x\to\pi/4}\frac{\ln tgx}{\cos 2x}$

5.19. $\lim\limits_{x\to\pi}\frac{e^{\pi}-e^{x}}{\sin 5x-\sin 3x}$

5.20. $\lim\limits_{x\to 2}\frac{\ln(9-2x^2)}{\sin 2\pi x}$

5.21. $\lim\limits_{x\to 2}\frac{1-2^{4-x^2}}{2(\sqrt{2x}-\sqrt{3x^2-5x+2})}$

5.22. $\lim\limits_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}$

5.23. $\lim\limits_{x\to 2}\frac{tg\pi x)}{x+2}1$

5.24. $\lim\limits_{x\to\pi}\frac{1-\sin\frac{x}{2}}{\pi-x}$

5.25. $\lim\limits_{x\to\pi/3}\frac{1-2\cos x}{\pi-3x}$

5.26. $\lim\limits_{x\to 2}\frac{arctg(x^2-2x)}{\sin 3\pi\ x}$

5.27. $\lim\limits_{x\to 1}\frac{1-x^2}{\sin\pi x}$

5.28. $\lim\limits_{x\to 1}\frac{\cos\frac{\pi x}{2}}{1-\sqrt{x}}$

5.29. $\lim\limits_{x\to 1}\frac{3-\sqrt{10-2}}{\sin 3\pi x}$

5.30. $\lim\limits_{x\to\pi}\frac{\sin 5x}{tg3x}$

6-masala. Funktsiyaning limiti hisoblansin.

6.1. $\lim\limits_{x\to 1}(\frac{3x-1}{x+1})^{\frac{1}{\sqrt[3]{x}-1}}$

6.2. $\lim\limits_{x\to a}(\frac{\sin x}{\sin a})^{\frac{1}{x-a}}$

6.3. $\lim\limits_{x\to 1}(\frac{2x-1}{x})^{\frac{1}{\sqrt[3]{x}-1}}$

6.4. $\lim\limits_{x\to 2}(\frac{\cos x}{\cos 2x})^{\frac{1}{x-2}}$

6.5. $\lim\limits_{x\to 1}\left(\frac{4x-1}{2x+1}\right)^{\frac{1}{\sqrt[3]{x}-1}}$

6.6. $\lim\limits_{x\to \pi/4}(tgx)^{\frac{1}{\cos\left(\frac{3\pi}{4}x\right)}}$

6.7. $\lim\limits_{x\to 1}\left(\frac{2x-1}{x}\right)^{\frac{1}{\sqrt[3]{x}-1}}$

6.8. $\lim\limits_{x\to a}\left(2-\frac{x}{a}\right)^{tg\frac{\pi x}{2a}}$

6.9. $\lim\limits_{x\to 2\pi}(\cos x)\frac{ctg2x}{\sin 3x}$

6.10. $\lim\limits_{x\to 2\pi}(\cos x)^{\frac{1}{\sin^2 2x}}$

6.11. $\lim\limits_{x\to 3}\left(\frac{6-x}{3}\right)^{tg\frac{\pi x}{6}}$

6.12. $\lim\limits_{x\to 4\pi}(\cos x)^{\frac{ctgx}{\sin 4x}}$

6.13. $\lim\limits_{x\to 1}(3-2x)^{tg\frac{\pi x}{2}}$

6.14. $\lim\limits_{x\to 4\pi}(\cos x)^{\frac{5}{tg5x-\sin 2x}}$

6.15. $\lim\limits_{x\to 3}\left(\frac{9-2x}{3}\right)^{tg\frac{\pi x}{6}}$

6.16. $\lim\limits_{x\to \frac{\pi}{2}}(\sin x)^{6tgx\cdot tg3x}$

6.17. $\lim\limits_{x\to 1}(2e^{1-1}-1)^{\frac{x}{x-1}}$

6.18. $\lim\limits_{x\to \frac{\pi}{2}}\left(tg\frac{x}{2}\right)^{\frac{1}{x-\frac{\pi}{2}}}$

6.19. $\lim\limits_{x\to 1}(2e^{x-1}-1)^{\frac{3x-1}{x-1}}$

6.20. $\lim\limits_{x\to \frac{\pi}{2}}(1+cos3x)^{\sec x}$

6.21. $\lim\limits_{x\to 2}(2e^{x-2}-1)^{\frac{3x+2}{x-2}}$

6.22. $\lim\limits_{x\to 1}\left(\frac{2x-1}{x}\right)^{\frac{\ln(3+2x)}{\ln(2-x)}}$

6.23. $\lim\limits_{x\to 1}\left(\frac{2-x}{x}\right)^{\frac{1}{\ln(2-x)}}$

6.24. $\lim\limits_{x\to \frac{\pi}{2}}\left(ctg\frac{x}{2}\right)^{\frac{1}{\cos x}}$

6.25. $\lim\limits_{x\to 1}(2-x)^{\frac{\sin\frac{\pi x}{2}}{\ln(2-x)}}$

6.26. $\lim\limits_{x\to 3}\left(\frac{\sin x}{\sin 3}\right)^{\frac{1}{x-3}}$

6.27. $\lim\limits_{x\to 1}\left(\frac{x+1}{2x}\right)^{\frac{\ln(x+2)}{\ln(2-x)}}$

6.28. $\lim\limits_{x\to \frac{\pi}{2}}(\sin x)^{\frac{18\sin x}{ctgx}}$

6.29. $\lim\limits_{x\to 1}\left(\frac{1}{x}\right)^{\frac{\ln(x+1)}{\ln(2-x)}}$

6.30. $\lim\limits_{x\to \pi}\left(ctg\frac{x}{4}\right)^{\frac{1}{\cos\frac{x}{2}}}$

7–masala. Funktsiya grafigining abtsissasi x_0 bo'lgan nuqtasiga o'tkazilgan normal (5.1–5.12 misollarda) yoki urinma (5.13–5.30 misollarda) tenglamasi topilsin.

7.1. $y=\frac{4x-x^2}{4}, \quad x_0=2$

7.2. $y=2x^2+3x-1, \quad x_0=-2$

7.3. $y=x-x^3, \quad x_0=-1$.

7.4. $y=x^2+8\sqrt{x}-32, \quad x_0=4$

8.11. $y = x^{21}, \quad x = 0{,}998$

88.12. $y = \sqrt[3]{x^2}, \quad x = 1{,}03$

8.13. $y = x^6, \quad x = 2{,}01$

8.14. $y = \sqrt[3]{x}, \quad x = 8{,}24$

8.15. $y = x^7, \quad x = 1{,}996.$

8.16. $y = \sqrt[3]{x}, \quad x = 7{,}64.$

8.17. $y = \sqrt{4x-1}, \quad x = 2{,}56$

8.18. $y = \dfrac{1}{\sqrt{2x^2+x+1}}, \quad x = 1{,}016$

8.19. $y = \sqrt[3]{x}, \quad x = 8{,}36$

8.20. $y = \dfrac{1}{\sqrt{x}}, \quad x = 4{,}16$

8.21. $y = x^7, \quad x = 2{,}002$

8.22. $y = \sqrt{4x-3}, \quad x = 1{,}78$

8.23. $y = \sqrt{x^3}, \quad x = 0{,}98$

8.24. $y = x^5, \quad x = 2{,}997$

8.25. $y = \sqrt[5]{x^2}, \quad x = 1{,}03$

8.26. $y = x^4, \quad x = 3{,}998$

8.27. $y = \sqrt{1+x+\sin x}, \quad x = 0{,}01$

8.28. $y = \sqrt[3]{3x+\cos x}, \quad x = 0{,}01$

8.29. $y = \sqrt[4]{2x - \sin\dfrac{\pi x}{2}}, \quad x = 1{,}02$

8.30. $y = \sqrt{x^2+5}, \quad x = 1{,}97$

***9–masala*. Berilgan funktsiyaning hosilasini toping.**

9.1. $y = \dfrac{2(3x^3+4x^2-x-2)}{15\sqrt{1+2}}$

9.2. $y = \dfrac{(2x^2-1)\sqrt{1+x^2}}{3x^3}$

9.3. $y = \dfrac{x^4-8x^2}{2(x^2-4)}$

9.4. $y = \dfrac{(2x^2-x-1)}{3\sqrt{2+4x}}$

9.5. $y = \dfrac{(1+x^8)\sqrt{1+x^8}}{12x^{12}}$

9.6. $y = \dfrac{x^2}{2\sqrt{1+3x^4}}$

9.7. $y = \dfrac{(x^2-6)\sqrt{(4+x^2)^3}}{120x^5}$

9.8. $y = \dfrac{(x^2-6)\sqrt{(4+x^2)^3}}{120x^5}$

9.9. $y = \dfrac{4+3x^3}{x\sqrt[3]{(2+x^3)^2}}$

9.10. $u = \sqrt[3]{\dfrac{(1+x^{\frac{3}{4}})^2}{x^{\frac{3}{2}}}}$

9.11. $y = \dfrac{x^6+x^3-2}{\sqrt{1-x^3}}$

9.12. $y = \dfrac{(x^2-2)\sqrt{4+x^2}}{24x^3}$

9.13. $y = \dfrac{1+x^2}{2\sqrt{1+2x^2}}$

9.14. $y = \dfrac{\sqrt{x-1}(3x+2)}{4x^2}$

9.15. $y=\frac{\sqrt{(1+x^2)^3}}{3x^3}$

9.16. $y=\frac{x^6+8x^3-128}{\sqrt{8-x^3}}$

9.17. $y=\frac{\sqrt{2x+3}(x-2)}{x^2}$

9.18. $y=(1-x^2)\sqrt[5]{x^3+\frac{1}{x}}$

9.19. $y=\frac{(2x^2+3)\sqrt{x^2-3}}{9x^3}$

9.20. $y=\frac{x-1}{(x^2+5)\sqrt{x^2+5}}$

9.21. $y=\frac{(2x+1)\sqrt{x^2-x}}{x^2}$

9.22. $y=2\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$

9.23. $y=\frac{1}{(x+2)\sqrt{x^2+4x+5}}$

9.24. $y=3\frac{\sqrt[3]{x^2+x+1}}{x+1}$

9.25. $y=3\sqrt[3]{\frac{x+1}{(x-1)^2}}$

9.26. $y=\frac{x+7}{6\sqrt{x^2+2x+7}}$

9.27. $y=\frac{x\sqrt{x+1}}{x^2+x+1}$

9.28. $y=\frac{x^2+2}{2\sqrt{1-x^4}}$

9.29. $y=\frac{(x+3)\sqrt{2x-1}}{2x+7}$

9.30. $y=\frac{3x+\sqrt{x}}{x^2+2}$

10–masala. **Berilgan funktsiyaning hosilasini toping.**

10.1. $y=((arctgx))^{\frac{1}{2}\ln arctgx}$

10.2. $y=(\sin\sqrt{x})^{\ln\sin\sqrt{x}}$

10.3. $y=(\sin x)^{5e^x}$

10.4. $y=(\arcsin x)^{e^x}$

10.5. $y=(\ln x)^{3^x}$

10.6. $y=x^{\arcsin x}$

10.7. $y=(ctg3x)^{2e^x}$

10.8. $y=x^{e^{tgx}}$

10.9 $y=(tgx)^{4e^x}$

10.10. $y=(\cos 5x)^{e^x}$

10.11. $y=(x\sin x)^{\sin(x\sin)}$

10.12. $y=(x-5)^{\sin x}$

10.13. $y=(x^3+4)^{tgx}$

10.14. $y=x^{\sin x^3}$

10.15. $y=(x^2-1)^{\sin x}$

10.16. $y=(x^4+5)^{ctgx}$

10.17. $y=(\sin x)^{\frac{5x}{2}}$

10.18. $y=(x^2+1)^{\cos x}$

10.19. $y=19^{x^{19}}x^{19}$

10.20. $y=x^{3x}2^x$

10.21. $y=(\sin\sqrt{x})^{\frac{1}{x}}$

10.22. $y=x^{e^{ctgx}}$

10.23. $y=x^{e^{\cos x}}$

10.24. $y=x^{2x}5^x$

10.25. $y=x^{e^{\sin x}}$

10.26. $y=(tgx)^{\ln tgx^{\frac{x}{4}}}$

10.27. $y=x^{e^{arctgx}}$

10.28. $y=(x^8+1)^{tgx}$

10.29. $y = x^{29^x} 29^x$

10.30. $y = (\cos 2x)^{\ln\cos\frac{x}{2}}$

11–masala. **Berilgan funktsiyaning n–tartibli hosilasini hisoblang.**

11.1. $y = xe^{ax}$

11.2. $y = \sin 2x + \cos(x+1)$

11.3. $y = \sqrt[5]{e^{7x-1}}$

11.4. $y = \dfrac{4x+7}{2x+3}$

11.5. $y = \lg(5x+2)$

11.6. $y = a^{3x}$

11.7. $y = \dfrac{x}{2(3x+2)}$

11.8. $y = \lg(x+4)$

11.9 $y = \sqrt{x}$

11.10. $y = \dfrac{2x+5}{13(3x+1)}$

11.11. $y = 2^{3x+5}$

11.12. $y = \sin(x+1) + \cos 2x$

11.13. $y = \sqrt[3]{e^{2x+1}}$

11.14. $y = \dfrac{4+15x}{2x+1}$

11.15. $y = \lg(3x+1)$

11.16. $y = 7^{5x}$

11.17. $y = \dfrac{x}{9(4x+9)}$

11.18. $y = \lg(1+x)$

11.19. $y = \dfrac{4}{x}$

11.20. $y = \dfrac{5x+1}{13(2x+3)}$

11.21. $y = a^{2x+3}$

11.22. $y = \sin(3x+1) + \cos 5x$

11.23. $y = \sqrt{e^{3x+1}}$

11.24. $y = \dfrac{11+12x}{6x+5}$

11.25. $y = \lg(2x+7)$

11.26. $y = 2^{kx}$

11.27. $y = \dfrac{x}{x+1}$

11.28. $y = \log_3(x+5)$

11.29. $y = \dfrac{1+x}{1-x}$

11.30. $y = \dfrac{7x+1}{17(4x+3)}$

12–masala. **Birinchi tartibli hosiladan foydalanib funktsiyaning grafigini yasang.**

12.1. $y = 2x^3 - 9x^2 + 12x - 9$

12.2. $y = 3x - x^2$

12.3. $y = x^2(x-2)^2$

12.4. $y = \dfrac{x^3 - 9x^2}{4} + 6x - 9$

12.5. $y = 2 - 3x^2 - x^3$

12.6. $y = (x+1)^2(x-1)^2$

12.7. $y = 2x^3 - 3x^2 - 4$

12.8. $y = 3x^2 - 2 - 3x^3$

12.9 $y = (x-1)^2(x-3)^2$

12.10. $y = \dfrac{x^3 + 3x^2}{4} - 5$

12.11. $y=6x-8x^3$

12.12. $y=16x^2(x-1)^2$

12.13. $y=2x^3+3x^2-5$

12.14. $y=2-12x^2-8x^3$

12.15. $y=(2x+1)^2(2x-1)^2$

12.16. $y=2x^3+9x^2+12x$

12.17. $y=12x^2-8x^{23}-2$

12.18. $y=(2x-1)^2(2x-3)^2$

12.19. $y=\frac{27(x-x^2)}{4}-x$

12.20. $y=\frac{x(12-x^2)}{8}$

12.21. $y=\frac{x^2(x-4)^2}{16}$

12.22. $y=\frac{27(x^3+x^2)}{4}-5$

12.23. $y=\frac{16-6x^2-x^3}{8}$

12.24. $y=-\frac{(x^2-4)^2}{16}$

12.25. $y=16x^3-36x^2+24x-9$

12.26. $y=\frac{6x^2-x^3-16}{8}$

12.27. $y=-\frac{(x-2)^2(x-6)^2}{16}$

12.28. $y=16x^3-12x^2-4$

12.29. $y=\frac{11+9x-3x^2-x^3}{8}$

12.30. $y=-\frac{(x+1)^2(x-3)^2}{16}$

13–masala. **Funktsiyaning asimptotalarini toping va grafigini yasang.**

13.1. $y=\frac{17-x^2}{4x-5}$

13.2. $y=\frac{x^2+1}{\sqrt{4x^2-3}}$

13.3. $y=\frac{x^3-4x}{3x^2-4}$

13.4. $y=\frac{4x^2+9}{4x+8}$

13.5. $y=\frac{4x^3+3x^2-8x-2}{2-3x^2}$

13.6. $y=\frac{x^2-3}{\sqrt{3x^2-2}}$

13.7. $y=\frac{2x^2-6}{x-2}$

13.8. $y=\frac{2x^3+2x^2-3x-1}{2-4x^2}$

13.9 $y=\frac{x^3-5x}{5-3x^2}$

13.10. $y=\frac{4x^3-3x}{4x^2-1}$

13.11. $y=\frac{2-x^2}{\sqrt{9x^2-4}}$

13.12. $y=\frac{x^2-6x+4}{3x-2}$

13.13. $y=\frac{3x^2-7}{2x-1}$

13.14. $y=\frac{x^2-16}{\sqrt{9x^2-8}}$

13.15. $y=\frac{x^3+3x^2-2x-2}{2-3x^2}$

13.16. $y=\frac{21-x^2}{7x+9}$

13.17. $y=\frac{2x^2-1}{\sqrt{x^2-2}}$

13.18. $y=\frac{2x^3-3x^2-2x+1}{1-3x^2}$

13.19. $y=\frac{x^2-11}{4x-3}$

13.20. $y=\frac{2x^2-9}{\sqrt{x^2-1}}$

13.21. $y=\frac{x^3-2x^2-3x+2}{1-x^2}$

13.22. $y=\frac{x^2-2x-1}{2x+1}$

13.23. $y=\frac{x^3+x^2-3x-1}{2x^2-2}$

13.24. $y=\frac{x^2-6x+9}{x+4}$

13.25. $y=\frac{3x^2-10}{\sqrt{4x^2-1}}$

13.26. $y=\frac{x^2-2x+2}{x+3}$

13.27. $y=\frac{2x^3+2x^2-9x-3}{2x^2-23}$

13.28. $y=\frac{3x^2-10}{3-2x}$

13.29. $y=\frac{-x^2-4x+13}{4x+3}$

13.30. $y=\frac{-8-x^2}{\sqrt{x^2-4}}$

14–masala. **Funktsiyani to'liq tekshiring va grafigini yasang.**

14.1. $y=\frac{x^3+4}{x^2}$

14.2. $y=\frac{x^2-x+1}{x-1}$

14.3. $y=\frac{2}{x^2+2x}$

14.4. $y=\frac{4x^2}{3+x^2}$

14.5. $y=\frac{12x}{9+x^2}$

14.6. $y=\frac{x^2-3x+3}{x-1}$

14.7. $y=\frac{4-x^3}{x^2}$

14.8. $y=\frac{x^2-4x+1}{x-4}$

14.9 $y=\frac{2x^3+1}{x^2}$

14.10. $y=\frac{(x-1)^2}{x^2}$

14.11. $y=\frac{x^2}{(x-1)^2}$

14.12. $y=\left(1+\frac{1}{x}\right)^2$

14.13. $y=\frac{12-3x^3}{x^2+12}$

14.14. $y=\frac{9+6x-3x^2}{x^2-2x+13}$

14.15. $y=-\frac{8x}{x^2+4}$

14.16. $y=\left(\frac{x-1}{x+1}\right)^2$

14.17. $y=\frac{3x^4+1}{x^3}$

14.18. $y=\frac{4x}{(x+1)^2}$

14.19. $y=\frac{8(x-1)}{(x+1)^2}$

14.20. $y=\frac{1-2x^3}{x^2}$

14.21. $y=\frac{4}{x^2+2x-3}$

14.22. $y=\frac{4}{3+2x-x^2}$

14.23. $y=\frac{x^2+2x-7}{x^2+2x-3}$

14.24. $y=\frac{1}{x^4-1}$.

14.25. $y=-\left(\frac{x}{x+2}\right)^2$

14.26. $y=\frac{x^3-32}{x^2}$

14.27. $y=\frac{4(x+1)^2}{x^2+2x+4}$

14.28. $y=\frac{3x-2}{x^3}$

14.29. $y=\frac{x^2-6x+9}{(x-1)^2}$

14.30. $y=\frac{x^3-27x+54}{x^3}$

VI BOB.
INTEGRAL HISOB.

1–§. Aniqmas integral va uni hisoblash.

$f(x)$ funktsiya biror (a,b) (chekli yoki cheksiz) intervalda aniqlangan bo'lsin.

1–ta'rif. Agar $f(x)$ funktsiya (a,b) intervalda differentsiallanuvchi $F(x)$ funktsiyaning hosilasiga teng, ya'ni $F'(x)=f(x)$, $x\in(a,b)$ bo'lsa, u holda $F(x)$ funktsiya (a,b) intervalda $f(x)$ funktsiyaning **boshlang'ich funktsiyasi** deyiladi.

1–misol: $F(x)=x^3$ funktsiya sonlar o'qida $f(x)=3x^2$ funktsiyaning boshlang'ich funktsiyasi bo'ladi, chunki

$$F'(x)=\left(x^3\right)'=3x^2=f(x).$$

$F(x)$ va $\Phi(x)$ funktsiyalarning har biri (a,b) intervalda bitta $f(x)$ funktsiya uchun boshlang'ich funktsiya bo'lsa, bu $F(x)$ va $\Phi(x)$ funktsiyalar (a,b) intervalda bir–biridan o'zgarmas songa farq qiladi. Shu narsani ta'kidlashimiz kerakki, $f(x)=3x^2$ funktsiyaning boshlang'ich funktsiyasi sifatida ixtiyoriy $\Phi(x)=x^3+C$ funktsiyani olishimiz mumkin. Bu yerda C –ixtiyoriy o'zgarmas son.

2–ta'rif. (a,b) intervalda berilgan $f(x)$ funktsiya boshlang'ich funktsiyalarning umumiy ifodasi $F(x)+C$ $C=const$, shu $f(x)$ funktsiyaning **aniqmas integrali** deb ataladi va $\int f(x)dx$ kabi belgilanadi.

Bunda $\int$–integral belgisi, $f(x)$ –integral ostidagi funktsiya, $f(x)dx$ esa **integral ostidagi ifoda** deyiladi. Demak,

$$\int f(x)dx=F(x)+C\quad (C=const)$$

2–misol. Ushbu $\int\left(x^2+2x+5\right)dx$ aniqmas integralni toping.

Quyidagi $F(x)=\frac{x^3}{3}+x^2+5x+C$ funktsiya uchun $F'(x)=(\frac{x^3}{3}+x^2+5x+C)'=x^2+2x+5$ bo'ladi. Demak, $\int\left(x^2+2x+5\right)dx=\frac{x^3}{3}+x^2+5x+C$.

Aniqmas integralning ta'rifidan bevosita uning quyidagi sodda xossalari kelib chiqadi.

1. $\frac{d}{dx}\int f(x)dx = f(x)$, demak $d\int f(x)dx = f(x)dx$
2. $\int F'(x)dx = F(x)+C$, ya'ni $\int dF(x) = F(x)+C$
3. $\int C\cdot f(x)dx = C\cdot\int f(x)dx$
4. $\int[f_1(x)+f_2(x)]dx = \int f_1(x)dx + \int f_2(x)dx$

Elementar funktsiyalarning aniqmas integrallari.

1. $\int x^{\mu}dx = \frac{x^{\mu+1}}{\mu+1}+C \qquad (\mu\neq -1)$,
2. $\int\frac{dx}{x} = \ln|x|+C$,
3. $\int a^x dx = \frac{a^x}{\ln a}+C$,
4. $\int e^{kx}dx = \frac{e^{kx}}{k}+C$,
5. $\int\sin ax dx = -\frac{1}{a}\cdot\cos ax + C$.
6. $\int\cos ax dx = \frac{1}{a}\cdot\sin ax + C$.
7. $\int\frac{dx}{\cos^2 ax} = \frac{1}{a}\cdot tgax + C.$
8. $\int\frac{dx}{\sin^2 ax} = -\frac{1}{a}\cdot ctgax + C.$
9. $\int\frac{xdx}{x^2+a} = \frac{1}{2}\ln|x^2+a|+C.$
10. $\int\frac{dx}{x^2+a^2} = \frac{1}{2}arctg\frac{x}{a}+C.$
11. $\int\frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}+C.$
12. $\int\frac{dx}{\sqrt{x^2+a}} = \ln\left|x+\sqrt{x^2+a}\right|+C.$
13. $\int\frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C.$

Integrallash usullari.

1. O'zgaruvchini almashtirib integrallash usuli.

Ushbu $\int f(x)dx$ aniqmas integralni hisoblash talab etilgan bo'lsin. Bunda $f(x)$ funktsiya biror $X=(a,b)$ intervalda aniqlangan va $f(x)=\varphi(g(x))\cdot g'(x)$ ko'rinishda yozish mumkin deylik. Agar $\varphi(t)$ funktsiya $T=(t_1;t_2)$ intervalda boshlang'ich $\Phi(t)$ ga ega bo'lib, $g(x)$ funktsiya $X\in(a,b)$ intervalda $(g(x)\subset T)$ differentsiallanuvchi bo'lsa, u holda

$$\int f(x)dx = \int\varphi(g(x))g'(x)dx = \Phi(g(x))+C$$

tenglik o'rinli.

3–misol. $\int \frac{xdx}{x^2+a^2}$ ni hisoblang $(a = const)$.

Yechilishi:

$x^2 + a^2 = t$ kabi almashtiramiz.

$$\int \frac{xdx}{x^2+a^2} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln(x^2+a^2) + C.$$

2. Bo'laklab integrallash usuli.

$u = u(x)$ va $v = v(x)$ funktsiya (a,b) intervalda uzluksiz $u'(x)$ va $v'(x)$ hosilalarga ega bo'lsin. U holda

$$\int u(x)dv(x) = u(x)v(x) - \int v(x)du(x)$$

tenglik o'rinli bo'lib, bu tenglikka **bo'laklab integrallash formulasi** deyiladi.

4–misol. $\int xe^x dx$ ni hisoblang.

Bu intervalda $u = x$, $dv = e^x dx$ deb olamiz. Bundan kelib chiqadiki $du = dx$, $v = e^x$.

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C = (x-1)e^x + C..$$

Ratsional funktsiyalarni integrallash.

1. Sodda kasrlarni integrallash.

Biror

$$P(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n \qquad (1)$$

ko'phad berilgan bo'lsin, bunda $a_0, a_1, a_2, ..., a_n$ – o'zgarmas haqiqiy sonlar, $a_n \neq 0$, $n \in N$ esa ko'phadning darajasi.

$$\frac{P(x)}{Q(x)} = \frac{a_0 + a_1x + a_2x^2 + ... + a_nx^n}{b_0 + b_1x + b_2x^2 + ... + b_kx^k} \qquad (k \in N)$$

ga **kasr ratsional funktsiya** deyiladi, agar $n<k$ bo'lganda to'g'ri kasr, aks holda **noto'g'ri kasr** deyiladi.

Ushbu

$$\frac{A}{(x-a)^m}, \quad \frac{Bx+C}{(x^2+px+q)^m}, \quad m=1,2,... \qquad (2)$$

ko'rinishdagi kasrlar **sodda kasrlar** deyiladi, bunda A,B,C hamda a,p,q lar o'zgarmas sonlar, x^2+px+q kvadrat uchhad haqiqiy ildizga ega emas.

Har qanday to'g'ri kasr (2) sodda kasrlar orqali ifodalanadi.

Agar $\frac{P(x)}{Q(x)}$ to'g'ri kasr maxrajidagi $Q(x)$ ko'phad, quyidagi

$$Q(x)=(x-\alpha)^m Q_1(x) \quad (m\in N)$$

ko'rinishda bo'lib, $Q_1(x)$ ko'phad esa $(x-\alpha)$ ga bo'linmasa, u holda berilgan to'g'ri kasr quyidagi

$$\frac{P(x)}{Q(x)}=\frac{A_m}{(x-\alpha)^m}+\frac{A_{m-1}}{(x-\alpha)^{m-1}}+...+\frac{A_1}{x-\alpha}+\frac{P_1(x)}{Q_1(x)}$$

ko'rinishda ifodalanishi mumkin, bunda $A_1,A_2,...,A_m$ — o'zgarmas haqiqiy sonlar, $P_1(x)$ — ko'phad.

Agar $Q(x)$ ko'phad

$$Q(x)=(x^2+px+q)^n\cdot Q_1(x)$$

ko'rinishga ega bo'lib (x^2+px+q kvadrat uchhad haqiqiy ildizga ega emas), $Q_1(x)$ ko'phad x^2+px+q ga bo'linmasa, u holda berilgan to'g'ri kasr quyidagi ko'rinishda ifodalanishi mumkin:

$$\frac{P(x)}{Q(x)}=\frac{B_nx+C_n}{(x^2+px+q)^n}+\frac{B_{n-1}x+C_{n-1}}{(x^2+px+q)^{n-1}}+...+\frac{B_1x+C_1}{x^2+px+q}+\frac{P_1(x)}{Q_1(x)}$$

bunda $B_1,B_2,...,B_n,C_1,C_2...C_n$ — o'zgarmas sonlar, $P_1(x)$ — ko'phad.

Sodda kasrlarning aniqmas integrallarini hisoblaymiz.

1. $\int\frac{A}{x-a}dx=\int\frac{d(x-a)}{x-a}=A\bullet\ln|x-a|+C$

2. $\int\frac{A}{(x-a)^k}dx=\int(x-a)^{-k}d(x-a)=A\bullet\frac{(x-a)^{1-k}}{1-k}+C$

3. $I=\int\frac{Ax+B}{x^2+px+q}dx$ bu integralni hisoblashda ikkita hol bo'lishi mumkin.

a) x^2+px+q kavadrat uchhad to'liq kvadrat bo'lsa, I – integral 1 va 2 hollarga keltiriladi.

b) Agar x^2+px+q kavadrat uchhad to'liq kvadratga kelmasa, bu holda uni to'liq kvadratga to'ldirib integrallash mumkin.

5–misol.

$$I=\int\frac{2x-2}{x^2-x+1}dx=\int\frac{2x-2}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}dx, \qquad x-\frac{1}{2}=t \text{ desak,}$$

$$I=\int\frac{2t-1}{t^2+\frac{3}{4}}dt=\int\frac{2t}{t^2+\frac{3}{4}}dt-\int\frac{dt}{t^2+\frac{3}{4}}=\ln\left(t^2+\frac{3}{4}\right)-\frac{2}{\sqrt{3}}arctg\frac{2t}{\sqrt{3}}+C=$$

$$=\ln(x^2-x+1)-\frac{2}{\sqrt{3}}arctg\frac{2t-1}{\sqrt{3}}+C$$

Izoh: Ma'lumki, elementar funktsiyaning hosilasi yana elementar funktsiya bo'lar edi, lekin integral olish uchun bu tasdiq o'rinli bo'lishi shart emas, ya'ni ba'zi bir elementar funktsiyalarning integrallari elementar funktsiya bo'lmay qolishi mumkin. Masalan, ushbu

1. $\int e^{-x^2}dx$, 2. $\int\cos x^2dx$,

3. $\int\sin x^2dx$, 4. $\int\frac{dx}{\ln x}$ $(x\geq 0, x\neq 1)$,

5. $\int\frac{\cos x}{x}dx (x\neq 0)$, 6. $\int\frac{\sin x}{x}dx$

integrallarning har biri elementar funktsiyalar yordamida ifodalanmaydi. Bu funktsiyalar amaliyotda ko'p uchraganligi sababli ularning qiymatlarini hisoblash uchun alohida jadval tuzilgan va ularning grafiklari yasalgan. Shu yo'l bilan elementar funktsiyalarda integrallanmaydigan funktsiyalar ham to'la o'rganilgan.

Nazorat savollari.

1. Boshlang'ich funktsiya.
2. Aniqmas integral.
3. O'zgaruvchini almashtirib integrallash usuli.
4. Bo'laklab integrallash usuli.
5. Ratsional funktsiyalarni integrallash.
6. Boshlang'ich funktsiyalar jadvali.

2–§. Aniq integral va uning tadbiqlari.

1. O'tilgan yo'l haqidagi masala. Biror moddiy nuqta to'g'ri chiziq bo'yicha $[t_0,T]$ vaqt oralig'ida $v=v(t)$ tezlik bilan $(t\in[t_0,T])$ harakat qilayotgan bo'lsa, uning bosib o'tgan yo'li S ni topish talab etilsin. $[t_0,T]$ vaqt oralig'ini $t_0<t_1<t_2<...t_n=T$ n–ta bo'lakka bo'lamiz va $\Delta t_k=t_k-t_{k-1}$, $(k=1,2,....,n)$ deb olamiz. Bu bo'laklarning eng kattasini $\lambda=\max\Delta t_k$ deb belgilaymiz. Agar bu oraliqlar kichik bo'lsa, u holda katta bo'lmagan xatolik bilan har bir oraliqda harakat bir xil deb hisoblashimiz mumkin.

$$S\approx v(\tau_1)\Delta t_1+v(\tau_2)\Delta t_2+.....+v(\tau_n)\Delta t_{n-1},$$

bu yerda $\tau_k\in[t_{k-1}:t_k]$. Yig'indini $\sum_{k=1}^{n}v(\tau_k)\Delta t_k$ – deb belgilaymiz va

$$S=\lim_{\lambda\to 0}\sum_{k=1}^{n}v(\tau_k)\Delta t_k$$

deb olamiz.

Aytaylik, $[a,b]$ kesmada chegaralangan $f(x)$ funktsiya berilgan bo'lsin. $[a,b]$ kesmada ushbu

$$a=x_0<x_1<x_2<...<x_n=b$$

ifodani qanoatlantiruvchi ixtiyoriy nuqtalarni olamiz.

$$\Delta x_k = x_{k+1} - x_k \quad \text{va} \quad \lambda = \max_{k=0,n-1} \Delta x_k$$

deb belgilaymiz.

$\forall \tau_k \in [x_k; x_{k+1}]$ nuqtalarni olib, quyidagi yig'indini tuzamiz $\sum_{k=0}^{n-1} f(\tau_k)\Delta x_k$.

Agar $[a,b]$ kesmaning bo'linish holatiga va τ_k nuqtani tanlashga bog'liq bo'lmagan $\sum_{k=0}^{n-1} f(\tau_k)\Delta x_k$. chekli limit mavjud bo'lsa, bu holatda bu limitga $f(x)$ funktsiyaning $[a,b]$ segmentdagi **aniq integrali** deb ataladi va quyidagicha belgilanadi.

$$\int_a^b f(x)dx = \lim_{\lambda \to 0} \sum_{k=1}^{n} f(\tau_k)\Delta x_k$$

$f(x)$ – ga **integral osti funktsiya** deyiladi. $a, в$ – larga mos ravishda integralning **quyi va yuqori chegarasi** deyiladi.

2. Aniq integral xossalari.

1⁰. O`zgarmas sonni integral belgisi ostidan chiqarib yozish mumkin:

$$\int_a^b Cf(x)dx = C\int_a^b f(x)dx.$$

2⁰. $\int_a^b [f_1(x) + f_2(x)]dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx$

3⁰. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

4⁰. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

$(a < c < b)$.

Teorema: *Agar $f(x)$ funktsiya $[a,b]$ kesmada uzluksiz va $F(x)$ bu kesmada $f(x)$ funktsiyaning boshlang'ich funktsiyasi bo'lsa, u holda*

$$\int_a^b f(x)dx = F(b) - F(a)$$

bo'ladi.

Bu formulaga **Nyuton–Leybnits formulasi** deb ataladi.

Misol.

$$\int_0^{\frac{\pi}{2}} \cos x dx = \sin x \Big|_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1.$$

Teorema: (O'rta qiymat haqidagi teorema). *Agar* $f(x)$ *funktsiya* $[a,b]$ *segmentda uzluksiz bo'lsa, u holda* $[a,b]$ *segmentda shunday* c *nuqta topiladiki,*

$$\int_a^b f(x)dx = (b-a)\cdot f(c)$$

tenglik o'rinli bo'ladi.

Isbot: Nyuton – Leybnits formulasiga ko'ra $\int_a^b f(x)dx = F(b) - F(a)$ deb yozishimiz mumkin. Lagranj teoremasiga ko'ra $F(b) - F(a) = (b-a)\cdot F'(c) = (b-a)f(c)$, bu yerda $a < c < b$ teorema isbot bo'ldi.

3. Yoy uzunligini hisoblash.

Ma'lumki, egri chiziq yoyining uzunligi shu egri chiziqqa chizilgan siniq chiziq perimetrining limiti sifatida ta'riflanadi. Siniq chiziq perimetri yig'indiga (integral yig'indiga) keladi va uning limiti aniq integralni ifodalaydi.

1. Faraz qilaylik, AB yoy $y = f(x)$ $(a \le x \le b)$ tenglama bilan aniqlansin. Bunda $f(x)$ funktsiya $[a,b]$ segmentda aniqlangan uzluksiz va uzluksiz $f'(x)$ hosilaga ega bo'lsin. AB yoyning uzunligi

$$l = \int_c^b \sqrt{1 + f'^2(x)dx} \quad (1)$$

bo'ladi.

1. Faraz qilaylik, AB yoy

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \qquad (\alpha \le t \le \beta)$$

tenglamalar sistemasi bilan aniqlansin (bu holda egri chiziq parametrik holda berilgan deyiladi). Bunda $x = x(t)$, $y = y(t)$ funktsiyalar $[\alpha, \beta]$ da aniqlangan, uzluksiz va uzluksiz $x'(t)$, $y'(t)$ hosilalarga ega.

AB yoyning uzunligi

$$l = \int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t)}dt$$

bo'ladi.

3. Faraz qilaylik, AB egri chiziq qutb koordinatalar sistemasida $\rho = \rho(\theta)$ $(\alpha \le \theta \le \beta)$ funktsiya bilan berilgan bo'lsin. Bunda $\rho = \rho(\theta)$ funktsiya $[a, b]$ segmentda uzluksiz va $\rho'(\theta)$ hosilaga ega. Bu holda AB egri chiziqning uzunligi

$$\rho = \int_\alpha^\beta \sqrt{\rho'^2(\theta) + \rho(\theta)d\theta} \quad (3)$$

bo'ladi.

4. Aniq integralning biologiyaga ba'zi tadbiqlari. Populyatsiya miqdori.

Vaqt o'tishi bilan populyatsiya miqdori o'zgarib turadi. Agar populyatsiya uchun sharoit yetarlicha yaxshi bo'lsa, u holda tug'ilish soni o'lishga nisbatan ko'p bo'ladi. Populyatsiya tezligini $v = v(t)$ (birlik t vaqt ichida) bilan belgilaymiz. Eski populyatsiya yashash joyida $v(t)$ tezlik kamayadi va asta–sekin nolga yaqinlashadi. Lekin populyatsiya yosh bo'lsa, o'zaro bir–biri bilan bo'lgan munosabatlar hali o'rnatilmagan bo'lsa yoki mavjud bo'lgan ba'zi tashqi ta'sirlar bunga ta'sir qilsa, misol uchun insonning aralashuvi, u holda $v(t)$ tezlik sezilarli darajada ko'payib yoki kamayib to'radi.

Agar populyatsiya tezligi $v(t)$ ma'lum bo'lsa, u holda biz t_0 dan T vaqtgacha bo'lgan oraliqda populyatsiya miqdorining o'sishini topa olamiz. Haqiqatdan ham t vaqt ichida $v(t)$ ta'rifidan, bu funktsiya N(t) populyatsiya miqdoridan olingan hosilaga teng va bundan kelib chiqadiki N(t) populyatsiya miqdori $v(t)$ funktsiyaning boshlang'ich funktsiyasi. Shuning uchun

$$N(T)-N(t_0)=\int_{t_0}^{T}v(t)dt \qquad (1)$$

Ma'lumki juda yaxshi sharoitda populyatsiya tezligining o'sishi $v(t)=ae^{kt}$ bo'ladi. Populyatsiya bu holda keksaymaydi. Bu holda (1) formuladan foydalanib, quyidagiga ega bo'lamiz:

$$N(T)=N(t_0)+a\int_{t_0}^{T}e^{kT}dt=N(t_0)+\frac{a}{k}e^{kt}\Big|_{t_0}^{T}=N(t_0)+\frac{a}{k}\left(e^{kT}-e^{kt_0}\right) \qquad (2)$$

Bu formula orqali ba'zi zambrug'larning (penitsillin ajratib chiqaradigan) miqdorini hisoblab chiqarish mumkin.

Populyatsiya biomassasi.

Shunday biomassalarni qaraymizki hayoti davomida massasi sezilarli o'zgaradi va shu populyatsiya umumiy biomassasini hisoblaymiz. Aytaylik, τ qaysidir birlik vaqtning o'sishini aniqlasin, N(τ) o'sishi τ ga teng populyatsiyaning maxsus miqdori. R(τ) − maxsus τ o'sishidagi o'rtacha massa, M(τ) − 0 dan τ o'sishgacha bo'lgan biomassa.

Shuni aniqlaymizki, N(τ)R(τ) ko'paytma τ o'sishdagi biomassaga teng. Ayirmani qaraylik,

$$M(\tau+\Delta\tau)-M(\tau)$$

Yuqoridagi ayirma τ dan $\tau+\Delta\tau$ o'sishdagi biomassa, quyidagini qanoatlantiradi:

$$N\left(\check{\tau}\right)P\left(\check{\tau}\right)\Delta\tau\le M(\tau+\Delta\tau)-M(\tau)\le N\left(\hat{\tau}\right)P\left(\hat{\tau}\right)\Delta\tau \qquad (3)$$

bu yerda $N\left(\check{\tau}\right)P\left(\check{\tau}\right)$ − etarlicha kichik, $N\left(\hat{\tau}\right)P\left(\hat{\tau}\right)$ − etarlicha katta miqdor ($[\tau,\tau+\Delta\tau]$ oraliqda $N\left(\hat{\tau}\right)P\left(\hat{\tau}\right)$ funktsiyaning). $\Delta\tau>0$ ni hisobga olib (3) tengsizlikdan quyidagiga kelamiz.

$$N\left(\check{\tau}\right)P\left(\check{\tau}\right) \leq \frac{M(\tau+\Delta\tau)-M(\tau)}{\Delta\tau} \leq N\left(\hat{\tau}\right)P\left(\hat{\tau}\right) \quad (4),$$

$N(\tau)R(\tau)$ funktsiyalar uzluksizligidan (yani $N(\tau)$ va $R(\tau)$ larning uzluksiz bulgani uchun) quyidagiga kelamiz.

$$\lim_{\Delta\tau\to 0}\left[N\left(\check{\tau}\right)P\left(\check{\tau}\right)\right] = \lim_{\Delta\tau\to 0}\left[N\left(\hat{\tau}\right)P\left(\hat{\tau}\right)\right] = N(\tau)P(\tau),$$

bundan $\lim_{\Delta\tau\to 0}\frac{M(\tau+\Delta\tau)-M(\tau)}{\Delta\tau} = N(\tau)P(\tau)$ yoki $\frac{dM(\tau)}{d\tau} = N(\tau)P(\tau)$ kelib chiqadiki, $M(\tau)$ biomassa $N(\tau)R(\tau)$ larning boshlang'ich funktsiyasi. Demak,

$$M(T)-M(0) = \int_0^{\pi} N(\tau)P(\tau)d\tau,$$

bu yerda T – berilgan biomassaning maksimal yoshi. M(0) aniqki nolga teng, u holda

$$M(T) = \int_0^{\pi} N(\tau)P(\tau)d\tau$$

bu ifoda orqali populyatsiyaning umumiy biomassasini hisoblaymiz.

Nazorat savollari.

1. Aniq integral.
2. Aniq integral xossalari.
3. O'rta qiymat haqidagi teorema.
4. Yoy uzunligini hisoblash.
5. Tekis shaklning yuzasini hisoblash.
6. Aniq integralning biologik masalalarga tadbiqi.

Mustaqil yechish uchun misollar va masalalar.

1–masala. Aniqmas integral topilsin.

1.1. $\int(4-3x)e^{-3x}dx$

1.2. $\int arctg\sqrt{4x-1}dx$

1.3. $\int(3x+4)e^{3x}dx$

1.4. $\int(4x-2)\cos 2xdx$

1.5. $\int(4-16x)\sin 4xdx$

1.6. $\int(5x+2)e^{3x}dx$

1.7. $\int(1-6x)e^{2x}dx$

1.8. $\int\ln(x^2+4)dx$

1.9. $\int\ln(4x^2+1)dx$

1.10. $\int(2-4x)\sin 2xdx$

1.11. $\int arctg\sqrt{6x-1}dx$

1.12. $\int e^{-2x}(4x-3)dx$

1.13. $\int e^{-3x}(2-9x)dx$

1.14. $\int arctg\sqrt{2x-1}dx$

1.15. $\int arctg\sqrt{3x-1}dx$

1.16. $\int arctg\sqrt{5x-1}dx$

1.17. $\int(5x+6)\cos 2xdx$

1.18. $\int(3x-2)\cos 5xdx$

1.19. $\int(x\sqrt{2}-3)\cos 2xdx$

1.20. $\int(4x+7)\cos 3xdx$

1.21. $\int(2x-5)\cos 4xdx$

1.22. $\int(8-3x)\cos 5xdx$

1.23. $\int(x+5)\sin 3xdx$

1.24. $\int(2-3x)\sin 2xdx$

1.25. $\int(4x+3)\sin 5xdx$

1.26. $\int(7x-10)\sin 4xdx$

1.27. $\int(\sqrt{2}-8x)\sin 3xdx$

1.28. $\int\frac{xdx}{\cos^2 x}$

1.29. $\int\frac{xdx}{\sin^2 x}$

1.30. $\int x\sin^2 xdx$

2–masala. Aniqmas integral hisoblansin.

2.1. $\int\frac{dx}{x\sqrt{x^2+1}}$

2.2. $\int\frac{1+\ln x}{x}dx$

2.3. $\int\frac{dx}{x\sqrt{x^2-1}}$

2.4. $\int\frac{x^2+\ln x^2}{x}dx$

2.5. $\int\frac{xdx}{\sqrt{x^4+x^2+1}}$

2.6. $\int\frac{(ar\cos x)^3-1}{\sqrt{1-x^2}}dx$

2.7. $\int tgx\ln Cosxdx$

2.8. $\int\frac{tg(x+1)}{Cos^2(x+1)}dx$

2.9. $\int\frac{x^3}{(x^2+1)^2}dx$

2.10. $\int\frac{1-Cosx}{(x-Sinx)^2}dx$

2.11. $\int\frac{Sinx-Cosx}{(Cosx+Sinx)^5}dx$

2.12. $\int\frac{xCos+Sinx}{(xSinx)^2}dx$

2.13. $\int \frac{x^3+x}{x^4+1}dx$

2.14. $\int \frac{xdx}{\sqrt{x^4-x^2-1}}$

2.15. $\int \frac{xdx}{\sqrt[3]{x-1}}$

2.16. $\int \frac{1+\ln(x-1)}{x-1}dx$

2.17. $\int \frac{(x^2+1)dx}{(x^3+3x+1)^5}$

2.18. $\int \frac{4arctgx-x}{1+x^2}dx$

2.19. $\int \frac{x^3dx}{x^2+4}$

2.20. $\int \frac{x+Cosx}{x^2+2Sinx}dx$

2.21. $\int \frac{2Cosx+3Sinx}{(2Sinx-3Cosx)^3}dx$

2.22. $\int \frac{8x-arctg2x}{1+4x^2}dx$

2.23. $\int \frac{\frac{1}{2\sqrt{x}}+1}{(\sqrt{x}+x)^2}$

2.24. $\int \frac{x}{x^4+1}dx$

2.25. $\int \frac{x+1/x}{\sqrt{x^2+1}}dx$

2.26. $\int \frac{x-\frac{1}{x}}{\sqrt{x^2+1}}dx$

2.27. $\int \frac{arctgx+x}{1+x^2}dx$

2.28. $\int \frac{x-(arctgx)^4}{1+x^2}dx$

2.29. $\int \frac{x^3dx}{x^2+1}$

2.30. $\int \frac{(arcSinx)^2+1}{\sqrt{1-x^2}}dx$

3–masala. Aniqmas integral hisoblansin.

3.1. $\int \frac{x^3+1}{x^2-x}dx$

3.2. $\int \frac{3x^3+1}{x^2-1}dx$

3.3. $\int \frac{x^3-17}{x^2-4x+3}dx$

3.4. $\int \frac{2x^3+5}{x^2-x-2}dx$

3.5. $\int \frac{2x^3-1}{x^2+x-6}dx$

3.6. $\int \frac{3x^3+25}{x^3+3x+2}dx$

3.7. $\int \frac{x^3+2x^2+3}{(x-1)(x-2)(x-3)}dx$

3.8. $\int \frac{3x^3+2x^2+1}{(x+2)(x-2)(x-1)}dx$

3.9. $\int \frac{x^3}{(x-1)(x+1)(x+1)}dx$

3.10. $\int \frac{x^3-3x^2-12}{(x-4)(x-3)(x-2)}dx$

3.11. $\int \frac{x^3-3x^2-12}{(x-4)(x-3)x}dx$

3.12. $\int \frac{4x^3+x^2+2}{x(x-1)(x-2)}dx$

3.13. $\int \frac{3x^3-2}{x^3-x}dx$

3.14. $\int \frac{x^3-3x^2-12dx}{(x-4)(x-2)^x}dx$

3.15. $\int \frac{x^5-x^3+1}{x^2-x}dx$

3.16. $\int \frac{x^5+3x^3-1}{x^2+x}dx$

3.17. $\int\frac{2x^5-8x^3+3}{x^2-2x}dx$

3.18. $\int\frac{3x^5-12x^3-7}{x^2+2x}dx$

3.19. $\int\frac{-x^5+9x^3+4}{x^2+3x}dx$

3.20. $\int\frac{-x^5+25x^3+1}{x^2+5x}dx$

3.21. $\int\frac{x^3-5x+5x+23}{(x-1)(x+1)(x-5)}dx$

3.22. $\int\frac{x^5+2x^4-2x^3+5x^2-7x+9}{(x+3)(x-1)^x}dx$

3.23. $\int\frac{2x^4-5x^2-8x-8}{x(x-2)(x+2)}dx$

3.24. $\int\frac{4x^4+2x^2-x-3}{x(x-1)(X+1)}dx$

3.25. $\int\frac{3x^4+3x^3-5x^2+2}{x(x-1)(x+2)}dx$

3.26. $\int\frac{2x^4+2x^3-41x^2+20}{x(x-4)(x+5)}dx$

3.27. $\int\frac{x^5-x^4-6x^3+13x+6}{x(x-5)(x+2)}dx$

3.28. $\int\frac{3x^3-x^2-12x-2}{x(x+1)(x-2)}dx$

3.29. $\int\frac{2x^4+2x^3-3x^2+2x-9}{x(x-1)(x+3)}dx$

3.30. $\int\frac{2x^3-x^2-7x-12}{x(x-3)(x+1)}dx$

4–masala. Aniq integral hisoblansin.

4.1. $\int_{-2}^{0}(x^2+5x+6)Cos2xdx$

4.2. $\int_{-2}^{0}(x^2-4)Cos3xdx$

4.3. $\int_{-1}^{0}(x^2+4x+3)Cosxdx$

4.4. $\int_{-2}^{0}(x+2)^2Cos3xdx$

4.5. $\int_{-4}^{0}(x^2+7x+12)Cosxdx$

4.6. $\int_{0}^{\pi}(2x^2+4x+7)Cos2xdx$

4.7. $\int_{0}^{\pi}(9x^2+9x+11)Cos3xdx$

4.8. $\int_{0}^{\pi}(8x^2+16x+17)Cos4xdx$

4.9. $\int_{0}^{2\pi}(3x^2+5)Cos2xdx$

4.10. $\int_{0}^{2\pi}(2x^2-15)Cos3xdx$

4.11. $\int_{0}^{2\pi}(3-7x^2)Cos2xdx$

4.12. $\int_{0}^{2\pi}(1-8x^2)Cos4xdx$

4.13. $\int_{-1}^{0}(x^2+2x+1)Sin3xdx$

4.14. $\int_{0}^{3}(x^2-3x)Sin2xdx$

4.15. $\int_{0}^{\pi}(x^2-3x+2)Sinxdx$

4.16. $\int_{0}^{\pi/2}(x^2-5x+6)Sin3xdx$

4.17. $\int_{-3}^{0}(x^2+6x+9)Sin2xdx$

4.18. $\int_{0}^{\pi/4}(x^2+17.5)Sin2xdx$

4.19. $\int_{0}^{\pi/2}(1-5x^2)Sinxdx$

4.20. $\int_{\pi/4}^{3}(3x-x^2)Sin2xdx$

4.21. $\int_{1}^{2} x\ln^2 x dx$

4.22. $\int_{1}^{e} \frac{\ln^2 x dx}{\sqrt{x}}$

4.23. $\int_{1}^{8} \frac{\ln^2 x dx}{\sqrt[3]{x^2}}$

4.24. $\int_{0}^{1} (x+1)\ln^2(x+1)dx$

4.25. $\int_{2}^{3} (x-1)^3 \ln^2(x-1)dx$

4.26. $\int_{-1}^{0} (x+2)^3 \ln^2(x+2)dx$

4.27. $\int_{0}^{2} (x+1)^2 \ln^2(x+1)dx$

4.28. $\int_{1}^{e} \sqrt{x}\ln^2 x dx$

4.29. $\int_{-1}^{1} x^2 e^{-\frac{x}{2}} dx$

4.30. $\int_{0}^{1} x^2 e^{3x} dx$

5−masala. Aniq integral hisoblansin.

5.1. $\int_{e+1}^{e^2+1} \frac{1+\ln(x-1)}{x-1}dx$

5.2. $\int_{0}^{1} \frac{(x^2+1)dx}{(x^3+3x+1)^2}$

5.3. $\int_{0}^{1} \frac{4arctgx - x}{1+x^2}dx$

5.4. $\int_{0}^{2} \frac{x^3 dx}{x^2+4}$

5.5. $\int_{\pi}^{2\pi} \frac{x + Cosx}{x^1 + 2Sinx}dx$

5.6. $\int_{0}^{\pi/4} \frac{2Cos + 3Sinx}{(2Sinx - 3Cosx)^3}dx$

5.7. $\int_{0}^{1/2} \frac{8x - arctg2x}{1+4x^2}dx$

5.8. $\int_{1}^{4} \frac{\frac{1}{2\sqrt{x}}+1}{(\sqrt{x}+x)^2}dx$

5.9. $\int_{0}^{1} \frac{xdx}{x^4+1}$

5.10. $\int_{\sqrt{3}}^{\sqrt{8}} \frac{x+\frac{1}{x}}{\sqrt{x^2+1}}dx$

5.11. $\int_{\sqrt{3}}^{\sqrt{8}} \frac{x-\frac{1}{x}}{\sqrt{x^2+1}}dx$

5.12. $\int_{0}^{\sqrt{3}} \frac{arctgx + x}{1+x^2}dx$

5.13. $\int_{0}^{\sqrt{3}} \frac{x-(arctgx)^4}{1+x^2}dx$

5.14. $\int_{0}^{1} \frac{x^3}{x^2+1}dx$

5.15. $\int_{0}^{Sin1} \frac{(arcSinx)^2+1}{\sqrt{1-x^2}}dx$

5.16. $\int_{1}^{3} \frac{1-\sqrt{x}}{\sqrt{x}(x+1)}dx$

5.17. $\int_{\sqrt{3}}^{\sqrt{8}} \frac{dx}{x\sqrt{x^2+1}}$

5.18. $\int_{1}^{e} \frac{1+\ln x}{x}dx$

5.19. $\int_{\sqrt{2}}^{2} \frac{dx}{x\sqrt{x^2-1}}$

5.20. $\int_{1}^{e} \frac{x^2+\ln x^2}{x}dx$

5.21. $\int_{0}^{1} \frac{xdx}{\sqrt{x^4+x^2+1}}$

5.22. $\int_{0}^{1} \frac{x^3 dx}{(x^2+1)^2}$

5.23. $\int_{0}^{\pi/4} tgx \ln Cosx dx$

5.24. $\int_{-1}^{0} \frac{tg(x+1)}{Cos^2(x+1)} dx$

5.25. $\int_{0}^{\frac{1}{\sqrt{2}}} \frac{(\arccos x)^3 - 1}{\sqrt{1-x^2}} dx$

5.26. $\int_{\pi}^{2\pi} \frac{1-Cosx}{(x-Sinx)^2} dx$

5.27. $\int_{\pi}^{\pi/4} \frac{Sinx-Cosx}{(Cosx+Sinx)^5} dx$

5.28. $\int_{\pi/4}^{\pi/2} \frac{xCosx+Sinx}{(xSinx)^2} dx$

5.29. $\int_{0}^{1} \frac{x^3+x}{x^4+1} dx$

5.30. $\int_{2}^{0} \frac{xdx}{\sqrt[3]{x-1}}$

6–masala. Aniq integral hisoblansin.

6.1. $\int_{\pi/2}^{2arctg2} \frac{dx}{Sin^2x(1-Cosx)}$

6.2. $\int_{0}^{\pi/2} \frac{Cosxdx}{2+Cosx}$

6.3. $\int_{\pi/2}^{2arctg2} \frac{dx}{Sin^2x(1+Cosx)}$

6.4. $\int_{2arctg\frac{1}{2}}^{\pi/2} \frac{Cosxdx}{(1-Cosx)^3}$

6.5. $\int_{0}^{\pi/2} \frac{Cosx-Sinx}{(1+Sinx)^2} dx$

6.6. $\int_{2arctg2}^{2arctg3} \frac{dx}{Cosx(1-Cosx)}$

6.7. $\int_{2arctg\frac{1}{3}}^{2arctg\frac{1}{2}} \frac{dx}{Sinx(1-Sinx)}$

6.8. $\int_{2arctg\frac{1}{2}}^{\pi/2} \frac{dx}{(1+Sinx-Cosx)^2}$

6.9. $\int_{0}^{\pi/2} \frac{Cosxdx}{5+4Cosx}$

6.10. $\int_{0}^{2\pi/3} \frac{1+Sinx}{1+Cosx+Sinx} dx$

6.11. $\int_{\pi/3}^{\pi/2} \frac{x+Cosxdx}{1+Sinx-Cosx}$

6.12. $\int_{0}^{\pi/2} \frac{(1+Cosx)dx}{1+Cosx+Sinx}$

6.13. $\int_{0}^{\pi/2} \frac{Sinxdx}{1+Cosx+Sinx} dx$

6.14. $\int_{0}^{2arctg\frac{1}{2}} \frac{1+Sinx}{(1-Sinx)^2} dx$

6.15. $\int_{0}^{2\pi} \frac{Cosxdx}{1+Cosx+Sinx}$

6.16. $\int_{0}^{2arctg1/3} \frac{Cosxdx}{(1+Cosx-Sinx)^2}$

6.17. $\int_{-\frac{2\pi}{3}}^{0} \frac{Cosxdx}{1+Cosx-Sinx}$

6.18. $\int_{-\frac{\pi}{2}}^{0} \frac{Cosxdx}{(1+Cosx-Sinx)^2}$

6.19. $\int_{0}^{\pi/2} \frac{Cosxdx}{(1+Cosx-Sinx)^2}$

6.20. $\int_{0}^{2arctg\frac{1}{2}} \frac{(1-Sinx)dx}{Cosx(1+Cosx)}$

6.21. $\int_{0}^{\pi/2} \frac{Sinxdx}{(1+Sinx)^2}$

6.22. $\int_{0}^{\pi/2} \frac{Sinxdx}{(1+Sinx+Cos)^2}$

6.23. $\int_{-\frac{\pi}{2}}^{0} \frac{Sinxdx}{(1+Cosx-Sinx)^2}$

6.24. $\int_{-\frac{2\pi}{3}}^{0} \frac{Cos^2xdx}{(1+Cosx-Sinx)^2}$

6.25. $\int_{0}^{\pi/2} \frac{Sin^2xdx}{(1+Cosx+Sinx)^2}$

6.26. $\int_{0}^{2\pi/3} \frac{Cos^2xdx}{(1+Cosx+Sinx)^2}$

6.27. $\int_{\pi/2}^{2arctg2} \frac{dx}{Sinx(1+Sinx)}$

6.28. $\int_{0}^{\pi/2} \frac{dx}{(1+Sinx+Cosx)^2}$

6.29. $\int_{0}^{\pi/2} \frac{Sinxdx}{2+Sinx}$

6.30. $\int_{0}^{\pi/4} \frac{dx}{Cosx(1+Cosx)}$

7–masala. Aniq integral hisoblansin.

7.1. $\int_{\frac{\pi}{2}}^{\pi} 2^8 Sin^8 xdx$

7.2. $\int_{0}^{\pi} 2^4 Sin^6 xCos^2 xdx$

7.3. $\int_{0}^{2\pi} Sin^4 xCos^4 xdx$

7.4. $\int_{0}^{2\pi} Sin^2 \frac{x}{4} Cos^6 \frac{x}{4} dx$

7.5. $\int_{0}^{\pi} 2^4 Cos^8 \frac{x}{2} dx$

7.6. $\int_{-\frac{\pi}{2}}^{0} 2^8 Sin^8 xdx$

7.7. $\int_{-\frac{\pi}{2}}^{\pi} 2^8 Sin^6 xCos^2 xdx$

7.8. $\int_{0}^{\pi} 2^4 Sin^4 xCos^4 xdx$

7.9. $\int_{0}^{2\pi} Sin^2 xCos^6 xdx$

7.10. $\int_{0}^{2\pi} Cos^8 \frac{x}{4} dx$

7.11. $\int_{0}^{\pi} 2^4 Sin^8 \frac{x}{2} dx$

7.16. $\int_{0}^{2\pi} Sin^8 \frac{x}{4} dx$

7.17. $\int_{0}^{\pi} 2^4 Sin^6 \frac{x}{2} Cos^2 \frac{x}{2} dx$

7.18. $\int_{-\frac{\pi}{2}}^{0} 2^8 Sin^4 xCos^4 xdx$

7.19. $\int_{\pi/2}^{\pi} 2^8 Sin^2 xCos^6 xdx$

7.20. $\int_{0}^{\pi} 2^4 Cos^8 xdx$

7.21. $\int_{0}^{2\pi} Sin^8 xdx$

7.22. $\int_{0}^{2\pi} Sin^6 \frac{x}{4} Cos^2 \frac{x}{4} dx$

7.23. $\int_{0}^{\pi} 2^4 Sin^4 \frac{x}{2} Cos^4 \frac{x}{2} dx$

7.24. $\int_{-\pi/2}^{0} 2^8 Sin^2 xCos^6 xdx$

7.25. $\int_{-\pi/2}^{\pi} 2^8 Cos^8 xdx$

7.26. $\int_{0}^{\pi} 2^4 Sin^8 xdx$

7.12. $\int_{-\pi}^{0} 2^8 Sin^6 xCos^2 xdx$

7.13. $\int_{\frac{\pi}{2}}^{\pi} 2^8 Sin^4 xCos^4 xdx$

7.14. $\int_{0}^{\pi} 2^4 Sin^2 xCos^6 xdx$

7.15. $\int_{0}^{2\pi} Cos^8 xdx$

7.27. $\int_{0}^{2\pi} Sin^6 xCos^2 xdx$

7.28. $\int_{0}^{2\pi} Sin^4 \frac{x}{4} Cos^4 \frac{x}{4} dx$

7.29. $\int_{0}^{2\pi} Sin^4 3xCos^4 3xdx$

7.30. $\int_{-\frac{\pi}{2}}^{0} 2^8 Cos^8 xdx$

8–masala. Aniq integral hisoblansin.

8.1. $\int_{0}^{16} \sqrt{256-x^2} - dx$

8.2. $\int_{0}^{1} x^2\sqrt{1-x^2}dx$

8.3. $\int_{0}^{5} \frac{dx}{(25+x^2)\sqrt{25+x^2}}$

8.4. $\int_{0}^{36} \frac{dx}{(9+x^2)^{\frac{3}{2}}}$

8.5. $\int_{0}^{\sqrt{\frac{5}{2}}} \frac{dx}{(5-x^2)^3}$

8.6. $\int_{1}^{2} \frac{\sqrt{x^2-1}}{x^4}dx$

8.7. $\int_{0}^{\frac{\sqrt{2}}{2}} \frac{x^4dx}{\sqrt{(1-x^2)^3}}$

8.8. $\int_{0}^{\sqrt{3}} \frac{dx}{\sqrt{(4-x^2)^3}}$

8.9. $\int_{0}^{1} \frac{x^4dx}{(2-x^2)^{\frac{3}{2}}}$

8.10. $\int_{0}^{2} \frac{x^2dx}{\sqrt{16-x^2}}$

8.11. $\int_{0}^{2} \sqrt{4-x^2}dx$

8.12. $\int_{0}^{4} \frac{dx}{(16+x^2)^{\frac{3}{2}}}$

8.13. $\int_{0}^{4} x^2\sqrt{16-x^2}dx$

8.14. $\int_{0}^{5/2} \frac{x^2dx}{\sqrt{25-x^2}}$

8.15. $\int_{0}^{5} x^2\sqrt{25-x^2}dx$

8.16. $\int_{0}^{4} x^2\sqrt{16-x^2}dx$

8.17. $\int_{0}^{4\sqrt{3}} \frac{dx}{\sqrt{(64-x^2)^3}}$

8.18. $\int_{\sqrt{2}}^{2\sqrt{2}} \frac{\sqrt{x^2-2}}{x^4}dx$

8.19. $\int_{0}^{2\sqrt{2}} \frac{x^4dx}{(16-x^2)\sqrt{16-x^2}}$

8.20. $\int_{-3}^{3} x^2\sqrt{9-x^2}dx$

8.21. $\int_{0}^{1} x^2\sqrt{1-x^2}dx$

8.22. $\int_{0}^{1} x^2\sqrt{1-x^2}dx$

VII BOB.
IKKI O'ZGARUVCHILI FUNKTSIYA TUSHUNCHASI.

1–§. Ikki o'zgaruvchili funktsiya tushunchasi.

1⁰. Ushbu $f: R^2 \to R$ akslantirish ko'p o'zgaruvchili (ikki o'zgaruvchili) funktsiya tushunchasiga olib keladi.

1–Ta'rif: Agar $E(E \subset R^2)$ to'plamdagi har bir (x,y) nuqtaga biror qoida yoki qonunga ko'ra bitta haqiqiy u son $(u \in R)$ mos qo'yilgan bo'lsa, E to'plamda **ikki o'zgaruvchili funktsiya berilgan (aniqlangan)** deyiladi.

Uni

$$f:(x,y) \to u \text{ yoki } u = f(x,y)$$

kabi belgilanadi. Bunda E funktsiyaning aniqlanish to'plami, x,y – funktsiya argumentlari, u esa x va y larning funktsiyasi deyiladi.

Masalan: f – har bir $(x,y) \in \{(x,y) \in R^2 : \rho((x,y),(0,0)) \le 1\} = E$ nuqtaga ushbu

$$(x,y) \to \sqrt{1-x^2-y^2}$$

qoida bilan bitta haqiqiy sonni mos qo'ysin. Bu holda $u = \sqrt{1-x^2-y^2}$ funktsiyaga ega bo'lamiz.

Aytaylik, $u = f(x,y)$ funktsiya $E(E \subset R^2)$ to'plamda berilgan bo'lsin. Ushbu $\{((x,y),u) : (x,y) \in E, u = f(x,y)\}$ to'plam **$u = f(x,y)$ funktsiyaning grafigi** deyiladi.

Masalan, $u = x^2 + y^2$, $u = \sqrt{1-x^2-y^2}$ funktsiyalarning grafiklari mos ravishda aylanma paraboloid hamda yuqori yarim sferani ifodalaydi.

Aytaylik, $u = f(x,y)$ funktsiya $E \subset R^2$ to'plamda berilgan, $(x_0, y_0) \in R^2$ nuqta E ning limit nuqtasi bo'lsin.

2–Ta'rif: Agar $\forall \{(x_n, y_n)\}$ $\quad ((x_n, y_n) \in E, \quad (x_n, y_n) \ne (x_0, y_0), n = 1,2,...)$ ketma–ketlik uchun $(x_n, y_n) \to (x_0, y_0)$ da $f(x_n, y_n) \to A$ $(A \in R)$ bo'lsa, u holda A **$f(x,y)$ funktsiyaning $(x,y) \to (x_0, y_0)$ dagi limiti (karrali limiti)** deyiladi va

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=A \quad \text{yoki} \quad \lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=A$$

kabi belgilanadi.

2⁰. Aytaylik, $u=f(x,y)$ funktsiya ochiq $E\subset R^2$ to'plamda berilgan, $(x_0,y_0)\in E$ nuqtalarga mos ravishda Δx va Δy orttirma beramiz, $(x_0+\Delta x,\ y_0+\Delta y)\in E$ bo'lsin. Ushbu

$$\Delta f(x_0,y_0)=f(x_0+\Delta x,y_0+\Delta x)-f(x_0,y_0) \quad (1)$$

ayirma $f(x,y)$ **funktsiyaning** (x_0,y_0) **nuqtadagi to'liq orttirmasi** deyiladi.

3–Ta'rif: Agarda $\Delta x\to 0$, $\Delta y\to 0$ da to'liq orttirma $\Delta f(x_0,y_0)$ nolga intilsa, ya'ni

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}=[f(x_0+\Delta x,y_0+\Delta x)-f(x_0,y_0)]=0$$

bo'lsa, $u=f(x,y)$ funktsiya (x_0,y_0) **nuqtada uzluksiz** deyiladi,

4–Ta'rif: Agar $(x,y)\to(x_0,y_0)$ da $f(x,y)$ funktsiyaning limiti mavjud bo'lmasa yoki

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=\infty \quad \text{yoki} \quad \lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=A\neq f(x_0,y_0)$$

bo'lsa, $f(x,y)$ funktsiya (x_0,y_0) **nuqtada uzilishga ega** deyiladi.

2–§. Ikki o'zgaruvchili funktsiyaning xususiy hosilalari.

1. Funktsiyaning differentsiallanuvchiligi tushunchasi.

1⁰. Ushbu

$$\Delta_x f(x_0,y_0)=f(x_0+\Delta x,y_0)-f(x_0,y_0)$$
$$\Delta_y f(x_0,y_0)=f(x_0,y_0+\Delta y)-f(x_0,y_0)$$

orttirmalar funktsiyaning (x_0,y_0) **nuqtadagi xususiy orttirmalari** deyiladi.

1−Ta'rif: Agar $\Delta x \to 0$ da $\frac{\Delta_x f(x_0, y_0)}{\Delta x}$ nisbatning limiti mavjud va chekli bo'lsa, bu limit $f(x,y)$ funktsiyaning (x_0, y_0) nuqtadagi x o'zgaruvchisi bo'yicha **xususiy hosilasi** deyiladi va $f'_x(x_0, y_0)$ yoki $\frac{\partial f(x_0, y_0)}{\partial x}$ kabi belgilanadi:

$$f'_x(x_0, y_0) = \frac{\partial f(x_0, y_0)}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta_x f(x_0, y_0)}{\Delta x}$$

Xuddi shunga o'xshash y o'zgaruvchi bo'yicha xususiy hosila ta'riflanadi:

$$f'_y(x_0, y_0) = \frac{\partial f(x_0, y_0)}{\partial y} = \lim_{\Delta y \to 0} \frac{\Delta_y f(x_0, y_0)}{\Delta y}$$

Ko'p o'zgaruvchili funktsiyaning xususiy hosilalarini hisoblashda bir o'zgaruvchili funktsiyaning hosilalarini hisoblashdagi ma'lum qoida va jadvallardan foydalanish mumkin.

Misol. Ushbu $f(x,y) = \sqrt{x^2 + y^2}$ funktsiyaning xususiy hosilalari $(x,y) \neq (0,0)$ nuqtada

$$f'_x(x,y) = \frac{2x}{2\sqrt{x^2 + y^2}} = \frac{x}{\sqrt{x^2 + y^2}}, \quad f'_y(x,y) = \frac{2y}{2\sqrt{x^2 + y^2}} = \frac{y}{\sqrt{x^2 + y^2}}$$

bo'ladi.

Berilgan funktsiya $(x,y) = (0,0)$ nuqtada xususiy hosilaga ega emas:

$$f'_x(0,0) = \lim_{x \to 0} \frac{\sqrt{x^2 + 0^2}}{x} = \lim_{x \to 0} \frac{|x|}{x}$$ — mavjud emas;

$$f'_y(0,0) = \lim_{y \to 0} \frac{\sqrt{0^2 + y^2}}{y} = \lim_{y \to 0} \frac{|y|}{y}$$ — mavjud emas.

2−Ta'rif: Agar $f(x,y)$ funktsiyaning (x_0, y_0) nuqtadagi to'liq orttirmasini quyidagi

$$\Delta f(x_0, y_0) = A \cdot \Delta x + B \cdot \Delta y + \alpha \cdot \Delta x + \beta \cdot \Delta y \quad (2)$$

ko'rinishda ifodalash mumkin bo'lsa, $f(x,y)$ funktsiya (x_0,y_0) **nuqtada differentsiallanuvchi** deyiladi, bunda A,B - o'zgarmas, α va β lar Δx va Δy larga bog'liq, $\Delta x \to 0$, $\Delta y \to 0$ da $\alpha \to 0, \Delta\beta \to 0$.

Agar $f(x,y)$ funktsiya $E \subset R^2$ to'plamning har bir nuqtasida differentsiallanuvchi bo'lsa, $f(x,y)$ funktsiya E to'plamda **differentsiallanuvchi** deyiladi. Shunday qilib, $f(x,y)$ funktsiyaning differentsiali quyidagicha yoziladi:

$$df = f'_x dx + f'_y dy \text{ yoki } df(x_0,y_0) = \frac{\partial f(x_0,y_0)}{\partial x}dx + \frac{\partial f(x_0,y_0)}{\partial y}dy \quad (3)$$

Misol. Agar $f(x,y) = (x+y)^2$ bo'lsa, u holda

$$f'_x = f'_y = 2(x+y) \text{ va } \qquad df = 2(x+y)(dx+dy).$$

Teorema: *Agar* $f(x,y)$ *funktsiya* $(x_0,y_0) \in E$ *nuqtada differentsiallanuvchi bo'lsa, funktsiya shu nuqtada uzluksiz bo'ladi.*

Isbot: $\Delta f(x_0,y_0) = A\cdot\Delta x + B\cdot\Delta y + \alpha\cdot\Delta x + \beta\cdot\Delta y$ munosabatdan, $\Delta x \to 0$, $\Delta y \to 0$ da $\Delta f(x_0,y_0) \to 0$ bo'lishini topamiz.
Demak, $f(x,y)$ funktsiya (x_0,y_0) nuqtada uzluksiz.

Teorema: *Agar* $f(x,y)$ *funktsiya* (x_0,y_0) *nuqtada differentsiallanuvchi bo'lsa, funktsiyaning shu nuqtadagi xususiy hosilalari* $f'_x(x_0,y_0)$, $f'_y(x_0,y_0)$ *mavjud bo'lib,* $f'_x(x_0,y_0) = A$, $f'_y(x_0,y_0) = B$ *bo'ladi.*

Isbot: $\Delta f(x_0,y_0) = A\cdot\Delta x + B\cdot\Delta y + \alpha\cdot\Delta x + \beta\cdot\Delta y$ da $\Delta x \neq 0$, $\Delta y = 0$ deb, $\Delta x \to 0$ da

$$\lim_{\Delta x \to 0}\frac{\Delta_x f(x_0,y_0)}{\Delta x} = \lim_{\Delta x \to 0}\frac{A\cdot\Delta x + \alpha\cdot\Delta x}{\Delta x} = A,$$

ya'ni $f'_x(x_0,y_0) = A$ bo'lishini topamiz.

Xuddi shunga o'xshash $f'_y(x_0,y_0)$ ning mavjudligi va $f'_y(x_0,y_0) = B$ bo'lishi ko'rsatiladi.

Endi murakkab

$$u = f(x,y) = f(\varphi(s,t),\psi(s,t)) = F(s,t)$$

funktsiyaning differentsialini hisoblaymiz.

(3) formulaga ko'ra

$$df = \frac{\partial f}{\partial s}ds + \frac{\partial f}{\partial t}dt$$

bo'ladi. Endi

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s},$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$$

ekanini e'tiborga olib topamiz:

$$df = \left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}\right)ds + \left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}\right)\cdot dt =$$
$$= \frac{\partial s}{\partial x}\left(\frac{\partial x}{\partial s}ds + \frac{\partial x}{\partial t}dt\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial s}ds + \frac{\partial y}{\partial t}dt\right) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Misol. Agar $f = xy$ bo'lib, bu yerda $x = t\sin 2\tau,\quad y = t^2\tau$ bo'lsa, u holda

$$f'_t = y\sin 2\tau + 2x\tau,\quad f'_\tau = -2yt\cos 2\tau + xt^2$$

bo'ladi.

2. Ko'p o'zgaruvchili funktsiyalarning yuqori tartibli hosila va differentsiallari.

Aytaylik, $f(x,y)$ funktsiya ochiq $E \subset R^2$ to'plamda berilgan bo'lib, $\forall (x,y) \in E$ nuqtada $f'_x(x,y),\ f'_y(x,y)$ xususiy hosilalarga ega bo'lsin.

3–Ta'rif: $f(x,y)$ funktsiya xususiy hosilalari $f'_x(x,y), f'_y(x,y)$ larning x va y o'zgaruvchilari bo'yicha xususiy hosilalari $f(x,y)$ funktsiyaning **ikkinchi tartibli xususiy hosilalari** deyiladi.

Demak,

$$(f'_x(x,y))'_x = f''_{x^2}(x,y),\quad (f'_x(x,y))'_y = f''_{xy}(x,y),\quad (f'_y(x,y))'_x = f''_{yx}(x,y),$$

$$(f'_y(x,y))'_y = f''_{y^2}(x,y)$$

yoki

$$\frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial x}\right)=\frac{\partial^2 f(x,y)}{\partial x^2};\quad \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right)=\frac{\partial^2 f(x,y)}{\partial x\partial y};$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right)=\frac{\partial^2 f(x,y)}{\partial y\partial x};\quad \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial y}\right)=\frac{\partial^2 f(x,y)}{\partial y^2}.$$

Odatda $f''_{xy}(x,y)$, $f''_{yx}(x,y)$ hosilalar **aralash hosilalar** deyiladi.

Xuddi shunga o'xshash $f(x,y)$ funktsiyaning uchinchi, to'rtinchi va hokazo tartibdagi xususiy hosilalari ta'riflanadi.

Ushbu

$$f(x,y)=\ln(x^2+y^2)$$

funktsiyaning ikkinchi tartibli xususiy hosilalarini toping.

Ravshanki,

$$f'_x(x,y)=[\ln(x^2+y^2)]'_x=\frac{2x}{x^2+y^2},$$

$$f'_y(x,y)=[\ln(x^2+y^2)]'_y=\frac{2y}{x^2+y^2}$$

bo'ladi. Ta'rifdan foydalanib funktsiyaning ikkinchi tartibli xususiy hosilalarini topamiz:

$$f''_{x^2}(x,y)=\left(\frac{2x}{x^2+y^2}\right)'_x=2\cdot\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}=\frac{2(y^2-x^2)}{(x^2+y^2)^2};$$

$$f''_{xy}(x,y)=\left(\frac{2x}{x^2+y^2}\right)'_y=-\frac{2x\cdot 2y}{(x^2+y^2)^2}=-\frac{4xy}{(x^2+y^2)^2};$$

$$f''_{yx}(x,y)=\left(\frac{2y}{x^2+y^2}\right)'_x=-\frac{2y\cdot 2x}{(x^2+y^2)^2}=-\frac{4xy}{(x^2+y^2)^2};$$

$$f''_{y^2}(x,y)=\left(\frac{2y}{x^2+y^2}\right)'_y=2\cdot\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}=\frac{2(x^2-y^2)}{(x^2+y^2)^2}.$$

3–§. Ikki o'zgaruvchili funktsiyaning ekstremumlari.

Aytaylik, $f(x,y)$ funktsiya ochiq $E\subset R^2$ to'plamda berilgan bo'lib, $(x_0,y_0)\in E$ bo'lsin.

Ta'rif. Agar shunday $U_\delta((x_0,y_0))\subset E$ atrof topilsaki, $\forall(x,y)\in U_\delta((x_0,y_0))$ uchun

$$f(x,y)\le f(x_0,y_0),\quad (f(x,y)\ge f(x_0,y_0))$$

bo'lsa, $f(x,y)$ funktsiya (x_0,y_0) **nuqtada maksimumga (minimumga) erishadi** deyiladi. (x_0,y_0) funktsiyaning **maksimum (minimum) nuqtasi** deyiladi. $f(x_0,y_0)$ miqdor funktsiyaning **maksimum (minimum) qiymatlari** deyiladi va quyidagicha belgilanadi:

$$f(x_0,y_0)=\max_{(x,y)\in U_\delta}\{f(x,y)\},\quad \left(f(x_0,y_0)=\min_{(x,y)\in U_\delta}\{f(x,y)\}\right)$$

funktsiyaning maksimum va minimumi umumiy nom bilan uning **ekstremumi** deyiladi.

Teorema. (Ekstremum mavjudligining zaruriy sharti). *Agar $f(x,y)$ funktsiya (x_0,y_0) nuqtada ekstremumga erishib, shu nuqtada chekli xususiy hosilalari f'_x va f'_y mavjud bo'lsa, u holda*

$$f'_x(x_0,y_0)=0,\quad f'_y(x_0,y_0)=0$$

bo'ladi.

Yuqorida keltirilgan shart yetarli shart bo'la olmaydi.

Misol. $z=x^3+y^3$, $z'_x=3x^2$, $z'_y=3y^2$. Bu funktsiya $(0,0)$ nuqtada hosilasi nolga teng, ammo bu funktsiya $(0,0)$ nuqtada ekstremumga ega emas, chunki bu nuqtaning ixtiyoriy atrofida har xil ishorali qiymat qabul qiladi, nuqtaning o'zida $z=0$.

Teorema. (Ekstremum mavjudligining yetarli sharti). *Faraz qilaylik, $f(x,y)$ funktsiya (x_0,y_0) nuqta atrofida birinchi va ikkinchi tartibli uzluksiz xususiy hosilalarga ega bo'lib, quyidagi shartni qanoatlantirsin:*

$$f'_x(x_0, y_0) = 0, \quad f'_y(x_0, y_0) = 0$$

va quyidagicha belgilash kiritamiz:

$$A = f''_{x^2}(x_0, y_0), \quad B = f''_{xy}(x_0, y_0), \quad C = f''_{y^2}(x_0, y_0), \quad Д = AC - B^2.$$

Unda 1) agar $Д > 0$ bo'lsa, u holda (x_0, y_0) nuqtada $f(x, y)$ funktsiya ekstremumga erishadi, agarda $A < 0$ bo'lsa maksimumga, $A > 0$ bo'lsa minimumga.

2) agar $Д < 0$ bo'lsa, u holda (x_0, y_0) nuqtada $f(x, y)$ funktsiya ekstremumga ega emas.

3) agar $Д = 0$ bo'lsa, u holda (x_0, y_0) nuqtada $f(x, y)$ funktsiya ekstremumga erishishi ham, erishmasligi ham mumkin.

Masala. Ximik reaktsiya tarkibida x, y va z moddalar qatnashadi. Reaktsiya tezligi V quyidagi qonun bilan ifodalanadi:

$$V = kx^2yz.$$

Shunday x, y va z moddalar tarkibini topingki, ximik reaktsiya tezligi V maksimal bo'lsin.

Yechilishi: Aytaylik,

$$x + y + z = 100 \ (\%), \text{ u holda}$$

$$z = 100 - x - y \quad \text{va}$$

$$V = kx^2y(100 - x - y) \ (1)$$

V funktsiyaning xususiy hosilalarini topamiz:

$$\frac{\partial V}{\partial x} = k\left(200xy - 3x^2y - 2xy^2\right)$$

$$\frac{\partial V}{\partial y} = k\left(100xy - x^3 - 2x^2y\right).$$

nolga tenglaymiz,

$$\begin{cases} 200xy - 3x^2y - 2xy^2 = 0 \\ 100x^2 - x^3 - 2x^2y = 0 \end{cases}$$

Bilamizki (1) funktsiya $x=0$ va $y=0$ da maksimumga erishmaydi, u holda

$$\begin{cases} 200-3x-2y=0 \\ 100-x-2y=0 \end{cases}$$

Sistemani yechib $x=50$, $y=25$ ni topamiz. U holda $z=25$. Bunda $(50{,}25)$ nuqtada $Д>0$ va $A<0$ ekanini topamiz.

Demak, $x=50\%$, $y=25\%$ va $z=25.\%$ bo'lgan tarkibda V tezlik maksimal bo'lar ekan.

Nazorat savollari.

1. Ikki o'zgaruvchili funktsiya tushunchasi.
2. Ikki o'zgaruvchili funktsiyaning limiti va uzluksizligi.
3. Ikki o'zgaruvchili funktsiyaning xususiy hosilalari.
4. Ikki o'zgaruvchili funktsiyaning differentsiali.
5. Ko'p o'zgaruvchili funktsiyalarning yuqori tartibli hosila va differentsiallari.
6. Ikki o'zgaruvchili funktsiyaning ekstremumlari.
7. Ikki o'zgaruvchili funktsiya ekstremumining zaruriy va yetarli shartlari.

4–§. Kompleks sonlar.

Quyidagi

$$z=x+iy \qquad (1)$$

ko'rinishdagi son **kompleks son** deyiladi, bunda x va y – ixtiyoriy haqiqiy sonlar, i esa $i^2=-1$ tenglik bilan aniqlanadigan mavhum birlik, x va y sonlar z kompleks sonning mos ravishda *haqiqiy qismi* va *mavhum qismi* deb ataladi va

$$x=\operatorname{Re} z, \quad y=\operatorname{Im} z$$

deb belgilanadi. Kompleks sonning (1) ko'rinishdagi yozuvi uning *algebraik shakli* deyiladi.

$z = x + iy$ kompleks son xOy dekart koordinatalar tekisligida abstsissasi x, ordinatasi y bo'lgan nuqta bilan yoki bu nuqtaning radius–vektori bilan tasvirlanishi mumkin. Bu vektorning uzunligi z kompleks sonning *moduli* deb ataladi va $|z|$ yoki r orqadi belgilanadi:

$$|z| = r = \sqrt{x^2 + y^2} \qquad (2)$$

Bu vektorning Ox haqiqiy o'qning musbat yo'nalishi bilan hosil qilgan burchagi z sonning *argumenti* deb ataladi va $Arg\, z$ orqali belgilanadi:

$$tg(Argz) = \frac{y}{x} \qquad (3)$$

$Arg\, a$ ko'p qiymatli kattalik va u 2π ga karrali songa qadar aniqlikda aniqlangan. $Arg\, z$ ning $-\pi$ dan π gacha bo'lgan oraliqda joylashgan qiymati unirng *bosh qiymati* deyiladi va $Arg\, z$ -yoki φ orqali belgilanadi:

$$-\pi < \arg z \le \pi.$$

Agar ikkita $z_1 = x_1 + iy_1$ va $z_2 = x_2 + iy_2$ kompleks sonning haqiqiy va mavhum qismlari mos ravishda teng ya'ni

$$x_1 = y_1, \qquad x_2 = y_2$$

bo'lsa, ular teng hisoblanadi.

Faqat mavhum qismining ishorasi bilan bir–biridan farq qiladigan $z = x + iy$ va $z = x - iy$ kompleks sonlar *qo'shma kompleks sonlar* deyiladi.

Algebraik shaklda berilgan kompleks sonlar ustida amallar quyidagi qoidalar bo'yicha bajariladi:

$$(x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2);$$

$$(x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_1y_2);$$

$$\frac{x_1+iy_1}{x_2+iy_2}=\frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)}=\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}+i\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}.$$

Kompleks sonlarni qo'shish (ayirish) bu sonlarni tasvirlovchi vektorlarni qo'shishga (ayirishga) keltiriladi.

z_1 va z_2 kompleks sonlar yig'indisining radius–vektori qo'shiluvchilarning radius–vektorlariga yasalgan parallelogramm diagonalidir.

z_1 va z_2 kompleks sonlar ayirmasining radius–vektori quyidagicha topiladi: ayriluvchining radius–vektori uchini kamayuvchining radius–vektori uchi bilan tutashtirish, so'ngra hosil qilingan vektorni o'z–o'ziga parallel ko'chirib, uning boshini O nuqtaga joylashtirish lozim.

$z=x+iy$ kompleks o'zgaruvchining asosiy transtsendent funktsiyalari quyidagi tengliklar bilan aniqlanadi:

ko'rsatkichli funktsiya

$$e^z=e^x(\cos y+i\sin y) \qquad (1)$$

trigonometrik funktsiyalar

$$\sin z=\frac{e^{iz}-e^{-iz}}{2i}, \qquad (2)$$

$$\cos z=\frac{e^{iz}+e^{-iz}}{2}; \qquad (3)$$

giperbolik funktsiyalar

$$shz=\frac{e^z-e^{-z}}{2}, \qquad (4)$$

$$chz=\frac{e^z+e^{-z}}{2}, \qquad (5)$$

$z=x+iy$ kompleks sonning *trigonometrik* va *ko'rsatkichli* shakli quyidagi ko'rinishga ega bo'ladi:

$$z=r(\cos\varphi+i\sin\varphi), \qquad (6)$$

$$r = re^{i\varphi}. \qquad (7)$$

bunda r va φ — mos ravishda z kompleks sonning moduli va argumentining bosh qiymati.

Trigonometrik va ko'rsatkichli shaklda berilgan kompleks sonlar uchun ko'paytirish, bo'lish, musbat butun darajaga ko'tarish, musbat butun darajadan ildiz chiqarish quyidagi formulalar yordamida bajariladi:

$$[r_1(\cos\varphi_1 + i\sin\varphi_1)]X[r_2(\cos\varphi_2 + i\sin\varphi_2)] = r_1r_2[\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)] \qquad (8)$$

$$(r_1e^{i\varphi_1})\cdot(r_2e^{i\varphi_2}) = r_1r_2e^{i(\varphi_1+\gamma_2)}, \qquad (8')$$

$$\frac{r_1(\cos\varphi_1 + i\sin\varphi_1)}{r_2(\cos\varphi_2 + i\sin\varphi_2)} = \frac{r_1}{r_2}[\cos(\varphi_1-\varphi_2) + i\sin(\varphi_1-\varphi_2)], \qquad (9)$$

$$\frac{r_1e^{i\varphi_1}}{r_2e^{i\varphi_2}} = \frac{r_1}{r_2}e^{i(\varphi_1-\varphi_2)}, \qquad (9')$$

$$[r(\cos\varphi + i\sin\varphi)]^n = r^n(\cos n\varphi + i\sin n\varphi), \qquad (10)$$

$$(re^{i\varphi})^n = r^ne^{in\varphi}, \qquad (10')$$

$$\sqrt[n]{r(\cos\varphi + i\sin\varphi)} = \sqrt[n]{r}\left(\cos\frac{\varphi+2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n}\right), \quad (k = 0,1,2,\dots,n-1) \qquad (11)$$

$$\sqrt[n]{re^{i\varphi}} = \sqrt[n]{re^{i\varphi}} = \sqrt[n]{r}e^{\frac{\varphi+2k\pi}{n}i}, \quad (k = 0,1,2,\dots,n-1) \qquad (11')$$

(10) va (11) munosabatlar **Muavr formulalari** deyiladi. Ko'rsatilgan tengliklarning o'ng tomonidagi ifodalarni trigonometrik yoki ko'rsatkichli shaklga keltirishda $\cos\varphi$, $\sin\varphi$, $e^{i\varphi}$ funktsiyalarning $2k\pi$ $(k = \pm1, \pm2, \dots)$ davrga egaligidan foydalaniladi.

1–misol. Ushbu tenglamaning haqiqiy yechimlarini toping:

$$(4+2i)x + (5-3i)y = 13 + i.$$

Yechilishi. Tenglamaning chap qismidan haqiqiy va mavhum qismini ajratamiz:

$$(4x+5y)+i(2x-3y)=13+i.$$

Bundan quyidagini yozib olamiz

$$\begin{cases} 4x+5y=13 \\ 2x-3y=1. \end{cases}$$

Bu tenglamani yechib quyidagini topamiz:
$x=2, \quad y=1.$

2–misol. Quyidagi kompleks sonning moduli va argumentini toping.

$$z=-\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}.$$

Yechilishi. Quyidagiga egamiz:

$$x=-\sin\frac{\pi}{8}<0, \qquad y=-\cos\frac{\pi}{8}<0.$$

Argumentning bosh qiymati quyidagiga teng:

$$\arg z=-\pi+arctg\left(ctg\frac{\pi}{8}\right)=-\pi+arctg\left(tg\left(\frac{\pi}{2}-\frac{\pi}{8}\right)\right)=-\pi+arctg\left(tg\frac{3}{8}\pi\right)=-\pi+\frac{3}{8}\pi=-\frac{5}{8}\pi.$$

Bundan kelib chiqadi,

$$Argz=-\frac{5}{8}\pi+2k\pi \qquad (k\in Z),$$

$$|z|=\sqrt{\sin^2\frac{\pi}{8}+\cos^2\frac{\pi}{8}}=1.$$

3–misol. Ushbu kompleks sonni trigonometrik ko'rinishda yozing.

$$z=-1-i\sqrt{3}.$$

Yechilishi. Quyidagini yozib olamiz,

$$|z| = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} = 2; \qquad tg\varphi = \frac{-\sqrt{3}}{-1} = \sqrt{3}, \quad \varphi = -\frac{2}{3}\pi.$$

Bundan kelib chiqadi,

$$-1 - i\sqrt{3} = 2\left[\cos\left(-\frac{2}{3}\pi\right) + i\sin\left(-\frac{2}{3}\pi\right)\right].$$

4–misol. $\left(-1+i\sqrt{3}\right)^{60}$ ni hisoblang.

Yechilishi. $z = -1 + i\sqrt{3}$ kompleks sonning trigonometrik ko'rinishini yozamiz,

$$-1 + i\sqrt{3} = 2\left(\cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi\right).$$

$$\left(-1+i\sqrt{3}\right)^{60} = 2^{60}\left[\cos\left(60\cdot\frac{5}{6}\pi\right) + i\sin\left(60\cdot\frac{5}{6}\pi\right)\right] = 2^{60}\left(\cos 50\pi + i\sin 50\pi\right) = 2^{60}.$$

Mustaqil yechish uchun misol va masalalar.

Quyidagi funktsiyalarning aniqlanish sohasini toping.

1. $u = 10 + 2x - y$ j: XOY tekisligi.
2. $u = \dfrac{5}{2x^2 + y^2}$ j: $(0;0)$ dan tashqarii XOY tekisligi.
3. $u = \dfrac{2}{\sqrt{xy}}$, j: I va III chorak:

$$x > 0, \quad y > 0; \quad ва \quad x < 0, \quad y < 0.$$

4. $u = \ln(x+y) + 2x - y + 7$, j: $x + y > 0$ yarim tekislik.

Quyidagi funktsiyalarning birinchi tartibli xususiy hosilalarini toping.

1. $u = x^2 + 5xy^2 - y^3$,

j: $u'_x = 2x + 5y^2$, $u'_y = 10xy - 3y^2$.

2. $u=\sqrt{2x+5y}$, j: $u'_x=\frac{1}{\sqrt{2x+5y}}$, $u'_y=\frac{5}{2\sqrt{2x+5y}}$.

3. $u=\ln(x+y)+2x-y+7$, j: $u'_x=\frac{1}{x+y}+2$, $u'_y=\frac{1}{x+y}-1$.

4. $u=x^2\sin y+y^3$, j: $u'_x=2x\sin y$, $u'_y=x^2\cos y+3y^2$.

Quyidagi funktsiyalarni ekstremumga tekshiring.

1. $u=x^3+8y^3-6xy+5$, j: $\left(1;\frac{1}{2}\right)$ nuqtada $u_{min}=4$; $(0;0)$ nuqtada ekstremumga ega emas.
2. $u=(x-1)^2+2y^2$ j: $(1;0)$ nuqtada $u_{min}=0$.
3. $u=2x^3-x^2+xy^2-4x+3$, j: $\left(-\frac{2}{3};0\right)$ nuqtada $u_{max}=4\frac{17}{27}$, $(1;0)$ nuqtada $u_{min}=0$, $(0;-2)$, $(0;2)$ nuqtalarda ekstremumga ega emas.
4. $u=3x+6y-x^2-xy-y^2$, j: $(0;3)$ nuqtada $u_{max}=9$.

Tenglamani yeching.

1. $(3x-i)(2+i)+(x-iy)(1+2i)=5+6i$, j: $x=\frac{20}{17}$, $y=-\frac{36}{17}$.
2. $\frac{1}{z-i}+\frac{2+i}{1+i}=\sqrt{2}$, bu yerda $z=x+iy$, j: haqiqiy yechimi yo'q.
3. $(4x-3y)+(3x+5y)i=10-(3x-2y-30)i$, j: $x=4$, $y=2$.
4. $(2-7i)x+(8+6i)y=(-6+5i)x-8$, j: $x=-\frac{1}{3}$, $y=-\frac{2}{3}$.

Quyidagi kompleks sonlarning moduli va argumentini toping.

5. $z=4+3i$; j: $|z|=5$, $\varphi=arctg\frac{3}{4}$.
6. $z=-2+2\sqrt{3}i$; j: $|z|=4$, $\varphi=\frac{2}{3}\pi$.
7. $z=-7-i$; j: $|z|=5\sqrt{2}$, $\varphi=arctg\frac{1}{7}-\pi$.
8. $z=-\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$; j: $|z|=1$, $\varphi=\frac{4}{5}\pi$.

Ushbu kompleks sonlarni trigonometrik ko'rinishda yozing.

9. $-2;$ j: $2(\cos\pi + i\sin\pi)$.

10. $2i;$ j: $2\left(\cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi\right)$.

11. $-\sqrt{2}+i\sqrt{2};$ j: $2\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)$.

12. $-1-i\sqrt{3};$ j: $2e^{-\frac{2\pi}{3}i}$.

Quyidagilarni hisoblang.

13. $\left(\frac{1+i\sqrt{3}}{1-i}\right)^{40};$ j: $-2^{19}\left(1+i\sqrt{3}\right)$

14. $(2-2i)^7;$ j: $2^{10}(1+i)$.

15. $\left(\sqrt{3}-3i\right)^6;$ j: 1728.

16. $\left(\frac{1-i}{1+i}\right)^8;$ j: 1.

VIII BOB
BIRINCHI TARTIBLI DIFFERENTSIAL TENGLAMALAR.
1–§. Asosiy tushunchalar.

$$F\left(x, y, y', y'', \dots, y^{(n)}\right)=0 \qquad (1)$$

ko'rinishdagi tenglama **n–tartibli oddiy differentsial tenglama** deyiladi.

Differentsial tenglamaning tartibi deb bu tenglamaga kiruvchi **eng yuqori tartibli hosila tartibiga** aytiladi. Masalan, $y - xy' = 0$ tenglama birinchi tartibli; $\frac{d^2x}{dx^2} + \frac{dy}{dx} = \frac{1}{1+e^x}$ tenglama ikkinchi tartibli; $y''' - 8y = 0$ tenglama esa uchinchi tartibli.

Tenglamadagi y ning o'rniga qo'yganda uni ayniyatga aylantiruvchi $\varphi(x)$ funktsiya **tenglamaning yechimi** deyiladi. Yechimning grafigi **tenglamaning integral egri chizig'i** deyiladi. Agar differentsial tenglamani qanoatlantiruvchi funktsiya oshkormas ko'rinishda, ya'ni $\varphi(x, y)=0$ ko'rinishdagi munosabat orqali berilgan bo'lsa, u holda tenglama integrali haqida gapiriladi.

(1) differentsial tenglamaning barcha yechimlarini, ixtiyoriy o'zgarmaslar $S_1, S_2, S_3, \dots, S_n$ ga bog'liq bo'lgan

$$y=\varphi\left(x, C_1, C_2, \dots, C_n\right)$$

munosabat bilan umumiy ko'rinishda ifodalash mumkin. Bu munosabatga (1) differentsial **tenglamaning umumiy yechimi** deyiladi.

Masalan, birinchi tartibli tenglama uchun umumiy yechim $y=\varphi(x, c)$ ko'rinishga, ikkinchi tartibli tenglama uchun esa $y=\varphi(x_1, c_1, c_2)$ ko'rinishga ega.

Umumiy yechimdan erkli o'zgarmaslarning turli son qiymatlarida hosil qilinadigan yechimlar bu **tenglamaning xususiy yechimlari** deyiladi.

Umumiy yechim geometrik nuqtai nazardan egri chiziqlar oilasini, xususiy yechim esa bu oilaning birorta egri chizig'ini aniqlaydi. Differentsial tenglamaning xususiy yechimini topish uchun boshlang'ich shartlar beriladi.

1–Misol. $y=2x$ funktsiya $x^2y''-2xy'+2y=0$ tenglamaning yechimi ekanligini tekshiring.

Yechilishi: $y'=2,\quad y''=0$ larni tenglamaga qo'yamiz:

$$x^2\cdot 0-2x\cdot 2+2\cdot 2x=0-4x+4x=0$$

ya'ni, $y=2x$ funktsiya haqiqatdan ham berilgan differentsial tenglamaning yechimi ekan.

2–Misol. $y'-2y=0$ differentsial tenglamaning umumiy yechimi $y=Ce^{2x}$ ko'rinishga ega. Uning $y(1)=e^2$ boshlang'ich shartni qanoatlantiruvchi xususiy yechimini toping.

Yechilishi: Ixtiyoriy o'zgarmas S ning izlanayotgan xususiy yechimga mos qiymati umumiy yechimining ifodasiga boshlang'ich shartlarni keltirib qo'yish natijasida hosil bo'ladi:

$$e^2=Ce^2$$

bu yerdan $C=1$. Hosil qilingan $C=1$ qiymatni umumiy yechimga qo'yib, berilgan boshlang'ich shartlarni qanoatlantiruvchi $y=e^{2x}$ xususiy yechimni hosil qilamiz.

1. O'zgaruvchilari ajraladigan differentsial tenglamalar.

$$M_1(x)M_2(y)dx+N_1(x)N_2(y)dy=0 \quad (2)$$

ko'rinishga ega bo'lgan tenglama, **o'zgaruvchilari ajraladigan differentsial tenglama** deyiladi. Tenglamani yechish uchun, tenglamaning birinchi hadlarini $M_2(x)N_1(x)$ (noldan farqli deb faraz qilamiz) ga bo'lib o'zgaruvchilari ajralgan $\frac{M_1(x)}{N_1(x)}dx+\frac{N_2(y)}{M_2(y)}dy=0$ tenglamani hosil qilamiz. Tenglamaning umumiy integrali hadma–had integrallash bilan topiladi:

$$\int\frac{M_1(x)}{N_1(x)}dx+\int\frac{N_2(y)}{M_2(y)}dy=C$$

3−Misol. $\left(\sqrt{xy}+\sqrt{x}\right)y'-y=0$ tenglamaning umumiy integralini toping.

Yechilishi: Berilgan tenglamani $y'=\frac{dy}{dx}$ ko'rinishda yozib olamiz va tenglamaning ikkala qismini dx ga ko'paytiramiz:

$$\left(\sqrt{y}+1\right)\sqrt{x}dy-ydx=0$$

o'zgaruvchilarni ajratamiz,

$$\frac{\sqrt{y}+1}{y}dy-\frac{1}{\sqrt{x}}dx=0$$

va integrallab quyidagiga ega bo'lamiz:

$$\int\left(y^{-\frac{1}{2}}+y^{-1}\right)dy-\int x^{-\frac{1}{2}}dx=C$$

bundan

$$2\sqrt{y}+\ln|y|-2\sqrt{x}=C$$

ni topamiz.

4−Misol. $ydx+ctgxdy=0$ tenglamaning $y\left(\frac{\pi}{6}\right)=-1$ boshlang'ich shartlarni qanoatlantiruvchi xususiy integralini toping.

Yechilishi: O'zgaruvchilarni ajratib quyidagiga kelamiz,

$$\frac{dx}{ctgx}+\frac{dy}{y}=0 \qquad \text{yoki} \qquad tgxdx+\frac{dy}{y}=0$$

ikkala qismini integrallaymiz va natijada

$$-\ln|\cos x|+\ln|y|=\ln C,$$

$$|y|=C|\cos x|; \quad y=\pm C\cos x=C_1\cos x$$

ga ega bo'lamiz. Boshlang'ich shartdan foydalansak, $(x=\frac{\pi}{6}, y=-1)$

$$-1 = C_1 \cos\frac{\pi}{6} = C_1 \cdot \frac{\sqrt{3}}{2}, \quad C_1 = -\frac{2}{\sqrt{3}}$$

Shunday qilib, izlanayotgan xususiy integral quyidagicha bo'ladi.

$$y = -\frac{2}{\sqrt{3}}\cos x$$

1–Masala. *(bakteriya ko'payishining tezligi haqida.)*

Bakteriya ko'payish tezligi uning soniga to'g'ri proportsional. Boshlang'ich t=0 vaqtda 100 ta bakteriya bo'lsin, 3 soatdan keyin ularning soni ikki barobar ko'payadi. Bakteriya sonining vaqtga bog'liqligini aniqlash kerak va 9 soatda bakteriya qancha marta ko'payadi?

Yechilishi. Aytaylik, x bakteriyalar soni bo'lsin. Masala shartiga ko'ra

$$\frac{dx}{dt} = kx$$

bu yerda k–proportsionallik koeffitsienti. Tenglamani o'zgaruvchilarga ajratib integrallasak, quyidagini hosil qilamiz:

$$x = Ce^{kt}$$

S ni aniqlash uchun t=0 va x=100 dan foydalanamiz. S=100 bo'ladi, demak,

$$x = 100e^{kt}$$

k–proportsionallik koeffitsientini t=3 va x=200 dan foydalanib topamiz:

$$200 = 100e^{3k} \text{ yoki } 2 = e^{3k}$$

bundan kelib chiqadiki $e^k = 2^{\frac{1}{3}}$. Shuning uchun qidirilayotgan funktsiya

$$x = 100 \cdot 2^{\frac{t}{3}}$$

bundan t=9 da x=800 ekanligini topamiz. Demak, 9 soat ichida bakteriya 8 marta ko'payar ekan.

2–Masala. *(Aralashmaning kontsentratsiyasi.)* Tarkibida 100l suv va 10 kg tuz bo'lgan idishga 30 $l/\min$ tezlik bilan suv to'xtovsiz quyib turiladi va idishdan 20 $l/\min$ tezlik bilan aralashma oqib chiqadi. Faraz qilamiz, suv bilan tuz tez aralashib ketadi. t vaqt ichida idishda qancha tuz qolishini aniqlang.

Yechilishi. Aytaylik, t vaqt ichida x – miqdorda tuz bor. dt vaqt ichida idishda dx – miqdorda tuz chiqib ketadi.(minus ishorasi x – kamayuvchi funktsiya ekanini bildiradi). t vaqtda idishda aralashma hajmi quyidagiga teng.

$$v = 100 + 30t - 20t = 100 + 10t$$

shuning uchun tuz miqdori (bir litr aralashmada) t vaqtda

$$\frac{x}{100+10t}$$

ga teng. Bundan kelib chiqadiki, dt vaqt ichida tuz

$$\frac{x}{100+10t} \cdot 20t$$

ga kamayadi.

Bundan quyidagi differentsial tenglamaga ega bo'lamiz.

$$-dx = \frac{20xdt}{100+10t}$$

yoki

$$-dx = \frac{2xdt}{10+t}$$

o'zgaruvchilarga ajratib integrallasak,

$$\frac{dx}{x} = -\frac{2dt}{10+t},$$

$$\ln x = -2\ln(10+t) + \ln C$$

bundan kelib chiqadiki

$$x = \frac{C}{(10+t)^2}.$$

Agar $t=0$, $x=10$ da $S=1000$ ga teng.

Shunday qilib, t vaqt ichida idishda tuzning kg hisobiga kamayish qonuni, quyidagi formula bilan beriladi:

$$x=\frac{1000}{(10+t)^2} \quad (1).$$

(1) formula orqali havzadagi tuz miqdorini bilgan holda yuqoridagi hodisaning boshlanganiga qancha vaqt o'tganini bilish mumkin. Mana shu fikr asosida dengiz va okean yoshi aniqlanadi.

3–Masala. *(Jismning sovishi.)* Atrofdagi havo temperaturasi 20^0 ga teng bo'lsin. Jismning sovish tezligi jism temperaturasi va atrofdagi havo temperaturasi ayirmasiga to'g'ri proportsional. Ma'lumki, 20 min ichida jism 100^0C dan 60^0C gacha soviydi. Jism temperaturasi θ ning t vaqt ichida o'zgarish qonunini aniqlang.

Yechilishi. Masala shartiga ko'ra, quyidagini yozamiz:

$$\frac{d\theta}{dt}=k(\theta-20),$$

bu yerda k – proportsionallik koeffitsienti. O`zgaruvchilarga ajratib integrallasak:

$$\frac{d\theta}{\theta-20}=kdt,$$

$$\ln(\theta-20)=kt+\ln c.$$

Bu ifodani potentsirlasak,

$$\theta-20=ce^{kt}.$$

c ni aniqlash uchun boshlang'ich shartdan foydalanamiz:

$$t=o \text{ da } \quad \theta=100^0.$$

Bundan $c=80$. Shuning uchun

$$\theta = 20 + 80e^{kt}.$$

Proportsionallik koeffitsienti k ni qo'shimcha shartlar yordamida aniqlaymiz, $t = 20$ $\theta = 60^0$. Bundan:

$$60 = 20 + 80e^{20k}$$

yoki

$$e^{20k} = \frac{1}{2}.$$

Demak,

$$e^k = (\frac{1}{2})^{\frac{1}{20}}.$$

Shunday qilib, natija quyidagicha:

$$\theta = 20 + 80\left(\frac{1}{2}\right)^{\frac{t}{20}}.$$

2. Bir jinsli differentsial tenglamalar.

Quyidagi tenglikni $f(tx, ty) = t^m f(x, y)$ qanoatlantiruvchi $f(x, y)$ funktsiyaga **m tartibli bir jinsli funktsiya** deyiladi. Masalan,

$$f(x, y) = x^2 + 2y^2 - xy$$

2 — tartibli bir jinsli funktsiya bo'ladi.

$$(tx)^2 + 2(ty)^2 - (txty) = t^2(x^2 + 2y^2 - xy)$$

Agarda M(x,y) va N(x,y) funktsiyalar bir xil tartibli bir jinsli funktsiyalar bo'lsa, unda ushbu

$$M(x, y)dx + N(x, y)dy = 0 \quad (3)$$

ko'rinishdagi tenglamaga **bir jinsli birinchi tartibli tenglama** deyiladi, Bir jinsli tenglama $y=ux$ (bu yerda u yangi izlanayotgan funktsiya) almashtirish orqali o'zgaruvchilarga ajraladigan tenglamaga keltiriladi. $y=ux$ tenglikni differentsiallab, topamiz: $dy=udx+xdu$.

4–Misol. $(y^2-3x^2)dx+2xydu=0$, $y(0)=0$ boshlang'ich shart bilan berilgan tenglamani yeching.

Tenglamani yechish uchun $y=ux$ almashtirish bajaramiz.

$$(u^2x^2-3x^2)dx+2x^2u(udx+xdu)=0$$

yoki

$$3(u^2-1)dx+2xudu=0,$$

$\frac{3dx}{x}+\frac{2udu}{u^2-1}=0$ ni integrallab, $3\ln x+\ln(u^2-1)=\ln C$ ni hosil qilamiz. Bu tenglikni potentsirlab, $x^2(u^2-1)=C$ ni topamiz. $u=\frac{y}{x}$ ni o'rniga qo'yib $x^3\left(\frac{y^2}{x^2}-1\right)=C$ yoki $x(y^2-x^2)=C$ boshlangich shartdan foydalanib S=0 ni topamiz. Qidirilayotgan xususiy yechim $y=\pm x$ bo'ladi.

Nazorat savollari.

1. Asosiy tushunchalar.
2. O'zgaruvchilari ajraladigan tenglamalar.
3. Bir jinsli tenglamalar.
4. Masalalar.

2–§. Birinchi tartibli chiziqli differentsial tenglamalar.

Ushbu

$$y'+py=q \qquad (1)$$

tenglamalarga **birinchi tartibli chiziqli differentsial tenglama** deyiladi. Bu yerda $p=p(x)$ va $q=q(x)$ lar (a,b) da uzluksiz funktsiyalar. $y=u\cdot\vartheta$ almashtirish bilan tenglama o'zgaruvchilari ajraladigan ikkita tenglamaga keltiriladi. $u=u(x)$ funktsiya yangi noma'lum funktsiya, $\vartheta=\vartheta(x)$ funktsiya esa ixtiyoriy tanlab olinadi. Bu almashtirish tenglamani

$$u'\vartheta+u\vartheta'+pu\vartheta=q$$

yoki

$$\vartheta\frac{du}{dx}+\left(\frac{d\vartheta}{dx}+p\vartheta\right)u=q$$

ko'rinishga keltiradi. ϑ ning ixtiyoriyligidan

$$\frac{d\vartheta}{dx}+p\vartheta=0$$

deb olamiz. O'zgaruvchilarga ajratib va integrallab:

$$\frac{d\vartheta}{\vartheta}=-pdx\ ,\ \ln\vartheta=-\int pdx$$

yoki

$$\vartheta=e^{-\int pdx}$$

bundan

$$e^{-\int pdx}\frac{du}{dx}=q$$

tenglamani hosil qilamiz. Bu tenglamani yechsak,

$$u=\int qe^{\int pdx}dx+C$$

ga kelamiz. Nihoyat, tenglamaning umumiy yechimi

$$y=e^{-\int pdx}\left[\int qe^{\int pdx}dx+C\right]\ (2)$$

bo'ladi.

Izoh: Agar (1) tenglamada $q(x)=0$ bo'lsa, u holda tenglama birinchi tartibli **chiziqli bir jinsli tenglama** deyiladi, aks holda **chiziqli bir jinsli bo'lmagan tenglama** deyiladi. Bundan kelib chiqadiki, birinchi tartibli bir jinsli tenglama

$$y' + py = 0 \quad (3)$$

ko'rinishda bo'ladi, yechimi esa quyidagicha:

$$y = Ce^{-\int p dx} \quad (4)$$

1–Misol. $x^2y^2y' + xy^3 = 1$ tenglamaning yechimini toping.

Yechilishi: Tenglamaning ikkala tomonini x^2y^2 ga bo'lamiz,

$$y' + \frac{y}{x} = y^{-2} \cdot \frac{1}{x^2},$$

agar $p = x^{-1}$, $q = x^{-2}$ deb olsak, tenglama

$$y' + py = qy^{-2}$$

Bernulli tenglamasiga keladi. $y = u \cdot v$ deb olsak, $y' = u'v + uv'$ deb tenglamaga qo'yamiz:

$$u' \cdot v + u \cdot v' + \frac{u \cdot v}{x} = \frac{1}{x^2u^2v}$$

yoki

$$u' \cdot v + u\left(v' + \frac{v}{x}\right) = \frac{1}{x^2u^2v}$$

Quyidagi ikkita tenglamani yechamiz:

$$1) \; v' + \frac{v}{x} = 0, \qquad 2) \; u' \cdot v = \frac{1}{x^2u^2v}$$

Birinchi tenglamani yechib v ni topamiz:

$$\frac{dv}{v} + \frac{dx}{x} = 0; \quad \ln v + \ln x = 0, \quad vx = 1; \quad v = \frac{1}{x}$$

v –ni 2–chi tenglamaga qo'yib, u ni topamiz:

$$\frac{u}{x}=\frac{1}{u^2}; \qquad u^2du=xdx; \quad \frac{u^3}{3}=\frac{x^2}{2}+\frac{c}{3}; \; u=\sqrt[3]{\frac{3}{2}x^2+C}$$

qidirilayotgan tenglamaning umumiy yechimi quyidagicha bo'ladi:

$$y=u*v=\sqrt[3]{\frac{3}{2x}+\frac{c}{x^3}}.$$

1. Chiziqli differentsial tenglamalarning biologiyaga tadbiqi.

Masala. Ovqatlanish resursi juda yaxshi sharoitda bo'lgan mikroorganizmlar jamoasini qaraylik. Vaqt o'tishi bilan jamoaning ko'payishi va nobud bo'lishi o'zgarib turadi. Ana shu o'zgarish qonunini toping.

Yechilishi. Aytaylik, $x=x(t)$ t vaqt ichidagi tirik organizmlarning soni bo'lsin, $x(t+\Delta t)$ esa – $t+\Delta t$ vaqtdagi soni. U holda ayirma

$$x(t+\Delta t)-x(t)=\Delta x$$

ni beradi. Δt vaqt ichida balog'atga yetganlarining bir qismi nasl qoldiradi, qolgan qismi nobud bo'lishi mumkin. Shunday qilib,

$$\Delta x=G-H$$

bu yerda G $t$dan $t+\Delta t$ vaqt o'tganda tug'ilganlari soni, H shu vaqt ichida nobud bo'lganlar soni.

Tug'ilganlar soni G Δt vaqt oralig'iga bog'liq va nasl qoldiruvchi "ota–ona" larning soniga bog'liq, chunki ular qancha ko'p bo'lsa tug'ilish shuncha ko'p bo'ladi.

Shunday qilib,

$$G=\Phi(x,\Delta t)$$

bu yerda $\Phi(x,\Delta t)$ funktsiya x yoki Δt ning o'sishi bilan o'sadi yoki x yoki Δt larning biri nolga intilsa nolga teng bo'ladi.

Δt o'zgaruvchiga kelsak eng oddiy tajriba shuni ko'rsatadiki, u chiziqli bo'lib agar kuzatishni ikki marta uzaytirsak, mikroorganizmlar nasli ham ikki marta oshadi. Shunday qilib,

$$\Phi(x, \Delta t) = f(x)\Delta t$$

$f(x)$ funktsiya xususiyati murakkabroq. Biz bilamizki, x o'sishi bilan $f(x)$ monoton o'sadi va $x=0$ bo'lsa nol bo'ladi. Ammo o'sish mikroorganizm turiga bog'liq. Biz nasl miqdorining "ota – ona" lar soniga to'g'ri proportsional bo'lgan holati bilan chegaralanamiz, yani $f(x) = \alpha x$ $(\alpha = const)$. Shunday qilib,

$$G = \alpha x \Delta t .$$

Shunga o'xshash,

$$H = \beta x \cdot \Delta t$$

va bundan kelib chiqadiki,

$$\Delta x = \alpha x \Delta t - \beta x \Delta t$$

yoki

$$\Delta x = \gamma x \Delta t \qquad (1)$$

bu yerda

$$\gamma = \alpha - \beta$$

(1) da tenglamaning ikkala tomonini Δt ga bo'lib, limitga o'tamiz:

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

natijada quyidagini hosil qilamiz

$$\frac{dx}{dt} = \gamma x \qquad (2)$$

yoki

$$\frac{dx}{dt}-\gamma x=0.$$

Birinchi tartibli chiziqli bir jinsli tenglamaga kelamiz. Bu tenglamani yechib,

$$x=Ce^{\gamma t} \quad (3)$$

ni hosil kilamiz.

$t=t_0$ da $x=x(t_0)$ (bu yerda t_0 boshlang'ich vaqtda $x(t_0)=x_0$ tirik mikroorganizmlar soni) boshlang'ich shart bilan S ni topamiz.

$$C=x_0e^{-\gamma t_0}$$

buni (3) ga qo'yib, vaqt davomida mikroorganizmlar o'zgarish qonunini topamiz.

$$x=x_0e^{\gamma(t-t_0)} \quad (4)$$

Ammo topgan bu qonuniyatimiz qanchalik haqiqiy hayotga to'g'ri kelish kelmasligini tajriba va kuzatishlar hal etadi. (4) formula shuni ko'rsatadiki, o'sish eksponentsial darajada, lekin hayotda birorta ham tirik organizm bu darajada o'smaydi. Chunki biz faraz qilgan (2) tenglamada ovqatlanish sharoiti yaxshiligi va tashqi faktorlarning ta'siri yo'qligi bu haqiqatga ziddir. Shunday qilib (2) tenglama yoki nazariy xarakterga ega (uzluksiz oziqlantirib turilganda va tashqi halaqit beruvchi kuchlar bo'lmasa, tirik organizmlar qanday ko'payishini ko'rish mumkin) yoki sun'iy ko'paytirishlar natijasini ko'rsatadi.

(2) tenglamani birinchi marta 1802 yil Mal'tus qo'llagan. Uning xatosi bu tenglamani nafaqat tabiatga, hatto insonlarga qo'llasa ham bo'ladi deb tushungan. Aslida tenglama tor doirada qo'llaniladi.

2. Tuliq differentsialli birinchi tartibli differentsial tenglamalar. Integrallovchi ko'paytuvchi.

Ushbu

$$M(x,y)dx + N(x,y)dy = 0 \qquad (1)$$

tenglama to'liq differentsialli tenglama deyiladi. Agarda tenglamaning chap tarafini $u(x,y)$ funktsiyaning to'liq differentsiali ko'rinishida yozish mumkin bo'lsa:

$$Mdx + Ndy \equiv du \equiv \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

(1) chi differentsial tenglamaning to'liq differentsialli tenglamasi bo'lishi uchun

$$\frac{\partial M}{\partial y} \equiv \frac{\partial N}{\partial x} \quad (2)$$

ning bajarilishi zarur va yetarli. (1) ning umumiy integrali

$$u(x,y) = C$$

yoki

$$\int_{x_0}^{x} Mdx + \int_{y_0}^{y} Ndy = C$$

ko'rinishda bo'ladi.

Ba'zi hollarda (1) chi tenglama tuliq differentsialli bo'lmasa, shunday $\mu(x,y)$ funktsiya tanlash mumkinki (1) ning chap tarafiga ko'paytirsak tenglama tuliq differentsialli ko'rinishga keladi.

$$du = \mu Mdx + \mu Ndy \quad (3)$$

bunday funktsiya $\mu(x,y)$ integrallovchi ko'paytuvchi deyiladi va quyidagilar kelib chiqadi.

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

yoki

$$N\frac{\partial\mu}{\partial x}-M\frac{\partial\mu}{\partial y}=\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\mu$$

hamda

$$N\frac{\partial\ln\mu}{\partial x}-M\frac{\partial\ln\mu}{\partial y}=\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x} \quad (4)$$

Agar $\mu=\mu(x)$ bo'lsa, u holda $\frac{\partial\mu}{\partial y}=0$ va (4) tenglama quyidagi ko'rinishga keladi:

$$\frac{d\ln\mu}{dx}=\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} \quad (5)$$

Agar $\mu=\mu(y)$ bo'lsa, yuqoridagidek $\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\frac{1}{M}$ ko'rinishda bo'ladi.

2–Misol: $2xy+3y^2+(x^2+6xy-3y^2)\frac{\partial y}{\partial x}=0$

tenglamani yeching.

Yechilishi: $N(x,y)=x^2+6xy-3y^2$, $\frac{\partial N}{\partial x}=2x+6y$

$M(x,y)=2xy+3y^2$, $\frac{\partial M}{\partial y}=2x+6y$

Shunday qilib, $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, ya'ni tenglamaning chap qismi $u(x,y)$ funktsiyaning to'liq differentsiali bo'ladi.

$$\frac{\partial u}{\partial x}=2xy+3y^2, \quad \frac{\partial u}{\partial y}=x^2+6xy-3y^2$$

Birinchi tenglamadan,

$$u(x,y) = x^2y + 3xy^2 + \varphi(y)$$

φ(y) funktsiyani aniqlash uchun, y bo'yicha oxirgi tenglikni differentsiallaymiz:

$$\frac{\partial V}{\partial y} = x^2 + 6xy + \frac{du}{dy} = x^2 + 6xy - 3y^2, \quad \text{ya'ni} \quad \frac{du}{dy} = -3y^2.$$

bundan $\varphi(y) = -y^3 + c_1$. Shuning uchun, $u(x,y) = x^2y + 3xy^2 - y^3 + c_1$.
Tenglamaning yechimi: $x^2y + 3xy^2 - y^3 = c$

3–Misol. $(1 - \frac{x}{y})dx + (2xy + \frac{x}{y} + \frac{x^2}{y^2})dy = 0$
tenglamani yeching.

Yechilishi: $M(x,y) = 1 - \frac{x}{y}$; $\quad N(x,y) = 2xy + \frac{x}{y} + \frac{x^2}{y^2}$.

$$\frac{\partial M}{\partial y} = \frac{x}{y^2}, \qquad \frac{\partial N}{\partial x} = 2y + \frac{1}{y} + \frac{2x}{y^2}.$$

Demak, tenglama to'liq differentsialli tenglama emas.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{x}{y^2} - xy - \frac{1}{y} - \frac{2x}{y^2} = -(2y + \frac{1}{y} + \frac{x}{y^2}).$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \quad \text{dan} \quad \frac{1}{\mu} \cdot \frac{d\mu}{dx} = -\frac{1}{x}, \quad \frac{d\mu}{\mu} + \frac{dx}{x} = 0$$

bundan, $\mu = \frac{1}{x}$ ni topamiz.

Berilgan tenglamani $\frac{1}{x}$ ga ko'paytiramiz:

$$\left(\frac{1}{x} - \frac{1}{y}\right)dx + \left(2y + \frac{1}{y} + \frac{x}{y^2}\right)dy = 0$$

Quyidagi ko'rinishda yozib olamiz:

$$\frac{dx}{x} + \left(\frac{1}{y} + 2y\right)dy - \frac{xdy - ydx}{y^2} = 0$$

$$d\left(\ln|x|+\ln|y|+y^2-\frac{x}{y}\right)=0 \quad \text{dan}$$

$\ln|x|+\ln|y|+y^2-\frac{x}{y}=C$ yechimga ega bo'lamiz.

3. Hosilaga nisbatan yechilmagan birinchi tartibli differentsial tenglamalar.

Agar $F(x,y,y')=0$ tenglama y' ga nisbatan 2–darajali bo'lsa, bu tenglama, biror sohada x va y ga nisbatan uzluksiz $y'=f_1(x,y)$ va $y'=f_2(x,y)$ echimga ega. Geometrik nuqtai nazardan bu tenglama shu sohaning ixtiyoriy (x_0,y_0) nuqtasida ikkita integral chiziqning yo'nalishlarini aniqlaydi.

Ushbu

$$y=x\varphi(y')+\psi(y')$$

tenglama **Langranj tenglamasi** deyiladi.

Bunda $y'=p$ almashtirish x ga nisbatan tenglamani chiziqli ko'rinishga keltiradi:

$$P=\varphi(p)+[x\varphi'(p)+\psi'(p)]\frac{dp}{dx}.$$

Uni yechsak,

$$x=\varphi(p)x+\psi(p)$$

bu yerda p

$$p=\varphi(p)$$

tenglamaning yechimi.

Ushbu

$$y=xy'+\psi(y')$$

tenglama **Klero tenglamasi** deyiladi. Bu tenglama Langranj tenglamasining xususiy holidir. Bu tenglama $y=Cx+\psi(c)$ umumiy integralga va $y = px + \psi(p)$ hamda $x + \psi'(p) = 0$ tenglamalardan p parametrni yo'qotishdan hosil bo'ladigan maxsus yechimga egadir.

4–Misol. $y' + y = x(y')^2$ tenglamani yeching.

Yechilishi: Bu tenglama Lagranj tenglamasi. Belgilash kiritamiz. $p = y'$, u holda $y = xp^2 - p$ bo'ladi.

x ga nisbatan differentsiallaymiz:

$$\frac{dy}{dx} = p^2 + 2px\frac{dp}{dx} - \frac{dp}{dx}$$

yoki

$$p = p^2 + 2px\frac{dp}{dx} - \frac{dp}{dx}$$

Quyidagi chiziqli differentsial tenglamani hosil qilamiz:

$$\frac{dx}{dp} + \frac{2}{p-1}x = \frac{1}{p(p-1)}$$

x ga nisbatan yechilib, $x = \frac{p - \ln p + C}{(p-1)^2}$ ni topamiz.

5–Misol. $\sqrt{(y')^2 + 1} + xy' - y = 0$ tenglamani yeching.

Yechilishi: Bu tenglama Lagranj tenglamasi. Belgilash kiritamiz. $p = y'$, u holda $y = xp + \sqrt{1 + p^2}$ bo'ladi.

x ga nisbatan differentsiallaymiz:

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{p\frac{dp}{dx}}{\sqrt{1+p^2}}$$

Bundan $\left(x + \frac{p}{\sqrt{1+p^2}}\right)\frac{dp}{dx} = 0, \quad x = -\frac{p}{\sqrt{1+p^2}}$ yoki $p = C$.

Shunday qilib, $y = Cx + \sqrt{1 + C^2}$ va

$$\begin{cases} x = -\dfrac{p}{\sqrt{1+p^2}} \\ y = px + \sqrt{1+p^2} \end{cases}$$

Bu sistemadan p ni yo'qotib quyidagini topamiz: $y = \sqrt{1-x^2}$.

Nazorat savollari.

1. Birinchi tartibli chiziqli differentsial tenglamalar.
2. Chiziqli differentsial tenglamalarning biologiyaga tadbiqi.
3. Tuliq differentsialli birinchi tartibli differentsial tenglamalar .
4. Hosilaga nisbatan yechilmagan birinchi tartibli differentsial tenglamalar.

Mustaqil yechish uchun misol va masalalar.

1.Quyidagi misollarda berilgan funktsiyalar mos ravishda tenglamalarning yechimi ekanligi tekshirilsin.

1. $y = \dfrac{\sin x}{x}$ $\qquad xy' + y = \cos x.$
2. $y = c\ell^{-2x} + \dfrac{1}{3}\ell^{x}$ $\qquad y' + 2y = \ell^{x}.$
3. $y = 2 + c\sqrt{1-x^2}$ $\qquad (1-x^2)y' + xy = 2x.$
4. $y = x\sqrt{1-x^2}$ $\qquad yy' = x - 2x^3.$
5. $y = \ell^{\arcsin cx}$ $\qquad xy' = ytg \ln y$
6. $y = (x-c)^3$ $\qquad y' = 3y^{2/3}.$
7. $y = \ell^{cx}$ $\qquad y = \ell^{xy'/y}.$
8. $y = cx^3$ $\qquad xy' = 3y.$
9. $y = \sin(x+c)$ $\qquad y^2 + y'^2 = 1.$
10. $x^2 + cy^2 = 2y$ $\qquad x^2y' = yy' + xy.$
11. $y^2 + cx = x^3$ $\qquad 2xyy' - y^2 = 2x^3.$
12. $y = c(x-c)^2$ $\qquad y'^3 = 4y(xy' - 2y).$
13. $cy = \sin cx$ $\qquad y' = \cos \dfrac{x\sqrt{1-y'^2}}{y}.$

14. $y = cx + \frac{c}{\sqrt{1+c^2}}$	$y - xy' = \frac{y'}{\sqrt{1+y'^2}}$.
15. $x^2 + y^2 - cx = 0$	$2xyy' + x^2 - y^2 = 0$.
16. $y = \sin x + c \cdot \cos x$	$y' \cos x + y \sin x = 1$.
17. $x + y + c(1 - xy) = 0$	$y' + \frac{1+y^2}{1+x^2} = 0$.
18. $x - y - c\ell^{\frac{x}{y-x}} = 0$	$yy' - 2y + x = 0$.
19. $y^2 = cx$	$2xy' - y = 0$.
20. $y = \frac{c}{\cos x}$	$y' - tgx \cdot y = 0$.
21. $y = -\frac{1}{3x+c}$	$y' = 3y^2$.
22. $y = \ell n(c + \ell^x)$	$y' = \ell^{x-y}$.
23. $x = y\ell ncy$	$y'(x+y) = y$.

2. O`zgaruvchilarga ajraladigan birinchi tartibli differentsial tenglamalar yechilsin.

1. $x(1+y^2)+y(1+x^2)y'=0$ j: $(1+x^2)(1+y^2)=c$.
2. $y'=xy^2+2xy$ j: $\frac{y}{y+2}=c\ell^{x^2}$.
3. $\ell^x dx-(1+\ell^x)ydy=0$ j: $\ell^{\frac{y^2}{2}}=\frac{\sqrt{\ell}}{2}(1+\ell^x)$.
4. $y'+y=2x+1$ j: $y-2x-1=-2+c\ell^{x}$.
5. $y'=\cos(x-y-1)$ j: $y=x-1-2avctg(c-x)+2n\pi \quad n\in Z$.
6. $(1+y^2)dx+(1+x^2)dy=0$ j: $x+y=c(1-xy)$.
7. $(1+y^2)dx+xydy=0$ j: $x^2(1+y^2)=c$.
8. $(y^2+xy^2)y'+x^2-yx^2=0$ j: $(x+y)(x-y-2)+2\ell n\left(\frac{1+x}{1-y}\right)+c$.
9. $(1+y^2)dx=xdy$ j: $y=tg\ell ncx$
10. $x\sqrt{1+y^2}+yy'\sqrt{1+x^2}=0$ j: $\sqrt{1+x^2}+\sqrt{1+y^2}=c$.
11. $x\sqrt{1-y^2}dx+y\sqrt{1-x^2}dy=0 \quad y(0)=1$ j: $\sqrt{1-x^2}+\sqrt{1-y^2}=1$.
12. $\ell^{-y}(1+y')=1$ j: $\ell^x=c(1-\ell^{-y})$
13. $y'=\sin(x-y)$ j: $x+c=ctg\left(\frac{y-x}{2}+\frac{\pi}{4}\right)$
14. $y'=ax+by+c \quad (a,b,c-const)$ j: $b(ax+by+c)+a=c\ell^{bx}$.

15. $(x+y)^2 y' = a^2$ — J: $x+y = a\,tg\left(c+\frac{y}{a}\right)$

16. $\left(x-y^2\right)dx + 2xydy = 0$ — J: $x\ell^{\frac{y^2}{x}} = c.$

17. $\left(1+x^2\right)dy - xydx = 0 \quad y(2) = 1$ — J: $y = \sqrt{\frac{x^2}{2} - 1}.$

18. $\left(xy^2 + x\right)dx + \left(y - x^2 y\right)dy = 0$ — J: $1 + y^2 = c(1 - x^2).$

19. $\left(xy^2 + x\right)dx + \left(x^2 y - y\right)dy = 0$ — J: $x^2y^2 + x^2 - y^2 = c.$

20. $xy' + y = y^2 \quad y(1) = 0{,}5$ — J: $y(1 - cx) = 1.$

21. $xy' = tgy$ — J: $\sin y = cx$

22. $2xyy' = y^2 - 1$ — J: $y^2 - 1 = cx.$

3. Birinchi tartibli bir jinsli tenglamalar yechilsin.

1. $y' = \frac{xy + y^2 \ell^{-\frac{x}{y}}}{x^2}$ — J: $\ell^{\frac{x}{y}} + \ell n|x| = c.$

2. $\left(x - y\cos\frac{y}{x}\right)dx + x\cos\frac{y}{x}dy = 0$ — J: $x = c\ell^{-\sin\frac{y}{x}}.$

3. $y' = \frac{y + \sqrt{x^2 - y^2}}{x}$ — J: $y = x\sin\ell ncx$

4. $y' = \frac{4x^6 - y^4}{2x^4 y}$ — J: $\frac{y^2 - x^3}{y^2 + 4x^3}x^5.$

5. $\left(x^2 + y^2\right)dx - xydy = 0$ — J: $x = c\ell^{y^2/(2x^2)}.$

6. $y' = 1 + \frac{y}{x}$ — J: $x = c\ell^{\frac{y}{x}}.$

7. $xy' = 3y - x$ — J: $x^3 = c(2y - x).$

8. $(x - y)dx + xdy = 0$ — J: $y = x\left(c - \ell n|x|\right)$

9. $y^2dx - \left(x^2 + xy\right)dy = 0$ — J: $y = c\ell^{-\frac{y}{x}}.$

10. $\left(x^2 - xy + y^2\right)dx - x^2dy = 0$ — J: $x = c\ell^{\frac{x}{x-y}}.$

11. $(x + 2y)dx - (2x + y)dy = 0$ — J: $(x - y)^3 = c(x + y).$

12. $xy' = y + x\ell^{\frac{y}{x}}$ — J: $\ell n|cx| = -\ell^{-\frac{y}{x}}.$

13. $xy' - x\cos^2\frac{y}{x} = y$ — J: $xc\ell^{tg(y/x)}.$

14. $y - xy' = \sqrt{x^2 - y^2}$ — J: $x = c\ell^{\arccos\frac{y}{x}}.$

15. $(x+y-2)dx+(x-y+4)dy=0$ J: $x^2+2xy-y^2-4x+8y=c.$

16. $(x+y+1)dx+(2x+2y-1)dy=0$ J: $x+2y+3\ln|x+y-2|=c.$

17. $(x^2y^2-1)dy+2xy^3dx=0$ J: $y=\frac{1+x^2y^2}{c}.$

18. $4x-3y+y'(2y-3x)=0$ J: $y^2-3xy+2x^2=c.$

19. $y'=\frac{2xy}{3x^2-y^2}$ J: $c(y^2-x^2)=y^3.$

20. $(y-xy')^2=x^2+y^2$ J: $c^2x^2=1+2cy;\ c^2-x^2=2cy.$

21. $3x+y-2+y'(x-1)=0$ J: $(x-1)(3x+2y-1)=c.$

22. $(3y-7x+7)dx-(3x-7y-3)dy=0$ J: $(x+y+1)^5\cdot(x-y-1)^2=c.$

4. Birinchi tartibli chiziqli differentsial tenglamalar va Bernulli differentsial tenglamalari yechilsin.

1. $y'-2xy=3x^2-2x^4$ J: $y=(c+x^3\ell^{-x^2})\ell^{x^2}=c\ell^{x^2}+x^3.$

2. $y'+y\cos x=\ell^{-\sin x}$ J: $y=(x+c)\ell^{-\sin x}.$

3. $y+ytgx=\frac{1}{\cos x}$ J: $y=c\cos x+\sin x.$

4. $y'\sin x-y\cos x=-\frac{\sin^2 x}{x^2}$ J: $y=c\sin x+\frac{\sin x}{x}.$

5. $y'\sin 2x=2(y+\cos x)$ J: $y=c\cdot tgx-\frac{1}{\cos x}.$

6. $y'-\frac{2y}{x}=2x^3$ J: $y=cx^2+x^4.$

7. $y'+2xy=2x\ell^{-x^2}$ J: $y=(x^2+c)\ell^{-x^2}.$

8. $y'+2y=x^2+2x$ J: $y=c\ell^{-2x}+\frac{1}{4}(2x^2+2x-1).$

9. $(x^2+2x-1)y'-(x+1)y=x-1$ J: $y=c\sqrt{x^2+2x-1}+x.$

10. $2xy'-y=3x^2$ J: $y=c\sqrt{x}+x^2.$

11. $xy'+y=y^2\ell nx$ J: $y=\frac{1}{1+cx+\ell nx}.$

12. $2yy'-x=-x^3\sin y$ J: $y=(c-\cos y)x^2.$

13. $y'+2y=\ell^x y^2$ J: $y(\ell^x+c\ell^{2x})=1.$

14. $y'-2xy=\ell^{x^2}$ J: $y=(x+c)\ell^{x^2}.$

15. $y'+xy+x=0$ J: $y=c\ell^{-\frac{x^2}{2}}-1.$

16. $xy'=2x\ell nx-y$ J: $y=\frac{c}{x}+x\ell nx-\frac{x}{2}.$

17. $(y+\ell^x)dx-dy=0$ J: $y=(c+x)\ell^x.$

18. $y'(1+x^2)-xy=\sqrt{1+x^2}$ J: $y=(arctgx+c)\sqrt{x^2+1}$.

19. $3xy'-2y=\frac{x^3}{y^2}$ J: $y^3=cx^2+x^3$.

20. $x^2y'+2x^3y=y^2(1+2x^2)$ J: $\frac{1}{y}=c\ell^{x^2}+\frac{1}{x}$.

21. $(xy+x^2y^3)y'=1$ J: $\frac{1}{x}=2-y^2+c\ell^{-\frac{y^2}{2}}$.

22. $y=xy'+y'\ell ny$ J: $x=cy-1-\ell ny$.

5. To'liq differentsialli 1–tartibli differentsial tenglamalar yechilsin.

1. $(\sin xy+xy\cos xy)dx+x^2\cos xy dy=0$ J: $x\sin xy=c_1$.

2. $(x^3+xy^2)dx+(x^2y+y^3)dy=0$ J: $x^4+2(xy)^2+y^4=0$.

3. $(x+y^2)dx-2xydy=0$ J: $x=c\cdot\ell^{\frac{y^2}{x}}$.

4. $2xy\ell nydx+\left(x^2+y^2\sqrt{y^2+1}\right)dy=0$ J: $x^2\ell ny+\frac{1}{3}\left(y^2+1\right)^{\frac{3}{2}}=c$.

5. $\left(2xy+3y^2\right)dx+\left(x^2+6xy-3y^2\right)dy=0$ J: $x^2y+3xy^2-y^3=c$.

6. $\left(3y^2+2xy+2x\right)dx+\left(6xy+x^2+3\right)dy=0$ J: $3xy^2+x^2y+3y+x^2=c$.

7. $2x\left(1+\sqrt{x^2-y}\right)dx-\sqrt{x^2-y}dy=0$ J: $x^2+\frac{2}{3}(x^2-y)^{\frac{3}{2}}=c$.

8. $\left(y^2+x^2+a\right)yy'+\left(y^2+x^2-a\right)x=0$ J: $(x^2+y^2)^2-2a(x^2-y^2)=c$.

9. $\left(1-\frac{x}{y}\right)dx+\left(2xy+\frac{x}{y}+\frac{x^2}{y^2}\right)dy=0$ J: $\ell n|x|+\ell n|y|+y^2-\frac{x}{y}$.

10. $(x^2-\sin^2 y)dx+x\sin 2ydy=0$ J: $x^2+\sin^2 y=cx$.

11. $ydx-(x+x^2+y^2)dy=0$ J: $arctg\frac{x}{y}-y=c,\ y=0$.

12. $(x+y-1)dx+(\ell^y+x)dy=0$ J: $\ell^y+\frac{1}{2}x^2+xy-x=c_1+1$.

13. $(2x+3x^2y)dx+(x^3-3y^2)dy=0$ J: $x^2y+3xy^2-y^3=c$.

14. $ydx-(4x^2y+x)dy=0$ J: $\frac{y}{x}+2y^2=c$.

15. $2xydx+(x^2-y^2)dy=0$ J: $3x^2y-y^3=c$.

16. $(2-9xy^2)xdx+(4y^2-6x^3)ydy=0$ J: $x^2-3x^3y^2+y^4=c$.

17. $\frac{y}{x}dx+\left(y^3+\ell nx\right)dy=0$ J: $4y\ell nx+y^4=c$.

18. $3x^2(1+\ell ny)dx=\left(2y-\frac{x^3}{y}\right)dy$ J: $x^3+x^3\ell ny-y^2=c.$

19. $\ell^y-(2y+x\ell^{-y})dy=0\to$ J: $x\ell^{-y}-y^2=c.$

6. Hosilaga nisbatan yechilmagan 1–tartibli differentsial tenglamalar. Lagranj va Klero tenglamalari yechilsin.

1. $y=x+y'-\ell ny'$ J: $p\neq 1,\ y=\ell^{x-c}+c;\ p=1,\ y=x+1.$

2. $y=2xy'+\ell ny'$ J: $y=\frac{2c}{\rho}+\ell n\rho-2\ \ x=\frac{c}{\rho^2}-\frac{1}{\rho}.$

3. $y=xy'+\frac{a}{2y'}\ (a=const)$ J: $y=cx+\frac{a}{c^2};\ 4y^3=27ax^2.$

4. $(y')^3-2x(y')^2+y'=2x$ J: $y=x^2+c.$

5. $(y')^2+y(y-x)y'-xy^3=0$ J: $\left(y-\frac{1}{x+c}\right)\left(y-c\ell^{\frac{x^2}{2}}\right)$

6. $(y')^2+(\sin x-2xy)y'-2xy\sin x=0$ J: $(y-\cos x-c)(y\ell^{-x^2}-c)$

7. $y=(y')^2\ell^{y}$ J: $y=0,\ y=\rho^2\ell^{\rho},\ x=(\rho+1)\ell^{\rho}+c.$

8. $\ell ny'+\sin y'-x=0$ J: $\begin{cases}x=\ell n\rho+\sin\rho\\ y=\rho+\cos\rho+\rho\sin\rho+c\end{cases}$

9. $y'\sin y'+\cos y'-y=0$ J: $y=1,\ x=\sin\rho+c.$

10. $(y')^2+(x+a)y'-y=0$ J: $y=(x+a)c+c^2$ ва $\begin{cases}y=\rho^2+(x+a)\rho\\ 2\rho+x+a=0\end{cases}$

11. $\sqrt{(y')^2+1}+xy'-y=0$ J: $y=cx+\sqrt{1+c^2}$ ва $\begin{cases}x=\frac{-\rho}{\sqrt{1+\rho^2}}\\ y=\rho x+\sqrt{1+\rho^2}\end{cases}$

12. $x^4(y')^2-xy'-y=0$ J: $y=x^4\rho^2-x\rho,\ 1-2\rho x^3=0$ ёки $y=-\frac{1}{4x^2}.$

13. $xy'=y+x\sqrt{1+(y')^2}$ J: $x=\frac{c(\rho+\sqrt{1+\rho^2})}{\sqrt{1+\rho^2}};\ y=x(\rho-\sqrt{1+\rho^2}).$

14. $y'+y=x(y')^2$ J: $x=\frac{(\rho-\ell n\rho+c)}{(\rho-1)^2}.$

15. $y(y')^2+2xy'-y'=0$ J: $z=y^2,\ y^2=cx+\frac{c^2}{4}.$

16. $(y')^2+y^2=1$ J: $y=\pm 1,\ y=\cos(x-c).$

17. $(y')^2-2xy'+y=0$ $\int$: $x=\frac{2}{3}\rho+\frac{c}{\rho^2}$; $y=2\rho x-\rho^2$, $y=0.$

18. $y=x+2y'-(y')^2$ $\int$: $y=x+1$, $x=-2\rho+c$, $y=x+2\rho-\rho^2$.

19. $2y=xy'+y'\ell ny$ $\int$: $\begin{cases} x=2c\rho-\ell n\rho-2 \\ y=c\rho^2-\rho \end{cases}$

20. $y=x(1+y')+y'^2$ $\int$: $\begin{cases} x=2(1-\rho)+c\ell^{-\rho} \\ y=\left[2(1-\rho)+c\ell^{-\rho}\right](1+\rho)+\rho^2 \end{cases}$

IX BOB.
IKKINCHI TARTIBLI DIFFERENTSIAL TENGLAMALAR.
1–§. Boshlang'ich shart bilan berilgan ikkinchi tartibli differentsial tenglamalar.

$$y''=f(x,y,y')=f(x) \qquad (1)$$

ko'rinishdagi ikkinchi tartibli differentsial tenglamalar ikki marta integrallash bilan yechiladi. $y'=z$ deb $y''=z'$ ni topamiz. Natijada o'zgaruvchilarga ajraladigan birinchi tartibli tenglamaga kelamiz:

$$z'=f(x) \text{ yoki } \frac{dz}{dx}=f(x) \text{ yoki } dz=f(x)dx.$$

Bu tenglamani integrallab quyidagini $\int dz=\int f(x)dx$ topamiz, bu yerdan

$$z=F(x)+C_1$$

ga kelamiz.

Buni ham yechsak, z ni $\frac{dy}{dx}$ ga almashtiramiz.

$$dy = [F(x) + C_1]dx$$

buni integrallasak,

$$y = \int F(x)dx + C_1x + C_2$$

bu yerdan

$$y = \Phi(x) + C_1x + C_2 \quad (2).$$

Bu $y'' = f(x)$ tenglamaning umumiy yechimidir. (2) tenglikda ikkita ixtiyoriy C_1 va C_2 koeffitsientlar hosil bo'ldi. Agar (1) tenglama uchun umumiy yechimdan qandaydir xususiy yechimga kelish kerak bo'lsa, $x = x_0$, $y = y_0$, $y' = y'_0$ boshlang'ich shart talab qilish zarur. U holda

$$y_0 = \varphi(x_0, C_1, C_2) \ ,$$
$$y'_0 = \varphi_x(x_0, C_1, C_2)$$

bo'ladi.

1–Misol. $y'' = \sin x - x$ tenglamaning $y(0) = -1$, $y'(0) = 1$ boshlang'ich shartlarini qanoatlantiruvchi xususiy yechimini toping.

Yechilishi: $\frac{dy}{dx} = z$ desak,

$$\frac{dz}{dx} = \sin x - x; \quad dz = (\sin x - x)dx; \quad z = -\cos x - \frac{x^2}{2} + C_1$$

Demak,

$$y' = -\cos x - \frac{x^2}{2} + C_1$$

$y'(0) = 1$ boshlang'ich shartdan foydalanib, topamiz:

$$1 = -\cos 0 + C_1$$

bu yerdan $C_1 = 2$. Shunday qilib,

$$\frac{dy}{dx} = -\cos x - \frac{x^2}{2} + 2 \quad \text{yoki} \quad dy = \left(-\cos x - \frac{x^2}{2} + 2\right)dx$$

bu tenglamani integrallaymiz:

$$y = -\sin x - \frac{x^3}{6} + 2x + C_2 .$$

Endi $y(0) = -1$ boshlang'ich shartdan foydalanib, topamiz:

$$-1 = C_2$$

Shunday qilib, izlanayotgan xususiy yechim ushbu ko'rinishga ega:

$$y = -\sin x - \frac{x^3}{6} + 2x - 1 .$$

1. Tartibini pasaytirish mumkin bo'lgan yuqori tartibli differentsial tenglamalar.

n – tartibli differentsial tenglamalarning umumiy ko'rinishi quyidagicha

$$\Phi\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0 \quad (1)$$

yoki n tartibli hosilaga nisbatan yechilgan differentsial tenglama quyidagicha

$$y^{(n)} = f\left(x, y, y', y'', \ldots, y^{(n-1)}\right) \quad (2)$$

bo'ladi. Bunda x – erkli o'zgaruvchi, $y = y(x)$ – noma'lum funktsiya, $y', y'', \ldots, y^{(n)}$ lar noma'lum funktsiyaning birinchi, ikkinchi va x.k., n – tartibli hosilalari. (1) differentsial tenglama xususiy holda ushbu

$$y^{(n)} = f(x) \quad (3)$$

ko'rinishga ega bo'lsin. Bunda (3) ni ketma – ket n marta integrallab umumiy yechimi topiladi. (1) differentsial tenglamada noma'lum funktsiya va uning dastlabki bir nechta tartibdagi hosilalari qatnashmasin:

$$\Phi\left(x, y^{(k)}, y^{(k+1)}, \dots, y^{(n)}\right) = 0 \qquad (4)$$

Bu holda $y^{(k)} = p = p(x)$ almashtirish natijasida (4) differentsial tenglamaning tartibi pasayib, ushbu

$$\Phi\left(x, p, p', \dots, p^{(n-k)}\right) = 0$$

ko'rinishga keladi.

(1) differentsial tenglamada erkli o'zgaruvchi x qatnashmasin:

$$\Phi\left(y, y', y'', \dots, y^{(n)}\right) = 0$$

bu holda $y' = p = p(x)$ almashtirish bilan differentsial tenglamaning tartibi bir birlikka pasayadi, ya'ni

$$y' = \frac{dy}{dx} = p$$

$$y'' = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = p \cdot \frac{dp}{dy}$$

$$y''' = \frac{d}{dx}\left(p \cdot \frac{dp}{dy}\right) = \frac{d}{dy}\left(p \cdot \frac{dp}{dy}\right) \cdot \frac{dy}{dx} = p^2 \frac{d^2 p}{dy^2} + p\left(\frac{dp}{dy}\right)^2$$

va h.k. bo'lishi e'tiborga olinadi.

2–Misol. $y''' = 60x^2$ tenglamani yeching.

Yechilishi: Tenglamaning ikkala tomonini dx ga ko'paytirib 2 – chi tartibli tenglamani hosil qilamiz. $y'''dx = 60x^2 dx$ dan $y'' = 20x^3 + C_1$ ni hosil qilamiz.

So'ngra yuqoridagini qo'llab 1 – chi tartibli tenglamani hosil qilamiz.

$y''dx = \left(20x^3 + C_1\right)dx$ dan $y' = 5x^4 + C_1 x + C_2$ ni hosil qilamiz.

$y'dx = (5x^4 + C_1x + C_2)dx$ dan $y = x^5 + C_1\frac{x^2}{2} + C_2x + C_3$.

2. O'zgarmas koeffitsientli bir jinsli chiziqli differentsial tenglamalar.

Ushbu

$$y^{(n)} + p_1y^{(n-1)} + ... + p_ny = 0 \quad (1)$$

tenglamaga **bir jinsli chiziqli differentsial tenglama** deyiladi. Tenglamaning umumiy yechimi

$$y = C_1y + C_2y_2 + ... + C_ny_n \quad (2)$$

dan iborat, bunda $y_1, y_2, ..., y_n$ lar (1) tenglamaning chiziqli bo'lmagan xususiy yechimlari, $C_1, C_2, ... C_n$ lar esa ixtiyoriy o'zgarmaslar .

O'zgarmas koeffitsientli *ikkinchi tartibli chiziqli bir jinsli tenglama* deb

$$y'' + py' + qy = 0 \quad (3)$$

ko'rinishdagi tenglamaga aytiladi.

(2) tenglamaning umumiy yechimini topish uchun

$$r^2 + pr + q = 0 \quad (4)$$

xarakteristik tenglama tuziladi, bu tenglama (3) tenglamadan izlanayotgan funktsiya hosilalarni r ning mos darajalari bilan funktsiyaning o'zini esa bir bilan almashtirish natijasida hosil qilinadi. U holda (3) tenglamaning umumiy yechimi (4) tenglama ildizlari r_1 va r_2 ning xarakteriga bog'liq holda topiladi. Bu yerda uch hol ro'y berishi mumkin.

1–hol. r_1 va r_2 ildizlar haqiqiy va har xil. Bunday holda (3) tenglamaning umumiy yechimi

$$y = C_1e^{r_1x} + C_2e^{r_2x}$$

ko'rinishga ega bo'ladi.

2–hol. r_1 va r_2 ildizlar haqiqiy va bir–biriga teng:

$$r_1 = r_2 = r\,,$$

bu holda (3) tenglamaning umumiy yechimi

$$y = (C_1 + C_2 x)e^{rx}$$

ko'rinishga ega bo'ladi.

3–hol. r_1 va r_2 ildizlari qo'shma kompleks:

$$r_1 = \alpha + \beta i\,,\ r_2 = \alpha - \beta i$$

Bunday holda umumiy yechim

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

ko'rinishda yoziladi.

3–Misol. $y'' + 3y' - 4y = 0$ tenglamani yeching.
Yechilishi: Xarakteristik tenglamani yozamiz:

$$k^2 + 3k - 4 = 0$$

echimlari: $k_1 = -4,\ \ k_2 = 1$
1 – holga ko'ra umumiy yechim: $y = C_1 e^{-4x} + C_2 e^x$

4–Misol. $y'' + 6y' + 9y = 0$ tenglamani yeching.
Yechilishi: Xarakteristik tenglamani yozamiz:

$$k^2 - 6k + 9 = 0$$

echimlari: $k_1 = k_2 = 3.$
2 – holga ko'ra umumiy yechim: $y = e^{3x}(C_1 + C_2 x)$

5–Misol. $y'' + 4y' + 13y = 0$ tenglamani yeching.
Yechilishi: Xarakteristik tenglamani yozamiz:

$$k^2+4k+13=0$$

echimlari: $k_1=-2+3i,\quad k_2=-2-3i$

3 – holga ko'ra umumiy yechim: $y=e^{-2x}(C_1\cos 3x+C_2\sin 3x)$.

3. O'zgarmas koeffitsientli va maxsus o'ng tomonga ega bo'lgan *ikkinchi tartibli* chiziqli bir jinsli bo'lmagan differentsial tenglamalar.

Ushbu

$$y''+py'+qy=f(x) \quad (1)$$

tenglama **o'zgarmas koeffitsientli** **chiziqli bir jinsli bo'lmagan differentsial tenglama** deyiladi. U mos chiziqli bir jinsli

$$y''+py'+qy=0 \quad (2)$$

tenglamadan o'ng tomonda turgan $f(x)$ funktsiya bilan farqlanadi.

(1) tenglamaning umumiy yechimini topish uchun (2) tenglamaning umumiy yechimi y_1 ni, so'ngra (1) tenglamaning birorta xususiy yechimi y^* ni topish kerak. Ularning yig'indisi berilgan bir jinsli bo'lmagan (1) tenglamaning umumiy yechimi bo'ladi

$$y=y_1+y^*$$

Quyidagi ikki holda (1) tenglamaning xususiy yechimi y^* ni topish qoidasini keltiramiz.

O'ng tomoni $f(x)$

$$f(x)=e^{kx}P_n(x) \quad (3)$$

ko'rinishga ega, bu yerda $P_n(x)$ ko'phad n – darajali ko'phad; $f(x)$ ning o'ng tomoni

$$f(x)=a\cos\lambda x+b\sin\lambda x \quad (4)$$

ko'rinishga ega.

Bu hollarning har qaysisini alohida – alohida ko'ramiz.

I. (1) tenglamaning o'ng tomoni $f(x)=e^{kx}P_n(x)$ ko'rinishga ega bo'lsin, bu yerda k soni (2) bir jinsli tenglamaga mos keluvchi xarakteristik tenglama

$$r^2+pr+q=0 \quad (5)$$

ning ildizi bo'lmasin. U holda (1) tenglamaning xususiy yechimini

$$y^*=e^{kx}Q_n(x) \quad (6)$$

ko'rinishda izlash kerak, bu yerda $Q_n(x)$ birorta n – darajali koeffitsientlari noaniq bo'lgan ko'phad.

Agar k soni (5) xarakteristik tenglamaning ildizi bo'lsa, u holda (1) tenglamaning xususiy yechimini

$$y^*=x^m e^{kx}Q_n(x)$$

ko'rinishida izlash kerak, bu yerda m son k ildizning karraligi (ya'ni, agar k – bir karrali ildiz bo'lsa, m = 1, agar k – ikki karrali ildiz bo'lsa, m = 2).

II. (1) tenglamaning o'ng tomoni

$$f(x)=a\cos\lambda x+b\sin\lambda x$$

ko'rinishga ega bo'lsin, bu yerda $\pm\lambda i$ sonlar (5) xarakteristik tenglamaning ildizlari emas. U holda (1) tenglamaning xususiy yechimini

$$y^*=a\cos\lambda x+b\sin\lambda x$$

ko'rinishda izlash kerak, bu yerda A va B – noaniq koeffitsientlar.

Agar $\pm\lambda i$ kompleks sonlar (5) xarakteristik tenglamaning ildizlari bo'lsa, (1) tenglamaning xususiy yechimini

$$y^*=x(a\cos\lambda x+b\sin\lambda x)$$

shaklida izlash kerak.

6–Misol. $y'' - 6y' + 9y = 3x - 8e^x$ tenglamaning umumiy yechimini toping.

Yechilishi: Xarakteristik tenglamani yozamiz:

$r^2 - 6r + 9 = 0$ yoki $(r-3)^2 = 0$ va yechimi $r_{1,2} = 3$ ga teng.

Bundan

$$u = e^{3x}(C_1 + C_2 x)$$

Berilgan tenglamaning xususiy yechimi u_1 ni topamiz. Bu yerda o'ng tomoni $f(x) = 3x - 8e^x$ ko'rinishga ega. Ko'phadning birinchi darajali qismi $3x$ va ko'rsatkichli qismi $-8e^x$ ga teng. Demak, xususiy yechimi $u_1'' = Ax + B + Ce^x$ ga teng.

$u_1' = A + Ce^x$ ga, $u_1'' = Ce^x$ ga tengligidan, tenglama quyidagi ko'rinishga keladi.

$$9Ax + (9B - 6A) + 4Ce^x = 3x - 8e^x$$

Bundan, $9a = 3,$ $9B - 6A = 0,$ $4c = -8$ ga tengligidan $A = \frac{1}{3},$ $B = \frac{2}{9}, C = -2$

Demak,

$$u_1 = \frac{1}{3}x + \frac{2}{9} - 2e^x,$$

$$y = u + u_1 = e^{3x}(C_1 + C_2 x) + \frac{1}{3}x + \frac{2}{9} - 2e^x.$$

Nazorat savollari.

1. Boshlang'ich shart bilan berilgan ikkinchi tartibli differentsial tenglamalar.
2. Tartibini pasaytirish mumkin bo'lgan yuqori tartibli differentsial tenglamalar .
3. O'zgarmas koeffitsientli bir jinsli chiziqli differentsial tenglamalar.

4. O'zgarmas koeffitsientli va maxsus o'ng tomonga ega bo'lgan ikkinchi tartibli chiziqli bir jinsli bo'lmagan differentsial tenglamalar.

2–§. Differentsial tenglamaning tatbiqlari.

I. Populyatsiya miqdorining dinamikasi.

Populyatsiya miqdorining dinamikasi (ya'ni, populyatsiya davrida tug'ilish va o'lish natijasida tirik mavjudotlar miqdorining o'zgarishi) ekologiyaning muhim masalalaridan biri. Birinchi tartibli differentsial tenglamalarni o'rganishda yuqoridagi masalaning eng oddiy holatini qaradik. Oziq–ovqat bilan ta'minlangan va tashqi muhit ta'siridan chegaralangan holda populyatsiya dinamikasi quyidagi differentsial tenglama bilan berildi:

$$\frac{dx}{dt} = \gamma x \quad (1)$$

Bu yerda $x = x(t) - t$ vaqtdagi populyatsiya miqdori. Bu tenglamaning yechimi quyidagicha

$$x = x_0 e^{\gamma(t-t_0)}$$

ekanligi kelib chiqdi.

Bu yerda x_0, t_0 boshlang'ich vaqtdagi populyatsiya miqdori. (1) tenglama yoki nazariy ahamiyatga ega yoki sun'iy sharoitdagi mavjudotlarning populyatsiyasini aniqlaydi.

Populyatsiyaning rivojlanishini 1845 yilda Ferxyulsta–Perla tenglamasi aniqroq yoritib beradi. Bu tenglamada mavjudotlarning ichki qarama–qarshiliklarini hisobga oladi, bu esa populyatsiya miqdorining tezligini sekinlashtiradi. Bu qarama–qarshiliklarga oziq–ovqat uchun kurash, jips yashaganda infektsiya tarqalishi va h.k. kiradi. Yuqoridagi faktlarni hisobga olib, Δx o'sishni hisoblashda $\gamma x \Delta t$ qiymatdan $h(x, \Delta t)$ qiymatni ayiramiz:

$$\Delta x = \gamma x \Delta t - h(x, \Delta t)$$

$h(x,\Delta t)$ funktsiya o'rniga ko'pchilik hollarda $\delta x^2\Delta t$ populyatsiyani qaraymiz:

$$h(x,\Delta t)=\delta x^2\Delta t$$

bu yerda δ — koeffitsient ichki qarama—qarshiliklar.

$h(x,\Delta t)$ qiymat — bu ichki qarama—qarshiliklar evaziga populyatsiya miqdori tezligining kamayishini ifodalaydi. Qarama—qarshiliklar qancha yuqori bo'lsa, urug'lanadiganlarning bir—biri bilan uchrashishi shuncha ko'p, bu uchrashishlar soni $x\cdot x$ ko'paytmaga to'g'ri proportsional, ya'ni x^2. Ikki xil mavjudotning uchrashishi xy ga to'g'ri proportsional. Bu turlar bir—birining joyini egallashi mumkin.

Shunday qilib,

$$\Delta x=\gamma x\Delta t-\delta x^2\Delta t \qquad (2)$$

bu tenglikni Δt ga bo'lamiz.

$$\frac{\Delta x}{\Delta t}=\gamma x-\delta x^2$$

va $\Delta t\to 0$ da limitga o'tamiz.

$$\frac{dx}{dt}=\gamma x-\delta x^2 \qquad (3)$$

bu tenglama **Ferxyulsta–Perla tenglamasi**.

Bu tenglamadan γx ni qavsdan chiqarib quyidagicha yozamiz:

$$\frac{dx}{dt}=\gamma x\left(1-\frac{\delta}{\gamma}x\right)$$

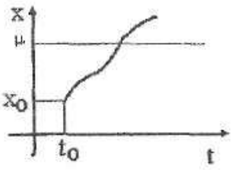
yoki

$$\frac{dx}{dt}=\gamma x\cdot\frac{\frac{\gamma}{\delta}-x}{\frac{\gamma}{\delta}}$$

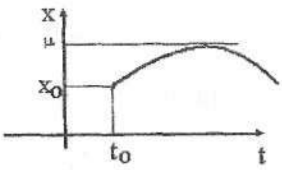
$\frac{\gamma}{\delta}=\mu$ deb belgilaymiz:

$$\frac{dx}{dt} = \gamma x \frac{\mu - x}{\mu} \qquad (5)$$

Agar $x_0 < \mu$ bo'lsa, u holda barcha $t > t_0$ vaqt uchun haqiqatan ham $x(t)$ differentsiallanuvchi funktsiya. (5) tenglamadan kelib chiqadiki, $x(t) < \mu$ da $\frac{dx}{dt}$ musbat bundan kelib chiqadiki, $x(t)$ o'sadi. Bundan shuni xulosa qilish mumkinki, $x(t)$ μ ga teng qiymatni qabul qilsa o'sadi. $x = \mu$ to'g'ri chiziqni kesib o'tadi (1 – rasm) yoki $x = \mu$ to'g'ri chiziqqa urinadi (2 – rasm).



1 – rasm



2 – rasm

Birinchi holda: $x(t) > \mu$ va $x'(t) > 0$. Bu (5) – tenglamaga zid. Ikkinchi holda, $x(t) < \mu$ va $x'(t) < 0$, bu ham (5) – tenglamaga zid. Shunday qilib, $x(t)$ μ ga teng bo'lishi mumkin emas, agar $x_0 < \mu$ bo'lsa.

O'zgaruvchilarni ajratib,

$$\frac{\mu dx}{x(\mu - x)} = \gamma dt$$

yoki

$$\frac{(\mu - x) + x}{x(\mu - x)} dx = \gamma dt$$

bundan

$$\left(\frac{1}{x} + \frac{1}{\mu - x}\right) dx = \gamma dt$$

Agar $x_0 < \mu$ deb hisoblasak, quyidagiga ega bo'lamiz:

$$\ln x - \ln(\mu - x) = \gamma t + \ln C$$

bundan

$$\frac{x}{\mu - x} = Ce^{\gamma t} \quad (6)$$

Qulaylik uchun $t_0 = 0$ va $x(0) = x_0 < \mu$ deb olib, (6) ga qo'ysak, C ni topamiz:

$$C = \frac{x_0}{\mu - x_0}$$

topilgan qiymatni (6) ga qo'ysak quyidagini hosil qilamiz:

$$\frac{x}{\mu - x} = \frac{x_0}{\mu - x_0} e^{\gamma t}$$

Bundan qidiralayotgan **Ferxyulsta–Perla** modelini hosil qilamiz:

$$x = \frac{x_0 \mu e^{\gamma t}}{\mu - x_0 + x_0 e^{\gamma t}}$$

2. **Epidemiya nazariyasida differentsial tenglama.**

Epidemiyaning eng oddiy turini qaraymiz. Aytaylik, o'rganilayotgan kasallik uzoq vaqt davom etadi, demak infektsiya tarqalishi kasallanishga nisbatan tezroq tarqaladi. Biz infektsiya tarqalish holati bilan chegaralanamiz. Aytaylik, a va n lar mos ravishda infektsiya yuqtirganlar va yuqtirmaganlar sonini aniqlasin. $x = x(t)$ t vaqtdagi sog'lom organizmlar, $y = y(t)$ t vaqtdagi infektsiyalangan organizmlar soni. Uncha katta bo'lmagan vaqt oralig'ida ya'ni $0 \le t \le T$ quyidagi tenglik o'rinli:

$$x + y = n + a \qquad (9)$$

Demak, uchrashish chog'ida infektsiya yuqtirilgan organizmdan sog'lom organizmga o'tadi, u holda sog'lom organizmlar soni vaqt o'tishi bilan kamayadi va u uchrashuvlar soniga proportsional bo'ladi (ya'ni, $x \bullet y$ ga proportsional). Bundan sog'lom organizmlar soni kamayadi va kamayish tezligi quyidagiga teng:

$$\frac{dx}{dt} = -\beta \cdot xy \qquad (10)$$

bu yerda β – proportsionallik koefitsienti. Bundan y ni topib (9) ga qo'ysak:

$$\frac{dx}{dt} = -\beta \cdot x(n + a - x).$$

O'zgaruvchilarga ajratib quyidagini topamiz:

$$\frac{dx}{x(n + a - x)} = -\beta \cdot dt$$

yoki

$$\frac{(n + a - x) + x}{x(n + a - x)} dx = -\beta(n + a)dt.$$

Bundan

$$\frac{dx}{x} + \frac{dx}{n - x + a} = -\beta(n + a)dt.$$

Integrallab, quyidagini hosil qilamiz:

$$\ln x - \ln(n - x + a) = -\beta(n + a)t + \ln c$$

yoki

$$\ln \frac{x}{n - x + a} = ce^{-\beta(n+a)t}$$

c ni topish uchun quyidagi boshlang'ich shartdan foydalanamiz. Agar $t=0$ da sog'lom organizmlar soni n bo'lsa (ya'ni $x=n$). Bundan $c=\frac{n}{a}$ ekanligi kelib chiqadi va

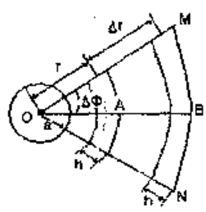
$$\frac{x}{n-x+a}=\frac{n}{a}e^{-\beta(n+a)t}.$$

Demak, qidirayotgan ifodamiz quyidagiga teng ekan:

$$x=\frac{n(n+a)}{n+ae^{\beta(n+a)t}}$$

3. Chumoli inidan tashqarida chumolining zichligi.

Chumoli hayotining umumiy yashash belgilari ko'pchilikka ma'lum. Chumoli topilgan oziq-ovqatni yoki qurilish materiallarini topilgan joyidan uyasiga tashiydi. Shuning uchun chumoli soni uyasining yaqinida uyidan o'zoqroqqa nisbatan ko'proq (3-rasm).



(3-rasm)

Oddiylik uchun chumoli uyasining markazi deb radiusi a ga teng doirani olamiz va bu doiradan tashqarida chumolilar uchun oziq-ovqat bir xilda taqsimlangan deb hisoblaymiz. Bu degani radiusi r ga teng aylanadagi muhit undan tashqarida ham bir xil. Shuning uchun radiusi r ga teng aylanada $(r>a)$ chumoli zichligi bir xil taqsimlangan deb faraz qilamiz. (Hashoratlar zichligi deganda ma'lum atrofdagi hashoratlar sonining shu atrof yuziga nisbatiga aytiladi). Bundan kelib chiqadiki, zichlik bu r masofaning funktsiyasi bo'lib, nuqtalari bitta nurda yotgan nuqtalar bilan chegaralansak kifoya.

Soddalik uchun statsionar holatni, ya'ni vaqt o'tishi bilan zichlik har bir nuqtada o'zgarmas deb hisoblaymiz. Ammo bu degani, chumoli bir—biri bilan aralashib ketmaydi degani emas. Oziq—ovqat qidirishda chumoli bir joydan ikkinchi joyga ko'chib yuradi. Biz bu qidirishni tasodifiy deb hisoblaymiz, yani birlik vaqt ichida ovqat qidirib biror atrofni tark etsa, xuddi shuncha sondagi chumolilar shu atrofga qo'shni atrofdan o'tadi. Shunday qilib chumoli uyasi atrofidan ovqat izlab boshqa joyga ko'chishsa, yana shu sondagi chumolilar uyasidan chiqadi.

Bir nurda yotgan ikkita qo'shni nuqtani qaraymiz: A nuqta chumolilar uyasining markazidan r masofada joylashgan bo'lsin, B nuqta $r+\Delta r$ masofada bo'lsin. Bu nuqtalar atrofida chumolilarning ko'chishini kuzatamiz. Aytaylik, $Q(r+\Delta r)$ B nuqtadagi chumolilar soni. Chumoli ovqat qidirayotganda birorta yo'nalish ikkinchisidan yaxshi bo'la olmaydi. Chunki atrof bir xil deb hisoblangan. Chumoli soni uning qaysi yo'nalishdan ovqat qidirishiga bog'liq emas. Bundan kelib chiqadiki, agarda A nuqta atrofidan B nuqtaga qarab $\alpha Q(r)$, $\alpha<1$ chumolilar chiqqan bo'lsa, u holda B nuqta atrofidan A nuqtaga $\alpha Q(r+\Delta r)$ sondagi chumolilar chiqishdi.

Bunda eng muhimi ikkala holatda ham α sonining bir xil ekanligidadir. α fiksirlangan yo'nalishdagi chumolilar qismini aniqlaydi. Ammo A nuqta atrofidan B nuqta atrofiga chiqqan chumolilarning barchasi ham B nuqtaga yetib bormaydi, chunki yo'lda ovqatni topganlari uyasiga qaytib ketadi. Bu toifadagi chumolilar A va B nuqtalar orasidagi masofa qancha katta bo'lsa shuncha ko'p bo'ladi. Shuning uchun A nuqta atrofidan chiqqan B nuqta atrofiga yetib boradigan chumolilar soni quyidagi ayirma bilan ifodalanadi:

$$Q_{AB} = \alpha Q(r) - \beta\alpha Q(r)\Delta r$$

Bu yerda β—proportsianallik koeffitsienti (manzilga yetmasdan qaytgan chumolilarni aniqlaydi). Bu koeffitsient atrofga bog'liq, lekin atrofda sharoit bir xil bo'lganligi uchun β shu atrofda o'zgarmas. $\alpha Q(r+\Delta r)$ qiymatga B nuqta atrofidan A nuqta atrofiga emas, boshqa yo'nalishga chiqib, yo'ldan ovqat topib orqaga qaytganlari ham kiradi. Bunday chumolilarga OMN sektordan hali chiqib ulgurmaganlari ham A nuqta atrofiga kiradi. Δr qancha katta bo'lsa, ular shuncha ko'p bo'ladi. Shunday qilib, B nuqta atrofidan chiqib A nuqta atrofiga tushadiganlari, quyidagi yig'indi orqali aniqlanadi.

$$Q_{AB} = \alpha Q(r+\Delta r) + \beta_1 \alpha Q(r+\Delta r)\Delta r$$

bu yerda β_1 – chumoli uyasiga qaytgan chumolilar sonini aniqlaydigan proportsianallik koeffitsienti. Biz statsionar holatda bo'lganimiz uchun, yani nuqta atrofida o'zgarmas sonda qolgani uchun quyidagi tenglik bajariladi.

$$Q_{AB} = Q_{BA}$$

ya'ni,

$$\alpha Q(r) - \beta \alpha Q(r)\Delta r = \alpha Q(r+\Delta r) + \beta_1 \alpha Q(r+\Delta r)\Delta r$$

Bundan biz differentsial tenglamaga kelamiz. Chumolilar soni uning zichligining yuza ko'paytmasiga teng bo'lgani uchun (α ga qisqartirib) oxirgi tenglikni quyidagicha yozamiz:

$$n(r)S_A - \beta n(r)S_A\Delta r = n(r+\Delta r)S_B + \beta_1 n(r+\Delta r)S_B\Delta r \qquad (12)$$

bu yerda $n(r)$ va $n(r+\Delta r)$ lar A va B nuqtalardagi mos ravishda zichligini belgilaydi; S_A va S_B shu nuqta atrofining yuzalari.

Yuzani qutb koordinatalar sistemasida hisoblab, quyidagini hosil qilamiz:

$$S_A \approx hr\Delta\Phi; \quad S_B \approx h(r+\Delta r)\Delta\Phi.$$

Buni (12) – tenglikka qo'yamiz:

$$n(r)hr\Delta\Phi - \beta n(r)hr\Delta\Phi\Delta r = n(r+\Delta r)\cdot h(r+\Delta r)\Delta\Phi +$$
$$+\beta_1 n(r+\Delta r)\cdot h(r+\Delta r)\Delta\Phi\Delta r$$

buni $h\Delta\Phi$ ga qisqartirib gruppalasak, quyidagini hosil qilamiz:

$$(r+\Delta r)h(r+\Delta r) - r\cdot n(r) = -[\beta_1 n(r+\Delta r)(r+\Delta r) + \beta n(r)r]\Delta r.$$

Bu tenglikni Δr ga bo'lib Δr→0 sak

$$\frac{d}{dr}(r \cdot n(r)) = -(\beta_1 + \beta) \cdot rn(r) \quad (13)$$

Qisqalik uchun $\beta_1 + \beta = \gamma$ deb belgilaymiz va

$$\frac{d(r \cdot n(r))}{r \cdot n(r)} = \gamma \cdot dr \quad (14)$$

Biz $n(r)$ zichlik uchun differentsial tenglama hosil qildik.

Aytaylik, $n(a)$ – chumoli uyasining chegarasidagi $(r = a)$ zichlik qiymati. (14) ni integrallab, quyidagini hosil qilamiz:

$$\ln[r \cdot n(r)] = -\gamma r + C \quad (15)$$

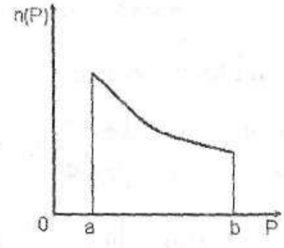
Boshlang'ich shartdan foydalanib, $C = \ln[a \cdot n(a)] + \gamma a$ ni hosil qilamiz. Buni (15) ga qo'yib,

$$\ln\frac{r \cdot n(r)}{a \cdot n(a)} = -\gamma(r - a)$$

bundan

$$n(r) = \frac{a}{r} n(a) e^{-\gamma(r-a)} \quad (16)$$

Bu qidirilayotgan egri chiziq tenglamasi. Agar a, $n(a)$ va γ larning qiymatlari ma'lum bo'lsa, grafigini chizish mumkin. (4 – rasm).



(4 – rasm).

r o'sishi bilan egri chiziq kamayadi. a va $n(a)$ larni tajriba yo'li bilan topish qiyin emas. Ammo γ koeffitsientni hisoblash qiyinroq. Agarda tuzilgan matematik modelni to'g'ri deb hisoblasak, bundan kelib chiqadiki qandaydir γ ni berilgan deb (16) dan zichlikning masofaga bog'liqligidan γ ni hisoblasak bo'ladi. Agar (16) o'rinli bo'lsa, u holda $r>a$ uchun ham o'rinli, xususan $r = b$ uchun

$$n(b) = \frac{a}{b} n(a) e^{-\gamma(b-a)},$$

bundan

$$\gamma = \frac{1}{b-a} \ln \frac{a \cdot n(a)}{b \cdot n(b)} \qquad (17)$$

γ ni hisoblash uchun n ni ham $n(a)$ kabi (16) formuladan hisoblashimiz kerak. Buning uchun quyidagilarni hisobga olishimiz kerak.

A) (16) formuladan ixtiyoriy katta r uchun $n(r) \neq 0$. Haqiqiy hayotda albatta bunday emas.

B) γ qiymat vaqtga albatta bog'liq, chunki kechasi va kunduzi quyosh aktiv vaqtda chumoli boshqa vaqtga nisbatan aktivligi kam bo'ladi.

Ammo kunning har xil vaqtda $n(a)$ va $n(b)$ qiymatlarni (17) formulaga asoslanib hisoblab, γ ni (16) formulaga qo'yib har xil vaqtda o'zining egri chizig'ini topa olamiz.

4. O'simlik bargining o'sishi.

Tuzilishi doira shaklida bo'lgan yosh yaproq yuzasining o'sish tezligi yaproq aylanasi uzunligi va unga tushgan quyosh nuri miqdoriga to'g'ri proportsional. Bu esa quyosh nuri va yaproq orasidagi burchak kosinusi va yaproq yuzasiga to'g'ri proportsional. Agar ertalab soat 6^{00} da yaproq yuzasi 1600 sm^2 va kech soat 18^{00}da shu kuni 2500 sm^2 bo'lsa yaproq yuzi S bilan t vaqt orasida bog'lanishni toping.

Quyosh nuri va yaproq orasidagi burchakni, ya'ni soat 6^{00} va 18^{00} ni (ishorasini hisobga olmagan holda) 90^0 ga teng, kun yarmida 0^0 ga teng deb qabul qilamiz.

Aytaylik, t vaqt yarim tun 00 dan boshlansin. Agar yaproq yuzasi S o'zgarsa, u holda yaproq o'sishining tezligi

$$\frac{ds}{dt} = k_1 2\pi r Q$$

bu yerda $2\pi r$ – yaproq aylanasining uzunligi, Q – yoruglik nurining soni, k_1 – proportsionallik koeffitsenti.

Yaproq yuzasi $S = \pi r^2$ dan quyidagini yozib olamiz

$$r = \sqrt{\frac{S}{\pi}}$$

U holda

$$\frac{ds}{dt} = k_1 \frac{2\pi}{\sqrt{\pi}} \sqrt{S} Q \qquad (18)$$

Quyidagi shartdan

$$Q = k_2 S \cos\alpha \quad (19)$$

(bu yerda α – nur va vertikal orasidagi burchak, k_2 – proportsionallik koeffitsenti) α burchak t argumentning chiziqli o'suvchi funktsiyasi ekanligi kelib chiqadi:

$$\alpha = k_3 t + b$$

k_3 va b parametrlarni qo'shimcha shartlar asosida topamiz:

agar $t = 6$ bo'lsa $\alpha = -\frac{\pi}{2}$,

agar $t = 12$ bo'lsa $\alpha = 0$,

agar $t = 18$ bo'lsa $\alpha = \frac{\pi}{2}$.

Oxirgi ikkita shartdan quyidagiga ega bo'lamiz:

$$\begin{cases} 0 = 12k_3 + b \\ \frac{\pi}{2} = 18k_3 + b \end{cases}$$

bu sistemani yechib

$$k_1 = \frac{\pi}{12}, \quad b = -\pi$$

ni topamiz. Bundan

$$\alpha = \frac{\pi}{12}(t-12)$$

buni (19) ga qo'yamiz.

$$Q = k_2 S \cos\left[\frac{\pi}{12}(t-12)\right]$$

buni (18) ga qo'yamiz.

$$\frac{ds}{dt} = k_1 k_2 \frac{2\pi}{\sqrt{\pi}} S\sqrt{S} \cos\left[\frac{\pi}{12}(t-12)\right]$$

$k_1 \cdot k_2 = k$ deb belgilasak. U holda o'zgaruvchilarni ajratsak

$$\frac{dS}{S\sqrt{S}} = k \frac{2\pi}{\sqrt{\pi}} \cos\left(\frac{\pi}{12}(t-12)\right)dt$$

buni integrallab

$$-\frac{2}{\sqrt{S}} = \frac{24k}{\sqrt{\pi}} \sin\left[\frac{\pi}{12}(t-12) + C\right] \qquad (20)$$

$t=6$ da S=1600 va $t=18$ da S=2500 shartlar bilan

$$\begin{cases} -\frac{1}{20} = -\frac{24k}{\sqrt{\pi}} + C \\ -\frac{1}{25} = \frac{24k}{\sqrt{\pi}} + C \end{cases}$$

Bu sistemani yechib

$$C = -\frac{9}{200}, k = \frac{\sqrt{\pi}}{24 \cdot 200}$$

ni topamiz. Buni (20) ga qo'yib quyidagiga ega bo'lamiz:

$$-\frac{2}{\sqrt{S}}=\frac{24\sqrt{\pi}}{24\cdot 200\sqrt{\pi}}\sin\left[\frac{\pi}{12}(t-12)\right]-\frac{9}{200}$$

Bundan esa

$$S=\frac{160000}{\left\{9-\sin\left[\frac{\pi}{12}(t-12)\right]\right\}^2}$$

ni topamiz.

5. Daraxt o'sishini hisoblash haqidagi masala.

Nega sharoit eng yaxshi bo'lganda ham daraxt ma'lum uzunlikdan oshmaydi? Nega daraxt turiga bog'liq bo'lmagan holda boshlang'ich vaqtda tez o'sadi, so'ngra o'sishi sekinlashib, asta—sekin o'sishi nolga teng bo'ladi?

Biroq biz bilamizki, daraxt tomirlarining o'sishi fotosintez yordamida energiyasi ko'payishiga olib keladi, ammo oziqlanish borgan sari qiyinlashib, bora—bora energiya yetishmay boshlaydi va daraxt o'sishdan tuxtaydi.

Shu fikrlarga asoslanib, energiya balansining taxminiy tenglamasini tuzamiz, ya'ni matematik modelini tuzamiz.

1. O'sayotgan daraxt o'sish davrida geometrik xususiyatini saqlaydi, ya'ni uzunligining daraxt diametriga nisbati o'zgarmaydi.
2. Erkin energiyani (yoki harakatdagi moddani) faqat fotosintez orqali oladi .
3. Erkin energiya tirik to'qima hosil qilishga va tuproqdan aralashmalarning ko'tarilishiga sarf bo'ladi.
4. O'rtacha hisobda katta vaqt oralig'ida birlik sirt yuzasiga o'zgarmas miqdorda yorug'lik tushadi va tarkibidagi moddalardan bir qismini yutishi mumkin.

Aytaylik, x daraxtning chiziqli ulchami. Bu degani daraxt balandligini x orqali, yaproqning yuzasini x^2 orqali va nihoyat daraxt hajmini x^3 orqali belgilaymiz. x ning o'zgarishini t orqali, ya'ni $x=x\ (t)$ orqali ifodalashga harakat qilamiz.

Aytaylik, $x(t_0)=0$ bo'lsin. Balans tenglamasini x bo'yicha ifodalasak, E erkin energiya daraxt tanasining yashil qismidan fotosintez orqali hosil bo'ladi, yashil qismi qanchalik ko'p bo'lsa shuncha energiya ko'p bo'ladi. Shunday qilib, E x^2 ga to'g'ri proportsional

$$E = \alpha x^2$$

bu yerda α – proportsionallik koeffitsenti (α yaproqning o'lchamiga va tuzilishiga hamda fotosintezga bog'liq). Bizning farazimizga ko'ra, boshqa energiya beruvchi omillar yo'q va biz energiyaning taqsimlanishini kuzatishimiz kerak. Energiya birinchidan, fotosintez sodir bo'lishi uchun sarflanadi. Bu sarflanish xam x^2 ga to'g'ri proportsional, yani βx^2, bu yerda β proportsionallik koeffitsenti α dan kichik.

Energiya ozuqaning butun daraxt tanasiga tarqalishi uchun sarflanadi. Ma'lumki, u energiya qancha ko'p sarflansa tana shuncha katta bo'ladi. Bundan tashqari bu sarflanish og'irlik kuchini yengishga bog'liq va bundan kelib chiqadiki, agar ozuqani qancha balandga ko'tarib sarflasa energiya shuncha ko'p sarflanadi. Shunday qilib, energiyaning bu sarfi hajmi x^3 ga va balandlik x ga to'g'ri proportsional, yani $\gamma \cdot x^3 \cdot x$.

Nihoyat energiya daraxtning massasini oshirishga sarflanadi (yani o'sishiga). Bu sarflanish o'sish tezligiga to'g'ri proportsional, yani $m = \rho x^3$ massadan vaqt bo'yicha hosila (ρ – daraxtning o'rtacha zichligi, x^3 – xajmi). Shunday qilib, oxirgi energiya sarflanishi quyidagicha ifodalanadi.

$$\delta \frac{d}{dt}\left(\rho x^3\right)$$

Energiyaning saqlanish qonunidan energiya sarfi quyidagiga teng bo'ladi.

$$E = \beta x^2 + \gamma x^4 + \delta \frac{d}{dt}\left(\rho x^3\right)$$

yoki

$$\alpha x^2 = \beta x^2 + \gamma x^4 + 3\delta \rho x^2 \frac{dx}{dt} \qquad (21)$$

bu esa biz qidirayotgan energiya balansining tenglamasi. Bu tenglamani $3\delta\rho x^2$ bo'lib quyidagicha yozamiz:

$$\frac{\alpha-\beta}{3\delta\rho}=a, \qquad \frac{\gamma}{3\delta\rho}=b$$

bundan:

$$\frac{dx}{dt}a-bx^2, \ a>0, b>0 \quad (22)$$

kelib chiqadi.

Daraxt o'sayotgan ekan, u holda $\frac{dx}{dt}>0$ bu demak, $a-bx^2\geq 0$, bundan $x^2<\frac{a}{b}$ kelib chiqadi. Shuning uchun (22) ni quyidagicha yozamiz.

$$-\frac{dx}{b\left(x^2-\frac{a}{b}\right)}=dt$$

buni integrallab

$$\frac{1}{2\sqrt{ab}}\ln\frac{\sqrt{\frac{a}{b}}+x}{\sqrt{\frac{a}{b}}-x}=t+c$$

boshlang'ich shart $x(t_0)=0$ dan foydalanib $c=-t_0$ ni hosil qilamiz.

$$\frac{1}{2\sqrt{ab}}\ln\frac{\sqrt{\frac{a}{b}}+x}{\sqrt{\frac{a}{b}}-x}=t+t_0 \quad (23)$$

bu tenglamani x ga nisbatan yechamiz

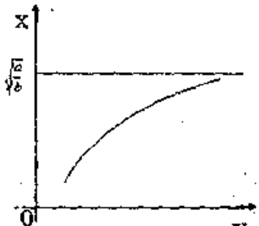
$$x=\sqrt{\frac{a}{b}}\cdot\frac{1-e^{-2\sqrt{ab}}(t-t_0)}{1+e^{-2\sqrt{ab}}(t-t_0)} \quad (24)$$

Bu daraxt o'sishini aniqlaydigan formula.

Agar a, b va t_0 qiymatlari ma'lum bo'lsa, bu formula yordamida vaqtga nisbatan o'rtacha o'sishini hisoblash mumkin. (24) formula bilan berilgan egri chiziqni tekshirish mumkin. $\frac{dx}{dt} > 0$ ligidan va (22) formuladan quyidagini hosil qilamiz:

$$\frac{d^2x}{dt^2} = -2bx$$

Shunday qilib, agar $t > t_0$, $x(t) > 0$ (bu daraxt balandligi) bo'lsa, u holda oxirgi tenglikdan $\frac{d^2x}{dt^2} < 0$ ga kelamiz. Demak, (24) – chiziq o'suvchi qavariq chiziq . (24) formuladan $t \to +\infty$ da $x(t) \to \sqrt{\frac{a}{b}}$ hosil bo'ladi. Grafigi quyidagicha bo'ladi.



(5 – rasm)

$\sqrt{\frac{a}{b}}$ balandlik daraxt o'sishining chegarasi, chunki energiya daraxtning fotosintez va ozuqa bilan ta'minlanishiga sarflanadi. Daraxt shuning uchun bu ko'rsatkichdan yuqori o'smaydi.

(24) formula bilan berilgan egri chiziq daraxt o'sishini qanchalik to'g'ri ifodalaydi? Bu savolga javob berish uchun dub daraxtini olib ko'ramiz. Yoshi 40 yildan 220 yilgacha bo'lgan dub daraxti uzunligi (24) formulada tekshirildi. a va b o'zgaruvchili ikkita tenglamalar sistemasi hosil bo'ladi va bu o'zgaruvchilarni topish mumkin. Topilgan a va b qiymatga qarab aniq egri chiziq chizildi. Bu egri chiziq tajribadagi dub daraxt o'sishining egri chizig'i bilan ustma – ust tushdi. Boshqacha qilib aytganda (40: x (40)) va (220: x (220)) nuqtadagi nazariy va tajribadagi egri chiziqlar ustma – ust tushdi. Demak, ko'rilgan matematik model ishonarli.

Mustaqil yechish uchun misol va masalalar.

1. Tartibini pasaytirish mumkin bo'lgan yuqori tartibli differentsial tenglamalar yechilsin.

1. $y'' = \cos x$ J: $y = -\cos x + c_1 x + c_2.$

2. $y'' = 1 - x^2$ J: $y = \frac{x^2}{2} - \frac{x^4}{12} + c_1 x + c_2.$

3. $y'' = \sin x - 1, y(0) = -1, y'(0) = 1$ J: $y = -\sin x - \frac{x^2}{2} + 2x - 1.$

4. $y'' = \cos 2x$ J: $y = -\frac{1}{4} coc 2x + c_1 x + c_2.$

5. $y'' = \frac{1}{x}$ J: $y = x\ln|x| - x + c_1 x + c_2.$

6. $y'' = \ln x$ J: $y = \frac{x^2 \ln|x|}{2} - \frac{3x^2}{4} + c_1 x + c_2.$

7. $y'' + y'^2 = 2e^{-y}$ J: $e^y + \frac{c_1}{4} = (x + c_2)^2.$

8. $y^{\text{Iv}} = x$ J: $y = \frac{x^5}{120} + c_1 x^3 + c_2 x^2 + c_3 x + c_4.$

9. $y''' = xe^x, y(0) = y'(0) = y''(0) = 0$ J: $y = (x-3)e^x + \frac{x^2}{2} + 2x + 3.$

10. $y''' = x\ln x, y(1) = 1, y'(1) = 0, y''(1) = -1$ J: $y = \frac{x^4}{24}\ln - \frac{13}{288}x^4 - \frac{3}{8}x^2 + \frac{8}{9}x + \frac{17}{32}$

11. $y''^3 - 2y'' - x = 0$ J: $y = \frac{9}{28}t^7; \frac{9}{10}t^5 + (\frac{2}{3} + c_1)\cdot t^3 - 2c_1 t + c_2; x = t^3 - 2t.$

12. $y''' = \sin x + \cos x$ J: $y = \cos x - \sin x + c_1\frac{x^2}{2} + c_2 x + c_3$

13. $y''' = \ln/x^2, y(1) = 0, y'(1) = 1, y''(1) = 2$ J: $\begin{cases} y = -\frac{x}{2}\ln^2 x + c_1\frac{x^2}{2} + c_2 x + c_3 \\ y = -\frac{x}{2}\ln^2 x + \frac{3}{2}x^2 - 2x + \frac{1}{2} \end{cases}$

14. $y''' = \sqrt{1 + (y'')^2}$ J: $y = \frac{e^{x+c_1} - e^{-(x+c_1)}}{2} + c_2 x + c_3$

15. $x = y''^2 + 1$ J: $x = z^2 + 1, y = \frac{4}{15}z^5 + c_1 z^2 + c_2$

16. $y'' + 2y' + 2y = 0$ J: $y = e^{-x}(3\cos x - 2\sin x)$

17. $y'' - 4y' + 8y = 0$ J: $y = e^{2x}(c_1 \cos 2x + c_2 \sin 2x)$

18. $y''+6y'+9y=0$ j: $y=(c_1+c_2x)e^{-3x}$
19. $y'y'''=3y''^2$ j: $x=y^2+y$
20. $y''=y'^2$ j: $x+c=e^{-y}$
21. $yy''+y'^3-y'^2=0$ j: $x=y+\ln y$
22. $y''=-y$ j: $y=c_1\sin x+c_2\cos x$
23. $yy''+y'^2=0$ j: $y=\sqrt{(x+c_1)^2+c_2}$

2. O`zgarmas koeffitsientli ikkinchi tartibli chiziqli bir jinsli tenglamalar yechilsin.

1. $y''-5y'-6y=0$ j: $y=c_1e^{-x}+c_2e^{6x}$
2. $y''-4y'+4y=0$ j: $y=e^{2x}(c_1+c_2x)$
3. $y''+9y=0$ j: $y=c_1\cos 3x+c_2\sin 3x$
4. $y''+6y'+25=0$ j: $y=e^{-3x}(c_1\cos 4x+c_2\sin 4x)$
5. $y''-9y=0$ j: $y=c_1e^{3x}+c_2e^{-3x}$
6. $y''-8y'=0$ j: $y=c_1+c_2e^{8x}$
7. $y''+6y'+9y=0$ j: $y=e^{-3x}(c_1+c_2x)$
8. $y''+16y=0$ j: $y=c_1\cos 4x+c_2\sin 4x$
9. $y''+2y'+2y=0$ j: $y=e^{-x}(c_1\cos x+c_2\sin x)$
10. $y''+5y'+6y=0$ j: $y=c_1e^{-2x}+c_2e^{-3x}$
11. $2y''-3y'-2y=0$ j: $y=c_1e^{-\frac{x}{2}}+c_2e^{2x}$
12. $y''-3y'+2y=0$ j: $y=c_1e^{x}+c_2e^{2x}$
13. $y''-6y'+9y=0, y(0)=2, y'(0)=7$ j: $y=e^{3x}(x+2)$
14. $y''-10y'+25y=0$ j: $y=e^{5x}(c_1+c_2x)$
15. $y''+49y=0$ j: $y=c_1\cos 7x+c_2\sin 7x$
16. $y''-4y'+13y=0$ j: $y=e^{2x}(c_1\cos 3x+c_2\sin 3x)$
17. $y''+3y=0, y(0)=1, y'(0)=3$ j: $y=2-e^{-3}$
18. $y''+4y=0, y(\frac{\pi}{2})=1, y'(\frac{\pi}{2})=-1$ j: $y=-\cos 2x+\frac{1}{2}\sin 2x$
19. $y''-y'-2y=0, y(0)=0, y'(0)=1$ j: $y=\frac{1}{3}e^{2x}-\frac{1}{3}e^{-x}$
20. $y''-2y'+5y=0, y(0)=1, y'(0)=-1$ j: $y=e^{x}(\cos 2x-\sin 2x)$
21. $y''+24y'+144y=0$ j: $[y=e^{-12}(c_1+c_2x)]$
22. $y''-5y=0$ j: $y=c_1e^{-\sqrt{5}x}+c_2e^{\sqrt{5}x}$

Foydalanilgan adabiyotlar ro'yxati

1. T.Jo'raev., A.Sadullaev., G.Xudoyberganov., X.Mansurov., A.Vorisov. **"Oliy matematika asoslari"**. 1-tom., Toshkent «O`zbekiston», 1995 yil, 2-tom. Toshkent «O`zbekiston», 1998 yil.
2. Minorskiy V.P. **"Oliy matematikadan masalalar to'plami"**. 1980 yil.
3. T.T.To'ychiev., D.X.Djumaboev. **«Matematik analiz fanidan 1–kurs talabalari uchun laboratoriya ishlari»**. Toshkent, 2003.
4. A.Sadullaev., X.Mansurov., G.Xudoyberganov., A.Vorisov, R.G'ulomov **"Matematik analiz kursidan misol va masalalar to'plami"**. 1 – tom, 1994 yil.
5. Баврин И.И. **"Высшая математика"**. Москва «Просвещение»,1980 г.
6. Бугров Я.С., Никольский С.М. **"Элементы линейной алгебры и аналитической геометрии"**. 1980 г.
7. Кудрявцов В.А., Демидович В.П. **"Краткий курс высшей математики"**. 1989 г.

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