CURRICULUM OF EDUCATION

B.Ed. (Hons.) Elementary
ADE (Associate Degree in Education)

Course Guide:
GENERAL MATHEMATICS

(Revised 2012)

HIGHER EDUCATION COMMISSION
ISLAMABAD-PAKISTAN

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Technical Support: Education Development Centre (EDC); Teachers College, Columbia University

How this course guide was developed

As part of nation-wide reforms to improve the quality of teacher education, the Higher Education Commission (HEC) with technical assistance from the USAID Teacher Education Project engaged faculty across the nation to develop detailed syllabi and course guides for the four-year B.Ed. (Hons) Elementary and two-year Associate Degree in Education (ADE).

The process of designing the syllabi and course guides began with a curriculum design workshop (one workshop for each subject) with faculty from universities and colleges and officials from provincial teacher education apex institutions. With guidance from national and international subject experts, they reviewed the HEC scheme of studies, organized course content across the semester, developed detailed unit descriptions and prepared the course syllabi. Although the course syllabi are designed primarily for student teachers, they are useful resource for teacher educators too.

In addition, participants in the workshops developed elements of a course guide. The course guide is designed for faculty teaching the B.Ed. (Hons) Elementary and the ADE. It provides suggestions for how to teach the content of each course and identifies potential resource materials. In designing both the syllabi and the course guides, faculty and subject experts were guided by the National Professional Standards for Teachers in Pakistan 2009 and the National Curriculum 2006. The subject experts for each course completed the initial drafts of syllabi and course guides.

Faculty and student teachers started using drafts of syllabi and course guides and they provided their feedback and suggestions for improvement. Final drafts were reviewed and approved by the National Curriculum Review Committee (NCRC).

The following faculty were involved in designing this course guide: Shabana Saeed, GCET (F) Rawalakot; Saima Khan, University of Education, Lahore; Khalid Pervez, GCET Kasur; Dr. Shahid Farooq, IER University of the Punjab, Lahore; Muhammad Zaman, BoC Sindh, Jamshoro; Muhammad Rauf, IER University of Peshawar; Noor Alam, GCET (M) Lalamusa; Shereen Taj, University of Balochistan, Quetta; Zakia Ishaq, GCEE (F) Pishin ; M. Nadeem, RITE (M) DI Khan; Zohra Khatoon, University of Sindh; Shoukat Usmani, GCET (M) Muzaffarabad; Ijaz-Ur-Rauf, GCET Shahpur Sadar, Muhammad Asim, University of Karachi; Rashid Ahmed Noor, RITE (M) Peshawar; Muhammad Rafique, GCET Mirpur; Farjana Memon, GECE (W) Hyderabad; Abdul Khaliq, BoC, Quetta; Muhammad Wasim Uddin, RITE (M) Haripur; Muhammad Afzal, University of Education, Lahore; Gul Muhammad, GCEE Quetta; Shabana Hyder, GECE (W) Hussainabad, Karachi; Dr. Iqbal Majoka, Hazara University; Ibad Ur Rehman, GCET (M) Jamrud; Ghulam Abbass, University of Education, Lahore; Safia Khatoon, GCET(F) Jamrud; Maria Akhtar, Fatima Jinnah Women University, Rawalpindi.

Subject expert guiding course design: Loretta Heuer, Senior Research and Development Associate, Education Development Center (EDC).

Date of NCRC review: 3 March 2012

NCRC Reviewers: Dr. Imran Yousuf, Arid Agriculture University, Rawalpindi; Dr. Tayyab, Foundation University, Islamabad.

Syllabus: General Mathematics

Subject: General Mathematics
Credit value: 3 credit hours
Prerequisite: SSC Mathematics

This course provides opportunities for prospective elementary teachers to strengthen their mathematical knowledge and skills and to gain confidence in their understanding of mathematics. An important outcome of this course is for prospective teachers to be able to teach mathematics successfully in the elementary grades.

Research-based knowledge about good math instruction provides a solid base of information for educators to use as they identify mathematics skills students need to develop, as well as teaching strategies and instructional approaches that best support the development of these skills. The course is designed based on what research tells us about good math instruction.

The overall organization of the course is divided into four units:

1. Number and Operations
2. Algebra and Algebraic Thinking
3. Geometry and Geometric Measurement
4. Information Handling

Each unit of study has a consistent design or organization and is meant to maximize time on learning for prospective teachers.

1. **Content:** Most one hour sessions will begin working on a math problem. Prospective teachers will engage in solving and discussing a math problem and sharing approaches and solutions. The content has been developed to so that prospective teachers will engage in mathematics *in depth* to help them connect concepts within and across the four units.

2. **Pedagogy:** In each lesson prospective teachers will actively engage in doing mathematics in order to experience approaches to teaching and learning math that they can use when they teach. They will recognize that there are often multiple ways of approaching a problem and in some instances more than one correct answer. The instructor will present questions that stimulate curiosity and encourage prospective teachers to investigate further by themselves or with their classmates.

   The course will also examine how children learn and develop mathematical understanding and skills and how the way children think influences the teaching of mathematics in the primary, elementary, and middle grades.

3. **Assignments:** Students are expected to continue learning about math and the teaching of math after class. There will be assignments to stretch prospective teachers content knowledge and to learn more about teaching math. Assignments will take many forms including independently solving math problems and school based tasks.

In summary, the General Mathematics course is a comprehensive effort to build and deepen maths content knowledge, to learn and use high-quality instructional practices, and to study ways in which young students approach and learn mathematics.

Course outcomes:

Students will:

- Increase their mathematical content knowledge for Number and Operations, Algebra and Algebraic Thinking, Geometry and Geometric Measurement, and Information Handling for teaching in the primary, elementary, and middle grades
- Increase their confidence, competence, interest, and enthusiasm for mathematics by exploring and doing mathematics
- Deepen an understanding of how children learn mathematics
- Build a variety of instructional techniques with clear purposes
- Enhance their use of questioning techniques to elicit children's understanding
- Learn ways to engage students in mathematical thinking through interactive activities

Semester Outline

Unit 1: Numbers and Operations (5 weeks/15 hrs)

The prospective teacher will:

- Differentiate between various types of numbers in our number system
- Know various models for arithmetic operations (addition, subtraction, multiplication and division) with natural numbers, rational numbers, and integers
- Understand Base-10 place value as it relates to natural numbers and eventually to decimals
- Be able to describe the relationship among and between fractions, decimals, ratios, rates, proportions, and percentages

Week #	Themes	Sub themes
1	Numbers and Operations	• Counting • Models for Addition & Subtraction with natural numbers • Addition and Subtraction as inverse Operations • Word problems involving addition and subtraction
2	Place Value Numbers and Operations	• Working in the base-10 system • Models for Multiplication with natural numbers • Multiplication and Division as inverse operations • Models for Division with natural numbers • Nature of the remainder in division • Factors, Prime and Composite Numbers
3	Fractions and Decimals	• Models of fractions (sets, number line, area, volume) • Types of fractions (proper, improper and mixed-number) • Decimals as fractions linked to base-10 place value • Concept of GCF and LCM • Operations with fractions and decimals
4	Percent Ratios and Proportion Rates	• Percent as related to fractions and decimals • Ratio and Proportion • Rates
5	Integers	• Integers, Operations with integers • Venn Diagrams

Unit 2: Algebra (4 weeks/12 hrs)

The prospective teacher will be able to:

- Describe the connection between Arithmetic and Algebra
- Identify the repeating and/or increasing unit in a pattern and express that pattern as a rule
- Understand what variables are and when and how variables are used
- Express algebraic relationships using words, tables, graphs, and symbols
- Use order of operations to solve for unknowns in algebraic equations

Week #	Themes	Sub themes
1	Algebra as Generalized Arithmetic Patterns	• Repeating patterns and growing patterns • Generalizing a pattern and finding a rule
2	Algebraic terminology, the concept of x as a variable, coordinate graphs, multiple representations, the concept of identity	• Creating coordinate graphs • Continuous, discontinuous, and discrete graphs • Equivalent expressions
3	Linear functions Order of Operations	• Interpreting tables, graphs and equations of linear functions • The concept of slope • Order of Operations
4	Square expressions and equations Symbol manipulation	• Interpreting tables, graphs and equations of quadratic functions • Solving for x, the unknown

Unit 3: Geometry and Geometric Measurement (5 weeks/15 hrs)

The prospective teacher will:

- Understand undefined terms in geometry
- Identify and construct different types of angles.
- Identify characteristics and measurable attributes of 2-dimentional figures and 3-dimentional objects
- Calculate area, perimeter, surface area, and volume
- Understand square numbers, square roots, and the relationships involved in the Pythagorean Theorem

Week #	Themes	Sub themes
1	Polygons	• Characteristics of Polygons with an emphasis on Triangles and Quadrilaterals,
2	Undefined terms in geometry Identification and construction of angles	• Point, line, line segment, ray • Models of angles • Benchmark angles • Classifying angles by measurement
3	Geometric Measurement: Area and Perimeter of polygons	• Perimeter and Area formulas

4	Geometric Measurement: Circumference and Area of Circles Surface Area of Cuboids and Cylinders	• Circumference and Area formulas • Surface Area formulas
5	Volume of Cuboids and Cylinders Introduction to the Pythagorean Theorem	• Volume formulas • Squares, square numbers, square roots (surds) • The Pythagorean Theorem

Unit 4: Information Handling (2 weeks/6 hrs)

The prospective teacher will:

- Recognize and construct various types graphs
- Determine which types of graphs best describe a given situation
- Analyze a graph and interpret its information
- Understand different measures of central tendency and determine which best describes a given situation

Week #	Themes	Sub themes
1	Graphic displays of information	• Collect & organise data via: tally marks, pictographs, line plot, bar graph, and line graphs (discrete and continuous) • Interpret the above graphic displays of data
2	Measures of dispersion and central tendency	• Range • Mean • Median • Mode

Course Grading Policy
A variety of assessments will be used to assign a final grade. It is recommended that course work be used to assign at least 50% of the final grade.

Your instructor will tell you at the start of the course how your final grade will be determined and which pieces of course work will be assessed.

Suggested Resources:

These resources provide additional information about maths education and the mathematical topics addressed during the course.

NCTM *Illuminations*http://illuminations.nctm.org/

New Zealand's Maths Currriculum: http://nzmaths.co.nz/

UK's N-Rich Maths site: http://nrich.maths.org/public/

How Students Learn: History, Mathematics, and Science in the Classroom www.nap.edu/catalog.php?record_id=10126#toc Published by National Academies Press.

What does Good Mathematics Instruction Look Like?: http://www.naesp.org/resources/2/Principal/2007/S-Op51.pdf

Mathematics for Elementary School Teachers, by Tom Basserear, published by Brooks Cole.

Elementary and Middle School Mathematics: Teaching Developmentally, by John A. Van de Walle, Karen Karp, and Jennifer Bay-Williams, published by Pearson Education.

Mathematics Explained for Primary Teachers, by Derek Haylock, published by SAGE Publications.

Faculty Notes
Unit 1 Number and Operations
Week 1: Addition, Subtraction, Equivalence

Weeklong Overview:

Session 1: Models for Addition
Session 2: Models for Subtraction
Session 3: Equivalence, Thinking Like Children

Faculty Preparation for Upcoming Week (1-2 hours)

Read the following articles and look through the following websites that address addition, subtraction, equivalence, and the way children think about mathematics:

- Addition and Subtraction in the Primary Grades http://tinyurl.com/AddSubtractPrimary
- "Children's Understanding of Equality: A Foundation for Algebra" Falkner, K., Levi, L., & Carpenter, T. Teaching Children Mathematics, Vol. 6, No. 4, Dec. 1999. http://ncisla.wceruw.org/publications/articles/AlgebraNCTM.pdf Can also be accessed at http://tinyurl.com/Children-Equality
- Models of addition and subtraction:
- Addition: http://tinyurl.com/IllumAddition
- Subtraction: http://tinyurl.com/IllumSubtraction
- The "balance" model for addition and subtraction using a ruler, pencil, and paperclips: http://tinyurl.com/IllumBalance

- Cognitively Guided Instruction (CGI):
- http://tinyurl.com/CGI-Joining
- http://tinyurl.com/CGI-Comparison
- http://tinyurl.com/CGI-Separate
- http://tinyurl.com/CGI-Joining-2

  Download and print out for student use:
  - Number Line 0-24: http://tinyurl.com/Number-Line-1-to-24
  - CGI Frameworks 3 pages): http://tinyurl.com/cgiFrameworks
  - Subtraction (Take Away, Appropriateness of Visual Representation): http://budurl.com/SubtractionCookies
  - Addition (Near Doubles): http://budurl.com/AdditionCandy

Read through the plans for this week's three sessions

The Number and Operations Unit begins with one equation (5 + 7 = 12) that students will explore in depth during all three sessions this week.

During Session 1, the emphasis will be on something that most of us adults take for granted: simple addition.

This idea will echo throughout this entire course: That while pre-service teachers need to understand maths as adults, they also need to understand mathematics <u>through the eyes of children</u>.

This session is designed to challenge students to think not about what they already know (such as number facts), but how children might begin thinking about addition. Thus, as basic as it seems, the first topic discussed in Session 1 is the strategy young children use when beginning to add: counting, "counting on," and even counting on their fingers as a readily available device for sums through 10.

The second topic for Session 1 is how a given number can be decomposed. For example, 12 can be expressed not only as 5 + 7, but 7 + 5 (which leads to children's becoming aware of the commutative property of addition at a very early age). But 12 can also be decomposed into doubles (6 + 6), more than two addends (5 + 4 + 3), and as a way to introduce place value (10 + 2).

The third topic for this session is that of four models for addition: joining sets, "counting on," moving forward on a number line, and balance/equivalency--which will be developed more fully in the third session of this week.

Session 2 will build on what students have just learned about models for addition to introduce models for subtraction. Besides adapting the above four addition models for subtraction: separating a sub-set, "counting back," moving backward on a number line, and balance/equivalency, a fifth model, comparison, is included. An example of this is "I have 3 brothers and 2 sisters. How many more brothers than sisters do I have?" In this situation, children need to deal with one-to-one correspondence and see how many "match up" and how many are "left over" (which is the solution to the problem).

This session will end with assigning two readings for students to do as homework. The first is an article on how children interpret the equals sign. The second is an introduction to Cognitively Guided Instruction, a rigorously researched method to organize the addition and subtraction models already discussed. Both articles will be used as starting points for discussion in Session 3.

Unit 1 Number and Operations
Week 1, Session 1: Addition

1. What are the important concepts?

a) Most students probably think of addition as "joining sets." However, there are three other models: 1) counting, 2) using a number line, and 3) the balance model, which will be developed more fully in the third hour of this week with an emphasis on the concept of equivalence. For more information on these different models, check out: http://tinyurl.com/IllumAddition

b) Decomposition of numbers. Young children's decomposition of a number into new combinations is the foundation for the associative and commutative properties of addition. It also allows students to build their number sense by creating alternative ways of thinking about a number.

2. How do children think about these concepts?

a) When thinking about the addition model of counting, young children move from "counting from one" to "counting on." These two short handouts explain the difference in young children's intellectual development when adding:

Counting from One: **http://tinyurl.com/Counting-From-One-NZ**

Counting "On": **http://tinyurl.com/Counting-On-NZ**

b) When using the number line model young children are often unsure if they should begin counting from 0 or from 1. With sets they need to begin at 1, but when using a number line, they need to begin at 0 and count "jumps." Eventually they will begin to notice that they can start at 5 and make 7 "jumps" (counting "on") to arrive at 12.

c) When decomposing numbers, as in the case of $5 + 7 = 12$, how might the 5 and the 7 be decomposed, then "recomposed" into new addends? Some children may be "looking for 10" and create $5 + (5 + 2) = 12$. Others may be thinking about doubles: $(5 + 1) + 6 = 12$. In either case children are beginning to manipulate numbers and develop their number sense. Later on, they will feel comfortable decomposing 2-digit numbers into tens and units, and eventually when working with multiplication, they will realize that they can decompose a number into its factors.

d) When asked to decompose a number (such as 12) young children often do this in a random manner: $8 + 4$, $3 + 9$. Although the teacher should record the children's responses as they are given, the next step is to enquire how their responses might be organized. Helping children rearrange their responses into an organized list will allow them to see a pattern, which will help them generalize a number's decomposition and build their number sense.
http://tinyurl.com/Decomp-Assess-NZ

3. What is essential to know or do in class?

a) Introduce the four models for addition
b) Explain decomposing (and then recomposing) numbers
c) Relate each of the above to children's thinking

4. Class Activities

a) Begin by asking students in pairs (or groups of no more than four) to brainstorm for five minutes about all the mathematics that may be implicit in the equation 5 + 7 = 12. Ask students to share their thoughts while you record the ideas on chart paper for future reference. After this activity, note with a check mark ideas that will be discussed during the rest of the week. Congratulate students for their willingness to go beyond their first impression and delve more deeply into the mathematics! (Add to the list any topics for the week that students did not mention.)

b) Introduce the four models for addition: joining sets, counting on, "hopping" forward on a number line, creating equivalence.

c) Give students the following scenario and ask them to "count on" by using their fingers (as young children would) to solve this problem. "I get up early and eat breakfast at 5 in the morning; 7 hours later I eat lunch. When do I eat lunch?" Ask how "counting on" is somewhat different from "counting from 0."

d) Distribute the number line handout and have students use the number line to add 5 + 7. Notice which students started from 0, hopped to 5, and then hopped 7 more places? Which students began at 5 and "counted on." Relate these two different ways of modeling the problem to the "starting from 0" and the "counting on" methods. (Also note that young children are often unsure about whether they should begin counting from 0 or from 1.) Have students think of a ruler as a model for a number line, one that includes not only whole numbers, but also fractions or decimals. How might they use rulers to help children think about adding (and later subtracting)?

d) Introduce the concept of decomposing a number.

e) Have students solve the problem: “In my apartment building there are 12 cats and dogs. How many might there be of each?” Ask for random answers and record these on chart paper. When all combinations have been given, have students create an organized list of the results in their notebooks. Discuss why helping young children organize mathematical information is an important step in noting patterns, not just for computation, but later on for patterns in algebra function tables (a + b = 12). Mention that they will use this chart in the next session to discuss subtraction.

5. Assignment (to be determined)

Unit 1 Number and Operations
Week 1, Session 2: Subtraction

1. What are the important concepts?

a) Just as students may have assumed that addition was limited to joining sets, they may consider subtraction only as "taking away." As with addition, however, subtraction can be conceptualized in four other ways: 1) counting backwards, 2) moving backwards on a number line, 3) the balance model, which will be developed more fully in the third hour of this week with an emphasis on the concept of equivalence, and 4) comparison. For more information on these different models, refer to: Models for Subtraction: http://tinyurl.com/IllumSubtraction

b) Linking the operations of addition and subtraction allows students to clarify their understanding of how these two operations are each other's inverse, which will "undo" what the other operation just "did." (The concept of the inverse will also be developed further when multiplication and division are discussed.) This also is be a good time to review the charts students created for the "7 cats and dogs" activity. How could the table, which originally charted addition, become a visual reference for subtraction?

2. How do children think about these concepts?

a) When using the comparison model for subtraction, children see the total number of items in both sets, e.g., 11 paperclips, 3 large and 8 small. But they need to deal with the items in those sets via one-to-one correspondence in order to discover which set contains more, which contains fewer, and what the "difference" is between the two. Again consider the "7 cats and dogs" activity from the vantage point of the question: "How many more?": I have 2 dogs and 5 cats. Of which do I have more? How many more? The answer in the comparison model, although numerically the same is substantively different from the "remainder" which is the answer to the "taking away" model. (There were 7 birds on a tree. Four flew away. How many remained?)

3. What is essential to know or do in class?

a) Introduce the five models for subtraction.
b) Relate subtraction to the decomposing (and then recomposing) of numbers that students did in the prior session on addition.
c) Link subtraction to addition by discussing how these two operations are inverses that "undo" each other.
d) Relate each of the above to children's thinking.

4. Class Activities

a) Begin by asking students to solve the equation 8 - 3 = ? by counting backwards. Suggest that they solve this problem as young children (who did not know the answer) asking them to count backwards to find the solution: use their fingers. Notice which students decompose the 8 into 4 fingers on each hand and which students decompose the 8 into 5 fingers on one hand and 3 on the other.

b) Have students recall the models for addition and introduce their counterparts in subtraction: 1) joining sets (taking away a set), 2) counting forward (counting backward), 3) hopping forward on the number line (hopping backward).

c) Introduce the idea that these three models highlight the idea of addition and subtraction being inverse operations where addition "undoes" subtraction and vice versa.

d) Have students experiment with the number line model for subtraction, “hopping backwards” from the starting number (the minuend) to find the answer to 8 - 3 = ?

e) Have students refer to the organized chart they created for the problem: “In my block of flats there are 12 cats and dogs. How many might there be of each?” Ask how their chart, which was used to model addition, can be used to describe subtraction.

f) Pose the following questions: “If there are 7 cats, then the remainder are dogs. How many are dogs? If there are 5 cats, then the remainder are dogs. How many are dogs?” Mention to students the use of the word “remainder," which implies separating a total quantity into two sets.

g) Use this work with the set model to introduce a fifth subtraction model (which did not have a corresponding model in the Addition session): comparison.

Use the 2-column organized chart saved from the prior session, adding a column to the right labeled “Difference," in order to begin working with the subtraction model of comparison. As students consider the chart, ask, “Which has more? How many more?” as you add entries into the "Difference" column. After the chart is complete ask students:

- Do you see any patterns as you look at the differences? (do you mean Difference?)
- Is there a symmetrical pattern? If so, why?
- Is there an odd or even pattern? If so, why?
- What does the chart show about the commutative property of addition?
- What does the chart show about the inverse operations of addition and subtraction?

h) To end the session, have students consider the numbers 13, 7, and 6, and have them create two *real life* subtraction scenarios:

- One where the subtraction model is "take away" and results in a remainder
- The other where the scenario involves a comparison, and results in a difference

5) Assignment

To prepare for the next class session have students read:

a) “Children’s Understanding of Equality: A Foundation for Algebra." Falkner, K., Levi, L., & Carpenter, T. Teaching Children Mathematics, Vol. 6, No. 4, Dec. 1999.
http://tinyurl.com/Children-Equality

Number and Operations
Unit 1 Number and Operations
Week 1, Session 3: Equivalence, Thinking Like Children

1. What are the important concepts?

a) The notion of Equivalence is one of the major, overarching concepts in all of mathematics. However, both adults and children often misinterpret the equals sign in an equation as meaning “the answer is….” rather than understanding it to be an indicator of the important mathematical notion: that there is a balance, an equivalence, on each side of the equation.

b) Cognitively Guided Instruction (CGI) word problems help teachers codify children's understanding of addition, subtraction, and the relationship between these two operations.

Recall the various models of addition and subtraction that were discussed in the prior sessions. Then consider the equation 3 + 4 = 7. Here are several ways this equation could be translated into CGI word problems:

- I have 3 raisins and get 4 more. How many do I have now? (This would be an example of a "joining" problem, where the RESULT (7) is unknown.)
- I have 3 raisins and get "some more." Now I have 7 raisins. (This would be an example of a "joining" problem where the CHANGE (+ 4) is unknown.)
- I have "some" raisins. I get 4 more and now have 7. (This is the third type of "joining" problem, only in this case the START (3) is unknown.)

c) One of the most useful ways of assessing students' understanding is analyzing their work samples.

2. How do children think about these concepts?

a) Especially important is a comment in the article CGI Newsletter: Other Types of Joining Problems: "This type of problem illustrates a *difference between child and adult thinking*. Adults would identify this as a subtraction problem but children do not. Children see this as a problem requiring a joining action."

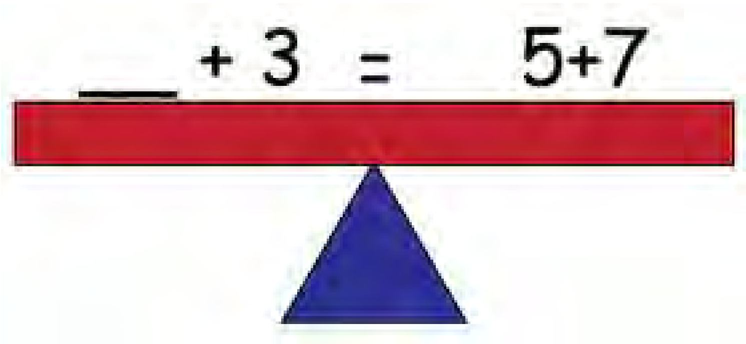
b) Misinterpretation about the equals sign is why many young children looking at the open equation 3 + ? = 7 and immediately conclude that the answer is 10. They see a 3, a 7, a plus sign, an equals sign, and then mentally rearrange those four elements into the more familiar 3 + 7 = ? where the answer is indeed 10.

c) Until children have a clear understanding of number sense combined with operations sense, we can expect them to assume that the answer for the ? in the equation 3 + ? = 7 is probably 10.

Pre-service teachers need to realize that it takes time, several models, and many hands-on experiences to help their future primary grade students coordinate number sense, operation sense, and computational fluency until they finally realize that for the equation 3 + ? = 7 the unknown is 4, not 10.

d) The following websites include the balance model for addition and subtraction
(http://illuminations.nctm.org/LessonDetail.aspx?ID=L55)
(http://illuminations.nctm.org/LessonDetail.aspx?ID=L40)
and describe how teachers can guide children's thinking toward understanding equivalence across the equals sign.

e) If a commercial balance is not available, the concept of balanced equations can be described to children as young as first grade by having them consider a teeter-totter or see-saw. Have children relate this visual model to an open addition sentence such as:



f) Children need to make generalizations about addition, subtraction, and equivalence that apply to many situations. This is why having several different models for these operations of addition and subtraction is important. However, the teacher's role is not only to introduce these models, but also to show how they are related to each other and how they are connected to the inverse operation.

g) Children eventually need to develop computational fluency, recalling their number facts with *automaticity*--which is not the same as memorization. For example, when asked for the sum of 5 + 6, a child may not have the answer (11) memorized, but he or she may quickly use the concept of "near doubles," recalling that since 5 + 5 equals 10, 5 + 6 must equal 11. Similarly, for any equation that involves subtracting 2, the child may quickly use "counting back 2" or visualizing hopping back 2 spaces on the number line. Thus, a child who has internalized these strategies and has learned about place value should be able to find the answer to 35 - 2 = ? Perhaps even more important, they can use this strategy to solve equations that bridge a decade, such as 31 - 2 = ? without resorting to "regrouping."

3. What is essential to know or do in class?

a) Introduce the concept of equivalence by using the model of a balance.
http://tinyurl.com/IllumBalance

b) Clarify the meaning of the equals sign.

c) Review and clarify participants' understanding of Cognitively Guided Instruction after they have read for homework about the different models for Result, Change, Start.

d) Have students compare and analyse two children's work samples as they begin to assess children's understanding of addition and subtraction.

4. Class Activities

a) Begin by asking students to describe their understanding of the equals sign. What does it mean to them? What do they think the equals sign means to young children? Give two examples: 5 + 7 = ?, and 5 + ? = 12. Note that for adults, these equations "look the same." But this is not necessarily true for children. How do students think children will solve these equations?

Introduce the balance model for addition and subtraction, demonstrating 2 + 3 = ? by drawing a balance beam like the one above on chart paper or the white board. Next, draw the balance model of for the equation 2 + ? = 5. Ask students how they could balance on both sides of the equation.

b) Distribute paper copies of the introduction to Cognitively Guided Instruction (CGI); students should have read to the introduction in an electronic format from the Internet to prepare for the this class session.

Ask students to describe their understanding of the different CGI models for addition and subtraction (Result, Change, Start), asking other students to help clarify areas of confusion. Ask them about page 3 of the handout, where the difficulty level of different types of problems are rated. Why do they think this is so? How does this relate to the different models of addition and subtraction they have studied this week.

Have students work in pairs or small groups using the basic equation 5 + 7 = 12 to generate word problems for Result, Change, Start. Make sure they create a story that illustrates Comparison. Have several students share their word problems, explaining why each story and its equation fit a particular CGI category.

c) End the session by distributing the two children's work samples: 1) Subtraction (Take Away, Appropriateness of Visual Representation) and 2) Addition (Near Doubles). Have students work in groups of three or four to compare, analyse, and discuss what they perceive about the children's understanding. Ask them to consider: Which models of addition and subtraction are being used? What can they infer about the child's understanding? How would they compare these two work samples with regard to mathematical sophistication?

5) Assignment

a) The article A Diagnostic Model: Working to Analyse Children's Thinking in Addition and Subtraction http://tinyurl.com/Diagnostic-Ch-Think asks teachers to consider children's thinking about addition and subtraction in depth, going beyond whether they have gotten the right or wrong answer. These questions allow for formative assessment, noticing what needs to be done to help a particular youngster or the class as a whole. The list also serves as a model for how teachers can develop their own list of questions for other mathematical topics, such as multiplication, fractions, area and perimeter, etc.
b) Review the following four samples of student work:

- Addition (Joining Sets, Figurative not Realistic Drawings): http://budurl.com/AdditionButterflies
- Addition (Joining Sets, Two Items Grouped to Reflect the Story, Not All Items in Set are Shown): http://budurl.com/AdditionCats
- Subtraction (Take Away, Erasures, Appropriateness of Story's Context): http://budurl.com/SubtractionFish
- Subtraction (Take Away, Many Drawings, Additional Note at the Bottom): http://budurl.com/SubtractionParrot

- Which addition and subtraction models are being used?
- What can you infer about each child's understanding?
- How would you rank these papers with regard to mathematical sophistication?

A Diagnostic Model: Working to Analyse Student Thinking in Addition and Subtraction Loretta Heuer

The following questions are designed to assist teacher reflection when analyzing student thinking about whole number operations:

1. Questions about student thinking - models for addition, different types of addition problems:

- When joining sets, do they need to begin counting at one, or can they hold the image of a number and "add on?"
- Can they use 10-frames to identify patterns of 5, 10, and ones?
- When using a number line, do they count "hops" rather than points, and relate the process to an addition equation?
- Can they use a balance to model equivalence and then write an equation to represent their work?
- Can they show an understanding of equivalence by generating several addition equations to describe the same number (e.g., 4 + 2 = 6, 3 + 3= 6, 5 + 1 = 6, 6 + 0 = 6)?

2. Questions about student thinking: models for subtraction, different types of subtraction problems:

- When subtracting, do they begin by counting the whole, or can they hold an image of the whole and "count back?"
- Can they use models of subtraction other than that of "taking away?"
- Can they use a number line to model subtraction by "jumping back" and relating the action to a subtraction equation?
- When comparing two sets, can they describe the difference as "more" or "fewer," and relate that to a subtraction equation?

3. Questions about students' *strategic* thinking: use of patterns, tools, derived facts:

- Can students generate "fact families" that show the relationship between addition and subtraction?
- Do they notice patterns such as doubling, "near doubling," "making ten," or
- adding/subtracting 1 or 0?
- Are they beginning to use the commutative (order) property of addition?
- Can they use a hundreds grid or number line to show addition by counting on"?
- Are they able to create and use an addition chart for the sums of 0 + 0 through 9 + 9?
- Can they see patterns in their addition chart that make remembering their facts easier?
- Which number facts are *easy* for your students to remember? Why do you think that is so? What patterns or strategies might help them become more efficient?
- Which number facts are *hard* for your students to remember? Why do you think that is so? What patterns or strategies might help them become more efficient?

Faculty Notes
Unit 1 Number and Operations
Week 2: Place Value in Base 10, Multiplication and Division of Whole Numbers

Weeklong Overview:

Session 1: Place Value for Whole Numbers in Base 10
Session 2: Multiplication of Whole Numbers
Session 3: Division of Whole Numbers

Faculty Preparation for Upcoming Week (1-2 hours)

Read the following articles/review the following websites that address place value, multiplication, and division:

http://tinyurl.com/ThinkMath-Mult
http://tinyurl.com/ThinkMath-Mult-Div
http://tinyurl.com/ThinkMath-MultVsAdd

Download and print out for student use:

Number Line 0-24: http://tinyurl.com/Number-Line-1-to-24
Ten Frame Mats: http://tinyurl.com/TensFrames
Hundred Chart: http://tinyurl.com/Chart-100
Multiplication Table: http://tinyurl.com/Mult-Chart-144

Bring to class:

A package of beans to be used as counters
Graph paper

Read through the plan for this week's three sessions

Session 1 begins addressing place value by using the equation from last week, 5 + 7 = 12. Session 1 then uses another equation (6 x 3 = 18) that students explore in depth in order to study the concepts of factors, multiples, multiplication, and division.

During the first session the students will focus on three things: 1) place value, 2) visual models for understanding place value, and 3) thinking about mathematical strategies for solving equations, which includes the following three issues to emphasize:

a) Although all equations are "equal," not all equations are equally *useful* as teaching tools. Selecting the equation 5 + 7 = 12 for exploration last week was a conscious instructional decision because the equation connects to the concept of place value this week. This connection indicates how teachers need to consider the implications of the numbers they use when using exemplars to introduce maths topics.

b) Many maths problems have "one right answer," such as 5 + 7 = 12. There are, however, different strategies that children might *use* to arrive at that "one right answer." Some of their strategies may seem more or less efficient to us adults. But efficiency is an end goal, once mathematical understanding is in place.

c) Valuing and discussing alternative solution strategies is an important way to help students make mathematical connections and see mathematical equivalency among different solution strategies. (This will become even more evident when considering proofs in algebra and geometry, which is why it is important to include discussions on alternative solution methods even at the beginning of the Number and Operations unit.)

Session 2 will address multiplication of whole numbers. Just as we began last week with a single equation to be explored in depth for its potential regarding addition and subtraction, we will do the same for multiplication and division, using the basic equation 6 x 3 = 18. This session will introduce several models for multiplication: array, intersections, area, partial products, and skip-counting.

Session 3 will introduce division of whole numbers as the inverse of multiplication. (This parallels the idea of addition and subtraction as inverse operations that "undo" each other.) Models for division of whole numbers will be introduced, and the issue of the remainder will be discussed as it relates to a problem's context.

Unit 1 Number and Operations
Week 2, Session 1: Place Value in Base 10

1. What are the important concepts?

a) Although all equations are "equal," not all equations are equally *useful* as teaching tools. Selecting the equation 5 + 7 = 12 for exploration last week was a conscious instructional decision, because the equation connects to the concept of place value this week. This connection indicates how teachers need to consider the implications of the numbers they use when using exemplars to introduce maths topics.

b) Many maths problems have "one right answer," such as 5 + 7 = 12. There are, however, different strategies that children might use to arrive at that "one right answer." Some of their strategies may seem more or less efficient to us adults. But efficiency is an end goal, once mathematical understanding is in place.

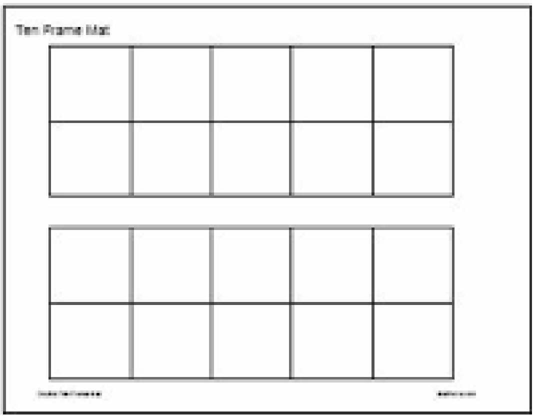
c) Valuing and discussing alternative solution strategies is an important way to help students make mathematical connections and see mathematical equivalency among different solution strategies. (This will become even more evident when considering proofs in algebra and geometry, which is why it is important for children to discuss alternative solution methods even in the Number and Operations unit.)

d) When young children need to go beyond the number 9 in counting and addition, they move toward multi-digit number sense, encountering the concept of tens and units.

e) There is a major difference between digits and numbers. For example, the "1" in the number 13 is simply a digit representing the number "10."

f) Young children can begin working with place value by using handfuls of small objects such as pebbles, beans, etc., arranging them into groups of 10 with perhaps some left over. At some point, however, the number of physical items becomes unwieldy, suggesting other models would be better, more efficient ways to further concept development.

g) To help children develop multi-digit number sense, two simple “ten frames” can be used to model why we need to use two digits to describe the number “12” in our base-10 number system, since each frame can only accommodate 10 items. Ten Frame Mats: http://tinyurl.com/TensFrames



h) The number line model, used earlier for simple addition, allows children to "count on" once they reach the end of a decade, such as when adding 28 + 3. Without much effort children will discover that they have moved from the 20s into the 30s. This model helps children avoid the assumption that they need to use an algorithm in order to solve this type of "bridging the decade" problem.

i) A hundred chart (which is really ten 1-10 number lines "stacked" upon each other), allows youngsters to model numbers from 1 to 100 (or 0 to 99) The hundred chart also permits children not only to move back and forth across *rows* to add and subtract units, but also to move up and down *columns* to add or subtract 10s. Hundred Chart: http://tinyurl.com/Chart-100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

j) When working with a Hundred Chart, teachers need to decide if they will use one that begins with 0 and goes to 99 (each row on the chart beginning with the same "tens number": 0-9, 10-19, 30-39, 90-99) or one that begins with 1 and goes through 100, with each row ending with the "next ten": (1-10, 11-19, 31-40, 91-100).

k) The basic number sentence 5 + 7 = 12 can also be transformed into 5 tens + 7 tens = 12 tens, or 50 + 70 = 120, moving beyond 100. That same 5 + 7 = 12 equation can also be used as a way for children to use patterns to use mental addition to solve 25 + 7, 35 + 7, 65 + 7, all without using a formal algorithm.

2. How do children think about these concepts?

a) When teaching children about place value in base 10, it is not only important to use a variety of physical and visual models, but also to help youngsters make connections among those various models. Sorting a pile of beans into groups of ten with "leftovers" (units) may come first. Next, small sticks, straws, or toothpicks can be "bundled" into groups of ten with a rubber band.

b) Once children have become comfortable with physical models they can use the Ten Frames as a way to organize items where there is 10 with some left over (e.g., 5 + 7 = 10 + 2). They do this first by actually placing small items on the squares in Ten Frames, and then they move from the "real" to the "representational" by putting dots into the Ten Frames to represent the 12 items.

This is a key opportunity to highlight that when children "bridge a decade" as in 5 + 7 = 12 there is a developmental sequence that occurs in chilren's thinking when they are given equation containing numbers and symbols: 1) a CGI type story context that explains the equation ("There were 5 ducks in a pond; there were 7 ducks on the shore. How many ducks in all?"), 2) the physical (the beans), 3) the representational (placing dots in Ten Frames), and 4) once again the symbolic (5 + 7 = 12).

c) More importantly, this sequence of how youngsters think about, understand, and learn mathematics will be repeated not only in Numbers and Operations, but in other areas of maths, especially algebra: 1) narrative, 2) physical, 3) representational (which will later include tables and graphs), and finally 4) the symbolic.

d) After children become familiar with the one-dimensional linear model of the number line from 0 through 20, they can begin to transfer this visualization to the two-dimensional Hundred Chart, which is simply 10 number line segments reorganized into a stacked, more compact, manageable format.

e) Children learn not only what they are taught, but what they "see" in their classroom. This is why it is important to have a number line from 0 through 100 (and later, from -20 to 120) prominently displayed in the room. The same is true for a Hundred Chart. These are mathematical references that children can use on a daily basis to solve maths problems. But more importantly, visuals seen day-after-day will help develop children's "number sense," transforming those visual models into *intellectual* models that will become a child's internal mental reference for understanding and working with mathematics.

3. What is essential to know or do in class?

a) Introduce the base-10 number system for whole numbers, linking this to the concept of decomposing two-digit numbers such as 12 into 10s with additional units.
b) Provide students with three visual models to help them think about how children can understand place value: the number line, Ten Frames, and the Hundred Chart.
c) Relate each of the above to children's thinking.
d) Mention that the base-10 number system does not relate only to whole numbers, but that it extends to negative numbers and decimals, both of which will be discussed later in the Number and Operations unit.

4. Class Activities

a) To introduce the concept of place value in our base-10 number system, distribute copies of the Ten Frames and a handful of beans. Explain the Ten Frame model, and then have students work in pairs to model the equation 5 + 7 = 12. After they have done this have a class discussion to discover how students used the Ten Frames to solve the problem.

Anticipate that some student may have placed 5 beans on one Frame, 7 beans on another, and then moved 5 of the beans from the second frame onto the first, which resulted in one Frame with 10 beans and another with 2.

Other students may have used the counting principle: Placing 5 beans on the first Frame, then taking 7 beans and distributing 5 of them onto the first Frame until it was full. At that point they would have placed the remaining two beans onto the second Frame showing that the sum was 12.

Ask if there were other strategies that students used. Ask if there was disagreement among partners as to which strategy was "the right one." Note that both of the above strategies were valid. But each is subtly different.

b) After they have explored ideas for 5 + 7 = 12, have them work in pairs with Ten Frames to find equations with sums that equal 11-20. Challenge students to include not only single digit addends (such as 8 + 3), but equations where one addend has two digits (such as 13 + 6).

In a whole class discussion ask how, for example, they modeled 13 + 6, and what the digits 1 and 3 mean for the number 13 and how the digits 1 and 3 relate to their Ten Frames.

Help students understand that *digits* are representations for parts of a *number*. Thus the digit "1" in the number 13 means the number "10," not the number "1." This distinction between digits and number is a crucial part of children's developing number sense. This distinction of digits vs. number will have major implications when youngsters begin to work with multi-digit calculations.

c) Introduce the Hundred Chart and have students articulate patterns they see. If no one mentions that the Hundred Chart is really a series of number line segments stacked upon one another, bring their attention to this.

d) Have students work with the Hundreds Chart to add multiples of 10 to a given number, perhaps starting with the number 7 and moving down the columns to model 7 + 10, 7 + 20, etc. What do the students notice about the patterns? How would they use the columns and rows in the Hundred Chart if they wanted to add 7 + 22? How is *that* different from adding 7 + 23 or 7 + 24 (where the sum moves into the next decade)?

Ask students what this pattern(?) would look like on a continuous number line from 1-100 where the sum moved to the next decade, such as with 7 + 23 becoming 30, or 7 + 24 becoming 31.

Finally, note the first item in the "What is Important" section above: that whereas all equations are "equal," not all equations are equally *useful* as teaching tools. Ask how selecting 5 + 7 = 12, then 7 + 10, 7 + 20, 7 + 22, and finally 7 + 23 and 7 + 24 built mathematical understanding better than using random equations.

Point out that the idea of patterns will be developed more fully in the Algebra unit, but that young children's work with patterns in Number and Operations is crucial to creating a coherent idea of mathematics as a logical system of thinking.

Emphasize that this is why teachers need to consider the implications of selecting specific numbers when creating exemplars to introduce maths topics.

e) End by mentioning that base-10 place value applies not only to whole numbers, but to negative numbers, decimals, percents, exponents, etc. Base-10 place value is a concept children will meet again and again as they learn mathematics.

5) Assignment

To prepare for the next class, which is on multiplication, have students read these two articles:

- http://tinyurl.com/ThinkMath-Mult
- http://tinyurl.com/ThinkMath-Mult-Div

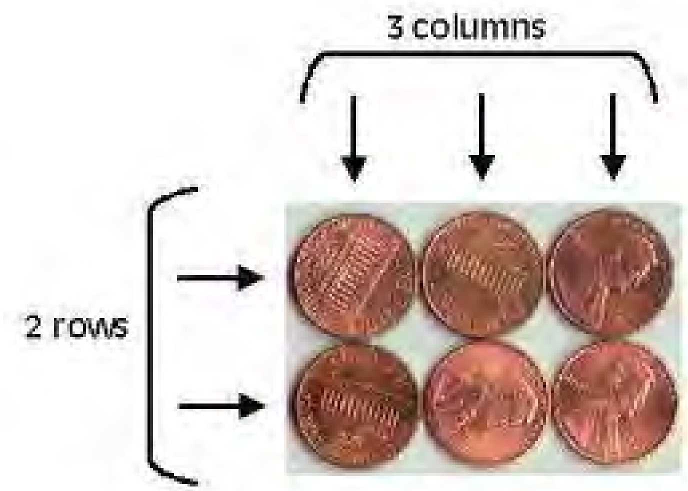
Unit 1 Number and Operations
Week 2, Session 2: Multiplication of Whole Numbers

1. What are the important concepts?

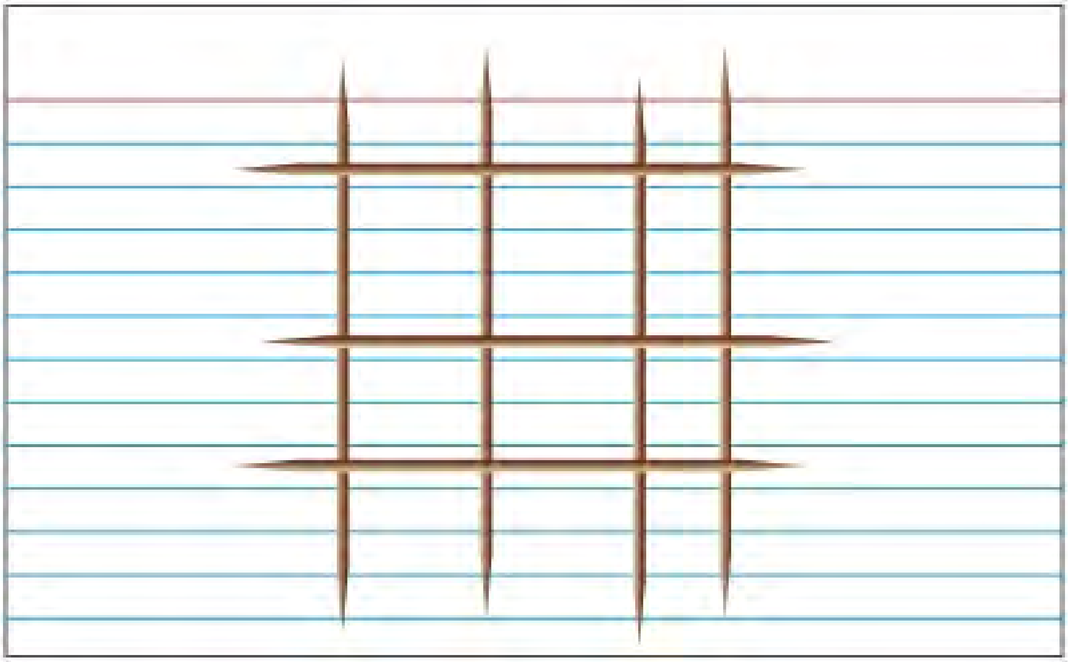
a) Just as addition is more than joining sets, and subtraction is more than "taking away," so, too, multiplication is more than repeated addition. Nor does multiplication always "make things bigger." This is one of the most important concepts to emphasize to your students this week: that what is implied by the multiplication of whole numbers does not hold true for integers, fractions, and decimals.

b) There are several models for multiplication, just as there were for addition and subtraction:

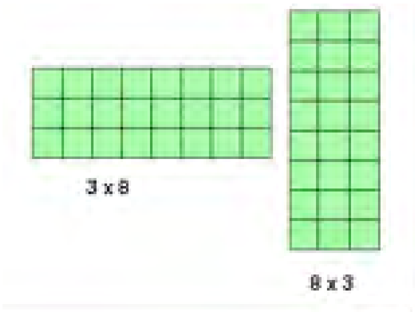
- Array: This is essentially a set model for multiplication, arranging discrete items in rows and columns and then counting to find the answer. Most students will be familiar with this model.



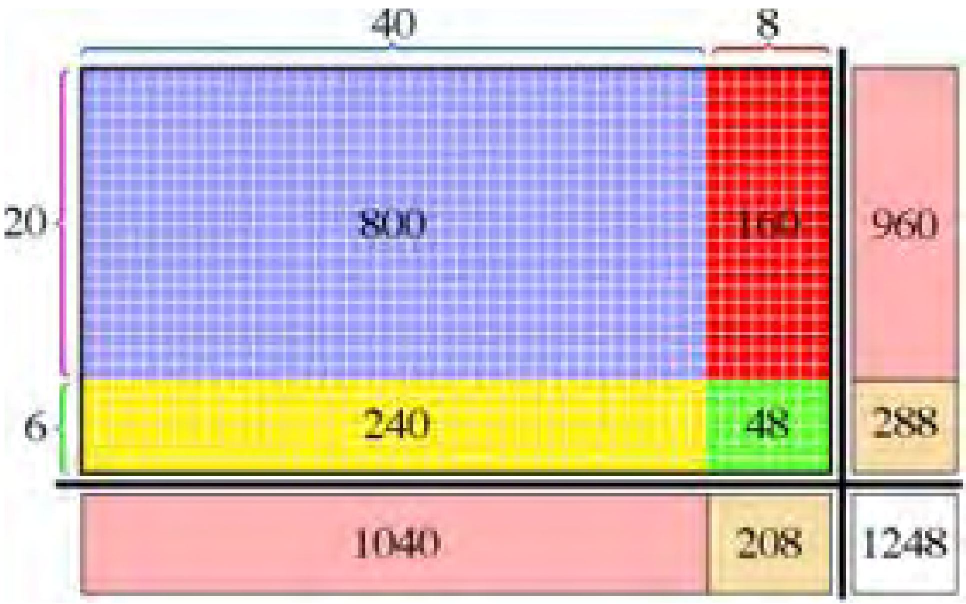
- Intersections: This model, which can be thought of as streets and intersections, uses lines to show multiplication. In the intersection model, the lines represent the factors (in this case 4 and 3) and the sum of the intersections (12) is the product. (This will probably be a new model for both adults and children, and it will usually take their doing several drawings to convince them that this really works.)



- Area: This model sets the stage for several important concepts, including geometric measurement, multiplication of fractions, and prime and composite numbers. Note that this is different from an array shown above in that the area model is continuous space, bounded by the rectangle, whereas the array was composed of discrete objects.



- This second area model, used for multi-digit multiplication and called the "partial products" model, builds on the simple area model described above, but now the factors are decomposed into tens and units. This will also be used as a visual model for multiplication in algebra. Although this picture shows the exact number of cells for the problem (indicating the magnitude of the partial products and solution) a simple hand-drawn set of labeled rectangles can suffice once students understand the process.



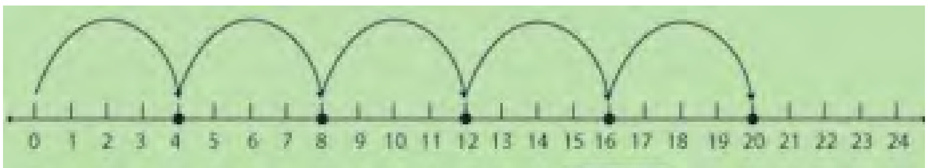
2. How do children think about these concepts?

a) Multiplicative thinking is substantively different from additive thinking, and it develops over time. At first, most young children think of multiplication as repeated addition for sets (2 eyes per person, 4 legs per elephant, etc.).

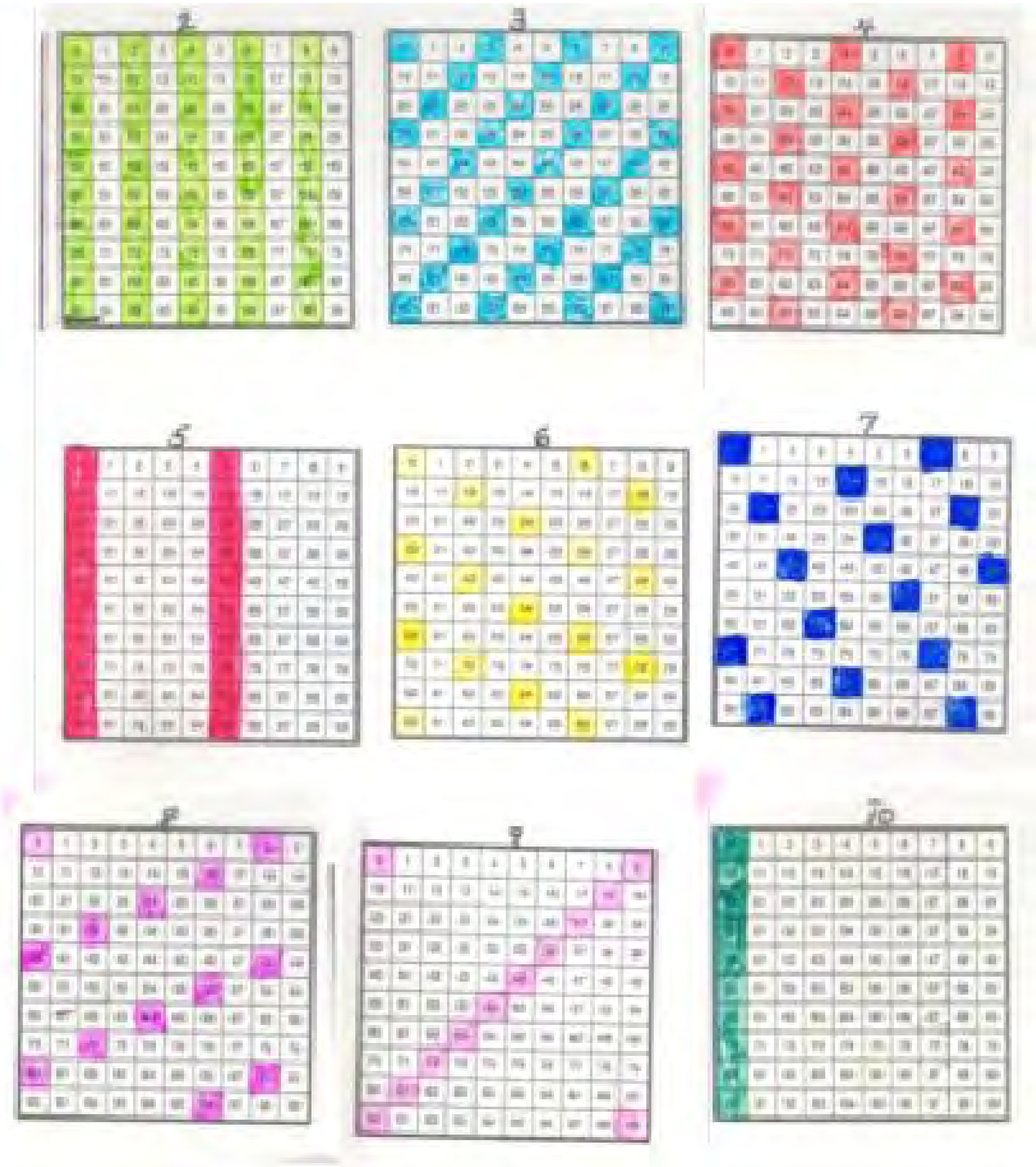
However, this "multiple sets" model is cognitively interesting since 4 legs (per elephant) x 2 elephants does not result in 8 elephants but 8 “elephant legs.” For young children this is conceptually different from addition and subtraction when they were working with sets with the same attribute. If you begin with 6 elephants and add 2 elephants, you have 8 elephants. Or, if you have 8 elephants and 2 elephants walk away, you have 6 elephants. Thus, when adding or subtracting the result will be...elephants.

When multiplying, however, instead of dealing with one variable (elephants), children need to hold *two* variables (the number of elephants *and* the number of legs per elephant) in mind at the same time.

Another way that young children begin to experience multiplication in a purely numeric sense is by “skip counting” (2, 4, 6, 8… 5, 10, 15, 20…) which can be modeled either by “hops” on the number line (in this case by 4s):



or by colouring cells representing those multiples on the hundred chart:



3. What is essential to know or do in class?

a) Introduce several models that can help students visualize multiplication of whole numbers.
b) Explain that multiplication is more than repeated addition or skip-counting.
c) Have students consider the fact that although multiplication of whole numbers makes the product greater than its factors, that this is not true for fractions, decimals, and integers.

4. Class Activities

a) Begin the session by asking students to consider the equation 6 x 3 = 18. Have them work in pairs to quickly write down all the mathematics they find implicit in this equation. Have students share out their thoughts, charting their responses and using the terms “factors” and “products” as you work with their ideas.

b) Introduce the array or set model. Have students draw 6 sets of 3 in an array, then have them create an array showing 3 sets of 6 in a different orientation. In both cases the factors are 3 and 6 and the product is the same: 18. Ask how the two arrays are different? Ask when this difference in orientation might matter. (E.g., If they needed to arrange 18 chairs in a narrow room in their home, they might create an arrangement of 6 short rows, each with 3 chairs. But if they were arranging 18 chairs in a classroom, they might decide to arrange the chairs in 3 rows, each with 6 chairs.) This is an opportunity to introduce and discuss the commutative property of multiplication and remind students of the commutative property of addition.

c) Introduce the intersection model for multiplication by having students draw a tic-tac-toe (noughts and crosses) grid. How many lines did they draw? Next, ask them NOT to focus on the cells (where they would they would enter an X or an O), but on the *intersections* of the lines. How many intersections are there? How does this number relate to the lines that they drew?

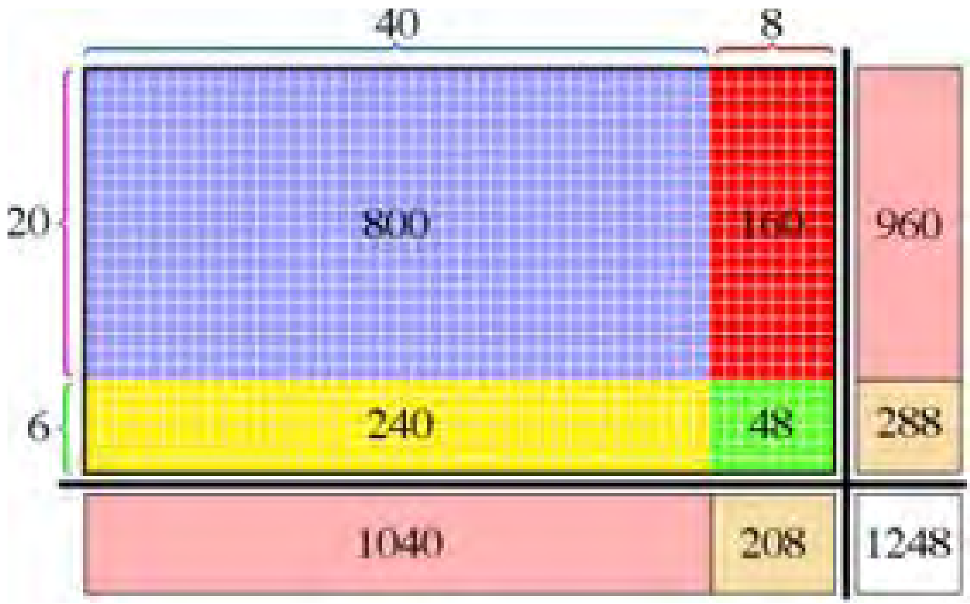
Have students draw the intersection model for 6 x 3 (6 lines crossed by 3 lines) and again ask about the number of intersections and how that number relates to the lines that they drew. Have them experiment using the intersection model for other equations. Ask students to explain why they think this model "works"?

d) Introduce the area model as shown above, where students draw a rectangle of 6 rows and 3 columns. Ask how this geometric model is different from the array. Have them turn their paper so that the orientation shows 6 columns and 3 rows. Remind them of commutative property of addition and how it applies to multiplication as well. Have them draw other area models to show different equations.

e) Allow students a few minutes to multiply 26 x 48, first by using the familiar algorithm, and then by using another method to arrive at the answer. Most likely, students will suggest using more "friendly" numbers (such as 25 x 48, or 26 x 50 and then adjusting their answer). This short calculation activity is designed to prepare students for using the area model for multi-digit multiplication. (This is also an opportunity to restate that although there is one "right answer" for 26 x 48, there are several strategies that can be used to find it.)

At this point introduce how the area model can be used for multi-digit multiplication. Note that this method will rely on something they did earlier in the unit: decomposing numbers.

Explain how they can combine place value with multiplication by decomposing each factor into tens and units and aligning those numbers on the top and side of the multiplication grid (as in this illustration).



After demonstrating how to multiply 26 x 48 by this "partial products" model, ask students to refer to how they solved 26 x 48 by using the traditional algorithm. Do they see any similarities? Is the answer the same? Do some of the same numbers appear in both methods? Why is this so? How do the two methods relate to each other?

f) Distribute a Hundred Chart to each student, so that they can work with skip counting. Have some students color in multiples of 2, others multiples of 3, etc. What patterns do they notice? How is this way of using a Hundred Chart to think about multiplication different from using a traditional multiplication chart? Where do they see skip-counting on the multiplication table?

X	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12
2	0	2	4	6	8	10	12	14	16	18	20	22	24
3	0	3	6	9	12	15	18	21	24	27	30	33	36
4	0	4	8	12	16	20	24	28	32	36	40	44	48
5	0	5	10	15	20	25	30	35	40	45	50	55	60
6	0	6	12	18	24	30	36	42	48	54	60	66	72
7	0	7	14	21	28	35	42	49	56	63	70	77	84
8	0	8	16	24	32	40	48	56	64	72	80	88	96
9	0	9	18	27	36	45	54	63	72	81	90	99	108
10	0	10	20	30	40	50	60	70	80	90	100	110	120
11	0	11	22	33	44	55	66	77	88	99	110	121	132
12	0	12	24	36	48	60	72	84	96	108	120	132	144

g) End the class by posing a challenge. Ask if multiplication always makes numbers "bigger," and if not, for what types of numbers does this not hold true.

<u>5) Assignment</u>

Have students read this blog about multiplication being more than repeated addition. Ask them to consider how they might teach multiplication in such a way that will avoid children assuming that multiplication always results in a product greater than the multiplicand and multiplier.

http://tinyurl.com/Mult-Repeat-Add

Unit 1 Number and Operations
Week 2, Session 3: Division of Whole Numbers

1. What are the important concepts?

a) Multiplication and Division are inverse operations. They "undo" the action of each other in the same way that addition "undoes" the action of subtraction (and vice versa).

b) Because multiplication and division are inverse operations, there are "fact families" of numbers in relationship with each other (such as 3, 6, and 18) that give rise to the following four equations:

$$3 \times 6 = 18$$
$$6 \times 3 = 18$$
$$18 \div 3 = 6$$
$$18 \div 6 = 3$$

This relationship between multiplication and division parallels the relationship for addition and subtraction (as in "fact family" of 5, 7, and 12 discussed last week):

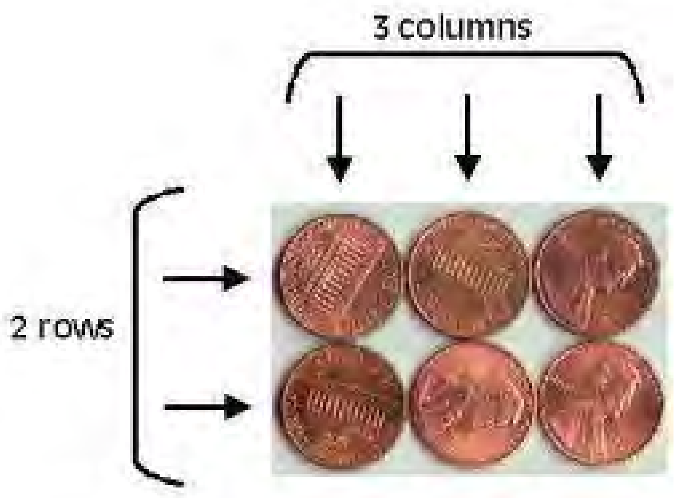
$$5 + 7 = 12$$
$$7 + 5 = 12$$
$$12 - 5 = 7$$
$$12 - 7 = 5$$

c) Multiplication (like addition) is commutative ($6 \times 3 = 3 \times 6$). However division, like subtraction, is <u>*not*</u> commutative ($18 \div 3 \neq 3 \div 18$).

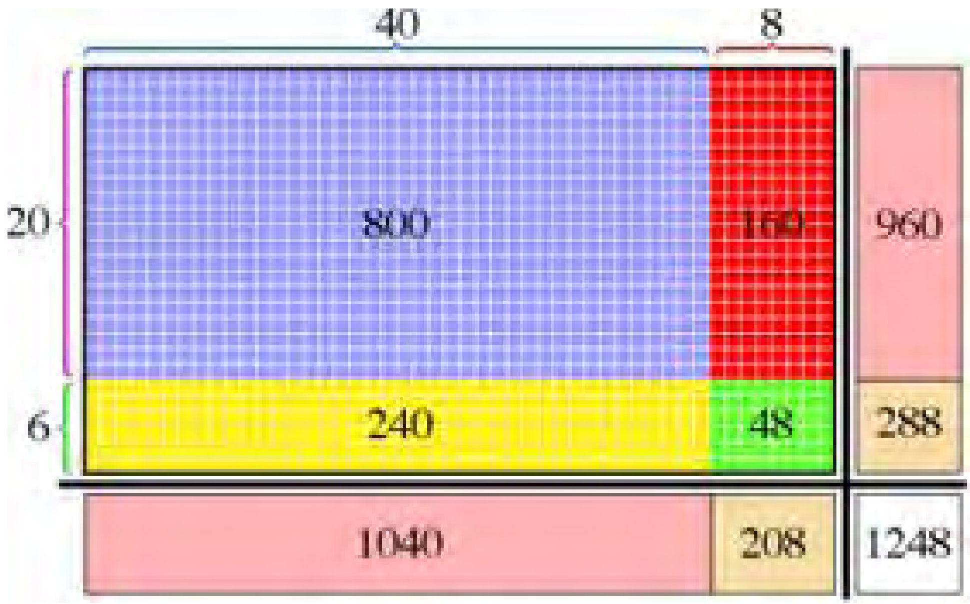
d) Models for Division:

- The *array and area models* that were used to visualize multiplication also can be used to visualize division. When using these two models for division, however, one needs to notice the whole (the product) first, and then notice the factors (the labels for the rows and columns).

- When multiplying, this array of 6 coins is product of factors: 3 (columns) x 2 (rows). But this same array also can be interpreted as 6 coins (the total number of coins) ÷ 3 (columns) = 2 (rows).



- The same is true for the following area diagram that was used to show multi-digit multiplication: 1248 (the total in the white box) ÷ 48 (on the horizontal) = 26 (on the vertical).



- When multiplying using the *number line model* you begin at 0, and then make equal sized "jumps" ahead. Thus, starting at 0 and making 6 jumps of "3" will "land" you on 18. The reverse would be true in this situation: starting at 18 and jumping "back" in groups of "3" six times will land you on 0.

  (However, if you were to use the number line to divide 19, you would make the same 6 jumps of "3," but in this case you would land on 1. This indicates two things: 1) that 6 and 3 are not factors of 19, and 2) that there is a remainder that needs to be considered.)

- There is an additional model for division: *sharing or distribution*. For young children, introducing this model could be as simple as a story about sharing 10 sweets amongst 5 children. Each child would receive 2. The sharing/distribution model also can raise the issue of remainders, for example sharing 11 sweets amongst 5 children.

e) Division as factoring vs. division as sharing: The sharing/distribution model is less of a visual model (such as in the array and area models) and more of a *dynamic model.* Children can act out story problems (such as 11 sweets for 5 children) and equations (11 ÷ 5 = ?) to help them understand how a quantity can be *divided amongst* a group. Ensure that pre-service teachers notice the difference between "dividing" a quantity into parts vs. "whole number factoring."

f) Nature of the Remainder: The sharing/distribution model, with its ability to generate remainders, allows children to make real life connections for division and to consider how to express the remainder.

For example, suppose you had 11 sweets to share amongst 5 children. When acting this out children will probably interpret the remainder of 1 as meaning each child will receive 2 sweets and there will be 1 left over. Whereas there may be a discussion about what to do with the leftover sweet (e.g., give it to the teacher), no young child will consider it sensible to unwrap a small sweet, try to cut it into 5 equal sized pieces, and give 1/5 or 0.2 of the remaining sweet to each of the 5 children! In this context, the remainder is simply "1."

However, in other contexts (and in abstract mathematical terms), the answer to 11 ÷ 5 = ? as either a fraction (2 1/5) or decimal (2.2) *would be* both sensible and appropriate.

g) Symbolic Representation for Multiplication and Division: While addition has its +, and subtraction has its –, multiplication and division have multiple symbolic representations. The expression 6 times the quantity 3 could be written as 6 x 3, 6*3 (especially on a calculator), and in algebra, 6 ◇ 3 or 6(3).

Similarly, 18 divided by 3 could be written as 18 ÷ 3, 18/3 (often found on a calculator), $\frac{18}{3}$ (with a horizontal fraction bar), or by using the long division "box": $\overline{)\quad}$

2. How do children think about these concepts?

a) After having used the array and area models for multiplication, youngsters should notice a connection between "the whole" (the product) and its factors. Building on what they already know, connecting these models to division is the next step.

b) However, confusion may occur when symbolic notation related to multiplication and division is introduced: How does 6 x 3 = 18 relate to 18 ÷ 6 = 3 ? This is why it is important to build on two concepts children learned earlier:

- Fact Families: Just as (5, 7, 12) related addition to subtraction, fact families such as (6, 3, 18) now relate multiplication to division
- Inverse Operations: Just as addition and subtraction were understood as inverse operations ("undoing" each other), so too are multiplication and division

c) For the sharing/distribution model it is important to give children problems to act out: “One for me, one for you, and one for you. Another for me, another for you, and another for you. Etc.” If there are any left over, note how the children try to interpret the meaning of the remainder.

In addition to providing objects (such as pencils or hard candy sweets) that *can’t* be subdivided, have children work with items (or pictures of items) such as bisquits that *could* be “cut into parts."

d) Children in the primary grades will not have the fractional or decimal vocabulary to describe the result of $11 \div 5 = ?$, but they should be able to say that for 11 bisquits shared amongst 5 children, each child will get 2 whole cookies and “a little bit more from the leftover one."

e) Symbolic Representation for Multiplication and Division: Teachers need to be aware of the confusion children encounter because of the different ways that multiplication and division equations are written. We adults move seamlessly amongst these representations in personal lives. However, children need to see one representation at a time, and teachers need to be precise when a new one is introduced.

When a teacher is working at the board after having taught children about the division sign ($18 \div 3$) he or she may unconsciously use the back slash (18/3) or the fraction bar with 18 above and 3 underneath the horizontal line (which is called a vinculum). Teachers need to be aware of these different representations that we adults take for granted but which can be confusing to children.

(Situations like this also might occur for older youngsters when introducing symbols for multiplication and division on a calculator.)

3. What is essential to know or do in class?

a) Introduce the idea of multiplication and division as inverse operations.
b) Link the array and area models for multiplication to these same models for division.
c) Have students consider how the number line model for multiplication may or may not model division of whole numbers.
d) Introduce the division model of sharing or distribution.
e) Clarify the difference between the division model of sharing or distribution and that of products with whole number factors.
f) Introduce different ways of interpreting the remainder.

4. Class Activities

a) Begin by reminding students how last week they used an addition chart to decompose the number 12 into (5 + 7), (6 + 6), etc. and how they then used that same chart to show subtraction (12 - 5 = 7, 12 - 6 = 6, etc.). Remind students that just as there is an *inverse relationship* between addition and subtraction, where one operation "undoes" the other, there is an inverse relationship for the operations of division and multiplication: one operation "undoes" the other as in:

$$3 \times 6 = 18$$
$$6 \times 3 = 18$$
$$18 \div 3 = 6$$
$$18 \div 6 = 3$$

b) Briefly review the array and area models for multiplication, noting how these models could be used for division: beginning with the product, and then finding the factors of that product.

c) Have students consider how the number line model for multiplication may or may not model division of whole numbers by using the equations 18 ÷3 and 19 ÷ 3.

d) As the main class activity, introduce the model of sharing or distribution by having students, in groups of 3, use counters to solve the following equations by distributing the counters to each other:

$$18 \div 3 =$$
$$17 \div 3 =$$
$$19 \div 3 =$$

In the class summary have students note that in the case of 18 when dividing by 3, the distribution created equal shares. For 17, the distribution could be termed either "2 more" or "1 less" of equal shares. 19 allows for "1 more" than equal shares. Have students describe what is happening with the remainder in each case.

e) As students work with the sharing or distribution model, have them explain how it is different from products in the array and area models that were based on whole number factors.

f) Introduce different ways of interpreting the remainder by using story problems such as:

- *Suppose our class had 26 students and one instructor. We want to visit a school 20 km from here. Several of you have access to cars and offered to drive. Each car can hold 6 people. How many cars and drivers do we need?* Students should note that although 27 ÷ 6 equals 4.5, it can also be thought of as 4 with a remainder of 3. However, in practical terms, there cannot be half a car. Or 4 cars with 3 people remaining behind. In this context, the remainder needs to *increase* by one more than the whole number quotient. Thus while the purely mathematical answer to the problem is 4.5, the realistic answer is: you need 5 cars.

- *Suppose I have 9 pencils to share amongst 5 students. How many pencils does each student get?*

The answer to 9 ÷ 5 is 1.8 or 1 4/5. But since pencils cannot be broken in order to be distributed, the answer is that each of the 5 students will receive 1 pencil. The remaining 4 pencils cannot be shared fairly.

- Suppose I have 6 large bisquits to share amongst 4 children. How many bisquits does each child get? The answer to 6 ÷ 4 could be interpreted as:
  - 1.5
  - 1 ½
  - 1 with a remainder of 2

  However, because we can cut the remaining 2 bisquits in half, it is sensible to say that each child would receive 1½ bisquits. In this case the remainder can be thought of as a fraction.

Ask how the contexts of these three examples imply different treatment of the remainder.

g) End the class by dividing the class into three groups. Ask each group to create a story problem where:

- the remainder should go to the next whole number
- the remainder can be disregarded
- the remainder should be treated as a fraction or a decimal

Have students report out their ideas.

5) Assignments

Have students:

1. Look at this video about multiplication and division as inverse operations: http://tinyurl.com/Relate-Mult-Divi
2. Use this multiplication table to discover patterns. http://tinyurl.com/Mult-Chart-144

Faculty Notes
Unit 1 Number and Operations
Week 3: Introduction to Rational Numbers

Weeklong Overview:

Session 1: Introduction to Fractions
Session 2: Introduction to Decimals
Session 3: Introduction to Least Common Multiple and Greatest Common Factor

Faculty Preparation for Upcoming Week (1-2 hours)

- View this PowerPoint presentation by Tad Watanabe. It covers models of fractions through operations with fractions (which will be discussed more fully in the Teaching Mathematics Course.):. http://tinyurl.com/Fraction-Watanabe-PPT

- Models of Fractions: http://tinyurl.com/Fraction-3Models

- Download and print out as handouts for class (1 per student):
    - Decimal Grid http://tinyurl.com/Decimal-Grid
    - Decimal Number Line PDF:
    - Area Model Fraction Strips: http://tinyurl.com/Area-Fraction-Strips

- Have ready to bring to class:
    - Rulers
    - Strips cut from copy paper to make fraction strips

- Read the plans for the upcoming three sessions

Weeklong Overview:

This week gives your student a very basic introduction to fractions and decimals--as *numbers*. *Operations* with fractions and decimals will be addressed in greater detail during the "Teaching Mathematics" course. Thus, it is not necessary for you to address operations with fractions and decimals this week. Just concentrate on students' conceptual understanding of these two types of numbers and how they are different from whole numbers.

Recognize that young children find these two types of numbers strange, because many of the properties of whole numbers no longer apply (such as, "Multiplication makes a number 'bigger'," or "The more digits in a number, the greater its value.")

Session 1 begins by defining a fraction as a *number* (on the number line). Students should not confuse this idea of a *fraction as a number* with the various visual and real life representations (such as segments of a circle), however useful those models can be in helping young children begin to understand fractions.

In order to do this, students will learn to distinguish between linear and area models by folding "fraction strips." By labeling the *folds* (as opposed to the segments) students will be creating a linear model (like a ruler). When the strips, which are of equal length, are laid out sequentially equivalent fractions will become evident.

Session 2 introduces decimals as a special type of fraction: a fraction whose denominator is a multiple of 10, but which is shown in a different format. 0.1 "looks" different from 1/10. However, it is the same number written in a different format. The *value* of the number remains the same. Young children find it hard to understand this.

Models for illustrating decimals will include both a grid and a linear model (a number line from 0 to 1 that is divided into hundredths).

Session 3, the Greatest Common Factor (GCF) and Least Common Multiple (LCM), builds on concepts introduced last week: factors and multiples.

Unit 1 Number and Operations
Week 3, Session 1: Introduction to Fractions

1. Maths Concepts to be studied

a) A Fraction is a Number: Just as many children (and many adults) assume that addition is joining sets, subtraction is "taking away," and multiplication is "making things bigger," they also consider fractions as "parts of things," such as an apple that has been cut into half or fourths. But when looking at models that clarify the real nature of "a fraction as number," the linear model (a number line, or in real life, a ruler) is perhaps the best way to make this important concept clear to children.

b) Models for Fractions: When we considered multiplication we looked at it from 3 "dimensions": 1) 0-dimension, the set model with items as "points in a plane," 2) 1-dimension, the linear model of intersecting lines, 3) 2-dimensions, the area model of cells on a grid (and the multi-digit multiplication mats). These three models hold true for fractions, but there is at least one additional model to consider: 4) 3-dimensions, volume.

c) What is "The Whole?": One of the most important concepts related to fractions is the nature of the whole. Suppose I have one apple and cut it into three equal pieces. Each piece is 1/3 of the whole. However, if I have two apples and cut each into thirds, I now have 6 pieces. What does 1/3 look like now? One-third of the first apple and 1/3 of the second apple have become 2/6 of the whole (which is now two apples).

This leads to another question about "the whole." Suppose the whole (in a recipe) is 1¼ kilos of flour. What is half of that? How might I model this? Would a drawing help?

The important issue here is that students' conception of fractions for one "item" (the single apple) will eventually need to move to fractions of multiple items (two apples) and fractional situations (1¼ kilos).

d) Equivalent Fractions: Consider the above example of the two apples cut into thirds, where 2/6 is 1/3 of "the whole;" or on a metric ruler, where ½ is equivalent to 5/10. Renaming a given fraction to one with a different denominator is the basis for the "least common denominator" that allows for operations with fractions.

How might we deal with that 1¼ kilo of flour when we try to divide it in half and then discover we need to deal with a denominator in eighths? Or when we double the recipe and find we need 2½ k of flour. How did our original number with its denominator of 4 move to denominators of 8 or 2?

2. How children think about these concepts

a) Young children's thinking about fractions is usually quite confused. When hearing "one-half" in adult conversations children often assume that all fractions are one-half, or that one-half means "any part of the whole." Thus, children need to learn that there are fractions other than one-half, and that fractions are *__equal__* parts of a whole (a connection to the "fair shares" model for division).
b) Children also apply whole number thinking fractions, assuming that if 6 is greater than 2, then 1/6 must be greater than ½.

c) Because fractions are usually introduced by the area model of a circle or rectangle, children often assume that ¼ of a circle *is* what ¼ means. This is why children need to see and work with multiple models (set, linear, volume) for fractions to see what ¼ looks like in different models

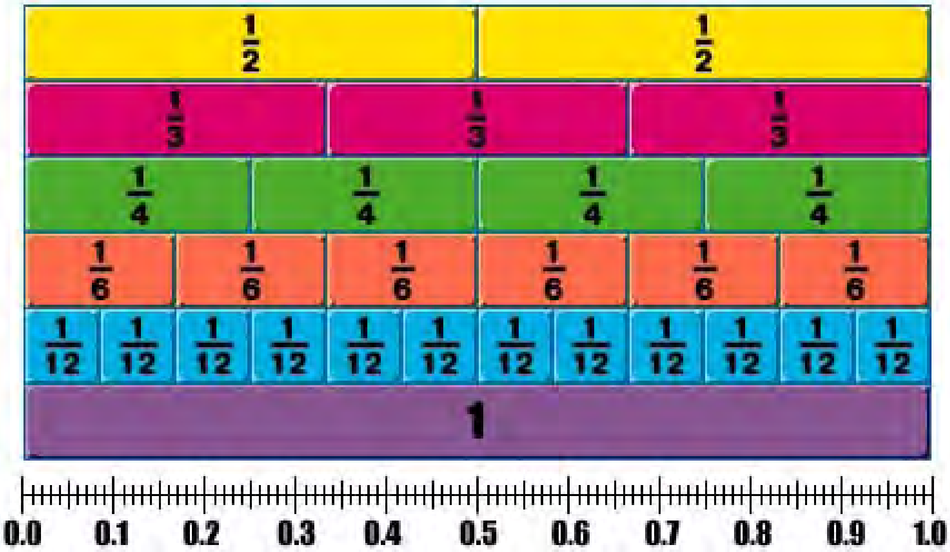
d) Exposing children to equivalent fractions, (e.g., ½ = 2/4 =3/6 = 4/8 = 5/10) allows them to notice a pattern indicating how the numerator and denominator are related.

3. What is essential to know or do in class

a) The conceptual model of fractions in various "dimensions"
b) The nature of "the whole"
c) Equivalent fractions
d) Relating each of the above to children's thinking

4. Class Activities

a) Begin by asking students in groups of no more than four to brainstorm for three minutes about all they know about fractions. Ask for and chart their responses.
b) Introduce the various dimensional models for fractions, asking students to give real-life examples for set, linear, area, and volume.
c) Have students experiment with the linear model for fractions by giving them narrow strips of paper that they can fold into halves, thirds, fourths, sixths, eighths, ninths, and twelfths. Challenge them to find a way to create fifths and tenths.
d) Ask students to label their fraction strips ***on the folds***, which creates a linear model like a ruler, so that the folds on the fourths strip would be labeled ¼, 2/4, and ¾. Note in the diagram below, that all the fractions are labeled on the ***segments,*** which translates into an area model, but the decimal equivalents are noted where the folds would lie.
e) Ask students about where zero-halves and two-halves are. Have them line up their fraction strips in order of increasing denominators to display patterns of equivalent fractions. Have them name the various equivalent fractions for ½. What about one-half on the thirds strip?
f) Introduce mixed numbers by having students lay their one-half strip end to end with that of another student. What is the whole now? What is one-fourth of 2? Three-fourths of 2?
g) Finally, have students compare various fractions. Which is greater: 4/10 or 4/6? 3/5 or 5/3? 5/6 or 5/8? How can you tell (without converting them to decimals)?



5. Assignments

Distribute this handout of an area model for fraction strips: http://tinyurl.com/Area-Fraction-Strips

Unit 1 Number and Operations
Week 3, Session 2: Introduction to Decimals

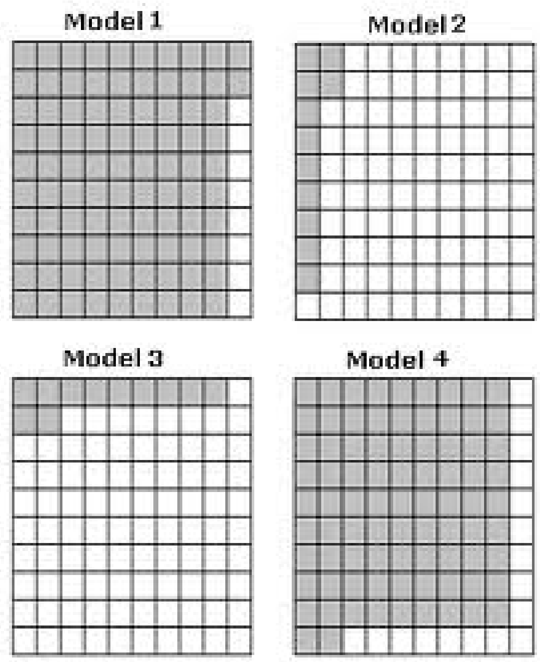
2. <u>Maths Concepts to be studied</u>

a. Decimals are a way to write fractions using a numeral's position in base-10 place value. When working with whole numbers, students saw that each numeral increased by a magnitude of ten moving to the left, and decreased by a magnitude of 10 moving to the right. A number such as 23 could be written with a decimal point as 23.0 without changing its value.

b. Students need to develop "decimal sense" to understand how decimals are named and the quantity they represent. One way to address this is to work with a number line that goes from 0.0 to 1.0 with points in between indicating tenths.

c. A second way is to use a 10 x 10 blank grid where a full row or column colored in indicates tenths. It is important that students understand how this grid is different from the Hundred Chart. In the Hundred Chart each cell was worth "1" and the "whole" was worth 100.

In the case of the decimal grid, each grid is only one unit, and each cell is 1/100 of the whole. Thus two decimal grids would be worth 2.0, but two small squares colored in would be worth 0.02.



d. Comparing and Ordering Decimals: Placing a group of decimal numbers in order is a challenging task for children. Consider: 6.6, 6.7, 6.08, 6.55, and 6.551. All five of these numbers have a six in the units place. But some are "shorter" numbers with only two digits while some are "longer numbers" with up to four digits. All of which gives rise to confusion (see below regarding how children think of these).

e. Fraction-Decimal Equivalents: Decimals, as mentioned before, are used as an alternative to the a/b model for writing fractions. As such, every fraction has its decimal equivalent, which is either truncated after several places (such 1/8's equivalent 0.125) or repeating (such as 1/3's equivalent of 0.33333…), which can be indicated by a short line over the repetend or repeating portion.

3. How children think about these concepts

a. Most of children's misconceptions about decimals are rooted in whole number thinking. For example, being familiar with the magnitude of natural numbers, they assume that a number with many digits must be a large number. Extending this to decimals, the child mistakenly believes that .283 > 2.5 because the first number has more digits, thinking that "longer" decimals have a greater value than "shorter" ones.

b. Other children may become focused on the denominator and think that any number of tenths must be greater than any number of hundredths (or thousandths) because tenths are greater than hundredths (or thousandths). These children believe, for example, that 6.45 > 6.731 because thousandths are very small parts of a number.

c. There is also confusion about how decimals relate to fractions. For example, some children believe that 1/8 and 0.8 are the same quantity—often because they learned that 1/10 and 0.1 are equivalent

4. What is essential to know or do in class

a. Decimals are a way to represent fractions whose denominators are multiples of 10 and is an extension of place value, and as such every fraction has a decimal equivalent
b. Comparing and ordering decimals can be visualized with a number line and a decimal grid in order to better understand the quantity a decimal fraction represents.
c. Relate each of the above to children's thinking

5. Class Activities

a. Begin by asking students in groups of no more than four to brainstorm for three minutes about what they know or recall about decimals. Chart their responses.
b. Ask students what a decimal point means and help them articulate extending numbers to the right of the decimal point as being a pattern consistent with what they already know about place value for whole numbers.
c. Distribute the decimal grid and discuss how it is only one unit, but subdivided into hundredths. Ask them to show one-tenth, then one-hundredth, and then one-thousandth. (For thousandths they will have to divide one of the cells into ten parts.) Then have them find 7 tenths and ask how they would write it in decimal format. What would 73 hundredths look like? How would they write it?
d. Have students begin to compare decimals by using the grids. Which is greater, 0.45 or 0.54? 0.08 or 0.8? How do they know?
e. Continue using the decimal grid, having students describe how they would find 7/50 or 1/4. How do they approach the task? How do they know their answer is correct?
f. Distribute the 0.0 to 1.0 number line page and have students label the tenths in decimal notation on one of the number lines, making sure they put a 0 to the left of the decimal point. After doing this, have them find various hundredths and label those.
g. Then have the students label one of the number lines with any fractional equivalents they know. Ask about 1/8 and 1/6. How would they know where to write those numbers?
h. Give students a series of numbers (such as above) and ask them to put them in order from least to greatest. Discuss the various types of erroneous thinking that are common

in children, and then ask how they think children might put this group of decimals in order.

6. Assignments

a. Have students read this article on student misconceptions about decimals: http://tinyurl.com/Dec-Misconcepts

b. Have students create a list of five decimal numbers that could highlight children's confusion about decimals and write them on five slips of paper. Have them ask a child to put the decimals in order while describing his or her rationale for the choices. This is a type of performance assessment called a clinical interview. Have the student take notes on what the child says and does.

Unit 1 Number and Operations
Week 3, Session 3: Greatest Common Factor, Least Common Multiple, Introduction to Operations with Fractions

7. Maths Concepts to be studied

a. Factors and Multiples were mentioned in the class session on multiplication and division. These concepts become especially important as students begin to add, subtract, multiply, and divide fractions.

b. To find the Greatest Common Factor (GCF) of two numbers, for example 16 and 24, we would list all the factors for each (for 16: 1, 16, 2, **8**, 4; for 24: 1, 24, 2, 12, 3, **8**, 4, 6) then look for the greatest one. In this case it would be 8.

c. The GCF is most often used when renaming fractions. For example for 5/20, the factors of 5 are 1, 5, and the factors of 20 are 1, 2, 4, 5, 10, 20. In this case, the GCF is 5. Both the numerator and denominator can be divided by 5, creating the equivalent fraction ¼.

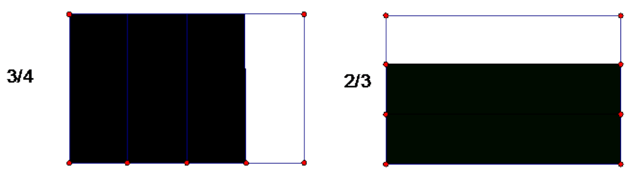
d. To find the Least Common Multiple (LCM) for 16 and 24, we would generate multiples for each (16, 32, **48**, 64), (24, **48**, 72, 96). In this case the LCM would be 48. Another way to find the LCM is to break each number into its prime factors (16 = 2 x 2 x 2), (24 = 2 x 3 x 4) and then cast out the duplicates. With a "2" cast out from each set of factors, we are left with the remaining factors 2, 2, 3, and 4 which, when multiplied, give 48.

e. In elementary school mathematics, the LCM is often referred to as the Lowest Common Denominator when adding fractions, such as 5/8 and 3/4. In this case the LCM is 8, which allows for renaming ¾ to its equivalent 6/8 with a resulting sum of 11/8. The same would hold true for subtraction: the renamed 3/4 (now 6/8) - 5/8 = 1/8.

f. Operations with Fractions: As mentioned above when adding and subtracting fractions, it is necessary to discover the LCM so that each fraction has the same denominator. This is also where a student's familiarity of equivalent fractions (as with the fraction strips laid out in rows) is important so that students begin to have an intuitive sense of common equivalents.

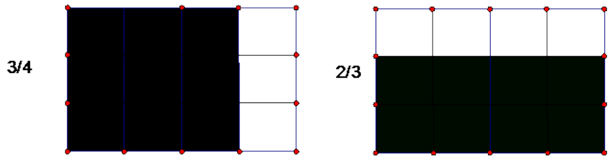
g. When multiplying fractions, students need to make sense of the fact that multiplication will not always "make things bigger." For example, in the case of 3/4 x 2/3, the answer is 1/2, which is less than either of the original numbers.

To help students understand this, a visual model can help. The following two rectangles are shaded to show 3/4 and 2/3.

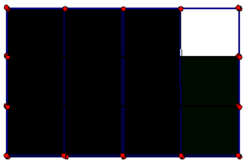


Now, partition each rectangle as shown below. Note that they now have the same number of cells (12). (This is, in fact, a way to represent the Lowest Common

Denominator. We can also use this method to compare fractions, noting here that 3/4 fills 9 of the cells whereas 2/3 only fills 8. Thus, 3/4 is greater than 2/3.)

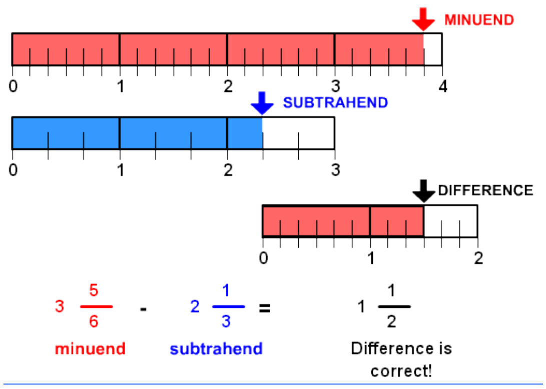


If we overlap the rectangles, the overlapping portion is 3/4 of the 2/3. Count the overlapping squares. The overlapping portion is 6/12, or its equivalent, 1/2.



8. <u>How children think about these concepts</u>

a. When working with fractions children commonly look at an addition or subtraction example and simply add (or subtract) the numerators, then add (or subtract) the denominators. Thus in the example above for 5/8 + 3/4 it would not be uncommon for a child to come up with a mistaken answer of 8/12. In subtraction, this same type of thinking might result in 3/4 - 5/8 = 2/4 with the youngster reversing the minuend and the subtrahend.



b. As mentioned in the session on multiplication, because of whole number thinking, children often assume that multiplication <u>*always*</u> "makes things bigger" than its factors.

c. When multiplying fractions (unlike when adding and subtracting them) you *do* operate across the numerators and across the denominators, multiplying each.

9. <u>What is essential to know, or do in class</u>

a. Present the concept of the GCF and LCM, how to find them, and how they are used when working with fractions.
b. Address how operations with fractions are both similar to and different from operations with whole numbers.
c. Provide students with visual models and strategies to help them conceptualize operations with fractions.
d. Relate each of the above to children's thinking

10. Class Activities

a. Have students add 1/4 and 1/3 using any method they choose. Have students share their methods and record on chart paper.
b. Introduce the Greatest Common Factor and ask students to find the GCF of various number pairs. Ask why they think this topic is relevant to this week's work on fractions.
c. Introduce the Least Common Multiple, and ask students to find the LCM of various number pairs. Again, ask about the relevance of this topic.
d. Ask if, when they added 1/4 and 1/3, they were intuitively using the GCF and LCM. Ask how the LCM might be related to denominators.
e. Note how the addition of and subtractions of fractions follow many of the rules for natural number addition and subtraction—as long as there is a LCM for the denominator
f. Move to the multiplication of fractions by introducing the grid model that helps explain why "fractions of fractions" are less than either of the two original numbers. Pose the problem 3/4 x 2/3. Ask students to estimate the answer. Then use the rectangular model to ask students why multiplication of these two fractions that are less than one resulted in an answer less then each of the original.
g. After focusing on the operation of multiplication, ask how this same visual model can be used to show the Least Common Denominator and equivalent fractions.

5. Assignments (to be determined by instructor)

Faculty Notes
Unit 1 Number and Operations
Week 4: Percents, Ratios, Rates

Weeklong Overview:

Session 1: Percent and Percentages
Session 2: Ratios
Session 3: Rates

Faculty Preparation for Upcoming Week (1-2 hours)

- Read Chapter 4 from the Mathematical Education of Teachers, available at http://tinyurl.com/Math-Educ-of-Teachers
- Review this website, which gives a detailed description for using a grid to understand percentages: http://tinyurl.com/Illum-Percent-Grid

- Download and print out as handouts for class (2 per student):
  Percent Grid: http://tinyurl.com/Illum-Percent-Grid-Handout

- Have ready to bring to class:
  Rulers
  Graph paper

- Read the plans for the upcoming three sessions

Weeklong Overview:

Session 1this week will extend last week's work with fractions and decimals to address the concept of percent and how to calculate percentages.

Session 2 will introduce the concept of ratio and proportion, emphasizing the difference between part-to-part and part-to-whole. This session will also question students' understanding of "cross-multiplying" and ask them to use their knowledge of equivalent fractions as an alternative strategy for solving ratio problems.

Session 3 will introduce rates, which will be dealt with more fully in the Algebra unit.

Unit 1 Number and Operations
Week 4, Session 1: Percents and Percentages

1. Maths Concepts to be studied

. Last week's emphasis on fractions and decimals laid the groundwork for understanding percents and percentages. Students should consider a particular decimal fraction such as 0.5 and ask how that representation translates into a fraction in a/b format (1/2) and then into a percent (50%).

a. Models for Percents: A Decimal Grid can be thought of as a "Percent Grid," where, instead of calling a shaded area 75 hundredths or 0.75, this same visual can be thought of as 75%.

b. As students move from fractions to decimals to percents, they need to consider "benchmarks" such as division by 10 being translated into multiplication by 1/10 or 0.1. Later they will relate this to "10% *of*" a number.

c. This idea of a benchmark fraction of 1/10 allows students to work from the basic "1/10" in order to consider either half of a tenth or double one tenth. Knowing that half of 0.1 is equal to 0.05 allows the student to calculate the sum of 0.1 + 0.05 as 0.15 and then translate that into 15%. Doubling 10% means 20%, tripling gives 30%, etc.

d. Changing decimals to percents involves "moving the decimal point two places to the right and then adding the % sign." For example, 0.02 would become 2% (not 20%), 0.25 would become 25%, and 0.125 (equivalent to the fraction 1/8) would become, by moving the decimal point two spaces to the right, 12.5% (not 125%, which children sometimes write by simply placing the percent sign after the decimal's last digit).

However, simply moving the decimal point is a procedure. Even if students can do it correctly, they may not understand why it works. Students need to realize that percent literally means parts "per hundred." Thus, in order to move from a decimal (which is based on 1.0) to a percent (which is based on 100) we need to multiply the decimal by 100:

$$\begin{array}{r} 0.75 \\ \underline{\times\ 100} \\ 75.00 \end{array}$$

Similarly to go from a percent to a decimal, we need to divide the percent by 100.

e. Percents in the real world often deal with prices and statistical data. When dealing with prices, there are two common scenarios: discounts and taxes.

In the case of a discounted price, the percent of the price is subtracted from 100% (the original price). Thus, for a $100 purchase with a discount of 20% I would end up paying only 80% of the original price. To compute this discount I would subtract the percent of the discount from 100% and use the remainder as the multiplier. ($100 x 80%, which I would calculate as $100 x 0.8 or $100 x 8/10.) If I needed to pay sales tax on my purchase, however, I would need to *add* that percent to the 100% of the original cost. Thus, in New York City where the sales tax rate is 8%, I would need to add 8% to 100% and then multiply my original $100 purchase by 108% (1.08) to determine my final cost.

2. How children think about these concepts

. Children's transition from decimals to percents rests on their familiarity with decimal notation and their ability to think about "moving the decimal point." For example, suppose that a child understands that 0.60 is somewhat "more than 1/2". As he or she transitions to the percentage model for that quantity (60%), it is helpful to create a classroom chart of three columns labeled Fraction, Decimal, Percent. Having a visual that shows how 6/10, 0.60 and 60% are simply different "names" for the same quantity can help children toggle between these three different representations that all describe the same number. Once again, this relates to the concept of equivalent fractions and fractions to decimals to percents. This can also be connected to earlier work done with whole numbers where a single quantity 12 could be represented by various expressions such as 5 + 7, 3 + 9, 13 - 1, etc.—all various ways to express the same number.

a. Youngsters often fail to consider the whole when working with percents. For example, they may think that 50% means that there are 50 objects in the whole. Or they may think that 50% of a year is 5 months when it is actually 6 months, again failing to recognize the whole.

b. As with fraction-to-decimal conversions, where children mistakenly assume that 1/6 is 0.60, they also may think that when working with fraction-to-percent conversions that 1/6 is 60%.

c. When using a calculator to work with percents, the input needs to be the percent's decimal equivalent. The need to know the percent's decimal equivalent is another reason why it is important to become fluent in moving between fractions, decimals, and percents, since even if children don't use calculators in the classroom, they may use them at home now and certainly they will need to know how to use them in the future.

3. <u>What is essential to know or do (in class)</u>

. Connect how three representations (fraction, decimal, percent) describe the same quantity.
a. Discuss how to calculate percentages, using the "discount" and "tax" models. Explain how this relates to the multiplication of fractions and decimals (80% of a given price will be less than the original, whereas 108% of the item's original cost will be more than the original).
b. Refer back to the nature of "the whole" by introducing the use of percentages in circle graphs.
c. Relate each of the above to children's thinking.

4. <u>Class Activities</u>

. Begin by asking students in groups of three or four to brainstorm for several minutes about what they know about percent. Chart their responses.
a. Refer to the decimal grids used last week and ask how their work with them might be rephrased in terms of percentages. Note any comments students make that relate percents to fractions and decimals and how they describe these relationships.
b. Briefly remind students about the work they did with multiplication of fractions and ask how this will relate to working with percents. If necessary, make clear the connection among all three representations of the same quantity.
c. To offer an opportunity to work with percents, present students with this problem, which they should solve in pairs or small groups:
You find a 20,000 PKR dress on sale at a discount of "60% off." What does the dress cost? (This is an occasion of using adult experiences to explore mathematics. Later

these pre-service teachers will need to create age-appropriate scenarios that will engage children in thinking about how mathematics relates to their real life experiences.)

This is an occasion of using adult experiences to explore mathematics. Later these pre-service teachers will need to create age-appropriate scenarios that will engage children in thinking about how mathematics relates to their real life experiences.

- Have students create a similar problem with a discount that has a context for students in the middle grades.

5. Assignment:

. Have students study this website which gives a detailed description for using a grid to understand percentages: http://tinyurl.com/Illum-Percent-Grid

Unit 1 Number and Operations
Week 4, Session 2: Ratios

6. Maths Concepts to be studied

a. Proportional thinking, like multiplicative thinking, develops over time with its roots in students' understanding of fractions, decimals, and percents. Ratios and proportions are used to compare two quantities in order to answer questions such as, "What is the ratio of men to women in our class?" where we are comparing two parts of the whole to each other (part-to-part).

   This is different from the type of comparison being made when asking the question, "What proportion of our class has a laptop computer?" where the comparison is part-to-whole. Note that these types of part-to-whole questions could be rephrased as, "What percent (or fraction) of our class has a laptop?"

b. Models for Ratios and Proportions: Ratios and proportions can be numeric, or they can be geometric. For a numeric example, consider the ratio of orange juice concentrate to water. Each can of concentrate has directions printed on the label that says it should be diluted with three cans of water. Thus, the resulting mix of 1 can of concentrate with 3 cans of water gives a total of 4 "cans worth" of juice.

   This mixture can be thought of in several different ways. If we consider the relationship "part-to-part," the mixture would have a ratio of 1:3 (1 part concentrate to 3 parts water).

   We can also think of the mixture as "part-to-whole" where the concentrate is 1/4 of the mixture (1:4) and the water is 3/4 of the mixture (3:4). Either relationship, part-to-part or part-to-whole, is valid, but we need to be clear about which type of relationship we are discussing.

7. How children think about these concepts

a. Youngsters often are not aware that the order of terms in a ratio is important. For example, if I have four children and only one is a girl, the ratio of girls to boys is 1: 3, whereas the ratio of boys to girls is 3:1.

b. Children also are confused by part-to-part vs. part-to-whole ratios that refer to the same situation. For example in the above scenario, I have three times as many sons as daughters (part-to-part) but 3/*4* of my children are boys, where the 4 represents the total number of children ("the whole") in my family.

c. As mentioned above, although youngsters can be taught cross-multiplying as a quick way to solve proportions, they usually have no idea why this works. Even though the written description of using equivalent fractions might seem involved, it actually makes the mathematics of solving proportions more sensible to students. Loretta, where above is this cross multiplying?

8. What is essential to know, and do in class

a. Compare and contrast the "part-to-part" and "part-to-whole" models for thinking about ratios and proportions, referring back to the nature of "the whole" that was discussed when studying fractions and percents

b. Have students devise ratio problems that relate to real life situations
c. Introduce solving proportions for an unknown by using equivalent fractions
d. Relate each of the above to children's thinking

9. Class Activities

a. Begin by describing the scenario of a family with 3 boys and 1 girl. Have students quickly write all the ways they can think of to express that relationship using both words (e.g., "I have three times as many sons as daughters." or "Three-fourths of my children are boys.") and in symbolic representation (3:1, 1/4, etc.).
b. Ask for and chart their responses. Be aware that some students will use the comparison model of subtraction to discuss the relationship (I have two more sons than I have daughters).
c. Use student responses to launch a comprehensive class discussion of 1) how part-to-part and part-to-whole relationships are different, 2) that part-to-whole relationships can be expresses as fractions or percents, and 3) that the order of numbers in a ratio is important.
d. As part of this discussion, have students generate several scenarios where ratios may be found in real life situations.
e. Ask students how they could solve 8/36 = 20/x. If they mention cross-multiplying, ask them to explain how it works. Then ask how they might use what they learned about equivalent fractions to solve for x. Have them work in partners and use this method to discover a solution.

10. Assignment:

Have students read Fractions, Units, and Unitizing (http://tinyurl.com/Fractions-Units) with its explanation of part-to-part, part-to-whole thinking.

What do they think about the last section and how it relates to adding fractions? Can they think of a real life scenario where adding part-to-part might occur?

Unit 1 Number and Operations
Week 4, Session 3: Introduction to Rates

11. <u>Maths Concepts to be studied</u>

a. Constant rates of change are ratios that show a relationship between an independent and dependent variable. In real life situations, this might be the unit price per loaf of bread or kilometres (km) per hour.

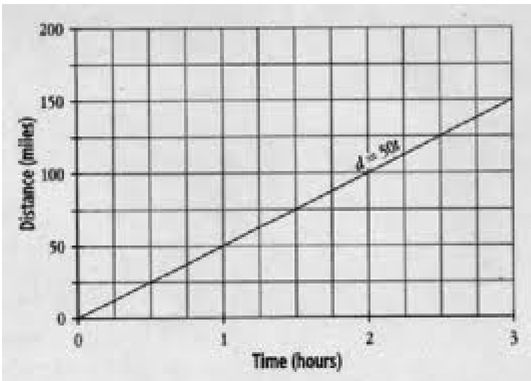
b. For example, if I drive at a constant speed (the independent variable), I can calculate the distance I drove for various lengths of time (the dependent variable).

c. Note that the word "per" separates the two units in the rate.

d. Multiple Representations: When working with rates, it is important for students to see various ways change can be represented: table, graph, and equation. In the km per hour scenario, a table would show the following distances:

Rate: 50 km per hour	
t: Time in hours (x)	d: Distance in km (y)
.5	25
1.0	50
1.5	75
2.0	100
2.5	125
3.0	150
5.0	250
n	50n

The graph of this scenario would look like this:



Notice that the graph shows a straight line, indicating a constant rate of change, and the distance traveled for any particular hour (or fraction of an hour) on the x-axis can be found by the line's location relative to the y-axis.

The third representation would be an equation, in this case d = 50 t where d is the distance and t is the time. Note that if the time and distance were known we could use simple algebra to discover the rate: d/t = 50.

- Understanding constant rates of change will become important later in algebra where they will be linked to linear equations and slope.

e. Rate problems in the real world might deal with prices (rate per unit), interest rates (rate per time period), or currency conversion rates. In fact, conversion tables (Celsius to Fahrenheit, for example) are all based on rates.

12. <u>How children think about these concepts</u>

a. Often students are presented with only the equation model when asked to work with rates. They are then asked to perform calculations, usually in the context of a word problem such as, "How many km will you drive in 2 hours if your average speed is 50 km per hour?" However, the very next problem assigned may ask them, "Calculate how many times your heart beats in an hour based on 110 heartbeats per minute." These two problems, focused only on one numeric answer do not illustrate, as a table does, that a *pattern* of change emerges for rate problems and the "variables" actually vary.

b. Without their seeing that ratio problems can be graphed on a coordinate grid, students will have difficulty understanding both the place of rates in linear relationships and how slope is related to a constant rate of change when they begin to study algebra.

c. Recall the scenario in the previous session, the family with 3 sons and 1 daughter, where the ratio of boys to girls was 3:1, but the ratio of girls to boys was 1:3. Order mattered. The order of the units for a rate matters, too. There is a major difference between finding the distance (km per hour) vs. finding the time it takes to go one km (hour per km).

d. Conversion rates are more complex than simpler rates such as km per hour. Thus, even if students understand what the rate means, they will need excellent computational skills to make the conversion. This is where in real life using even a simple calculator makes the conversion quick and easy.

13. <u>What is essential to know or do (in class)</u>

a. Explain how constant rates of change are related to ratios and proportions.
b. Have students construct and interpret tables and graphs to solve a rate problem.
c. Have students create an equation to express the rate.
d. In a whole group summary discussion, modelling how a similar discussion would take place in a middle grade classroom, compare and contrast the three different representations.
e. Relate each of the above to children's thinking.

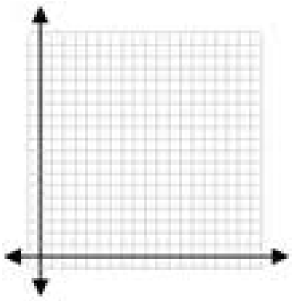
14. <u>Class Activities</u>

a. Begin by asking students to share how rates they use in their daily life. As each rate is shared, note the use of the word "per" to name the two units involved. Also show how rates can be written with a " / " sign as in km/hour. Ask how this representation relates to division. For each example that has been shared, ask the student what a reasonable rate might be (such as 3 km/hour walking, 30 km/hour driving). Do they know any specific rates, such as converting centimetres to inches or vice versa?

b. Note the rates that they shared are called "constant rates of change" and that they will explore several ways to represent them. Mention the terms "independent and dependent variables" and ask students why these are accurate descriptions for the units in the rate.

c. Introduce a rate problem such as the one above, having students work in pairs, one student using the rate of 30 km/hr, the other with a rate of 40 km/hr. First they will create a table beginning with various numbers for x, finding y, and then determining the value of y for any x. Ensure they know the traditional format for setting up this 2-column chart with the rate labeling the top.

d. Next, ask them to create a graph using the data from their table. Ensure that students know what a Quadrant I graph is and how to scale it appropriately to fit the problem's numbers.



e. Have each pair of students compare their two graphs and discuss with each other:
    i. How are they alike?
    ii. How are they different?
    iii. What is the "equation of the line" on each of the two different graphs?

f. End the session by having a whole class discussion that summarizes their answers to the above questions and asks additional questions such as:
    i. How could you find the distance for times between your last numerical entry and "n" by using the table?
    ii. How would distances beyond your last numerical entry be shown on the graph?
    iii. How are the table, graph, and equation related to each other?
    iv. Predict what your graph would look like if you were walking at 3 km/hour. Quickly add that data to your graph. Was your prediction correct? What two things does this graph compare?
    v. How does the difference between the y-values in your table relate to the rate?
    vi. Show the table above with the rate of 50 km/hour that includes fractional values for x, and missing entries between 3.0 and 5.0. Ask what students notice about your table that might be different from theirs.

vii. Ask them if there is a consistent difference between the y-values in their table? If so, how does this pattern relate to the rate? If not, why not?

15. <u>Assignment (to be determined by instructor)</u>

Faculty Notes
Unit 1 Number and Operations
Week 5: Integers, Integer Operations, Reflection on Mathematical Processes

Weeklong Overview:

Session 1: Introduction to Integers
Session 2: Addition and Subtraction of Integers
Session 3: Multiplication and Division of Integers, Reflection on Maths Processes

<u>Faculty Preparation for Upcoming Week (1-2 hours)</u>

- Look through the following websites that address integers and operations with integers:
    - An explanation of negative numbers, models for them, and operations with them: http://tinyurl.com/ThinkMath-Integers
    - http://tinyurl.com/Integer-NumberLines
    - http://tinyurl.com/Integers-Interactive

- Do the math:
    - Practice adding integers with two-colour counters using this interactive applet: http://tinyurl.com/Integer-Chip-Trading
    - Practice subtracting with two-colour counters using the tutorial "Integer Chips Teacher Script," which you will also use as a class handout in Session 2

- Print out the following handouts for class:
    - Integer number lines (1 page, 2 per student): http://tinyurl.com/Integers-NumLine
    - "Integer Chips Teacher Script" (2 pages, 1 per student) http://tinyurl.com/Integer-Script
    - "Integer Chips Student Worksheet" (a 6-page packet, 1 packet per student) http://tinyurl.com/Integer-SubtrWorksheet
    - Integer Addition and Subtraction Reference Sheet http://tinyurl.com/Integer-Reference (1 page 1 page per student)
    - Multiplication and Division of Integers Reference Sheet http://tinyurl.com/Integer-Mult-Div-Reference
    - End of Unit Reflection worksheet (2 pages, 1 per student) http://tinyurl.com/Unit-1-Reflection

- Prepare and have ready to bring to class:
    - Large white beans and markers (or crayons) for students to create 2-colour counters

    - A 6-meter strip of paper folded into twelfths, to be used as “walk on” integer number line
    - Graph paper
    - Post-it Notes of two different colours

- Read the plan for the upcoming three sessions

Weeklong Overview:

Session 1 will introduce a new type of number: integers. Several models for integers will be shown: a number line that extends "to the left" beyond zero and two-coloured counters. (Although two-coloured counters can be purchased, it is easy and less expensive to use large white beans, which students can colour using felt tip markers or crayons.)

Session 2 will address addition and subtraction with integers using the number line model and the two-coloured counter model.

Session 3 will be devoted to multiplication and division of integers using the model of arithmetic patterns.

Because this is the last session of the Number and Operations unit, there needs to be time allotted for students to reflect on the mathematical *process standards* that were used during this unit:

- Modelling and multiple representations
- Mathematical communication
- Problem solving
- Connections: both to real life situations and to other areas of mathematics (algebra, geometry, and information handling)

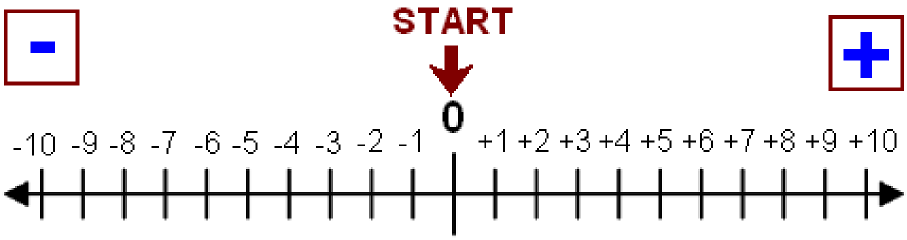
Unitt 1 Number and Operations
Week 5, Session 1: Introduction to Integers

16. <u>Maths Concepts to be studied</u>

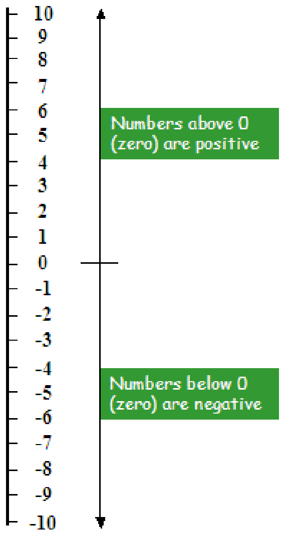
- Integers, like fractions, are another type of number in our number system. The following PowerPoint presentation is a short introduction to integers' place in our number system and the integer number line model that can be used to think about it: http://tinyurl.com/Integers-Num-Line-PPT

- Models for Integers:

    - The number line: Up to this point students have worked with number lines that began at 0 and went on indefinitely in a positive direction. They also created and labeled fraction strips to show numbers on the number line between whole numbers (especially those between 0 and 1). Now students will extend the number line beyond zero in order to deal with negative integers.



- We can also think of a vertical number line



- Notice that simply by introducing the integer number line, notation for the negative sign also has been introduced. Although published materials rarely make this distinction, it is helpful when writing on the board or creating worksheets to use a short dash (-) or a superscript dash (¯) in front of the number to indicate the "negative" sign, and a longer dash (—) to signify the operation of subtraction such as in the expression 2 — (-3).

Students will use an integer number line like the ones above not only to experiment with addition and subtraction of integers in the next session, but also to explore several other topics today: opposites, absolute value, a four-quadrant coordinate plane, and the use of integers in real life situations.

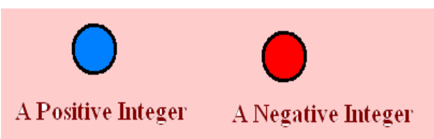
- Two-colour counters: This is a model that will be used in the next two sessions on operations with integers. If you have not worked with 2-colour counters, it is *crucial* that you practice with them before introducing and using them in the next two sessions.

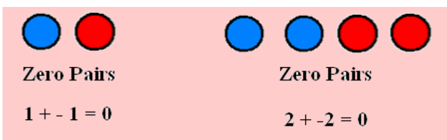
  Commercial two-colour counters usually have black or red on one side and white or yellow on the other. However, two-colour counters can be made by simply taking a bag of large white beans and marking one side with a crayon or felt tip marker. Detailed instructions on using 2-colour counters to model addition and subtraction with integers are included in the article above and in the "Integer Chips Teacher Script" document that will be used in the next two sessions.

- Numerical patterns: Although not a hands-on way of working with integers, number patterns are a way to stress that our number system is an orderly and logical one.

$4 - 1 = 3$
$4 - 2 = 2$
$4 - 3 = 1$
$4 - 4 = 0$
$4 - 5 = ?$
$4 - 6 = ?$

- For each positive integer there is a negative integer that is called its "opposite." For example, the opposite of 2 is -2; the opposite of -5 is 5. In a pair of opposites, each number is equidistant from 0 on opposite sides of the number line. More important, when two opposites are added their sum equals zero. This concept of "zero-sum pairs" is the foundation of using two-colour counters to model integer addition and subtraction in the next session.





- It is also important that students recognize that zero is neither positive nor negative.

a. Finally, although this section has been on integers, eventually students will need to realize that in order for our number system to be consistent, there will be numbers (fractions such as – 1/2½, decimals such as –0.3, and later irrational numbers) that lie between negative integers.

17. <u>How children think about these concepts</u>

a. When youngsters begin to work with integers, they need to deal with multiple concepts and integrate them. This is because they are being exposed to a new kind of number in our number system, a type of number that differs from whole or natural numbers.

   Recall that dealing with and integrating concepts happened when children first learned about fractions. In order to develop "fraction sense," they needed to connect new terminology ("fourths" rather than "four"), a new symbol for notation (the fraction bar), a new relationship between two numbers (a/b), and several new visual models to help them understand this new concept. And all of this needed to be done before they could meaningfully compute with fractions, decimals, and percents.

   Similarly, this same long list of tasks relates to integers as youngsters begin to develop "integer sense." This integrating is a complex procedure that means, just as with fractions, students will need both multiple models for integers as well as time and relevant activities to make sense of this new type of number.

b. Just as children often use whole number thinking when considering the number of digits in decimals (thinking incorrectly 3.0001 must be greater than 3.1), they tend to apply whole number thinking to integers. A common misconception is that a number like -14 must be greater than 3 because -14 has more digits. Children need to work extensively with number lines in order to realize that <u>*any*</u> number to the right is greater than any number to its left, and that any positive number (even 1) is *always* greater in value than any negative number (even -100).

c. Giving youngsters rules for integer operations before they understand integer concepts is almost always ineffectual. Not only do children not comprehend the meaning of what they are doing, but the various rules simply become a list to memorize—and often forget. If students continually need to refer to a reference sheet when doing operations with integers, it is a sign that they have not yet internalized basic concepts about integers.

d. As mentioned above, youngsters often become confused by the subtle distinction in notation of the dash (indicating a negative integer) and the minus sign (indicating the operation of subtraction). And as such, <u>*students need reminders to call -3 "negative 3," not "minus 3."*</u>

e. Even very young children can be introduced to negative numbers simply by adding an extension to the left of their classroom number line. (Some commercial number lines for the primary grades actually have the negative numbers from -1 to -10 written in red.)

3. What is essential to know or do (in class)

f. Introduce integers as a new type of number with a new type of notation.
g. Introduce three integer models: number line, 2-colour counters (and zero-sum pairs), and mathematical patterns.
h. Clarify vocabulary associated with integers, especially negative versus minus, and zero-sum pairs.
i. Relate each of the above to children's thinking.

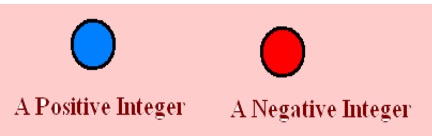
4. Class Activities

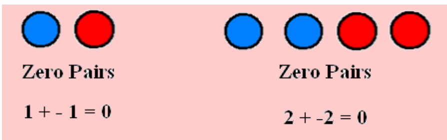
j. Begin by asking students to share what they know and remember about integers. Chart their responses. New ideas can be added over the next two sessions when they deal with operations involving integers.

Introduce integers as a new type of number. If possible show the PowerPoint presentation at http://tinyurl.com/Integers-Num-Line-PPT. Note that this presentation introduces the model of the integer number line and notation for negative numbers. If you cannot use this presentation in class, create a Venn Diagram of our number system on the board.

k. Distribute one copy of the integer number line to each student, asking how this number line is different from the one they used before. Using the number line as a tool, ask about number pairs such as 2 and -2. What do they notice about them? As students respond, build their ideas into a discussion of opposites. If no one mentions it, ask about the sum of 2 and -2 in order to lead into a discussion of zero-sum pairs.

Briefly introduce the idea of "zero-sum pairs" and ask how they relate to the concept of opposites. (To demonstrate zero-sum pairs on the white board use Post-it-Notes of two different colours.)





- For the last activity, have students use graph paper to create a four quadrant coordinate grid with both x and y parameters as -10 to 10. Have students think of the grid as the intersection of the x-axis (the horizontal integer number line) and the y-axis (the vertical integer number line). Have them plot the points: (5, 4) (-5, 4), (-5, -4), and (5, -4). What do they notice?

- Briefly show the model of pattern continuation as described above and ask for a rationale for why it is a valid way to think of negative numbers.

5. Assignment:

l. Have students review the PowerPoint presentation on integers' place in our number system and how the number line can model positive and negative integers: http://tinyurl.com/Integers-Num-Line-PPT

m. To prepare for the next class, have students work with the following 2-colour chip applet for the addition of integers. Ask them to consider how the applet models zero-sum pairs. producthttp://tinyurl.com/Integer-Chip-Applet

Unit 1 Number and Operations
Week 5, Session 2: Operations with Integers

18. Maths Processes to be studied

- The goal for this lesson is to have students develop algorithms for operations with integers by engaging in various activities. This is in contrast to simply memorizing the rules versus understanding underlying concepts so well that students can reconstruct the rules when necessary.

- Operations are *actions*. In the models and activities described below, there is a distinction between the negative sign of a number and the minus sign indicating the operation of subtraction. Thus, students will physically move objects or act out the operations to build their understanding. These would include actually walking on a number line in prescribed ways to model addition and subtraction with integers, and moving 2-colour chips into zero-sum pairs.

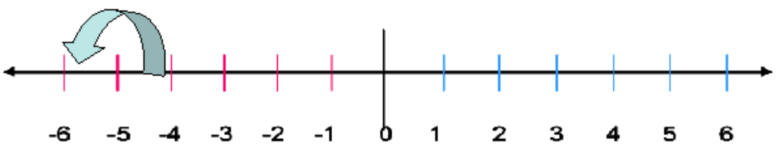
- Different operations are best served by different models:

  - When using (and actually "walking" on) a number line to model an equation, there are two "directionalities" that we need to consider: a) the sign for the number, the direction to which we "face": positive (right), negative (left) versus b) the sign for the *operation*, an action (in which direction we "walk," forwards or backwards).

We will be using variations of one basic equation to explore the concept of adding and subtracting integers with the goal of students seeing patterns and coming to generalizations.

1. For example, to model the equation (-4) + (-2) = x, I would start by standing at -4, the first number in the equation looking toward the left of 0, in the negative direction. Then I consider the second term. Because it is -2, I would stay turned in that negative direction.

Once I have the signs of my numbers in place, I need to consider the operation. Since the operation is addition (+) I *walk forwards* and land on -6.



2. To model the equation (-4) + (+2) = x, I would begin on -4, looking in the negative direction. Then because 2 is positive, I "turn " to face the positive direction, toward the right of 0. Since the operation is addition, I *walk forwards* and land on -2.

3. On the other hand, if the operation was subtraction, (-4) – (+ 2), I would begin at -4, looking in the negative direction, turn to model positive 2, and then *walk backwards* 2 steps and land on –6.

4. Finally, if the equation were (-4) – (- 2) = x, I would start at -4 facing to the left. Then I would stay turned to the negative direction because of -2. However, because the operation says subtract, I would *walk backwards* 2 steps and land on -2.

As students "walk the talk" and record their results they will begin to realize that there is a pattern, a relationship among the four equations:

(-4) + (+ 2) = - 2
(-4) – (+ 2) = - 6
(-4) + (- 2) = - 6
(-4) – (- 2) = - 2

This exercise should also give rise to the generalization about rules for adding and subtracting integers as well as coming to the realization that subtracting a number is the same as adding its opposite.

This is another example of "fact families" where, for example, the whole numbers 3, 4, and 7 could be organized into the following four equations:

3 + 4 = 7
4 + 3 = 7
7 – 3 = 4
7 – 4 = 3

You might also show this applet (also listed in the assignments at the end of this class) to help clarify this "turn and walk" idea:
http://tinyurl.com/Integer-Runner

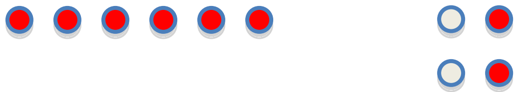
- Recall from Week 1, that two models for the subtraction of whole numbers were "finding the difference" and "taking away." Although an equation such as -4 –(-2) can be modelled by either the number line or 2-colour chips, the number line is a more useful model for problems that have a context that requires finding a difference (such as temperature), while the 2-colour chips model may be more effective in situations such as "taking away" (such as dealing with sums of money).

  (There is a full tutorial on how to use 2-colour chips included in the material for the first hour of the week. Please be sure to work through all the problems yourself before introducing this model in class.)

Briefly, the idea is that when we add opposites such as 2 and -2, the sum is zero. Hence we have the term “zero-sum pairs.” And zero, as we know, can be added to any number without changing the value of the original number.

Consider - 6 – (+2) = x where negative numbers are represented by red chips and positive numbers are represented by white chips. I set up -6:

However, since there are no white (positive) chips to take away I need to add enough zero sum pairs to allow me to subtract a positive two.



When I take away the two white (positive) chips the remainder is -8.

19. <u>How children think about these concepts</u>

a. All the concerns cited in the previous session, Introduction to Integers, are of concern when youngsters add and subtract integers.
b. When using 2-colour counters, students need to understand that adding any number of zero-sum pairs does not change the value of a number.

20. <u>What is essential to know or do (in class)</u>

a. Develop algorithms for the operations of addition and subtraction with integers by engaging in meaningful activities that foster in-depth understanding of these processes operations.
b. Relate above to children’s thinking.

21. <u>Class Activities</u>

a. Begin by asking students to recall a) what they remember about operations with integers and b) how they learned this topic when in school.
b. Let students know that they will be experimenting with integer activities that may be new to them and that they will need to be patient with themselves as learners and that they may experience the same feelings of uncertainty that youngsters experience when working with these activities for the first time.
c. Have several students create a “walk on” integer number line using the 6m strip of paper you brought to class.

d. Introduce the idea of directionality—that we can use the number line to model facing in a negative or positive direction, and then walk either forwards to show addition or backwards to show subtraction.
e. Have several students "walk the line" in response to integer equations like those above. Record the equations and their answers on the board so students can consider a pattern that might develop into an algorithm.
f. Distribute a copy of the integer number line and work through several addition problems and subtraction problems, with the students using arrows (such as shown above) to solve each problem.

- Distribute the beans and have students colour them with crayons or markers to create 2-colour counters. Also distribute the 2-page tutorial *Integer Chips – Teacher Script.*

g. Review the idea of zero-sum pairs and how to use this concept to solve the equations on the tutorial sheet. Work through the tutorial slowly, while students do actual hands-on work with the beans, modelling each scenario.
h. When students have completed the tutorial, distribute the 6-page packet *Integer Chips – Student Worksheets*. Have students work in pairs or small groups to solve the problems and write their own "subtraction rules" after comparing and contrasting various types of equations.
i. At the end of class, have students report on patterns they have discovered (with both the number line and with the 2-colour counters) that led to their developing rules for the addition and subtraction of integers.

22. <u>Assignment:</u>

a. Distribute the following handout which organizes the rules for addition and subtraction with integers: http://tinyurl.com/Integer-Reference

- To help students understand the number line model for addition and subtraction of integers, have them look at this website of a "virtual runner": http://tinyurl.com/Integer-Runner

- To help students understand the 2-colour chip model for addition and subtraction of integers, have them work with these interactive virtual manipulatives:

  - Addition: http://tinyurl.com/Integer-Add-Applet
  - Subtraction: http://tinyurl.com/Integer-Subtract-Applet

Unit 1 Number and Operations
Week 5, Session 3: Multiplication and Division of Integers, Reflection on Practice

This last session of this Number and Operations unit will be a time for students to consider the mathematical processes that were used during the unit:

- Modelling and multiple representations
- Mathematical communication
- Problem solving
- Connections both to real life situations and to other areas of mathematics (algebra, geometry, and Information handling)

Please allow time after the mathematics content of this session so that students can complete the reflection pages and have a whole group discussion about their thoughts.

23. Maths Concepts to be studied

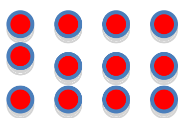
- While physical and visual models were helpful when learning about the addition and subtraction of integers, they are less helpful when dealing with integer multiplication and division. To introduce integer multiplication and division, have students build on what they already have learned about addition and subtraction of integers, multiplication and division of whole numbers, pattern continuation, and the overall consistency of our number system. Thus, this session will be somewhat more abstract than the prior session.

- Models for multiplication of integers:

  Although we discussed earlier that multiplication was more than repeated addition, the repeated addition model is helpful in understanding integer multiplication. This may be shown by:

  a) Repeated leaps of the same size on a number line

  b) Laying out 3 rows of -4 (red) chips:



  c) Repeated addition equations such as:

$$3 \times (-4) = (-4) + (-4) + (-4) = -12$$

  Notice that the sign of the product is negative, as it would be if we started at 0 and took 3 leaps of (-4) in the negative direction.

  Since multiplication is commutative, the expression (-3) x 4 can be rewritten as 4 x (-3), and then solved in the same manner as above. Again, the sign of the product is negative.

  After doing several of these it should be obvious one can simply multiply the two numbers and if one is positive and the other negative, the sign of the product is negative.

It is also important for students to understand that when multiplying a negative number by a positive number, the result is never "greater" than the factors. Have them look at the results of their multiplication and see if they notice this pattern before mentioning it.

- However, when two negative numbers are multiplied, the result is *positive*. In this situation, the number line and repeated addition are less useful models to help students understand why this is so.

  One model that is helpful is to look at patterns with the understanding that our number system is logical and consistent.

  (-3) x 4 = - 12
  (-3) x 3 = - 9
  (-3) x 2 = - 6
  (-3) x 1 = - 3
  (-3) x 0 = 0
  (-3) x (-1) = ?

  The answer cannot be -3; that was the answer to (-3) x 1. So it must be something else. To maintain the pattern of the answer becoming 3 greater each time, (-3) x (-1) must be +3.

  Recall that in Session 1 this week we discussed the concept of opposites. Since (-1) is the opposite of 1, then the answer to (-3) x (-1) must be the opposite of (-3) x 1.

  However logical this is, students may not be convinced. Here is where a real life example can make this concept easier to understand. One useful negative context is time past. For example, I have a $200 car payment taken out of my savings account each month. This can be thought of as -$200. How much more was in my savings account 3 months ago (-3 for time past). The answer is that 3 months ago I had + $600 more in my account.

- Models for division of integers: Since division is the inverse of multiplication, the same models described above apply. Although repeated addition is useful in explaining integer multiplication, students find it more difficult to apply that idea to the division of integers. Thus, beginning by building on what they learned about patterns and the idea of the inverse may be more helpful. Note that by creating the chart in this format, the idea of dividing a negative by a negative immediately results in four examples of a positive product.

(-3) x (+ 4) = - 12	so - 12 ÷ (-3) = + 4
(-3) x (+ 3) = - 9	so - 9 ÷ (-3) = + 3
(-3) x (+ 2) = - 6	so - 6 ÷ (-3) = + 2
(-3) x (+ 1) = - 3	so - 3 ÷ (-3) = + 1
(-3) x 0 = 0	so 0 ÷ (-3) = 0
(-3) x (-1) = + 3	so + 3 ÷ (-3) = - 1

- Fact Families: Just as we could build a fact family for the inverse operations of multiplication and division of whole numbers:

  3 x 4 = 12
  4 x 3 = 12
  12 ÷ 4 = 3

$$12 \div 3 = 4$$

we also can create fact families for integers:

$$(-3) \times 4 = -12$$
$$(-3) \times (-4) = 12$$
$$-12 \div (-3) = 4$$
$$12 \div (-3) = -4$$

24. How children think about these concepts

- All the concerns noted in this week's Session 1 still apply to the multiplication and division of integers.
- Youngsters become confused by the apparent contradiction that when you add two negative numbers the result is negative. But when you multiply two negative numbers the result is positive.
- Even if youngsters can apply the rules for operations with signed numbers, it is no guarantee they understand what those rules mean.

25. What is essential to know or do (in class)

- Introduce models for multiplication of integers, beginning with visuals such as the number line and 2-colour counters, then moving to patterns
- Introduce models for division of integers, continuing the emphasis on patterns and the consistency of our number system.

a. Relate each of the above to children's thinking.
b. Have students reflect in writing and then in a whole class discussion on their learning during Unit 1.

26. Class Activities

a. Begin by asking what students recall about multiplication and division of integers and by what methods they learned them.

b. Introduce integer multiplication by reminding students how they could model negative numbers with 2-colour counters. Ask them how they might multiply 4 x (-3) by using the 2-colour counters. How might they model the same 4 x (-3) by using a number line?

c. Remind students that although multiplication is more than repeated addition, repeated addition can help them understand integer multiplication. Ask them how they would multiply 4 x (-3) by using the repeated addition method. Then ask how we could multiply (-3) by 4, noting the commutative property of multiplication if no one suggests it.

d. As you begin to introduce the multiplication of two negative numbers, ask students how they might model this. Note where there may be misconceptions. Also note anyone who suggests using patterns and ask for elaboration. If necessary, recreate the pattern list above on the board and ask what is happening to the products. Emphasize that our number system is consistent and that because of opposites, the product of (-3) x (-1) cannot be equal to the product of (-3) x (+ 1).

e. Ask if anyone can share a real life example of multiplication of two negative integers. If not, offer the idea of past time (if modelled on a time line); past time can be considered negative if today were at point 0. Present the banking example above to help students understand the concept

f. Introduce the division of integers by building on the patterns developed for multiplication above.

g. Mention inverse operations and how the operations of multiplication and division of whole numbers are related. Ask if students can create a multiplication and division fact family for the whole numbers 3, 4, 7. Then have them create a fact family for integers.

h. Distribute the 2-page reflection sheet and give students about 5 minutes to fill it out.

i. Finally, have a whole group discussion of how the different mathematical processes helped them better understand the concepts they will need to teach.

27. Assignment:

a. Have students download the following fact sheet,that organizes the rules for multiplication and division with integers:

   - Multiplication and Division of Integers Reference Sheet
     http://tinyurl.com/Integer-Mult-Div-Reference

Name: ____Date: ____________

End of Unit Reflection

Teachers (including pre-service teachers) need to reflect on their practice on a routine basis.

This last session of Unit 1 Numbers and Operations is designed to help you experience that type of reflection.

During Unit 1 you worked with many content areas:

- Addition and Subtraction of Whole Numbers
- Multiplication and Division of Whole Numbers
- Fractions, Decimals, Percents
- Operations with Fractions and Decimals
- Proportion, Ratios, Rates
- Integers and Operations with Integers

At the same time you were also engaging in deep mathematical thinking. As you reflect on each of the four processes below, recall *something specific* that you found important because of this class:

- Modelling and multiple representations
- Mathematical communication
- Problem solving
- Connections

1. Modelling and multiple representations (a new way to think about a familiar topic)

2. Mathematical communication (something heard in class discussions or observed in student work samples)

3. Problem solving (an activity in class where you were unsure of how to begin—but figured out how to find a solution)

4. Connections (a surprising connection either to real life situations or to other areas of mathematics: algebra, geometry, or information handling)

Faculty notes
Unit 2 Algebra
Week 1: Patterns, Algebra as Generalized Arithmetic, Children's Algebraic Thinking

Weeklong Overview:

Session 1: Patterns as fundamental to understanding algebra

Session 2: Algebra as generalized arithmetic
Session 3: The algebraic thinking of young children

Faculty Preparation for Upcoming Week:

- Read the following articles and look through the following websites that address patterns and algebraic thinking:
  - The Algebra of Little Kids: Language, Mathematics, and Habits of Mind: http://tinyurl.com/ThinkMath-Early-Algebra
  - Patterns in Numbers and Shapes: http://tinyurl.com/Patterns-Numbers-Shapes
  - Exploring Patterns: http://tinyurl.com/Exploring-Patterns
  - Powerful Patterns: http://tinyurl.com/Powerful-Patterns
  - Patterns in Pascal's Triangle: http://tinyurl.com/Patterns-Pascal-1

- Download and print out for student use:
  - Pascal's Triangle:
    - http://tinyurl.com/Patterns-Pascal-2
  - Centimetre grid paper: http://tinyurl.com/Cm-Grid-Paper
  - Two articles for students to read:
    - What Do Students Struggle with When First Introduced to Algebra Symbols? (3-page pdf) http://tinyurl.com/Algebra-Struggles
    - Algebra in the Elementary Grades? Absolutely! http://www.mathsolutions.com/documents/2002_Algebra_Instructor.pdf or available at http://tinyurl.com/EarlyAlgebra-Absolutely
  - One copy per 4-person group (reflection on reading assignment, "What Do Students Struggle with..."): http://tinyurl.com/AlgebraSymStrug

- Bring to class:
  - Crayons, coloured pencils, or markers
  - Various small objects that can be arranged into patterns

- Read through the plans for this week's three sessions

In Session 1 students will come to understand the importance of patterns as fundamental to algebraic thinking, and why exploring patterns is introduced to children in the early grades. Children first begin work with repeating patterns. They need to notice a pattern and determine the "pattern unit" such as AB AB or ABB ABB. Then they are asked to duplicate the pattern and finally to extend it. Note that children are also exposed to patterns in songs, stories, and physical activities.

Later children are introduced to "growing" patterns of both pictures and numbers as described in the document *Patterns in Numbers and Shapes.*

Young children should also look for number patterns in tables and charts, such as in this addition table of sums. How many 2s and 12s are there inside the grid? Compare that to sums of 11 and 3. What about the number of 7s? What would the table look like if we coloured in sums that were only represented once in one color, twice in another color, etc.? What pattern would emerge?

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Session 2 is devoted to the relationship between arithmetic and algebra, with algebra being a form of generalized arithmetic. This session is unlike most of the prior sessions. It will include far more discussion and fewer maths activities.

Students will need to read the following article, "What Do Students Struggle with When First Introduced to Algebra Symbols?" (3-page pdf) http://tinyurl.com/Struggle-Algebra
which can either be read on its website for homework or printed out by the instructor for distribution at the end of the Session 1.

Session 3 will address young children's algebraic thinking in a more activity-centred mode.

Unit 2 Algebra
Week 1, Session 1: Patterns and Algebraic Thinking

1. What are the important concepts?

a) Patterns are found in nature, in numbers, and in everyday products and activities. They may be as concrete as an AB AB row of coloured blocks or as intangible as the repeating days of the week.

b) There are repeating patterns and growing patterns.

c) Patterns can be duplicated and extended.

d) Although patterns are emphasized in the Algebra unit, they were also part of the Number and Operations unit.

e) It is especially important for children to look for and discover number patterns.

2. How do children think about these concepts?

a) Children begin by noticing that there *is* a patterns and then take the next step, identifying the repeating unit. (In the upper grades, youngsters may be asking *if* a pattern exists in a given data set.)

If given a pattern of coloured cubes arranged AB AB AB, children easily may be able to duplicate the pattern on grid paper using the same colors. However, if they are asked to duplicate the AB AB pattern by using two different types of coins, they may become confused since the attributes of color and shape are different. They are not yet seeing the abstraction of AB as the repeating pattern unit.

b) When asked to extend a block pattern (by adding more coloured blocks to the row) young children usually are able to do this once they have determined the repeating unit.

However, they fail to realize that the pattern can be extended forever--if only there were an infinite number of blocks. They tend to assume that the pattern ends when the materials run out.

c) Teachers need to point out less obvious types of patterns such as the surface design of fabric, the daily schedule (which is a pattern of routine), the repeated refrain of songs, etc.

d) Growth patterns are less obvious to children. Once again, the child needs to be able to notice and duplicate the pattern. When the child is young, he or she is usually working with a visual arrangement of shapes.

The challenge comes when the child is asked to extend the pattern, a more sophisticated task. After working with visual growth patterns, it is important for children to see numerical growth patterns. This can be as basic as the multiples of 2 in skip-counting.

3. What is essential to know or do in class?

a) Identify patterns in the immediate environment.

b) Describe, represent, and extend repeating patterns.

c) Introduce the concept of growing patterns and have students describe and represent them both graphically and numerically, and then extend them.

d) Introduce the idea of a "pattern rule" that sets the foundation for functions in more formal algebra later on.

e) Work with Pascal's Triangle to discover patterns of numerical relationships.

4. Class Activities

a) Begin by having students identify visual patterns in the classroom environment. Then ask about other types of patterns that they perceive in their everyday life. Prompt them to go beyond visual patterns to numerical ones and intangible ones. Notice if all the examples given are repeating patterns. If anyone suggests a growth pattern, ask them to explain this idea and how it differs from repeating patterns. (If no one gives an example of a growth pattern, simply reserve discussion until you introduce the concept later in the session.)

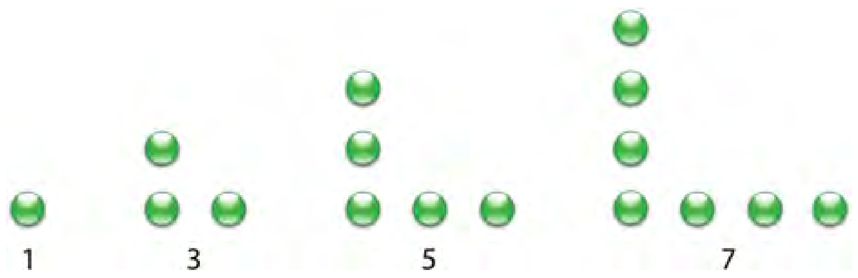
b) Distribute the centimetre grid paper. Using small objects such as coins or cubes, create a pattern that students can translate into coloured squares on their grid paper. Ask about extending the pattern. What did they need to perceive in order to do this? The grid paper is only several centimetres wide. Does the pattern end at the end of the grid paper?

Ask what other kind of repeating patterns are there? (E.g., the repeating decimal of 1/3, the repetition of TV shows from week to week, the months of the year, etc.)

c) Introduce the concept of growth patterns by drawing a simple pattern such as:

Have students duplicate and extend the pattern on their grid paper.

Then ask them to consider numerical growth patterns, something as simple as counting by 2s starting from 1 (not 0). How could they represent those numbers visually? If they need a hint, draw the first two figures in the following sequence:



d) Introduce the idea of a pattern rule. How can we express mathematically the growth that is happening in the triangle and the "Ls"? Mention that these simple rules of +1 and + 2 will take on significant development in formal algebra.

e) Distribute copies of the Pascal's Triangle worksheet. Have students work independently to identify both repeating and growth patterns. If they need a prompt, suggest that they look on the diagonals.

f) To end the session, have students share out and discuss the patterns that they found. Remind students that just as they were challenged by this activity of discerning patterns in a complex numerical and visual format such as Pascal's Triangle, young children can be equally confused when faced with what appears simple to adults: an AB AB pattern of coloured cubes.

5. Assignment

To prepare for the next class session students should read the following article:
http://tinyurl.com/Struggle-Algebra

Unit 2 Algebra
Week 1, Session 2: Algebra as Generalized Arithmetic

1. What are the important concepts?

a) Algebra is a symbolic way to express what students already know from their work with arithmetic, number, and operations.

b) "x" is not only an "unknown" in equations but also a variable in expressions.

c) Equivalent formulas can be expressed in different formats.

d) Adults' past experience learning algebra affects their definition of algebra and how it is taught. Addressing these experiences may require "unlearning" images of algebra, replacing them with images of broader algebraic thinking rather than just symbolic notation and algebraic formulas.

2. How do children think about these concepts?

a) From the article "What Do Students Struggle with When First Introduced to Algebra Symbols?": "It was found that only a small percentage of students were able to consider algebraic letters as generalized numbers or as variables, with the majority interpreting letters as specific unknowns."

Youngster's formal introduction to algebra usually consists of finding x as the unknown, a particular "answer" for a given equation. However, x is not only an unknown. It can also represent a variable where, for example, there is an infinite number of answers for $y = x + 1$. This distinction between the unknown and the variable is crucial for pre-service teachers to understand.

b) Youngsters think the terms *x* and *y* are somehow both mysterious and unchangeable. (I had a school principal once ask, "Why x and y?" To which I responded that x and y could be *any* letters (sometimes n) and that they were purely a mathematical convention. I also noted that x usually represents the independent variable (the input) whereas y usually represents the dependent variable (the output).

c) Youngsters usually are not clear about the relationship among equivalent relationships such as these descriptions for the area of a square. Youngsters tend to see them as totally different.

- Measurement from a numerical formula (the area of a square with a side length of 3 can be expressed as 3 x 3 = 9)
- A symbolic formula (A = s x s)
- The algebraic formula for the area of a square expressed in x and y notation, $y = x^2$

d) Note the difference between the visual orientation for notation in algebra and arithmetic. In most arithmetic equations, the answer comes "to the right." Instead, most algebraic formulas have the dependent variable y ("the answer") on the left.

Later, when y is replaced by function notation f(x) students become confused thinking that this means "f" is multiplied by x since they recently have been introduced to the algebraic format for multiplication such as 2x or ab.

<u>3. What is essential to know or do in class?</u>

a) Have a small group, then whole class, discussion about the article, “What Do Students Struggle with When First Introduced to Algebra Symbols?”

b) Have students work with familiar algebraic equations by using “n” as a variable to develop patterns that illustrate how algebra can be thought of as generalized arithmetic.

<u>4. Class Activities</u>

a) Begin the session by dividing students into small groups to discuss “What Do Students Struggle with When First Introduced to Algebra Symbols?”, the article read for homework, Assign each group Question 1 and one of the other discussion prompts. Ask them to discuss both the article as a whole and the questions their group has been given. Allow about 8-10 minutes for the small group discussion.

1) How does this article relate to the way you learned algebra as a student? How can x be both "the unknown" and a variable?

2) What was your first instinct when solving the word problem about sharing money? Did you use arithmetic or algebra? What difference do you see between these two methods? How might you help youngsters begin to shift from arithmetic to algebraic ways of thinking?

3) When you saw the two-column chart and looked for patterns, what did you see? What type of thinking led youngsters to come up with other patterns? Were those patterns valid? Why or why not?

4) When you saw the growth pattern in Figure 2, how did you extend the pattern to find the number of sticks in Figure 25?

5) What do you think of the author's comment that "*Moving from arithmetic to algebraic generalizations is a process that has been found to take time*." If this is so, when should the "algebraicifacation" of arithmetic begin?

6) How does the difference between the minus sign for the operation of subtraction and the negative sign for numbers less than 0 influence students’ work in algebra?

b) Have a large group discussion, starting with their overall thoughts of how they were taught algebra. Did they "struggle when first introduced to algebraic symbols"? Was algebra connected to arithmetic when they began studying it? At what point did they notice a connection? Did they see algebra as generalized arithmetic or as equations into which they substituted numbers in order to come up with an answer?

Ask what they thought of the idea of x as a variable, not just as an unknown. Was this a new idea for them?

Then have each group share out their thoughts for the other question they were assigned. Ask for other group's input on the ideas expressed.

To end the discussion, note that all these questions are fundamental to the teaching of algebra and ask if algebraic thinking begins in the early grades. Which model for x (unknown or variable) might younger children find easier to understand? How in later years would they as teachers integrate these two models?

c) For the remainder of the class, have students investigate the link between algebra and arithmetic. In secondary school most students were given generalized equations such as $(n + 1) \times (n - 1) = n^2 - 1$. Have them create a 5-column chart with the column headings:

- *n* (the numbers 1 through 10)
- the expression "*(n + 1) x (n - 1)*"
- the arithmetic computation using *n* in the expression "*(n + 1) x (n - 1)*"
- the expression "*n(squared) -1*"
- the arithmetic computation using *n* in the expression "*n(squared) -1*"
- what do they notice?

d) Have them create a second chart for $(n + 1) \times (n + 1) = n^2 + 2n + 1$. Why does $(8 + 1) \times (8 + 1) = 64 + 16 + 1$, or 9^2 or 81? Does this pattern hold true for all the numbers in their chart? Would this growth pattern continue for all ns?

e) End the session by having students discuss how this activity relates to the idea of *x* as a variable and how algebra can be thought of as generalized arithmetic.

5. Assignment

Have students work the problem on the difference of squares from this website: http://tinyurl.com/Diff-of-Squares Do they understand how the algebraic notation relates to the arithmetic?

Unit 2 Algebra
Week 1, Session 3: The Algebraic Thinking of Young Children

1. What are the important concepts?

a) Children can be introduced to growth patterns at an early age and can learn to extend growth patterns.

b) A problem can give rise to equivalent expressions describing correct solutions.

c) Knowing how to use the distributive property of multiplication over addition is often a key strategy to understanding why expressions are equivalent.

d) Teachers can monitor young children's algebraic thinking and use it to highlight important algebraic generalizations.

2. How do children think about these concepts?

a) As mentioned earlier, young children notice repeating patterns earlier than growth patterns. Because growth patterns are less obvious to them, children benefit from working with real objects before being asked to represent the pattern by drawing.

It is also important that the teacher listen to individual children's way of describing the pattern, since it is likely that they will have different methods for thinking about the problem.

Having students chart their numerical findings on a T-chart is another way to represent the growth pattern.

The idea of multiple representations for the same algebraic function will continue into later grades when youngsters will be able to use graphs as a fifth representation.

b) The distributive property of multiplication over addition is not something that is usually formally introduced to young children. It is likely, however, that children can make informal use of the distributive property in early algebra activities.

Helping students articulate what they are doing in these informal situations paves the way for their having a firm sense of the distributive property when it is formally introduced in later years.

3. What is essential to know or do in class?

a) Introduce the article "Algebra in the Elementary Grades? Absolutely!" and have students scan it to note its format: the author's expositional text and her commentaries on student thinking.

b) Have students engage in the article's activity, extending a growth pattern and coming up with a "pattern rule."

c) Note the distributive property of multiplication over addition and the idea of equivalency.

c) Discuss what they learned about children's mathematical thinking from reading the article.

4. Class Activities

a) Begin the session by distributing copies of the article “Algebra in the Elementary Grades? Absolutely!” Allow students to scan the article, noting that they will go into the article in greater detail during the remainder of the class.

Mention that this is not just an article that presents student *work* for analysis, but that it provides insight into 7 year-old children's thinking as they are *working* on an algebraic activity involving patterns.

As the students continue to refer to the article, have them note the role of the teacher. What does the teacher notice? What does she say? How might she use what she sees and hears during student work time to have a discussion at the end of class to bring students' ideas into sharper focus?

b) Although you might not have enough coloured blocks for students to simulate the problem, you can draw the first two examples on the board and then have students create patterns of trees on blank paper.

Note that this is *really* a "growing pattern" since it's a simulation of how a small tree grows from year to year. (On a practical note, keep in mind that all trees reach a maximum height and that even after several hundred years redwoods and giant sequoias in the US grow wider, not taller. But even these trees show a growth pattern in the cross section of their trunk. Notice how the width of each year's growth layer decreases as the years go by.)

c) After students have drawn about 10 years growth for their trees, ask them to stop and consider a mathematical expression for the tree's growth. E.g., for x = years, (x trunk sections + x leaf sections + the top triangle). Ask them how many shapes they drew for any particular year.

d) The summary is an opportunity to discuss three important mathematical concepts:

- The idea of a pattern rule

- The distributive property of multiplication over addition [x (trunk shapes + leaf shapes) + 1 top shape]

- The idea of equivalent expressions, that [x (trunk shapes + leaf shapes) + 1 top shape] is equivalent to (x trunk shapes + x leaf shapes + 1 top shape).

Early attention to equivalence and equivalent expressions is important because it will be featured in later sessions of this unit.

e) End the session by having students refer back to the article and read through the photo scenarios of the teacher's observation of individual students. What was the teacher thinking about student thinking?

If you were the teacher in this classroom and heard all these student ideas, how would you prepare an end-of-class summary that moved from rudimentary to more sophisticated thinking about growth patterns?

5. Assignments (to be determined by instructor)

Faculty Notes
Unit 2 Algebra
Week 2: Variables, Coordinate Graphs, Multiple Representations, Equivalence

Weeklong Overview:

Session 1: Developing an understanding of x as an unknown and as a variable, coordinate graphing, discrete graphs
Session 2: The importance of multiple representations in algebra, continuous graphs
Session 3: Introduction to symbolic representation and mathematical equivalence

Faculty Preparation for Upcoming Class (1-2 hours)

- Read the following articles and look through these websites that address variables, multiple representations, coordinate graphs, and equivalence:

- The *College Preparatory Mathematics* teaching guide for its Multiple Representations chapter. You'll note on page 17 that high school students are given the same "tree growth" problem that 7 year-olds worked on in the Marilyn Burns article last week. There are also excellent diagrams of growth patterns linked to how they can be represented in various formats: http://www.cpm.org/pdfs/information/conference/AC_Con_Mult_Rep.pdf (Also available at: http://tinyurl.com/Algebra-Mult-Rep)

- Introduction to Mark Driscoll's *Fostering Algebraic Thinking*. Note the list of questions on page 3 that you can use as discussion prompts during class to raise issues about the teaching of algebra with your pre-service teachers:

www.heinemann.com/shared/onlineresources/e00154/intro.pdf (Also available at: http://tinyurl.com/Fostering-Algebra)

- Annenberg Institute at Brown University's *Insights to Algebra's* online teacher education unit, "Variables and Patterns of Change": http://tinyurl.com/Algebra-Insights

- Free online graphing calculator. This free on-line graphing calculator is available 24hrs /7days a week. It requires no subscription, no downloading, no software, and has no advertising.") In line with the idea of multiple representations, this calculator also creates a table of values next to the graph. http://tinyurl.com/Free-Graph-Calc

- Download and print out for your use in lesson planning:

    - "The Coin Graph" activity: http://tinyurl.com/CoinGraph
    - *A colour transparency* of solutions for "Tiling the Pool": http://tinyurl.com/TilingPoolSolutions

- Download and print out copies for student use:

    - Kitchry (Moong Dal Rice) Recipe: http://tinyurl.com/Kitchry-Rice
    - Graph Stories: http://tinyurl.com/GraphStories
    - Tiling the pool problem: http://tinyurl.com/PoolHandout

- Bring to class:

    - Coins (enough so that each student can receive 2)
    - Chart paper (ideally chart-sized graph paper) and markers
    - Graph paper
    - Rulers
    - Crayons or coloured pencils

- Read through the plans for this week's three sessions

Weeklong Overview:

Session 1 starts with an introduction to x. Although most students have been taught to think of x as the unknown to solve for in an equation, this course begins with another meaning for x: a variable. Students will engage in a short discussion about what they think this new interpretation of x means, leading to an emergent understanding of what variables are, when they are used, and how they express change.

Talking about variables in the abstract, however, is not particularly helpful to students' understanding of what variables are, how they are used, and how they express change. In order to help make sense of variables, this session will also introduce coordinate graphs to help students visualize how change in one variable affects a change in the other. The idea of direct variation will be introduced along with a definition of a discontinuous graph.

Students may also need to refresh their memory of how to set up a first quadrant graph, label the graph, label the axes, and plot points. This will be one of the times that the instructor will use the pedagogical strategy of whole class demonstration, having students co-create a table of values (T-chart) and coordinate graph.

Session 2 continues with the idea of how multiple representations of the same algebraic situation need to be recorded. After having explored a discrete graph in the prior session, the students will now work with continuous graphs where a function is continuous.

The session will begin with students looking at two graphs on the handout "Graph Stories" and then asking them what they think these graphs mean. How are they different from discrete graphs? Both of the graphs on the handout involve the idea that time is continuous, but that *change over time* is continuous *only for a specific interval.* The session will then move on to look at a graph of a linear function.

Session 3 will address the concept of equivalence by using the classic problem, "Tiling the Pool." The focus here will be to integrate a narrative, drawing, table of values on a T-chart, and a coordinate graph with the goal of discovering a symbolic representation for a pattern rule. Students will work in pairs to come up with a generalized function to describe iterations on the visual representation, again thinking of how to extend a growth pattern, but this time by a function rule, not by a counting method. Note that there are multiple equivalent expressions as solution to this problem. (My pre-algebra students have discovered 10 equivalent expressions!)

Unit 2 Algebra
Week 2, Session 1: Variables and Coordinate Graphs

1. What are the important concepts?

a) Variables are symbols (often letters) used to represent patterns of change. The input variable is called the independent variable; the output variable is called the dependent variable.

b) Variables are used in "pattern rules" (later called functions) to indicate a relationship and a rate of change between variables.

c) Patterns of change can be represented on a coordinate graph that is created from data collected on a T-chart.

d) The coordinate plane is divided horizontally by the x-axis (for the independent variable) and vertically by the y-axis (for the dependent variable). This is why the columns of a T-chart are often labeled with the variables x and y.

2. How do children think about these concepts?

a) Usually a youngster is first introduce to algebra by being asked to find the value of x as the unknown, a unique "answer" for a given equation. However, x is not only "the unknown," it can also represents a variable, as where there are infinite answers for the expression x + 1. The distinction between these two meanings for x is important for teachers to understand so that they can communicate this difference to the youngsters in their classrooms.

b) When setting up a graph, students need to create a scale on each axis with a consistent interval between each cell on the grid. Youngsters often "scale the data," by using the numbers, which may not have a consistent interval between them, from their T-chart.

c) Youngsters have a tendency to connect the points they have plotted on their graph. When changes are continuous (as for the circumference of a circle), the points should be connected to show that continuous change is happening between the data points. In other cases, often involving counting objects, there is *no change* between the data points and the student should *not* connect the points. This kind of graph is termed a discrete graph.

3. What is essential to know or do in class?

a) Introduce variables, what they are and when and how they are used.

b) Remind students how to set up a first quadrant coordinate graph.

c) Have students collect data, record it on a T-chart, and then transfer that information to a coordinate graph.

d) Help students understand how to:

1) Set the range and appropriate scale for each axis
2) Label each axis and title the graph
3) Plot the data points (x, y) from the chart they created

e) Introduce the idea of a pattern rule that can be used to describe the change between two variables.

f) Introduce the concept of direct variation where the variables change at the same rate. (Note that this is often not true of situations where data is collected in the real world.)

4. Class Activities

a) Begin by having students give examples of things that change over time. Ask how those changes might be represented. Students may, for example, refer to the "tree" problem done in the last session where the number of blocks used changed according to the age of the tree.

Other changes such as distance as a function of time related to speed (d= rt) or the change in a circle's circumference as a function of its diameter [C = pi(d)] may be mentioned. Note: what students often call formulas are actually pattern rules put into equations.

b) Use "The Coin Graph" activity. Do this activity as a whole class demonstration while students create the same T-chart and graph.

During the coin collecting activity, create a T-chart of the data, adding x and y to the names of the two columns. Note that x is the independent variable, and y is the dependent variable.

When this activity is complete, refer to the empty grid paper on which you and they will create a graph together. Demonstrate how to set up a first quadrant coordinate graph. Note that you labeled the axes to reflect issues for this particular activity, but that the conventional way of discussing the two axes is to call the horizontal axis x and vertical axis y.

Next, ask how you should add a scale to each axis. What would be a reasonable interval to get all the data from the T-chart onto the graph? Proceed to label the axes with numbers.

Have students come up to the graph, refer to their T-chart, and plot the points. When finished, ask about a "latecomer" scenario. How would the graph look if someone added their coins after class was in session, extending the table and graph? What if this activity were done in an auditorium with many more people? How would that affect how they might scale the graph?

What about a pattern rule? What in the table and on the graph suggest one? Is the pattern rule the same for both the table and the graph?

Note that this activity involved what is called direct variation, where the numbers changed at a constant rate. Mention that not all change between two variables is constant.

Ask students if they think they should connect the points on the graph. This will give rise to the difference between discrete and continuous graphs.

e) End the session by noting that in the next session they will continue to explore various ways to represent algebraic data and that they will look at data that is not discrete but continuous.

5. Assignments (to be determined by instructor)

Unit 2 Algebra
Week 2, Session 2: Continuous Graphs, Multiple Representations

1. What are the important concepts?

a) The appearance of a graph can imply a story of change.

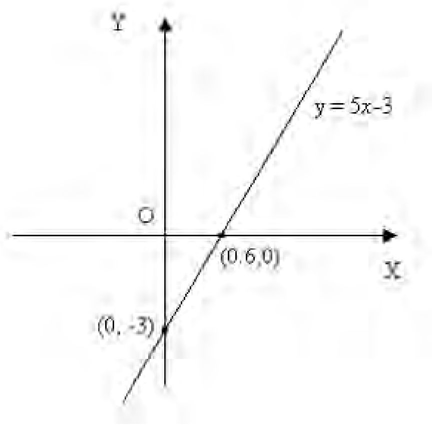
b) Change on a graph is shown in certain intervals.

c) Some continuous graphs are composed of only one straight line, which shows a linear function.

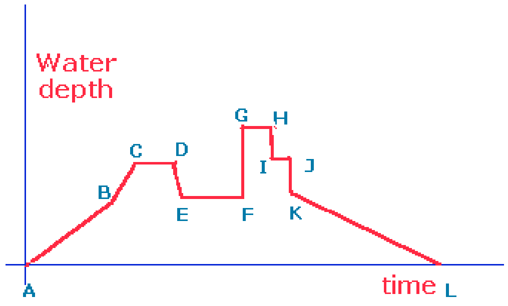
Tables can also show this by having a constant "first difference" between consecutive y values.

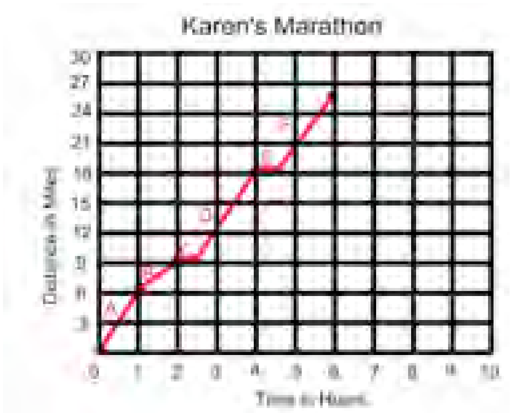
2. How do children think about these concepts?

a) Many youngsters think that graphs are simply a mathematical arrangement of points and lines on a grid, done for their maths teacher, and created in a procedural manner. To help them expand their thinking about graphs, it is important that they analyse graphs that do *not* look like this abstract textbook illustration (which is not even presented on a coordinate plane):



Using graphs such as the water level and marathon graphs, allows them to think about how a graph can be interpreted to tell a story of real world events.





b) Similarly, because algebraic graphing is often introduced in a unit on linear functions, youngsters may assume that algebraic graphing is just a matter of drawing a straight line between two points, and that all algebraic graphs are "line graphs."

c) By using multiple forms of representation, youngsters who formerly had this vision of algebraic graphs can begin to see important connections between diagrams, tables, graphs, equations, and written and oral narratives.

Although all of these representations can model the mathematics in real world situations, the most important reason for using multiple representations is to illustrate a mathematical connection. When students realize this, they begin to understand the power of algebra more deeply.

d) When entering data into a table of values (T-chart), youngsters often create column headings that use the first initial of the variable. Thus, when exploring the relationship between the side length of a square and its area, they may label the first column "s" and the second column "A." At some point youngsters need to generalize the idea of the variable and understand why those columns are conventionally labeled "x" and "y."

This becomes especially important when students have access to graphing calculators (either hand-held or on the Internet at: http://tinyurl.com/Graph-Calc-Free) because calculators *only* accept x and y as variables (unless a hand-held one has been programmed with a specific formula).

3. What is essential to know or do in class?

a) Introduce the idea that a graph can be interpreted to tell a story of change over time.

b) Have students use the real world context of increasing and decreasing ingredients in a recipe to illustrate the difference between a discrete and continuous graph.

c) Have students develop a symbolic expression that could show the relationship between x and y.

d) Have students discuss the connections among all the various representations used in solving the problem.

4. Class Activities

a) Begin the session by distributing the handout of the water level and marathon graphs.

Ask students how these differ from the discrete graphs they explored in the prior session. Noting that they probably learned about line graphs as a secondary school student, ask if these two graphs could be considered line graphs. Have them consider that each of these is a graph composed of several line segments, each constituting something that was continuous, but which happened within a particular interval or time frame.

Ask them to pose a timeline about what different lines in the graph of water in a tub might mean. Why does it begin and end at 0? What might account for the fall and rise in the depth of the water.

When analysing the marathon graph, draw attention to question 2, which asked about what happened during Interval C. What does a horizontal line on a graph mean?

b) Distribute the handout of the Kitchry recipe, which asks students to consider how the recipe will change if they needed to decrease or increase each ingredient in order to serve different numbers of people. Have students chart and graph two different types of ingredients: the lentils (a measurement expressed in pounds) versus the cardamom (expressed as a countable item). Can they predict what each graph will look like?

When they have completed the assignment, ask questions to help students analyse and compare the two graphs. Then compare them to the coin, water level, and marathon graphs.

Can the students have predicted what the graph would look like by using the table? What patterns do they notice? Introduce the idea of first difference, the difference between successive values for y. Note the lentil graph is called a linear function, whereas the graph describing the data is a single straight line.

Finally, ask students to write an expression in symbolic terms that could serve as a rule for this linear relationship.

e) End the session by referring to the use of multiple representations (narrative, table, graph, and expression) that were used to solve the problem. Ask what connections they saw among these different approaches to solving the problem. Ask what they thought were the advantages and disadvantages of each representation.

Ask which one of these representations seemed most useful. If students offer a strategy that relies more on arithmetic than algebra try to steer the conversation toward an algebraic approach.

5. Assignments

Have students experiment with this free online calculator to see how the work they did by hand today is interpreted by technology: http://tinyurl.com/Graph-Calc-Free

Why does their cardamom graph (a discrete graph) show up as a line when using the calculator?

Unit 2 Algebra
Week 2, Session 3: Equivalence, Multiple Representations

1. What are the important concepts?

a) The idea of the equals sign, discussed in Week 1 Session 3 of the Number and Operations unit takes on new and extended meaning in this Algebra unit where symbols (not just numbers) are added to students' thinking about equivalence

b) Not only can numerical equations such as $3 + 5 = 7 + 1$ be proved to show equivalence, so can algebraic equations such as $4s + 4 = 4 (s + 1)$.

This idea of determining proof of equivalence, justified by symbol manipulation, is one of the cornerstones of algebra, moving young children's informal algebraic thinking to more formal ways to think about equivalence.

c) Multiple representations of a problem show how the a table of values and its corresponding graph can give rise to different symbolic expressions that are equivalent to each other.

d) The distributive property of multiplication over addition becomes an important solution strategy when comparing expressions to investigate their equivalence.

e) Symbolic representation allows students to move from the calculation of specific data to a generalized formula that can be expressed in variables.

2. How do children think about these concepts?

a) When youngsters are given the "tiling" problem for a pool with a side length of 5, their first instinct usually is to assume that the solution must be either:

> 1) The perimeter surrounding the border tiles (28) or
> 2) The perimeter of the pool (20) (not considering that there are 4 corner tiles).

Because of what they have learned about perimeter in earlier grades, they think that one of these two *lengths* can be translated into the *number* of tiles surrounding the pool.

Teachers need to be aware of and anticipate such common student responses so that they can sensitively redirect a youngster's thinking to the question posed.

b) In the middle grades, students will begin to have a formal sense of the distributive property of multiplication over addition. They will need to be reminded of the informal way they have used the distributive property in earlier grades and then be directed to the procedural way in which it is applied when using symbols.

c) In addition to the use of multiple representations, the Tiling the Pool problem involves the informal use of symbolic notation and symbol manipulation. Research shows that students need opportunities to solve problems involving symbols and symbol manipulation informally *before* being introduced to these procedures in a formal manner.

3. What is essential to know or do in class?

a) Have students solve the Tiling the Pool problem, which involves a growth pattern, by using multiple representations as solution strategies. After working with one-digit numbers for the side length of the pool, can they extend their pattern rule to a pool with the side of 50 units? For a pool with a side length of unspecified units?

b) Have students represent a variety of solutions in symbolic format, then use symbol manipulation to demonstrate that all their correct solutions were equivalent.

c) Introduce the idea that although all their solutions were equivalent, they might not look the same if the problem was solved without a corresponding coloured diagram. Use the coloured transparency to emphasize that while solutions may be mathematically equivalent, they are not necessarily the same in a real world situation.

d) Have students connect the various representations they used in solving this problem.

4. Class Activities

a) Introduce the Tiling the Pool problem by distributing the student handout. Refer to the directions, answer any general questions, and have students work in pairs to solve the problem.

Allow plenty of time for this, since students will be creating diagrams, a table, a graph, and several symbolic expressions that all represent the same situation.

Listen to their conversations during this assignment as they pose tentative ideas and resolve them.

b) After students have finished finding symbolic expressions, bring the group together and ask about the symbolic expressions that their pictures, table, and graph helped them discover. Have students report out their expressions and note them on the board. Some student contributions may be:

- $4s + 4$
- $4(s + 1)$
- $2s + 2(s + 2)$
- $2s + 2s + 4$
- $4(s + 2) - 4$
- $s + s + s + s + 4$

Ask how these (correct) expressions, which all look different, can be proved equal. If someone mentions the distributive property, follow through on this. If not, this is the time to formally introduce the concept. If a student suggests an expression that is not on the above list, have them try to show equivalence.

Ask if they saw "first differences" in their table. How did that contribute to how they envisioned the graph?

c) If no one has suggested it, ask about the following expression:

- $(s + 2)^2$ squared - s^2

Is it equivalent to the ones above? (It is.) Moreover, how can a quadratic expression involving two square numbers result in a linear solution?

d) Ask students to think decoratively. What would some of the above patterns look like if a designer used more than one colour tile for the border? Ask how the following expressions (tile designs) would look in colour. Have students draw three 4 x 4 pools plus their surrounding tiles on graph paper. Have them colour in a border that could show these three different border designs:

- $N = 4s + 4$
- $N = 4(s + 1)$
- $N = 2s + 2(s + 2)$

e) End the session by asking how this week's focus on multiple representations has influenced their own algebraic thinking. How might this have changed their assumption as to how they would introduce algebraic concepts, graphing, and multiple representations to their future students?

5. Assignments (to be determined by instructor)

Kitchry (Moong Dal Rice)

For 2 servings:

- 1 cup rice
- ½ cup dal chana
- 1 ½ teaspoon salt
- 1 teaspoon turmeric powder
- ½ cup sliced onions
- 1 whole dry chili, (optional)
- 2 cardamom pods
- 3 sticks of cinnamon
- 2 star anise
- 4 tablespoons of canola oil

Method

- Pick lentils and rice. Soak lentils in a dish of water
- Wash onion and chili. Slice onion
- Drain water off lentils and add all ingredients into pressure cooker
- Add water to just cover contents of pot
- Switch on stove. When the pressure rises, time it for 8 minutes.
- Release pressure, open pot and taste. Adjust salt if you would like more flavor

Think about the amount of ingredients in this recipe designed for four servings.

What if you wanted to serve only one person? How would that change amount of ingredients?

On the other hand, if you have a large family and were having a gathering you might want to make 8 or even 16 servings. How would that change the amount of ingredients?

Create a T-chart of <u>*how much*</u> dal you would need for 1, 4, 8, and 16 people.

Create another T-chart chart showing <u>*how many*</u> cardamoms you would need for the same number of people.

What do you think is a pattern rule for each?

Translate your two charts into graphs.

How did you scale your axes? What patterns do you see? Is one graph continuous and the other discrete? How do the phrases "how much" and "how many" differ mathematically? What do they mean when graphing a pattern rule?

Faculty Notes

Unit 2 Algebra
Week 3: Linear Functions, Slope, Order of Operations

Weeklong Overview:

Session 1: Introduction to linear functions
Session 2: Introduction to the concept of slope
Session 3: Introduction to the conventional order of operations

Faculty Preparation for Upcoming Class (1-2 hours)

- Look through the following websites on linear functions and order of operations:
    - Annenberg Institute at Brown University's *Insights to Algebra's* online teacher education unit, "Linear Functions": http://tinyurl.com/Insights-Linear
    - Rise-Run Triangles (a homework assignment includes "Just Slope" below): http://tinyurl.com/Rise-Run-Triangles

- Download and print out for student use the following documents:
    - Taxi Cab Fares: http://tinyurl.com/NewYork-Taxi-Fares
    - Stairs According to Code: http://tinyurl.com/Stairs-Slope
    - Just Slope: http://tinyurl.com/Just-Slope

- Bring to class:
    - Graph paper
    - Rulers
    - Crayons or coloured pencils

- Read through the plans for this week's three sessions

Weeklong Overview:

Session 1 builds on last week's introduction to linear functions, briefly reviewing the characteristics of the table, graph, and symbolic representation of the recipe problem.

Students will be guided to notice that when using conventional x and y notation their equation resulted in the generalized equation of $y = mx$ where m was a multiplier of ingredients in the original recipe.

It is also important that they understand that the multiplier (m) can be not only a whole number but a fraction, decimal, or a mixed number.

After this review, students will engage in a problem where the y-intercept (which has not been formally discussed) is *not* 0. The informal definition of the y-intercept will be that of a "starting point," such as in a problem involving finances where there is a basic initial cost.

For example, a taxicab in New York City imposes an initial cost for the ride. Then a cost is added for each fraction of a mile travelled:

- \$2.50 upon entry into the cab
- \$0.40 for each one-fifth of a mile travelled

This can be modeled by a linear equation where the "starting cost" (the y-intercept) is 2.50, to which is added 0.40*x* with x being the number of "one-fifths of a mile" travelled.

Although there are certain problems, such as tiling the pool, that lend well to diagrammatic representations, the taxicab problem is more likely to be solved by talking through it and then representing it in a table and graph before coming up with an equation.

This taxicab problem, like the recipe problem, is an introduction to linear functions set in a real life context to which adults like pre-service teachers can relate. Finding linear function problems that relate to middle grade students can be challenging. However, it is important that pre-service teachers begin to understand linear functions more deeply by solving problems that are meaningful to them as adults.

Session 2 will emphasize how linear equations in the form of $y = mx$ are related to slope. The idea of "first difference" mentioned earlier will become important, since it relates to the conventional formula for slope, which is expressed as the "difference in y related to the difference in x."

Your mentioning "first difference" in last week's session sets a foundation for students understanding how subtraction in the following symbolic representation is related to the first differences they noted on their chart.

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Session 3 begins by looking at the conventional order of operations in arithmetic and the use of clarifying devices such as parentheses when evaluating more complex algebraic expressions. As usual, this will be approached not by presenting a series of rules, but by giving students an arithmetic problem requiring several operations that may perplex them, resulting in more than one answer. When discussing their answers, students will be introduced to the conventional order of operations, and then move to discover how algebra uses additional notation to make the order of operations much more clear.

Unit 2 Algebra
Week 3, Session 1: Linear Functions

1. What are the important concepts?

a) There is a connection between the various representations for linear functions.

b) All linear functions have certain characteristics in common: a straight line graph, a constant difference on a table, and an equation written in the form of y = mx + b.

c) The "b" in y = mx + b is the y-intercept on a graph.

d) You can recognize a linear function simply by looking at a graph, a table, or an expression.

e) Differences in the appearance of a graph showing the same data points are due to a difference in scale.

2. How do children think about these concepts?

a) When introducing linear relationships, textbooks often present the formula y = mx + b, give a table of values, and ask youngsters to create a graph.

In this course, we take the opposite approach:

1) Learn about a table of values by creating what is called a T-chart
2) Create a graph by using the table of values.

Only when these two skills are in place do we introduce symbolic notation. This sequence of events helps students make sense of the equation for a linear function:

$$y = mx + b$$

b) When presented with a linear function problem where the y-intercept is *not* 0, middle grade students are often surprised or confused. This is because the emphasis in their prior work may have been with situations where graphs began at (0, 0).

c) Similarly, if youngsters have worked only with first quadrant graphs, they will need the teacher’s help so that they can envision the entire coordinate plane. This is where their integer work in the Number and Operations unit with horizontal and vertical number lines can help them connect with a four-quadrant graph.

3. What is essential to know or do in class?

a) Introduce a linear function problem where the y-intercept is not 0.

b) Have students create a table, graph, and symbolic expression to represent fares over various distances.

c) Have a whole class discussion that brings out similarities and differences between the taxicab problem and the recipe problem from last week.

d) Introduce the conventional format for linear equations, y = mx + b.

4. Class Activities

a) Begin the session by distributing copies of Taxicab Fares and ask students to work in pairs to create a table, chart, and symbolic expression to represent fares over various distances.

Note how students set up their graph. Do they scale the y-axis in whole number amounts? What interval did they use for distances, which are expressed in "one-fifth of a mile" units? To how many miles did they extend their table of values?

It is important that pairs of students make these decisions independently, and that you do not give them specific requirements. This is to illustrate later that the same data can look different when plotted on graphs that are scaled differently.

When students have finished the assignment, begin a discussion about the "look" of each graph. Are they all the same? If not, how are they different? Why is this so?

Have students recall the recipe problem from last week, asking how the taxicab problem is similar to it.

Students should note that the pulse graph showed a straight line, the table showed a constant first difference, and the expression was in the form of mx.

Ask how the two problems differ.

Continue by discussing that because there is an initial fee, the graph begins at 2.50 on the y-axis. This is different from most graphs they have encountered, which begin at (0, 0).

Tell students that there is a special name for the place where a line crosses the y-axis: the y-intercept.

Ask how this is represented in the expression they wrote. Remind students of the expression they wrote for the recipe problem last week, that it was in the form of y = mx.

What is the form of the taxicab problem's expression? Ask if there is a connection between the y-intercept on their graph and the 2.50 in their expression.

Note that in the recipe problem the graph started at 0, so the expression could be thought of as mx + 0. However in the taxicab problem, the expression is mx + 2.50. Tell students that the conventional notation for all linear equations is y = mx + b where b is a constant and represents the y-intercept.

5. Assignments (to be determined by instructor)

Unit 2 Algebra
Week 3, Session 2: Linear Relationships and Slope

1. What are the important concepts?

a) Slope is a characteristic of linear functions, the constant rate between two variables.

b) In a linear equation in the form of y = mx + b, the slope is the coefficient "m."

c) Slope is informally described as "rise over run." It can be more formally thought of as "the vertical change divided by ('over') the horizontal change" or expressed by the following formula:

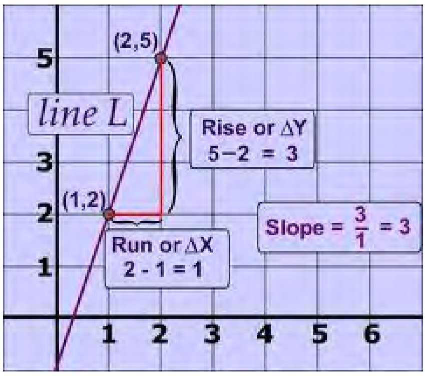
$$\frac{y_2 - y_1}{x_2 - x_1}$$

d) Slope implies steepness, but the same slope on graphs scaled differently may look different.

e) The slope of a line can be positive (going "up"), negative ("going down"), or 0 (a horizontal line where there is no change).

2. How do children think about these concepts?

a) In many textbooks, slope is introduced via a "rise-over-run" visual on a graph. Although the following diagram may seem self-explanatory to adults, it is too abstract to be useful in introducing slope to middle grade students.



Youngsters eventually will need to interpret this visual explanation of slope, but it has too many terms and symbols to be an effective introduction to the concept.

b) Although students usually interpret slope as a measure of "steepness," the scale of a graph can be misleading. Graphs with different scales will present different visual images of steepness. This also true of most graphing calculators where the cell on the graph is rectangular, not square. The rectangular cell means that a line with a slope of 1 does not go up at a 45-degree angle to match the graph that a youngster created on graph paper where the cells are square.

c) Youngsters need to connect the concept of "rise" to the vertical distance on the y-axis and "run" to the horizontal distance on the x-axis in order to understand the conventional formula for finding slope (shown below). However, teachers often assume that students can translate this "rise over run" into the formula "delta y over delta x."

Youngsters are usually confused as to what "delta" means, and how delta relates to the changes in y and x in the conventional formula for calculating slope:

$$\frac{y_2 - y_1}{x_2 - x_1}$$

3. What is essential to know or do in class?

a) Introduce the concept of slope by having students work with the Stairs According to Code problem.

b) Have students discuss the features of their graphs that involve slope.

c) Clarify the difference between a graph's steepness and its slope.

d) Introduce the term "coefficient" and point out its role in indicating slope (m) in y = mx + b.

e) Introduce three ways to talk about slope:

1) Rise over run
2) Change in vertical versus horizontal change, and
3) The formula:

$$\frac{y_2 - y_1}{x_2 - x_1}$$

f) Introduce several equations in y = mx + b format that have a positive slope, a negative slope, and a negative y-intercept.

4. Class Activities:

a) Begin by distributing copies of the Stairs According to Code problem. Have students follow the directions, first by creating a graph, then making a table of values, and finally developing an equation in y = mx + b format. (Again, do not give parameters for the graph so students will create graphs with different scales that can be used for comparison.)

b) Once students have completed the assignment, have them discuss their graphs in informal terms, noting that the ratio (slope) between any two points is 7 "up" versus 10 "over," or 7/10. (It is important that students realize that the same 7 to 10 ratio exists between *any* two points on the line, not just two adjacent points.) Have students use crayons or coloured pencils to colour in a "slope triangle" made by the line as in the picture above.

c) Since students usually equate slope with steepness, point out the different appearances of the graphs they created. Why does the same data have different "steepness" but the same slope?

d) When you move to the symbolic expressions that students developed, introduce the term "coefficient," relating it to the "m" in the equation y = mx + b.

This completes students introduction to y = mx + b: "b" is the y-intercept (a constant), "m" is the slope of the line, x is the input (independent) variable, and y is the output (dependent) variable.

e) Help students see the connection between "rise over run" and "vertical change divided by horizontal change" before introducing the following formula. (Do not assume that they will have a clear understanding of the connection among all three ways to express slope by the end of the lesson.)

$$\frac{y_2 - y_1}{x_2 - x_1}$$

f) Present the following set of equations in y = mx + b format. Ask students to predict what the graph of each will look like. Note that one of the equations has a negative slope whereas another has a negative y-intercept. Do not attempt to go into depth about these two new ways to think about linear relationships on the coordinate plane, but have students use the free online calculator to explore these four equations before the next class.

- y = 5x
- y = 5x + 2
- y = 5x – 2
- y = -5x + 2

g) End the session by distributing the Just Slope handout that will further help students explore lines with a negative slope. They should go to the website below to get more information on how to complete the assignment.

<u>5. Assignments</u>

a) Use the Rise-Run Triangles handout to explore slope.

b) Use the Just Slope handout to explore linear relationships that result in negative slopes.

c) Use the free online calculator to explore these four equations before the next class.

- y = 5x
- y = 5x + 2

- $y = 5x - 2$
- $y = -5x + 2$

Unit 2 Algebra
Week 3, Session 3: Order of Operations

1. What are the important concepts?

a) There are conventions when multiple operations are required to evaluate an expression.

b) These conventions are made easier in algebra by using parentheses to clarify which operations need to be done before others.

c) The context of a given problem can provide clues for using the order of operations to model the problem.

2. How do children think about these concepts?

a) The order of operations can be strange and confusing to children. They assume that they should calculate all numbers from left to right.

In fact, since a basic calculator adds numbers in sequential order, the resulting answer is often incorrect for a given problem. This only strengthens children's belief that this method must be correct because "the calculator says so."

b) When told the correct order of operations, children either do not remember it or they rely on a mnemonic. Remembering the mnemonic, however, is purely procedural and does not help youngsters understand *why* the conventional order of operations is crucial not just in basic calculations but also in real world situations.

c) Once parentheses are introduced into an expression, it makes calculation using the order of operations much easier for children to understand and use.

3. What is essential to know or do in class?

a) Review the homework assignment that helped students use "slope triangles" to visualize slope, especially negative slope and negative y-intercepts. Have students develop an equation in $y = mx + b$ form for several of the graphs.

b) Have students work in pairs to solve an arithmetic equation designed to assess their understanding of the conventional order of operations without parentheses.

c) Introduce the conventional order of operations. Then add how parentheses can clarify how to evaluate expressions that appear confusing.

d) Have students work on a problem that involves order of operations involving a real life situation, then generalize this by using variables.

4. Class Activities

a) Begin class by having students share what they learned by working the homework assignment. Since the assignment focused only on the graph, extend the conversation by asking students to consider how they might use the slope they discovered and the y-intercept to create an equation in the form of y = mx + b.

Do a few of these together, then have them see how quickly they can write equations for the rest of the graphs on the page.

b) Move to this session's focus topic, Order of Operations, by writing the following expression on the board: 5 x 8 + 6 ÷ 6 - 12 x 2. Have students work in pairs to evaluate it. (The correct answer is 17.)

As they work on the problem, notice how they engage with their partner. What rationales do they have for the way they think the expression should be evaluated? Are students negotiating with each other as to how to proceed? For students who move from left to right their answer will be 0.66666... (This is the answer that entering the numbers sequentially into a basic calculator would give--which is not the right answer!)

Notice how long students are working to solve the problem. Students operating from this sequential left-to-right assumption will likely take more time since they will be dealing with fractions or decimals as opposed to students who are working with integers.

c) Ask students to share their answers and how they determined them. This should give rise to several alternative ways of thinking about the problem. Ask why this occurred.

Honour the fact that all students were clever and resourceful when thinking about how to deal with such a confusing calculation but that some of their procedures did not follow the conventional way for dealing with these types of equations.

At this point, tell students that there is a conventional way to deal with these types of equations: order of operations.

d) Introduce the idea that order of operations insists on doing multiplication and division first. Only then can we add and subtract. Have students use this information to re-evaluate the expression so that it results in 40 + 1 - 24.

e) Proceed to say that algebra uses a particular notation, parentheses, to make all of this easier. Parentheses group certain numbers and operations together so it is clear what operation to perform first. By using parentheses, the above expression becomes (5 x 8) + (6 ÷ 6) - (12 x 2). Make clear to students that when looking at an expression, they need to perform the operation in the parentheses *before doing anything else*. Then they would do any remaining multiplication/division, and finally any addition/subtraction.

(At this point do not mention how the order of operations deals with exponents. This will be included in next week's focus on quadratic [square] equations.)

f) Present the following equation where C is cost: C = 100 + 15 x 20. How would students calculate the cost given what they *now* know about order of operations?

g) Extend the discussion about C = 100 + 15 x 20 in several ways. First have students insert parentheses to make the order of operations clear.

Next, ask (given what they now know about start-up costs from the taxi problem) what real world situation this equation might refer to. (E.g., a gathering where there was a $100 room rental plus dinner for 15 people at $20 apiece.) *Just as a graph can tell a story, so can an equation.*

Finally, ask students to disregard the specifics of C as "cost" and translate the above equation into the generalized y = mx + b format. How could this equation be used to calculate an initial deposit of $100 in a bank account with $15 deposited for each of 20 weeks?

5. Assignments (to be determined by instructor)

Faculty Notes
Unit 2, Algebra
Week 4: Quadratics, Solving for x as the Unknown

Weeklong Overview:

Session 1: Introduction to Quadratics (1) Tables and Graphs
Session 2: Introduction to Quadratics (2) Equations, Connecting Algebra and Arithmetic
Session 3: Finding x, the Unknown

Faculty Preparation for Upcoming Class (1-2 hours)

Read the following articles and look through these websites that address quadratic equations and finding x, the unknown:

- Nguyen, D. M. (2004). "Persistent student misconceptions about Algebra and symbolisms: What is that x anyway?": http://distance-ed.math.tamu.edu/Precalculus_home/resources/Misconceptions_in_Algebra_Symbolism.pdf (Also available at: http://tinyurl.com/x-in-Algebra)
- Teacher Notes connecting graphing and Algebra Tiles (Geometry Graphing Connection): http://tinyurl.com/GeomGraph
- Partial product method for multi-digit multiplication (video): http://tinyurl.com/Partial-Prduct-Method (You will need to wait for a commercial to finish.)
- Henri Picciotto's Algebra Lab Gear Website: http://tinyurl.com/Algebra-Lab-Gear-1
- Algebra Tiles Quick Demo: http://tinyurl.com/Algebra-Lab-Gear-Demo

Download and print out the following documents for student use:

- Rectangle Graph Activity: http://tinyurl.com/RectangleGraph
- Quick Guide to various types of equations: http://tinyurl.com/Functions-Quick-Guide
- Graphing Quadratic Functions: http://tinyurl.com/Graphing-Quad-Functions
- Homemade Algebra Tiles: http://tinyurl.com/HomeMade-Algebra-Tiles
- *Colour* copy of "Connecting Graphing and Algebra Tiles" (including page 1 of the teacher notes: http://tinyurl.com/Connect-Geom-Graph

Bring to class:

- Graph paper
- Rulers
- Scissors

Read through the plans for this week's three sessions

Weeklong Overview:

Session 1 starts by students developing a table of values and a graph for what they probably anticipate is another linear function activity.

Unlike in past sessions where you were asked not give students guidelines about parameters and scaling, you will need to be quite directive about this so that students can focus on the attributes of the resulting graph without considering conflicting images.

Regarding their data table: it will have more than four columns, and you will need to tell students which two columns to use when setting up their graph.

Once they have used their table of values to construct the graph they may be in for a surprise: it's not a line. It's a parabola!

At this point you and the students will do two things that will occupy the rest of the class period: 1) analyse their graphs and then 2) analyse their table of values.

When analysing their graphs, listen to the features that students notice and be prepared to ask follow-up questions and comment about intersection with the x axis, a line of symmetry, the parabola's orientation (opening up or opening down), etc.

When analysing their table of values, students will discover that what they learned about "first differences" for linear equations no longer holds true. What does a "second differences" mean?

Students also should notice patterns in their tables with regard to symmetry, similar to this chart:

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	25	16	9	4	1	0	1	4	9	16	25

It is important to ask a variety of questions about how their table relates to their graph.

(A note about limited time: Last week when introducing slope, the focus was on positive slope because there was not enough class time available to discuss negative slope and negative y-intercepts. The same holds true here for quadratic graphs. There will be a homework assignment with a handout so that students can see that some parabolas are "upturned" whereas others are "downturned," some have a line of symmetry on the y axis (as the above table implies) whereas others do not, etc. This is a good opportunity for them to use the free online graphing calculator (http://tinyurl.com/Graph-Calc-Free) to explore the eight different quadratic graphs on the homework sheet. This approach to using technology, where students spend time analysing eight computer-drawn graphs, may be a more valuable learning experience than having them hand-draw each one. The homework sheet has equations in a format that allows students to enter them into the online calculator and see not only the graph but also the corresponding table of values.)

Session 2 continues the idea of multiple representations in algebra by adding quadratic expressions to the tables and graphs that students produced in the prior session. Thus they will need to bring the prior session's work back to class to extend the work they did with table and graphs.

Session 2 will show how the expanded format for a quadratic equation ($y = ax^2 + bx + c$) and the factored format are two different ways to express the same function.

The session will also relate these two formats to the table of values, graph, and algebra as generalized arithmetic. This will be an opportunity for students to revisit the partial products method for multiplication that they studied in the Number and Operations unit, noting how this method relates to quadratic equations.

A key aspect of this session will be the introduction of a manipulative called Algebra Tiles to model how the partial product method applies to quadratics, showing the link between the expanded and factored forms. It is imperative that you read through the Algebra Tiles handouts and websites to prepare for this session.

Session 3 will end this Algebra unit where most algebra textbooks begin: solving for "x the unknown."

Imagine the confusion of this student who says to his teacher: *But yesterday you said 'x equals 2.'*

By this time your students will have experienced x as a variable for over three weeks. Now it is time to shift to the other meaning for x, and deal with the symbol manipulation.

Phrases such as "like terms," have not been specifically noted during this unit. However, by the time students have reached the end of this unit they should have an intuitive grasp of them. When combined with their understanding of equations from the Number and Operations unit, students should be well positioned to use the commutative, associative, distributive, and identity properties to solve for x.

This final session of the unit will also ask students to reflect on what they can do as teachers of young children in the early and middle grades to foster the algebraic thinking that youngsters will need in middle grades and in secondary school.

Unit 2 Algebra
Week 4, Session 1: Introduction to Quadratics (1): Tables and Graphs

1. What are the important concepts?

a) Quadratic (square) equations can be modeled geometrically by rectangles.

b) Quadratic equations appear on a graph as a parabola.

c) Quadratic equations have a constant "second difference" when seen in a table of values.

d) When placed on a coordinate graph, a parabola can be inspected for its maximum/minimum points, line of symmetry, and the places where it may cross the x-axis.

e) It is important to link drawings, tables, graphs, and symbolic ways of representing quadratic equations so that students have a comprehensive picture of how quadratic equations differ from linear ones.

2. How do children think about these concepts?

a) Youngsters seem intrigued by a graph that shows a parabola. Up to now they have seen picture graphs, bar graphs, "line graphs," graphs of linear functions, etc.

This curved-shaped graph is something new. Because of this, youngsters need to create (not just be shown) multiple representations of the same quadratic function to solidify their understanding of this new concept.

b) The point at which a parabola has a minimum or a maximum is called its "vertex." This is a different interpretation of "vertex" that youngsters know from geometry lessons in their earlier years. In their geometry lessons, the word “vertex” described a "corner" of a polygon or polyhedron.

c) When analysing their tables and graphs, youngsters need to continually go back to their drawings. This will help them interpret how the maximum area of the rectangle is shown as the maximum point on the graph.

Numerically, when looking at their table they need to see that the maximum area of the rectangle was s^2 and that their drawing showed a square.

3. What is essential to know or do in class?

a) Have students complete the sketches, table of values, and graph for a rectangle of varying dimensions but with a constant perimeter.

b) Have a whole class discussion about the characteristics of a parabola and the second differences in a table of values.

c) Leave plenty of time to discuss the homework assignments, which will extend today's work by introducing second-degree equations and foreshadowing the next session's work with Algebra Tiles and the links between algebra and arithmetic.

4. Class Activities

a) Begin by distributing the Rectangle Graph worksheet. Have students work in pairs to sketch and label the dimensions of the various rectangles, then create a table of values and a graph to model the problem.

b) When students have finished the assignment, begin by discussing the graph.

- How is it different from the graphs of linear functions that they worked with last week?
- Did the shape of the graph surprise them?
- How would they describe the graph's shape? If no one mentions the word "parabola" introduce it now.
- Continue to probe about the parabola's features.
  - Does it cross the x-axis? If so, where?
  - What are the coordinates of those points?
  - How "tall" is the parabola?
  - What point on the parabola shows its "maximum" height?
  - Does the parabola appear symmetrical?
  - If so, could you draw its line of symmetry?
  - How could you express that?

c) After discussing the graph, discuss the table. Remind students of the "first difference" they saw in the tables of values for linear equations. Have them write the first differences next to their table. Is there a first difference here? Is there another pattern? It is likely that some students will notice a "second difference." At this point distribute the handout *A Quick Guide to Functions* that shows how degrees of "difference" can tell us about the shape of a function's graph.

d) Remind students that they used three different representations today and that in the next class session they will be using two other representations, manipulatives and equations, to explore similar problems.

Also mention that the Rectangle problem resulted in a parabola with specific characteristics. Not all parabolas look like this. Their homework assignment will help show this.

e) Give students the handouts and URLs below for homework. Allow time to explain the assignments.

5. Assignments

a) Distribute:

1) Graphing Quadratic Functions (first two pages of the PDF): http://tinyurl.com/Graphing-Quad-Functions
2) Homemade Algebra Tiles: http://tinyurl.com/HomeMade-Algebra-Tiles

b) Give students the URLs for:

1) The free online graphing calculator: http://tinyurl.com/Free-Graph-Calc
2) Algebra Tiles demonstration: http://tinyurl.com/Algebra-Lab-Gear-Demo
- 3) The video on the partial product method for multi-digit multiplication: http://tinyurl.com/Partial-Prduct-Method (You will need to wait for a commercial to finish.)

c) Have students bring their Algebra Tiles, Graphing Equations assignment, and today's work on rectangles to the next class session.

d) For the Graphing Quadratic Functions assignment, students will explore eight different quadratic equations expressed in the expanded form $y = ax^2 + bx + c$, ideally by using the free online graphing calculator to "draw" the graphs.

Have them:

- Sketch the graphs on the worksheet
- Analyse the graphs for patterns by looking at the coefficients of a, b, and c, and by making conjectures.

e) Have students colour and cut out a set of Algebra Tiles to bring to the next class session.

The x and x(squared) pieces should be coloured blue and the "1" or unit pieces should be coloured yellow. These are the manipulatives they will use to model quadratic equations and show a link between arithmetic and algebra.

Have students look at the Algebra Tiles demonstration to see how the how the tiles are used to model the factored and expanded forms of quadratic equations.

f) The third assignment is for students to look at this video to review the partial product method of multi-digit multiplication: (Let students know there will be an advert at the beginning for several seconds before the video continues.)

Unit 2 Algebra
Week 4, Session 2: Introduction to Quadratics (2): Equations, Connecting Algebra and Arithmetic

1. What are the important concepts?

a) Quadratic Functions can be represented by equations in the form of $y = ax^2 + bx + c$.

b) The coefficients a, b, and c in $y = ax^2 + bx + c$ determine the shape of the equation's parabolic graph.

c) The "second differences" on a table of values indicate a "second-degree equation," where the greatest power of x is 2. (Similarly, a "first difference" indicates a "first-degree" or linear equation where the power of x is 1.)

d) A quadratic equation can be expressed either in its expanded form or its factored form.

e) Quadratic equations can be modelled by a manipulative called Algebra Tiles.

f) The partial products method of multi-digit multiplication, introduced in the Number and Operations unit, can be directly linked to Algebra Tiles.

2. How do children think about these concepts?

a) Because Number and Operations is the major focus of early education, youngsters need strategies to help them link their *number sense* to algebra concepts.

Youngsters who are visual learners need ways to link algebra to their *geometric sense*.

Children who are tactile learners need to work with manipulative materials, moving them around in space, in order to make sense of mathematical concepts.

All these different kinds of learners can benefit from using Algebra Tiles to connect algebra to what they already know, as well as how they need to learn it.

b) When presented with various representations of a quadratic function, youngsters need their teacher to make connections among various representations.

For example, students may not notice that the maximum or minimum of a graph is evident from the table of values. Or that the "c" in the expanded form of $y = ax^2 + bx + c$ is the y-intercept.

Rather than telling youngsters about these characteristics teachers need to ask probing questions that force their students to look for patterns. This means that teachers must find specific examples to elicit youngster's thinking about quadratic patterns.

3. What is essential to know or do in class?

a) Have a whole class discussion of the "Graphing Equations" homework, exploring the relationship between a quadratic equation in its expanded $y = ax^2 + bx + c$ form and the shape of its graph.

b) Introduce Algebra Tiles as a way to model quadratic equations in factored form.

c) Link the Algebra Tile model (using x as a variable) to the partial product multiplication model used in arithmetic.

4. Class Activities

a) Begin by reviewing the homework assignment, "Graphing Equations."

Students were asked to draw the graph of equations by using a graphing calculator, sketch the graphs, and then make conjectures between a graph's shape and the coefficients of the terms in its equation.

Ask questions such as:

- What happened when the coefficient of "a" was negative?
- Will the parabola open “up” or “down”?
- Did any of the equations shift to the right or left of (0,0)?
- Which coefficient seemed to cause that?
- Which coefficient made a graph wider or narrower?
- What does the coefficient "c" show?
- Did you find any pattern for "b"? (You may need to point out that c, which looks as if it has no variable, really has x to the 0-power, which is equal to 1. A discussion on this point is not warranted now, but it does show students that there is a pattern to the exponents of x in the equation.)

Students may have been confused by the factored form in the last two equations, since last week's lesson on order of operations did not mention exponents. This is an opportunity to ask them how they interpreted $2(x - 4)^2$. Mention that the factored form will become clear once they have worked with their Algebra Tiles.

b) Review the partial product method for multi-digit multiplication by having students decompose 23 x 17 onto graph paper: 20 units + 3 units horizontally, and 10 units + 7 units vertically. What does their filled-in grid look like? (200 units + 30 units + 140 units + 21 units = 391 units) Now ask them to draw a quick diagram on blank notebook paper with *just the numbers* 20 and 3 horizontally and 10 and 7 vertically. Can they quickly multiply those numbers to fill in the grid? What numbers did they come up with? Ask if it mattered in finding the answer that they used actual units vesus numbers once they understood the process.

Remind students that when they decomposed the numbers 23 into (20 + 3) and 17 into (10 + 7), they were creating factors that will help them think about quadratic equations when they use Algebra Tiles.

c) Have students work in pairs for this activity. Introduce the Algebra Tiles by having students model the expression (2x + 3) x (x + 7) by arranging on their desktop two "x" pieces and three “unit” pieces horizontally, and one "x" piece and seven “unit” pieces vertically

Next, have them fill in the products with their x^2, x, and unit tiles. Discuss:

- What did they notice?
- How does this algebraic result relate to the work they just did with numbers?
- How does the factored form relate to the total number of Algebra Tiles (the expanded form)?

d) Next, have students use their Algebra Tiles to create a rectangle that contains $2x^2$ tiles, 5 x tiles, and 3 unit tiles. (This is a model for $2x^2 + 5x + 3$.)

What are the dimensions of this rectangle? What are the factors of $2x^2 + 5x + 3$? (This task, working backwards from the expanded form to its factors is more challenging than the previous task.)

e) If time allows, have students solve $y = x^2 + 3x + 4$, substituting 10 for x. Then solve the following equations, substituting 10 for the value for x:

- $2x^2 + 3x + 4$
- $5x^2 - 3x - 4$
- $(x + 8) * (x + 2)$
- $(x - 3) * (x + 5)$

How do the above expressions relate to the partial products method for 2-digit multiplication?

f) Distribute colour copies of the homework assignment, Connecting Graphing and Algebra Tiles.

5. Assignments

Colour copy of "Connecting Graphing and Algebra Tiles" (3 pages, include teacher notes): http://tinyurl.com/connect-geom-graph

Unit 2 Algebra
Week 4, Session 3: Solving for x, the Unknown

1. What are the important concepts?

a) "*x* as an unknown" is a different concept from "*x* as a variable."

b) The properties of arithmetic that students have already learned (commutative, associative, distributive, and identity) also hold true for (and can be applied to) algebraic equations.

c) An expression *without* a given value for x relates to functions, where x is an unspecified variable.

d) On the function's table of values, any given value for x has a corresponding "y-value."

e) Evaluating *expressions* with a given value for x by using substitution is a starting point for finding x as the unknown in an *equation*.

f) Solving an equation for x relates to the "balance model" of equivalency.

g) In order to create this balance students need to use 1) symbol manipulation, 2) arithmetic properties, and 3) order of operations.

2. How do children think about these concepts?

a) In arithmetic, equations are usually written so that the "answer" comes on the right hand side of the equation with the typical syntax of a number sentence being "4 plus 3 equals 7."

In algebra's use of symbolic notation, that syntax is often reversed so that the "answer" is on the left of the equals sign, which would result in the arithmetic equation "7 = 4 + 3."

After many years of working with equations in the format of a + b = c, it can be difficult for youngsters to shift to a syntax where the answer is on the left hand side of the equation and where the expressions and operation signs are on the right hand side, such as in 7 = 4 + 3 (or its corresponding algebraic equation, 7 = 2 x + 3).

b) When working with linear equations in the format y = mx + b, youngsters may not understand that the dependent variable "y" can have a coefficient. For example in the equation 4y = 2x, youngsters may not realize they need to divide both sides of the equation by 4 in order to maintain equivalence. Instead, they may interpret the equation as either y = 2x + 4, or 4y = 4(2x).

c) Even if youngsters know that variables can have a coefficient, they may not realize that a coefficient does not need to be a whole number. They need to understand that coefficients can be any type of the numbers they have studied: integers, fractions, or decimals.

d) As in the cartoon below, youngsters may assume that once they have found the value for what x "is" in a given equation, that x has that same specific numerical value for x in other equations. For example, suppose there are three problems on a worksheet, the first being x + 3 = 7. The teacher expects a youngster will correctly solve for x and discover that its value is 4. If the next problem is x + 5 = 9, the value for x is still 4. However, if the third problem is x + 3 = 9, youngsters may be looking for a pattern and determine that x must once again be 4.

3. What is essential to know or do in class?

a) Review the Algebra Tiles homework assignment using prompts from the Teacher Notes section.

b) Introduce a new definition for x: x, the unknown. Note common misconceptions that youngsters may have about this.

c) Review the "balance model" of equivalence and how it works in algebra

d) Review the commutative, associative, distributive, and identity properties and note how they work in algebra.

e) Introduce the idea of "like terms" and how they can be combined to solve for x.

f) Note the strategy of isolating x by using the arithmetic properties and the concept of equivalence.

g) To end this Algebra unit, allow time for a discussion about students' reflections on what they think about algebraic thinking and the teaching of algebra—which may be quite different from how they learned this subject when they were in school.

4. Class Activities

a) Begin by reviewing the homework assignment, A Geometry Graphing Connection. How did the questions about the Algebra Tiles as a visual model relate to the past assignment, Graphing Equations? What did students notice about the equations of "incomplete rectangles" using the Algebra Tiles? What does that imply?

Use the discussion prompts in the section Parabolas. (Since you have given the Teacher Notes page to students they should be prepared to discuss these questions.)

b) Shift the focus to x as the unknown in a given equation. If possible, use this cartoon to illustrate the way youngsters may think about this:

But yesterday you said x equals 2.

c) Have students quickly create a table of values for the linear function 2x + 3 beginning with x = 0 and ending with x =5. Ask students how they would use their table to evaluate the expression when x = 2. Did their answer reflect the y-value for when x was 2?

Ask students how they learned to evaluate expressions and solve equations when they were in middle and secondary school. How is this table-of-values method different from the method of "substituting" 2 for x in the expression 2x + 3?

d) Have students think about what they learned about equivalence in the Numbers and Operations unit. How does the "balance" model help them understand algebraic equations across the equal sign? How can they use what they've learned about properties, order of operations, and equivalence to "isolate x" and find the unknown?

e) End the session by having students respond to this prompt for their reflection, then have a whole class discussion about their thoughts.

- What mathematical or pedagogical ideas from the Algebra strand stand out as I think about my future students, whether kindergartners or youngsters taking a pre-algebra course?
- What steps would I take to prepare my students to be algebraic thinkers ready for their future algebra courses?

5. Assignments (to be determined by the instructor)

Faculty Notes
Unit 3 Geometry
Week 1: Pre-assessment, Polygons, Similarity, Benchmark Angles

Weeklong Overview:

Session 1: Unit Pre-assessment
Session 2: Characteristics of Polygons, Regular and Irregular Polygons, Classifying Polygons, Hierarchy for Polygons
Session 3: Introduction to Similarity, Benchmark Angles

Faculty Preparation for Upcoming Week (1-2 hours)

Look through the following websites that address polygons:

- Hierarchy of Polygons: http://tinyurl.com/Polygon-Hierarchy
- Sorting Polygons: http://tinyurl.com/Sorting-Polygons-1
- Venn Diagram about Quadrilaterals: http://tinyurl.com/Quads-Venn

Download and print out for student use:

- 25 Polygons: http://tinyurl.com/25-Polygons
- "Attributes of Polygons": http://tinyurl.com/Polygons-Attributes

Bring to class:

- Lined paper
- Plain paper
- Graph paper
- Rulers
- Scissors

Read through the plans for this week's three sessions

Session 1 begins with a pre-assessment that helps the instructor discover students' current understanding of geometry topics that are included in the elementary grades.

Session 2, the first instructional session of the Unit, begins with polygons. Many texts begin with angles (a more abstract concept) and then move to polygons (which are more recognizable in the environment).

As with Unit 1, which focused on Number and Operations by moving from the concrete to the abstract, we will begin by exploring polygons so that students will begin to notice the role of angles in forming these shapes.

This method, based on research that helps teachers understand how children think about geometry, is a helpful model in a university course supporting pre-service teachers.

During this session, students will learn to recognize the characteristics of polygons and begin to classify them according to number of sides, side lengths, and eventually equal versus non-equal angles.

Session 3 will address similar polygons and use a 90-degree right angle as a "benchmark" from which other angles can be devolved and to which other angles can be compared.

Unit 3 Geometry
Week 1, Session 1: Unit Pre-assessment

1. What are the important concepts?

This first day of the Geometry Unit is focused on assessment. It is an opportunity for the instructor to discover what students know, think, and remember about geometry.

It addresses (and pre-assesses) the vocabulary, concepts, and skills that students will study during the next five weeks:

- Vocabulary and Terminology
- Polygons and Circles
- Angles
- Cuboids and Cylinders
- Area, Volume, and Surface Area
- Right Triangles and the Pythagorean Theorem

It is important to take time to assess students' prior knowledge, since it is likely that most of them equate geometry with their high school coursework rather than thinking of geometry as part of their every day lives: wrapping a package (surface area), taking a shorter diagonal route through a park (Pythagorean Theorem), or holding a soccer ball (a sphere covered with pentagons and hexagons).

Similarly, if students think of geometry as a high school subject dealing with proofs and formulas, they may not realize how accessible geometry is for young children.

The questions suggested in the pre-assessment below are not meant to be "scored." Rather, they offer a window on student thinking that will be addressed in subsequent sessions over the next five weeks.

2. What is essential to do in class?

The following whole class discussion should be lively and engaging, with a focus on 1) discerning what students already know and 2) stimulating student thinking by asking probing questions and providing clues rather than giving answers.

As you pose the following types of questions, note the vocabulary and terminology that students use when describing their thoughts. Is their terminology formal or informal? Consider how the phrasing of *your* questions can model for students how to become more formal and precise in their geometric language.

Geometry is visual mathematics. Thus, during the discussion ask students to come to the board to draw what they are trying to communicate. Students should also be drawing and labelling diagrams in their notebooks, and noting points of confusion that can be followed up in later sessions during the unit.

Since this class meeting is a pre-assessment, try not to go in depth when dealing with individuals' misconceptions. There will be ample time and opportunities to deal with student confusion during the remainder of the unit.

3. Class Activities

Here are some sample "prompts" that are designed to help students think about the many concepts they will encounter during the Geometry Unit.

1) When did you first start learning geometry? If students say the first learned about geometry in high school, ask about what they think young children might *already* know about geometry.

2) What geometric shapes and forms do you see in this room? Do you think young children would be able to notice those things, too? How could you help youngsters begin to see the geometry in their environment (shape of the windows, tiles on the floor, stripes on clothing, the circle of a clock, polygons on a soccer ball, etc.).

Extend questions about these 2-dimensional shapes by asking about 3-dimensional forms (a book (cuboid), a pencil (cylinder or hexagonal prism), a ball (sphere), a cardboard packet (cuboid), etc.).

3) After this informal discussion about polygons and polyhedra, ask about the generic names for polygons, which are dependent on the number of a polygon's sides: triangle, quadrilateral, pentagon, hexagon, (you may omit the 7-sided septagon or heptagon) and octagon.

Ask why is there no 2-sided polygon.

4) Continue by asking:

- What do you think is implied by the term "regular" polygon? Why might a given polygon be called "regular" versus "irregular"?

When talking about polygon shapes, do students mention parallelograms, rhombuses (rhombi), and trapezoids?

Do they realize that a square is a rectangle? If so, can they explain why? Is a square a rhombus? If so, can they explain why?

5) Ask about circles? Circles are closed shapes, but not polygons. This might give rise to a definition of a polygon as well as descriptions of shapes that are *not* polygons.

Is a "ball" a "circle"? If so, why? If not, why not?

What about a cube versus. a square?

Is a "box" a 2-dimensional shape (such as a box to tick on a survey) or a 3-dimensional form (such as a packet of rice)?

Why do you think young children (and even adults) use these kinds of terms incorrectly and interchangeably?

6) Have students consider a square and a cube.

What is a side? What is an edge? A face? A corner? A vertex?

Which terms apply to 2-dimensional shapes? Which ones apply to 3-dimensional forms? How do we use these terms informally? (E.g., ingredients are listed on the "side" of the "box" when we're shopping. But that "side" is really the "face" of a cuboid when we consider that same packet in geometric terms.)

7) Once students have considered these concrete aspects of geometry, move to more abstract, undefined terms asking them to describe their thinking about the following words: point versus. "dot", line versus line segment, ray, degrees, plane, parallel, perpendicular.

8) When addressing angles, ask which angle comes to mind first. (Most likely it will be a 90-degree right angle.) Ask about other angles that they could derive from a 90-degree angle. Do they mention a 180-degree straight angle? A 360-degree angle?

9) Sketch (no need to construct) different angles on the board. Ask students to describe them. What terminology do they use?

10) Move to geometric measurement and ask what students remember learning about area and volume. Do they respond with formulas? If so, ask how young children (who don't know formulas) might be thinking about area. How might these same young children think about volume? What about surface area? What kind of real life examples do students give when discussing geometric measurement?

11) Finally, ask about measurement as applied to right triangles. What do student recall learning about right triangles? Do they offer the formula for the Pythagorean Theorem? Ask how the formula might be *proved*. (Which is very different from simply recalling the formula.) Ask where the Pythagorean Theorem might be used in everyday life.

12) At this point you have introduced all the major topics to be addressed during this 5-week unit.

You also have an overall picture of your students' understanding about these topics. Are your students wedded to formulas? Do they see geometry around them? Do they consider how young children can approach geometry?

Unit 3 Geometry
Week 1, Session 2: Characteristics of Polygons, Regular and Irregular Polygons, Classifying Polygons, a Hierarchy for Polygons

1. What are the important concepts?

a) Polygons are closed 2-dimensional shapes formed by line segments that meet at a "corner" or vertex.
b) A polygon is named by the number of its sides.
c) If all sides of a polygon are equal, their interior angles are equal, and thus the shape is termed a regular polygon.
d) The shape of a polygon depends not only on its side lengths, but also on the measures of its interior angles.
e) Polygons can be classified according to their side length and angle attributes.

2. How do children think about these concepts?

a) When considering triangles, most young children think of an equilateral triangle standing on its base. If the same triangle is rotated so that a vertex is pointed downward, they often do not "see" it as a triangle.

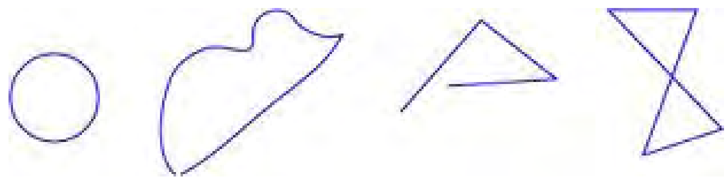
Similarly, if shown a scalene triangle, with three different side lengths, they also might not perceive it as a triangle. This is because they have equated their concept of a triangle with a single model. They have not yet discovered that a triangle is any 3-sided closed figure.

b) Quadrilaterals can be organized into a hierarchy, from an irregular quadrilateral to the most specialized quadrilateral, a square.

Most youngsters do not realize that a square is also both a rectangle and a rhombus, as well as a parallelogram. Just as with triangles, this can be because they have not had experience discussing the characteristics of quadrilaterals.

However, they also lack the vocabulary and terminology (such as *parallel* and *opposite*) as well as the idea of angle measurement. Thus, once they realize a polygon has four equal sides but doesn't "look" square they may think of it as a "squashed square" where two of the vertices look "pointier" than the other two. This informal description of a rhombus is a starting point from where the teacher can begin to formalize how to describe the shape's characteristics.

c) Children may assume that any closed figure is a polygon, or that an open figure composed of line segments is a polygon. Thus it is important to offer counterexamples such as those below so that students can begin to refine their definition of a polygon's required characteristics.



3. What is essential to know or do in class?

a) Introduce polygons and several counterexamples, having students develop a working definition for what constitutes a polygon.

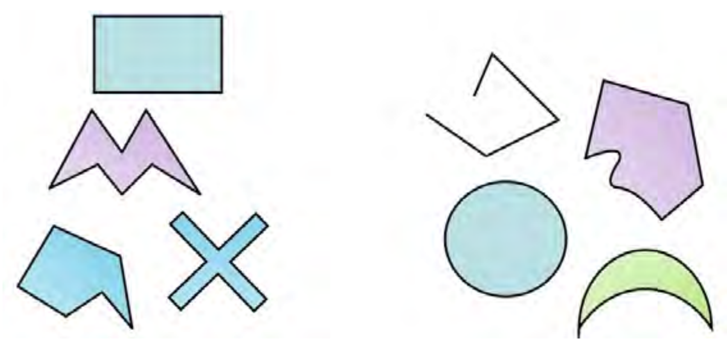
b) Introduce regular and irregular polygons, having students develop a working definition (including both side length and angle measure) for what constitutes a regular polygon.

c) Have students classify polygons according to their attributes.

4. Class Activities

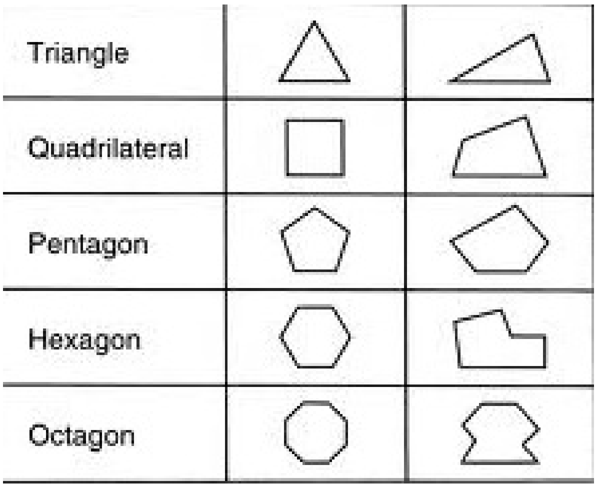
a) Begin by referring to the comments about polygons students made in the prior session, noting that those ideas will become more precise during today's class.

b) On the board, draw figures of polygons and counterexamples such as the ones in the picture below. Ask for thoughts as to why you divided the shapes into two columns:



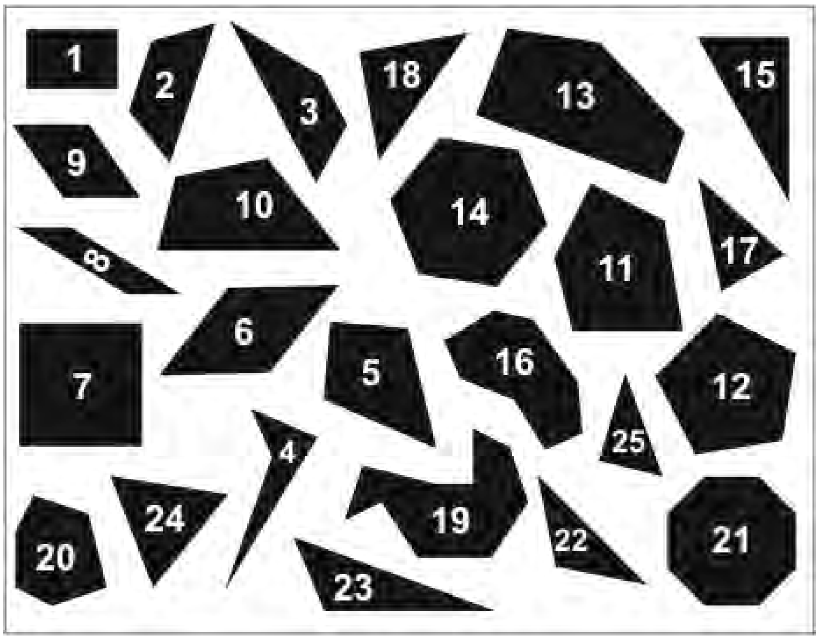
As students talk about their ideas, press for greater precision in the way they describe the different shapes in order to come up with a working definition of what is, and is not, a polygon. Note that students may not think of concave polygons in the same way they think of convex polygons. Or that may call the rectangle a "regular" polygon, when the word "regular" has a special mathematical meaning that will be explored in the next activity.

c) Since all of polygons in the picture above are *irregular* polygons, introduce the idea of regular polygons by drawing on the board another two-column chart similar to that below:



Again, listen to the way students attempt to describe the differences between the two columns. Do they note not only the side lengths but also the angles? If they say, for example, that the triangle on the left is equilateral, ask what they mean by that word. Is a square "equilateral," too? What about the word "equiangular"? What might that mean? And how does it apply to the two pentagons? By the end of this discussion, students should have come up with a working definition of regular and irregular polygons. Finally, refer back to the rectangle in the coloured picture above. Ask why a rectangle is <u>*not*</u> a regular polygon. Does their working definition help clarify this?

d) Distribute the page of polygons from http://tinyurl.com/25-Polygons and the worksheet *Attributes of Polygons*.



Divide the students into five groups; give each group one of the following sets of polygon characteristics in order to classify the 25 shapes on the handout. Note that several polygons may fit the same category or that a particular polygon may fit into several categories. Notice the term "congruent," which you may need to define for students if they haven't used the word already.

Group 1
1. PENTAGON
2. OPPOSITE SIDES PARALLEL
3. AT LEAST ONE RIGHT ANGLE

Group 2
1. HEXAGON
2. ALL SIDES CONGRUENT
3. REGULAR POLYGON

Group 3
1. OCTAGON
2. OPPOSITE ANGLES CONGRUENT
3. PARALLELOGRAM

Group 4
1. OPPOSITE SIDES CONGRUENT
2. RHOMBUS
3. QUADRILATERAL

Group 5
1. ALL ANGLES CONGRUENT
2. TRAPEZOID
3. CONCAVE POLYGON

Have a spokesperson for each group report out the results of their particular discussion. Have students note omissions and areas of confusion in each others' presentations, justifying their responses.

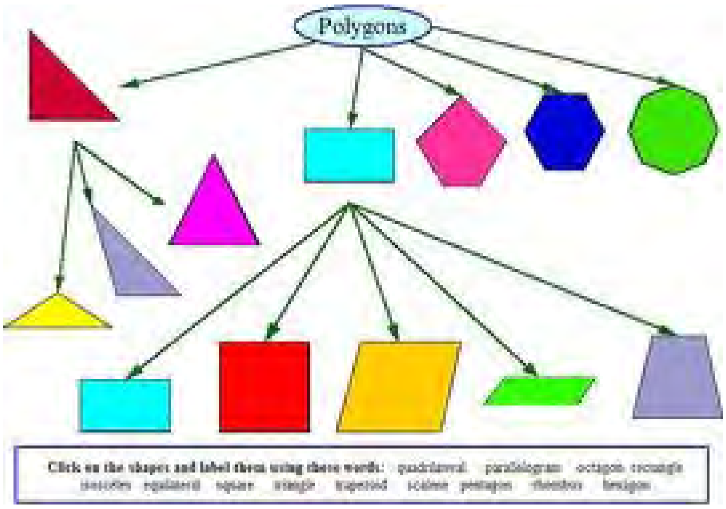
e) End the session by reviewing and acknowledging some of the more formal terminology students had begun using as the session progressed.

5. Assignments

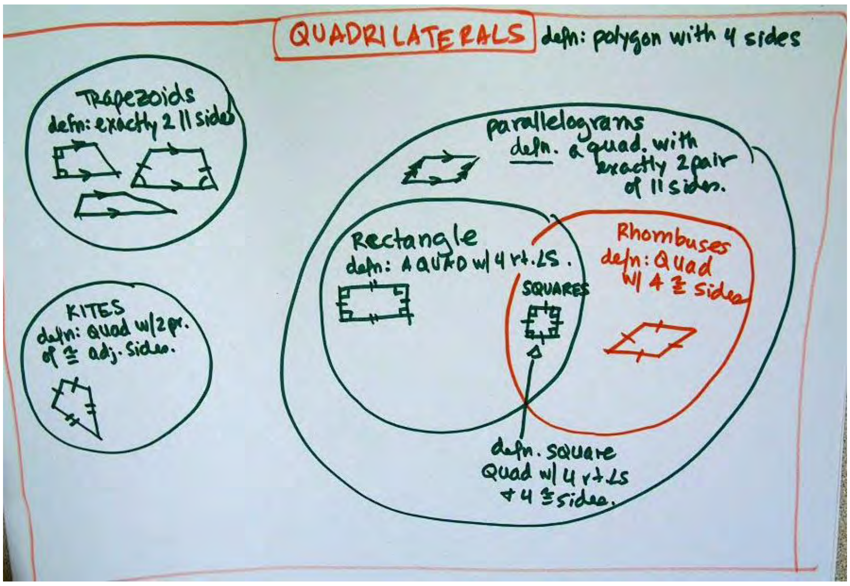
a) Have students bring their 25-polygon sheet to class for the next session, Session 3.

b) Have students visit these two websites for visual models that show how various polygons are related:

- Hierarchy: http://budurl.com/PolygonHierarchy



- Quadrilateral Venn Diagram: http://budurl.com/QuadrilateralVenn



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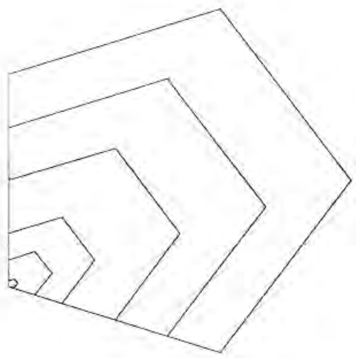
Unit 3 Geometry
Week 1, Session 3: Introduction to Similarity, Reference Angles

1. What are the important concepts?

a) Similar figures are those whose side lengths are in direct proportion to each other, with the corresponding angles equal in degrees. Consider this drawing of a tiger:

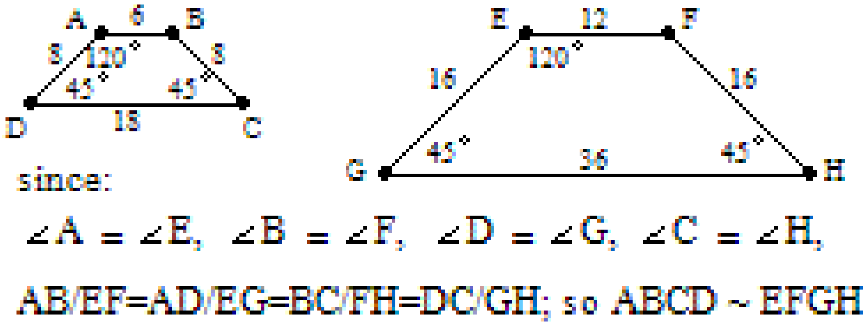
The larger drawing resulted by using a "scale factor" of 2, doubling both the length *and* the width of the original picture.

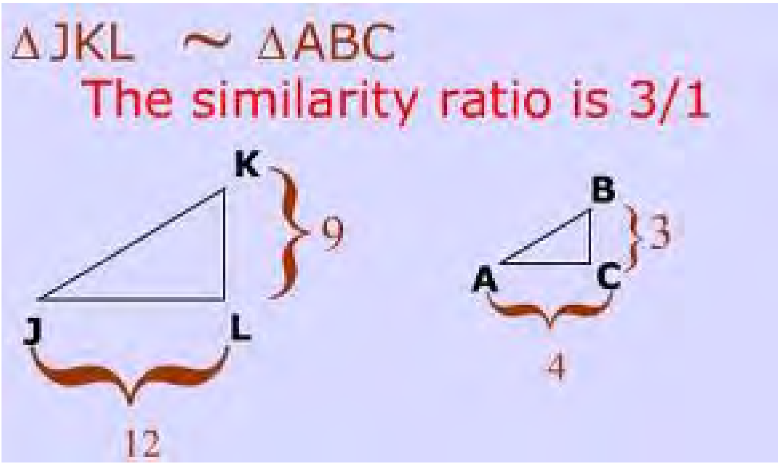
When dealing with geometric shapes (which are far simpler than the drawing of the tiger) each side of the figure is multiplied by the same scale factor while *the corresponding angles remain the same*. Look at this example of a regular pentagon that retains its shape regardless of the scale factor used:



b) Scale factors are just that: factors. As such, the operations used when creating similar figures are multiplication and division (or multiplying by a scale factor less than 1).

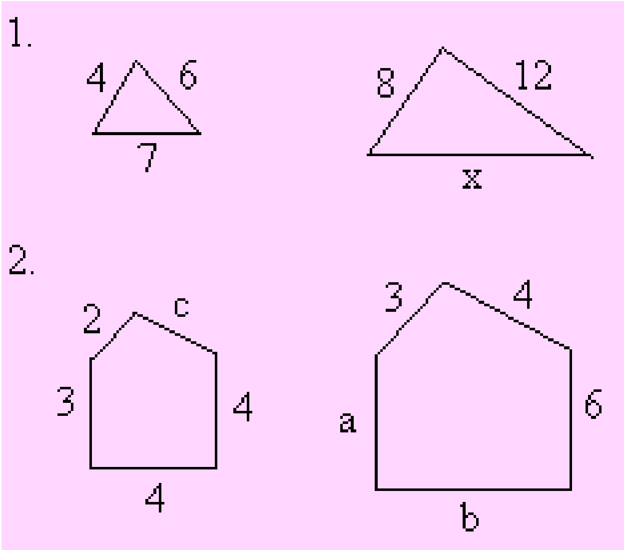
c) The mathematical sign to indicate that two shapes are similar is $\sim$. Thus the final sentence for the first set of diagrams would be read, "Figure ABCD is similar to figure EFGH."





d) Knowing that corresponding side lengths are in proportion to each other can help solve for missing information. Consider these two sets of figures with missing information. How can the relationship between the side lengths of 6 and 12 help solve for x?

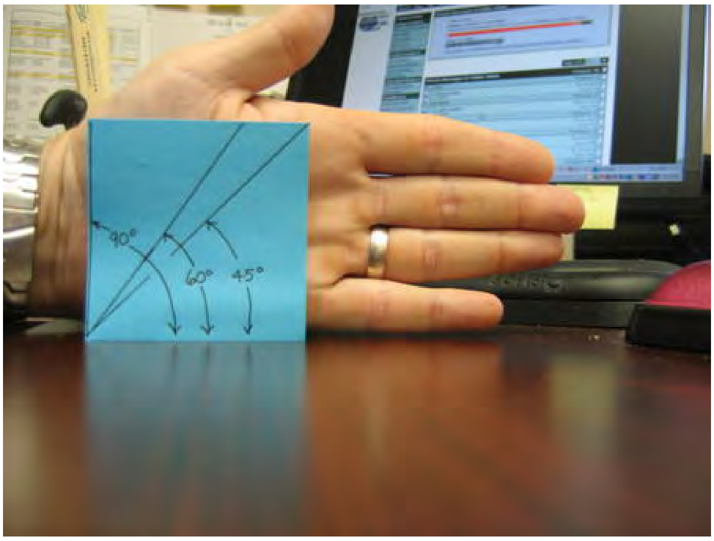
A somewhat more difficult situation is involved in the second set of pictures. What is the scale factor here?



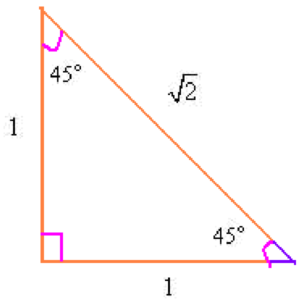
e) Finally, for an amusing real life example of similarity:

f) Benchmark angles are those that serve as a reference point. They can be used both to devolve other angles and to compare angles.

The most common benchmark angle is the 90° right angle. Using the 90° angle as a reference point, one can devolve the 180° straight angle and the 360° revolution angle, and estimate the 45°, 22.5°, 135°, 60°, 30°, and 120° angles.



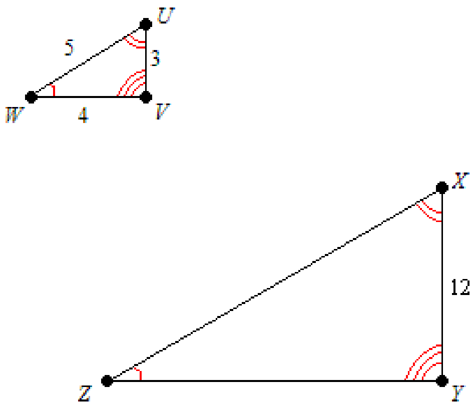
g) Note that certain polygons include these specific angles. For example, an equilateral triangle has three 60° angles for a total of 180°. A square, with four 90° angles, has an interior angle sum of 360°. If a square is cut on the diagonal, it will have one 90° angle and two 45° angles. (Again, each of the resulting two triangles has an angle sum of 180° degrees.)



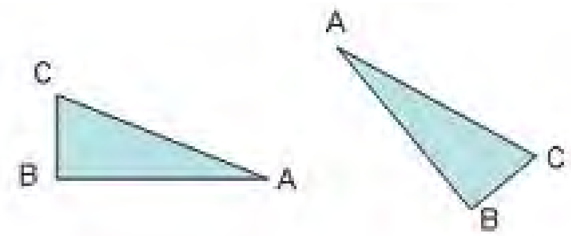
h) Other benchmark angles devolved from the 90° angle can be used for comparing angles and estimating their measurement. For example, if one knows how to create a 45° angle, one can determine if a given angle is greater or less than 45°. The same would be true for an angle that looks to be between 90° and 135°.

2. How do children think about these concepts?

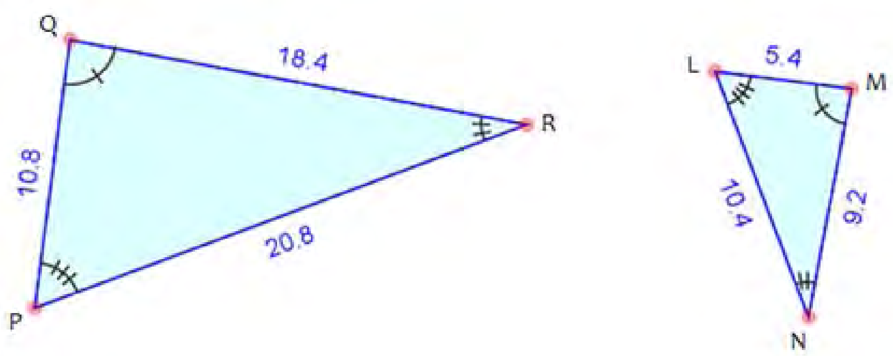
a) One of the most important considerations when trying to establish similarity between two figures is their *corresponding sides*. This is relatively easy if the figures have the same orientation as in this diagram (which also allows for establishing the length of other sides based on the 3:12 ratio of the vertical side):



But if the *orientation* of the shapes is not aligned, as in the figure below, children may fail to see which are the corresponding sides. In this picture, the two triangles are exactly the same (congruent), but the orientation is different.



In this second set of similar triangles, both the orientation and the scale factor are different. In testing for similarity, it is helpful for youngsters to rotate their worksheets and notebooks or use thin paper for tracing one figure and placing it over the other.



(This is much like a younger child's not realizing that an equilateral triangle with its base on the horizontal is congruent to the same triangle with a vertex "down.")

b) Youngsters (and pre-service teachers) may not understand that to create similar figures a scale factor involves multiplication or division, not addition or subtraction.

"For example, suppose a triangle has side lengths of 2 cm, 4 cm, and 7 cm. *Multiplying* these side lengths by 2 produces 4 cm, 8 cm, and 14 cm. Therefore, a triangle with these side lengths would be similar to the original. However, *adding* 2 to each original side length will produce a triangle that is 4 cm, 6 cm, and 9 cm. It is ***not*** similar to the original, although students often mistakenly think that it is....After several semesters of attempting to explain, in abstract terms, why multiplication and division are necessary where similar figures are concerned, I realized that many pre-service teachers need visual and hands-on experiences." (Johnson, G. November, 2010. " Mathematical Explorations: Similar Triangles." *Mathematics Teaching in the Middle School*. Reston, Virginia, National Council of Teachers of Mathematics: 16(4), pp. 248-254.)

c) Even when children have had experiences that prove that the degrees in the corresponding interior angles of two similar figures are equal, they often hold onto the mistaken belief that a larger figure (with its longer sides) should also have more degrees in its angle sum than the smaller figure.

The fact that *side lengths can vary* (in proportion to the original) but that the measurement in degrees of the *corresponding angles remains the same* is much like the idea (to be discussed later) that the *length* of an angle's rays is irrelevant to the angle *between* those two rays.

3. What is essential to know or do in class?

a) Corresponding sides and corresponding angles are the key elements of similarity. Corresponding sides must be in proportion; corresponding angles must have the same measure in degrees.

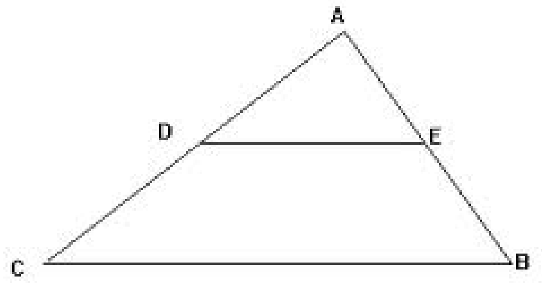
b) Scale factors to create similar shapes use the operations of multiplication and division, not addition and subtraction.

c) Using a 90° benchmark angle allows for a relatively accurate estimation and comparison of angles. This informal way of estimating angle measurements is a precursor to greater accuracy when measuring and constructing angles using a protractor.

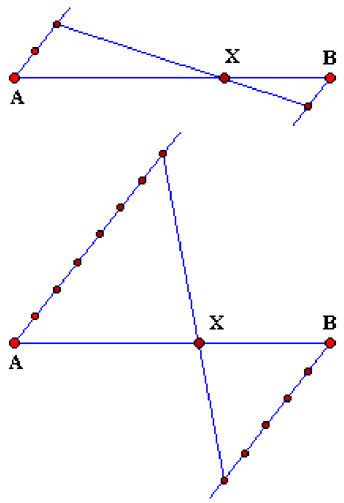
4. Class Activities

a) Begin by reviewing any thoughts students voiced during the pre-assessment about similarity and angle measurement. Let students know that they will be exploring the concept of similar figures in several drawing activities.

b) Have students take a sheet of *lined* paper and draw a large triangle, with the vertex on the top line, and its base on the bottom line. Have students draw a horizontal line parallel to the base from one side of the triangle to the other. What do they notice about the new triangle that they created inside the original one?



c) On a sheet of *plain* paper, have students draw two 20 cm. intersecting lines (a large "X"). It is not necessary that the lines intersect at their halfway points. Rather, it is preferable for the understanding of similar triangles that the lines cross at a point *other than* the halfway point, such as:



Once students have done this, have them join the endpoints as shown above. What do they notice about the two triangles they have created?

Next, have them fold the paper on the point of intersection and hold it up to the light. The two triangles will be overlapping. What do they notice now? Why is this so?

As you listen to their explanations, help students move toward mathematically precise terminology:

1) The "vertical angles" (a new term introduced here) created by intersecting lines are equal.

2) The principle of corresponding-side-angle-corresponding-side is a way of confirming similarity.

(These are formal, secondary school ways of discussing similarity. However, pre-service teachers should be able to justify similarity in this way even though they will describe it in simpler terms for children.)

Make sure students realize that a similar shape's corresponding angles are the same even though its corresponding sides differ in length. The next activity will focus on how the proportionality of side lengths is a key "factor" in similarity.

d) On a sheet of *graph* paper have students draw three rectangles:

- 1 units x 2 units
- 5 units x 6 units
- 4 units x 8 units

Do they think these three rectangles "look similar"? (Each has four 90° angles so the angle principle of similarity holds true.)

What do students notice as they compare the side lengths of these rectangles? Do students notice differences in proportion that occurred when adding 4 to both dimensions of the original 1 x 2 rectangle? Can *addition* of equal units to corresponding sides justify similarity (1 + 4 = 5, and 2 + 4 = 6)?

What happened when the original 1 x 2 rectangle had each of its side lengths *multiplied* by 4 resulting in the 4 x 8 rectangle? Are rectangles 1 and 3 similar? If so, why? If not, why not?

e) Summary: Summarize this section of today's lesson on similarity by asking the following questions:

- How does the side length property relate to the term "scale *factor*"?
- How can knowing the scale factor help solve for missing sides on two similar figures?
- In the rectangle problem, there were still four 90° angles, yet all the figures were not similar. What is the role of angles in similarity?
- What does the word "corresponding" imply? Can one double the side length *and* double the angle measurement and still have two similar figures?
- How do the two components of similarity (corresponding angles and corresponding side lengths) provide a working definition of similarity?

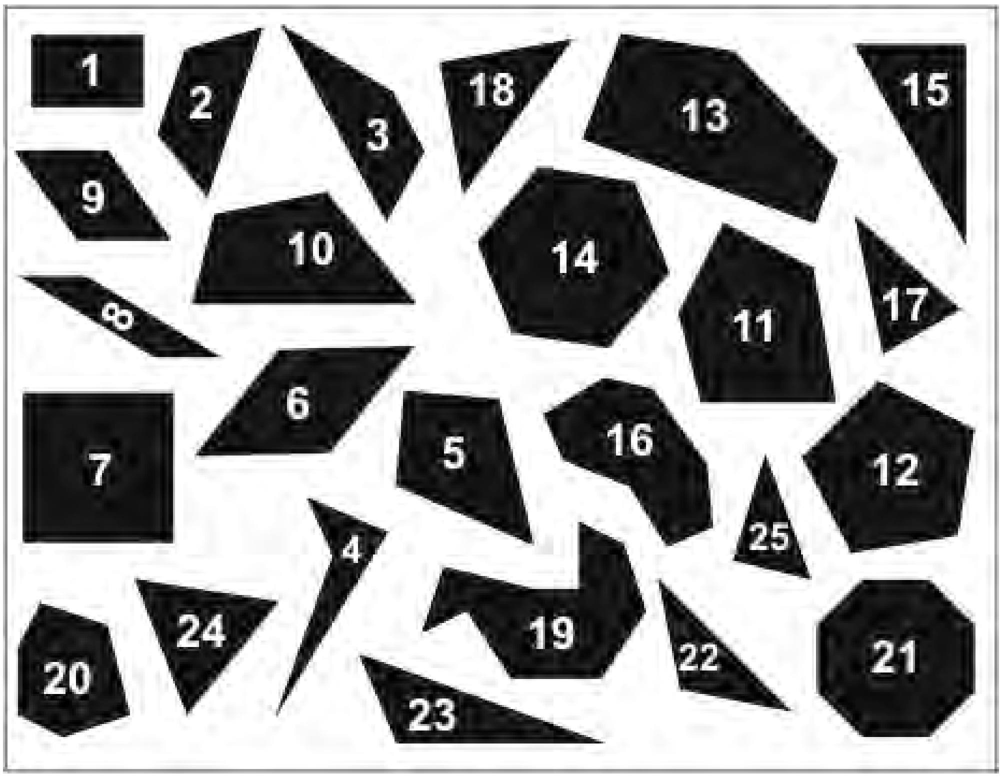
f) Have students sketch three 90° angles on grid paper. These will become their reference angles. How might they:

- Find 45°, 22.5°, and 135° angles by using one of the 90° angles that they drew?
- Use another of their 90° angles to estimate angles of 60°, 30°, and 120°?
- Use their third 90° angle to find a 180° (straight) angle and a 360° (full rotation) angle?

Once students have done this, ask them to describe the usefulness of a 90° angle as a reference or "benchmark" angle to help estimate and compare angles.

In the prior session, students were told to bring their 25 polygon shapes worksheet to class today. Have them look at the following polygons:

- Triangles: 18, 22, 24
- Quadrilaterals: 1, 2, 5, 6, 7, 9
- Pentagon: 12
- Hexagon: 14
- Octagon: 21



5. Assignment

Use angles derived from the 90° reference angle to estimate the interior angles (and then the shape's angle sum) for each of the above 12 polygons. Bring results to the next class session where angles and angle sums will be explored more fully.

Faculty Notes
Unit 3 Geometry
Week 2: Angles, Angles in Polygons, 360° Angles, Tessellations

Weeklong Overview:

Session 1: Angles: Types of angles, Measurement of angles, Size of line segments not affecting Angle Size, Interior angles of Polygons, Angle Sums in Triangles.

Session 2: Angle Sums in Polygons, 360° around a Point
Session 3: Tessellations and Tiling a Plane

Faculty Preparation for the Upcoming Week (1-2 hours)

Read the following article and lesson plan:

- "From Tessellations to Polyhedra": www.aug.edu/~lcrawford/Readings/From_Tesselations_to_Polyhedra.pdf Also available at http://budurl.com/AnglesTessel
- "What's Regular About Tessellations?" http://tinyurl.com/Regular-Tessell

Look through the following websites:

- Naming and Measuring angles: http://tinyurl.com/Benchmark-Angles-VIdeo
- Angles around a Point: http://tinyurl.com/Angles-Around-Point
- Virtual pattern blocks (interactive applet): http://tinyurl.com/Virtual-Pattern-Blocks
- Consistency of angle sum in a polygon with "n" sides (interactive applet): http://tinyurl.com/Angle-Sum-Applet
- Naming Tessellations: http://tinyurl.com/Tessel-Naming
- Tessellation Creator (interactive applet): http://tinyurl.com/Tessellation-Applet
- Distorting triangles, squares, and hexagons to design tessellations (4 pages + interactive applet): http://tinyurl.com/Tessellation-Applet-2

Download and print out for your own use:

- What's Regular About This Polygon? http://tinyurl.com/Regular-Polygon (Print out page 1 as a transparency for use in class)
- Angles around a Point: http://tinyurl.com/Angles-Around-Point
- Creating a new tessellation: http://tinyurl.com/Tessellations-PDF-1
- Coloured triangle-square tessellations (* Make 3 or 4 *colour* copies of these two pages to use as samples in small groups): http://budurl.com/ColorTessell
- Answer key to "What's Regular About Tessellations" assignment: http://tinyurl.com/Regular-Answer-Key Links as code

Download and print out for class use and assignments (1 copy per student):

- Pattern blocks: http://tinyurl.com/Pattern-Blocks-PDF
- Angles Around a Point: http://tinyurl.com/Angles-Around-Point
- Two Semi-regular Tessellation Colouring Sheets: http://tinyurl.com/Tessel-Coloring-Sheets
- Creating Tessellations: http://tinyurl.com/Tessel-CutOut-Method
- "What's Regular About Tessellations" Assignment: http://tinyurl.com/Tessell-Assign
- "What's Regular About Tessellations" Cut-outs: http://tinyurl.com/Tessell-Cutouts

Bring to class:

- Analog clock
- Scissors

- Paste or glue sticks
- Chart paper

Read through the plans for this week's three sessions

Weeklong Overview

This second week builds on students' previous work with polygons in order to make explicit the types and characteristics of angles.

During Session 1, students will learn to recognize and categorize angles based on their type as well as on their measurement in degrees. They will also consider why the length of the rays or line segments creating a given angle is not relevant to the size of the angle itself (which is the measurement in degrees of the interior space *between* the two rays or line segments that form the angle).

During Session 2, students building upon their work with angles to inform their understanding of polygons, will explore the interior angle sum of regular and irregular triangles, which eventually will lead to their finding the interior angle-sum of any polygon.

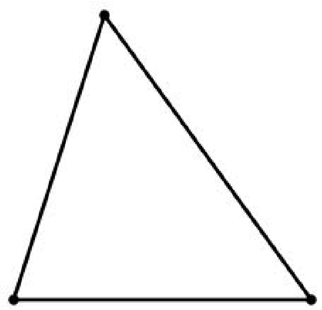
In Session 3 of the week, students will engage in activities that will help them discover why certain polygons (or combinations of polygons) can tessellate (or tile a plane surface) without having any gaps or overlaps.

Unit 3 Geometry
Week 2, Session 1: Angles—Types of angles, Length of line segments not affecting the size of the angle, Measurement of angles, Interior angles and angle sums of triangles

1. What are the important concepts?

a) Types of angles: Angles are usually categorized by their measurement (acute, right, obtuse, straight, reflexive, whole). However, it is illuminating for students to shift their thinking and consider angles from a different perspective: wedge, branch, and dynamic.

A "wedge" angle is one whose two sides are "closed" by either additional sides (as in the first two pictures), or an arc (as in the bread). Once an additional "side" is added, the angle has become static.



A second type of angle can best be understood by looking at a tree branch. The angle formed has no additional side. It might be thought of as being composed of two rays that could go on forever if tree branches could grow to infinity.

Like the "wedge" angles, these "open" branching angles are static. However, there is a third type of angle that is dynamic: it moves around its vertex, sometimes acute, sometimes straight, and sometimes obtuse. Think of a non-digital clock or watch. Note how the hands move around the center, and as they do the angle between the hands changes minute by minute.

Take a look at this clock face, where the hours are not indicated in numerals, but in the angles the hour hand will make when the minute hand reaches 12. Notice the look of symmetry on the left- and right-hand sides of the clock face, and the gradual increase in the angle from 12 to 6, and its angles from 6 to 12 as reflex angles that go beyond the 6 o'clock straight angle.

b) Angles are the space *between* two rays or line segments that meet at a vertex. A clock face can help students understand that the size of the angle formed by two rays or line segments converging at (or emanating from) a vertex is NOT dependent on the length of the line segments.

Consider two clocks: Big Ben and a small bedside alarm clock. The pictures below might *imply* the two clocks are the same size, but Big Ben ("the largest four-faced chiming clock in the world") has a clock face of 7 metres (23 ft) in diameter. Compare that to the little alarm clock that probably has a diameter of 6 cm.

Yet both display the 30° angle indicating 1 o'clock. When considering angles, the size of the clock face and the length of the hands are immaterial. The 30° angle remains constant.

c) Angles can be categorized by measurement. Angles can be described by their relationship to two important benchmark angles, the 90° right angle and the 180° straight angle.

- Angles that measure more than 0° and less than 90°are termed *acute*. Thus, the benchmark angles of 30°, 45°, and 60°are acute.

- An angle that measure exactly 90° is called a *right* angle.

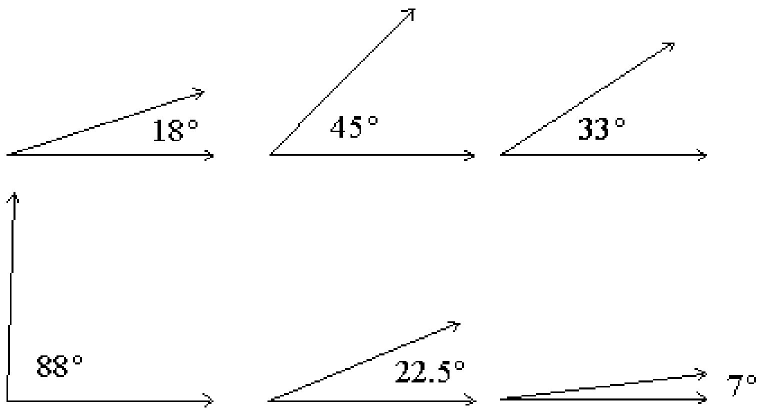
- An angle more than 90° but less than 180° is called *obtuse*.

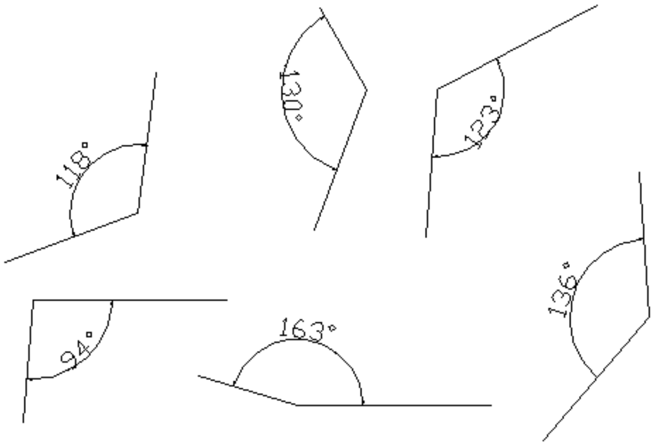
- An angle that is exactly 180° is called a *straight* angle.

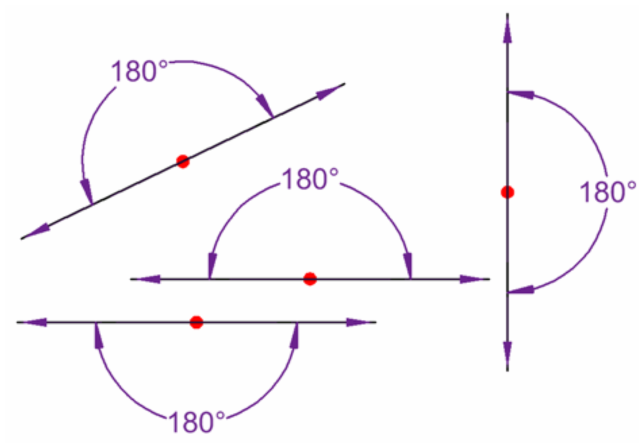
- Angles greater than 180° but less than 360° are called *reflex* angles.

In one hour, the minute hand on an analog clock moves through 360° to create what is termed a "whole angle," a "round angle," or a "complete revolution." The more formal term, "perigon," is not necessary for students to know, but the idea of the 360° angle sum around a point will become important when considering circles and tessellations.

Here are images that display acute, obtuse, straight, and reflex angles. Note the angle measurement indicated for each.


18°
45°
33°
88°
22.5°
7°


118°
130°
123°
94°
163°
136°


180°
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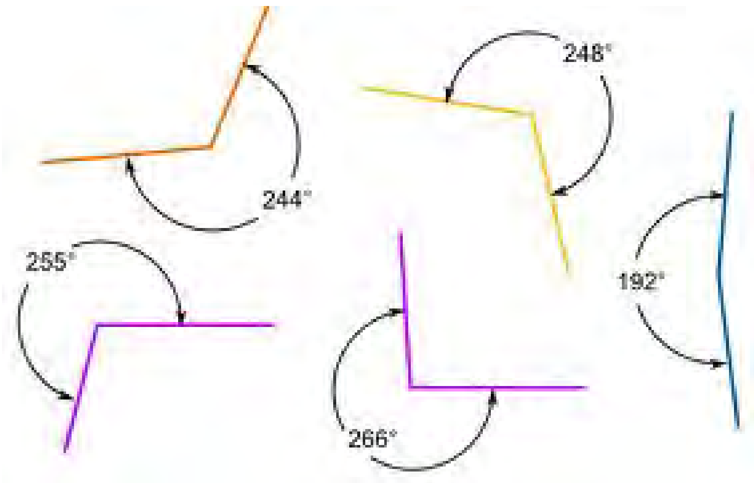


d) Interior angle sums of triangles

After students have explored polygons and angles, they need to consider how these two geometric concepts are related.

If students consider the interior angles of a regular polygon (such as an equilateral triangle or a square), what might they think?

Since they have been introduced to benchmark angles, they will probably conclude that an equilateral triangle has an angle sum of 180°, since each of its three angles measures 60°. They also will assume that a square and a rectangle have interior angle sums of 360° because each figure has four 90° right angles.

What about a triangle with a 90° angle (a right angle triangle) or a scalene triangle with 3 different angles? What is the interior angle sum of those triangles? Is it also 180°? Is the interior angle sum of 180° true for *all* triangles?

Students may understand that a square and a rectangle are both quadrilaterals with an angle sum of 360°. Is an angle sum of 360° true for *all* quadrilaterals, such as rhombuses and trapezoids?

What about polygons with more than four sides? What is the angle sum of a regular hexagon?

How can students use what they know about triangles to determine both the total number of interior degrees in a regular hexagon as well as the number of degrees in each interior angle?

Will this angle sum hold true for *irregular* hexagons as well?

As you pose these questions to your students, realize that they do not need to know the answers now. There will be a homework assignment as well as a follow-up activity during the next class session to help students make generalizations and formalize their ideas.

2. How do children think about these concepts?

a) Seeing angles: Although children may notice angles in their every day experience, they may need to be alerted to what they informally know.

For example, young children may walk down a street and see right angles in the street signs or in buildings. What about right angles in fabric? Why are right angles so prevalent in the man-made world around us?

What about the branching of trees as shown above? Or the movement of a clock's hands around a dial?

How can teachers help children notice and name the angles that are in their everyday environment?

b) Angles are the space *between* two rays or line segments that meet at a vertex.

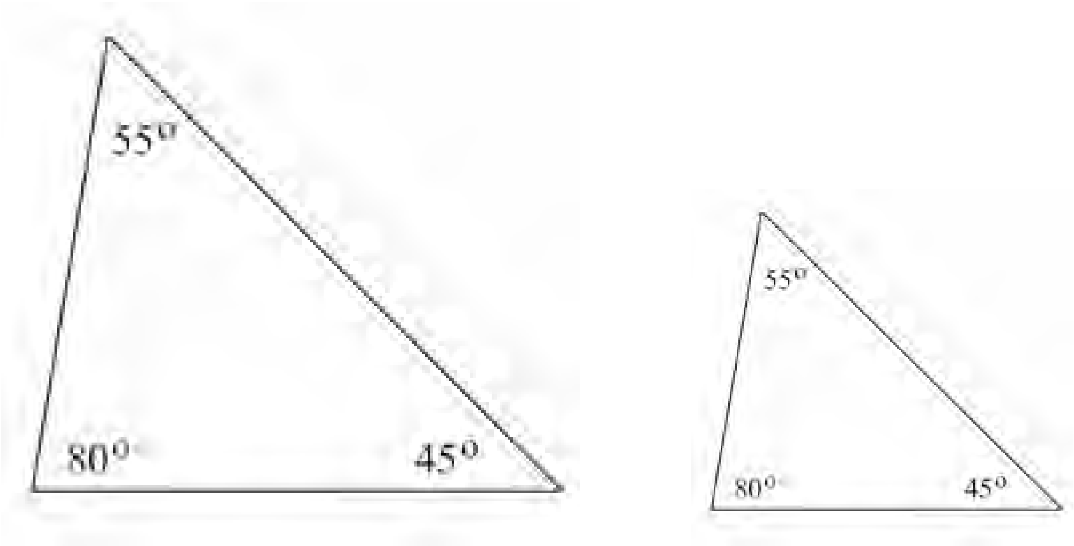
Children often incorrectly assume that the longer the line segments, the bigger the angle. This is because up to now children's experience with the words "bigger" and "larger" related to linear (1-dimensional) or area (2-dimensional) measurement.

If the following two triangles were drawn on a grid, the *area covered* by each angle in the first triangle would be "bigger" than the area covered by the second triangle's corresponding angles.

Applying their prior knowledge about size to angles, it is logical for children to assume that the angles in the first triangle below are "bigger" than the angles in the second.

It requires a major shift in their thinking so that children can begin to consider a new kind of measurement (degrees) and a new definition of "bigger."

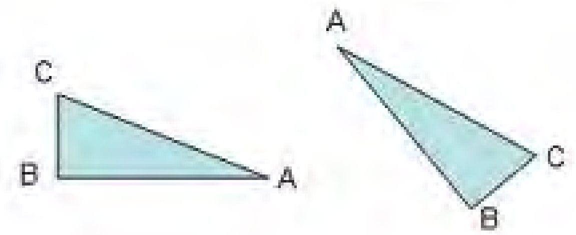
To help children understand this, they can cut out two similar triangles and lay the corresponding angles atop each other. They will see that the angles match up perfectly to each other and are therefore the "same size"—in degrees.



c) Angles can be categorized by measurement. When given pictures of acute, right, and obtuse angles, children intuitively know that they "look" different.

Children may describe a right angle as a "square." (Note that the symbol for a right angle is a small square within the angle.) They may also say it looks "like a corner" since many of the "corners" they see (pages of a book, table, windows, etc.) are right angles.

Just as young children needed to "see" that any three-sided polygon regardless of its orientation is a triangle, they now need to see that a right angle remains a right angle, even if its orientation is changed.



d) When describing acute angles, children often call them "pointy," especially if they are less than 30°.

e) Children usually lack ways to describe an obtuse angle, perhaps because these angles are less obvious in their everyday environment. Teachers need to identify real life examples of obtuse angles such as the hands of a clock at 10:10 or a stop sign.

f) Having children discuss their ideas about angles in simple shapes such as pattern blocks, can also help them differentiate among acute, right, and obtuse angles.

g) Older children will be introduced to the idea of the 180° straight angle. This can be difficult if students' informal definition of an angle means that an angle "has a corner."

This is where the activity, ripping the corners off a triangle and rearranging them to form a straight angle, can illustrate that while a straight angle does not have a corner, it indeed has a vertex, which is where the three angles ripped from the triangle come together.

h) Interior angle sums of triangles: A discussion of straight angles by older children is an appropriate lead-in to determining the angle sum of a triangle.

Children may be comfortable knowing that triangles, even those that look different from each other, all have: 1) three sides and 2) three angles.

But there is a third attribute of all triangles: the sum of the angles in *any* triangle equals 180°. Telling children this fact is not enough. They need to experiment with a variety of triangles to prove that the three angles of any triangle will form a straight angle of 180°.

This fact will become important as older children realize that they can subdivide any polygon into triangles and calculate the polygon's angle sum.

3. What is essential to know or do in class?

a) Introduce the three models for angles: wedge, branch, and dynamic/rotational.

b) Assess if students understand that the size of an angle is determined by its measurement in degrees, *not* the length of its two line segments.

c) Angles can be categorized by measurement: while students may be able to describe acute, right, and obtuse angles, they may not be familiar with the idea of straight, reflex, and whole angles.

d) Have students conjecture about the sum of the interior angles of a triangle, then have them prove or disprove their initial ideas in order to reach the generalization that all triangles have an angle sum of 180°, not just special cases such as equilateral (60°-60°-60°) or isosceles right (90°-45°-45°) triangles.

4. Class Activities

a) Types of angles: Have students recall what they said when describing angles in Session 1's pre-assessment.

Most likely they will refer to types of angles characterized by measurement in degrees. By drawing pictures similar to the ones above, introduce the idea of wedge angles within a closed figure (such as the triangle and bread slices) and angles that radiate out from a vertex like branches of a tree.

Use the analog clock to ask about angles formed when the minute hand moves around the clock's center. Ask how they would determine the time if the clock face did not have numbers.

Finally, ask students to name a real life example for each of these three types of angles.

Ensure that students understand the vocabulary required for the rest of the lesson, including "line segments," "rays," and "vertex."

b) Have students draw two squares of different sizes, then ask which square has the bigger angles. Students may be puzzled by this question, and may ask what you mean by "bigger." Ask what *they* think and to predict what *a child* would think. After hearing their answers, ask why a child may think that the angles in the larger square are bigger than the ones in the smaller square.

Ask students why the angles in both squares are the same even though the size of the squares is different.

Draw two similar triangles on the board and ask the same questions.

Finally, draw two "branching" angles whose angle measurements are the same but have different length rays emanating from the vertex. Ask if these angles are the same. As the discussion develops, elicit from students that angles are the space *between* two rays or line segments but that the length of the line segments or rays do not determine the size of the angle.

c) After students understand that the size of an angle is not determined by the length of the rays or line segments, ask how they would determine if one angle is "bigger" than another.

Draw two acute angles of different sizes and ask which is the "bigger" angle, and why.

Draw a right angle, then an obtuse angle and ask the same question.

Notice how students describe their perceptions.

If the angles look different, how can we quantify their size, and what sort of measurement system should be used? If students mention degrees, remind them of the benchmark angles they worked with in the prior session.

Many will know the terms "acute," "right," and "obtuse" and the angle measurements of less than 90°, exactly 90°, and between 90° and 180°.

When describing an obtuse angle, students may say that it is "greater than 90°. To challenge this, draw a straight line with a point (vertex) on it and ask them what sort of angle they see.

To demonstrate, draw two clock faces on the board and set one to show 3 o'clock. Ask what angle the hands make. Draw 6 o'clock on the second clock and ask what size angle the hands make.

Clarify that this is called a straight angle, and its measure is 180°. Ask through how many degrees the hour hand will have gone by 9 o'clock. Note that angles greater than 180°are called reflex angles, and when the hour hand has gone from midnight to noon, it will have make a complete rotation of 360°.

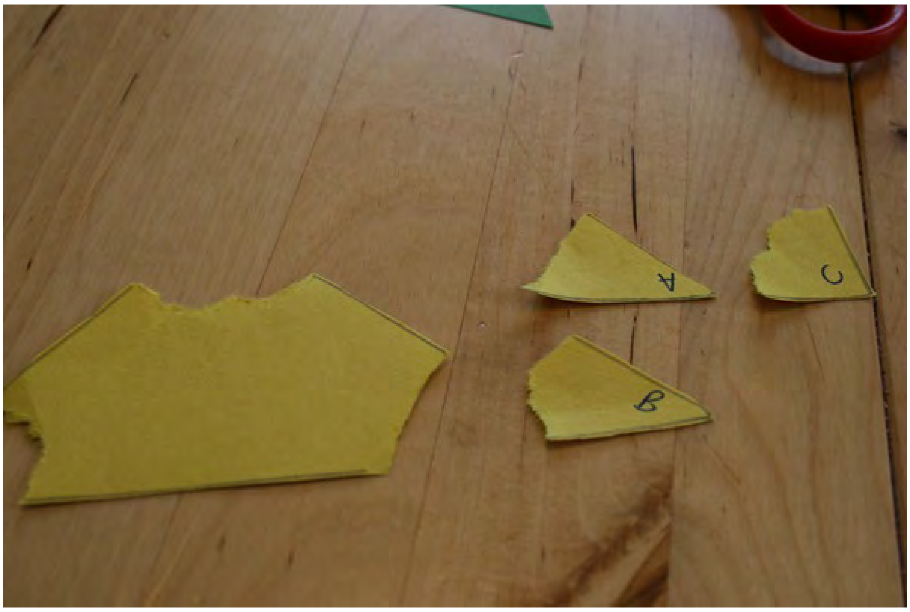
d) Ask students how many total degrees there are in the four angles of a square. What about in a rectangle?

Do they have any idea of how many degrees there are in an equilateral triangle? In a right isosceles triangle?

At this point distribute scissors and ask students to cut out a triangle from notebook paper, label its vertices A, B, and C, and then rip off (*not cut off*) its vertices.

Their triangles will be of various shapes and sizes. This will be helpful for comparison purposes and support a generalization.

http://tinyurl.com/Ripping-Corners



Ask them to rearrange the pieces. What do they notice? Is there more than one arrangement where the three angles are adjacent to each another and still form a 180° (straight) angle?

Ask if this is happening for ***all*** triangles regardless of size? Did anyone have a triangle for which the angle sum is ***not*** 180°? Why or why not?

Finally, ask what generalizations they can make about the angle sum in a triangle. What do they believe to be the total number of degrees in any triangle? Why do they believe this is so?

e) Distribute copies of the following for homework:

1) Pattern Blocks: http://tinyurl.com/Pattern-Block-Template
2) Angles Surrounding a Point: http://tinyurl.com/Angles-Around-Point

Assign the following activities for homework and share the following five websites to review concepts from today's lesson and prepare for the next session.

5. Assignments

a) Have students colour the polygons on the pattern block page:

- Hexagon (yellow)
- Trapezoid (red)
- Square (orange)
- Triangle (green)
- Thin rhombus (tan)
- Other rhombus (blue)

Then have them cut out the shapes and record two things:

- The measure in degrees for each polygon's angles—without using a protractor
- Different ways that various shapes can be arranged around a single point to make a 360° angle. Have them use the angles-around-a-point handout to record discoveries.

  (Students can also experiment with this activity by using the virtual pattern blocks from the website below.)

b) Have students cut out a variety of quadrilaterals (squares, rectangles, rhombuses, trapezoids, parallelograms, and non-defined 4-sided figures).

Using the same process that was done with triangles, have them label and rip off the corners and arrange them. What patterns do they notice?

c) Cut out more quadrilaterals. Cut them in half on a diagonal. What do they notice about the resulting shapes? What about their angle sum? How does that angle sum relate to the angle sum they found in other quadrilaterals? Why do they think this is so?

d) Look through the following resources:

- Angle descriptions with photographs: http://tinyurl.com/Angle-Types
- Angle descriptions with animation: http://tinyurl.com/Angle-Animation
- Angle tutorial: http://tinyurl.com/Angle-Tutorial
- Naming and Measuring angles (video): http://tinyurl.com/Angle-Name-Measure
- Virtual pattern blocks: http://tinyurl.com/Virtual-Pat-Blocks

e) Have students bring the coloured-in cutout pattern blocks to the next class session.

Unit 3 Geometry,
Week 2, Session 2: Angles in Polygons, 360-degrees around a Point

1. What are the important concepts?

a) Any polygon can be dissected into triangles in order to determine the sum of the polygon's interior angles.

This continues the discussion of all triangles having a 180° angle sum from the prior session.

In order to calculate the number of degrees in a given polygon, lines are drawn from one vertex to all the others. These are termed *diagonals*. The construction of these diagonals results in a series of triangles inside the polygon.

This process is called "triangulating the polygon." Since each of these triangles contains 180°, the angle sum of the polygon can be calculated by multiplying 180° by the number of triangles.

b) By calculating (and charting) the angle sum of 3- through 8-sided polygons, a pattern will emerge. This pattern results in a formula that can be used to find the angle sum for a polygon of any number of sides.

This will result in a completed chart that looks something like this:

Polygon	Number of Vertices (n)	Number of triangles	Angle Sum (m)
Triangle	3	1	1(180) = 180
Quadrilateral	4	2	2(180) = 360
Pentagon	5	3	3(180) = 540
Hexagon	6	4	4(180) = 720
Heptagon	7	5	5(180) = 900
...	...	...	...
decagon	10	8	8(180) = 1440
100-gon	100	?	?
n-gon	**n**	**n - 2**	**(n – 2)180**

Note: It is <u>*crucial*</u> that the completed chart <u>*not*</u> be given to the students beforehand. Because pattern detection is such an important mathematical trait, the goal is that the students should not only *learn* the triangle sum theorem, but, that they should *develop* this concept. Adequate time should be allotted for students to develop this concept.

c) All polygons with the same number of sides have the same interior angle sum.

For homework, students cut various quadrilaterals on the diagonal in order to form two triangles, each with an angle sum of 180-degrees. From this activity, students should come to a generalization that this applied not only to squares (the regular quadrilateral) but to *any* quadrilateral.

When students dissect polygons with various number of sides into triangles, they will come to the generalization that regardless of a polygon's shape or the measure of its individual angles, any polygon of n sides will have the same angle sum because it contains the same number of triangles.

For an interactive applet that demonstrates this angle sum consistency, see http://tinyurl.com/Angle-Sum-Applet

d) If several polygons can be arranged around a point without any gaps, the sum of the angles surrounding the point is 360-degrees.

This continues the discussion of angles in the prior session where acute, right, obtuse, and straight angles were emphasized. For homework, students engaged in an activity using pattern blocks (whose angles measured 45°, 60°, 90°, 120°, and 135°) to surround a point. Generalizations from this activity will help students understand the concept of a "whole angle." This concept is important as students begin to work with tessellations in the next session.

2) How do children think about these concepts?

a) When dissecting a polygon, youngsters may not realize that the results need to be triangles. They may assume that any "cut" will be valid.

b) Although most youngsters would understand the word "diagonal" as related to the single line drawn between opposite vertices of a quadrilateral, they may not be aware of a more complete definition of diagonal as it applies to polygons with more than four sides.

This is an opportunity for teachers to clarify the more extended meaning of the word to mean two non-consecutive vertices of a polygon or polyhedron.

c) Children may assume that a "skinny" rhombus (such as the tan pattern block) has a smaller angle sum than the blue rhombus pattern block. This is because their eyes are more apt to notice acute angles rather than obtuse ones.

This is another instance where, in order to build toward a generalization about the angle sum of all quadrilaterals, it is useful to have youngsters actually rip off and arrange all four angles of a paper quadrilateral to show how its angle sum still is 360°.

d) If youngsters are asked to memorize the angle sum formula out of context there is no guarantee that they will remember it.

Thus it is important that they, like the pre-service teachers in this course, actually chart data, notice patterns, come to a generalization, and only then develop a formula.

Not only will this process help them more fully understand the theorem, but it will allow them to reconstruct and use the formula when needed in the future.

e) There *is* an angle of 360° around any point. Students may limit their thinking about angles to acute, right, and obtuse. Perhaps older children may consider straight and reflex angles.

But if they are given a point and asked to describe its surrounding angle, even adults are likely to say 0°. (Just as they may have said 0-degrees when introduced to a straight angle.) To their eyes, the angle around a point is invisible...unless given activities (such as the ones arranging pattern blocks around a point) to help them "see" 360°.

3) What is essential to know or do in class?

a) Dissect polygons into triangles in order to develop (not just provide) the angle sum formula.
b) When given a variety of polygons with n sides, students can demonstrate why all the polygons have a consistent angle sum.
c) Show that there are always 360° around a point.

4) Class Activities:

a) Begin by reviewing the homework.

Discuss the quadrilateral activity first. Ask about student discoveries. Have students extend their thinking by asking what would have happened if a square were not cut in half diagonally, but "crosswise" into two rectangular halves. What was the angle sum of the original square? What is the angle sum of the two resulting rectangles?

Why did the two triangles, which resulted from cutting on the diagonal, have an angle sum equal to that of the original square whereas cutting it crosswise doubled the angle sum?

b) Ask students about their assignment to find the angles and angle sums for each of the pattern blocks. What was the angle sum of the square, trapezoid, and the two rhombuses? Why do they think that was so? What was alike about those four shapes?

c) Ask students about the regular hexagon. What was the angle measurement at each of its vertices? How did they determine that? Did they use benchmark angles? Lay the corners of two equilateral triangles on the hexagon's internal angle? How did they find the hexagon's angle sum? Do they think this angle sum would be true for all hexagons (just as 180-degrees was true for all triangles and 360-degrees was true for all quadrilaterals)?

d) In this next activity, students will develop the angle sum theorem. Again, it is crucial that they experiment and not be given the formula beforehand.

It may be necessary to define what is meant by the term "diagonal" and to review the angle sum of any triangle being 180-degrees. Divide the class into three groups. Have one group work with triangles and hexagons, the second group work with pentagons and decagons, and the third group work with quadrilaterals and octagons. (Note that each group has a second polygon with the number of sides double that of the first. Later you will ask if the angle sum doubles when you double the number of sides. If not, why not?)

Have student groups draw various polygons with a given number of sides, divide them into triangles with diagonals, and calculate the angle sum.

Polygon	Number of Vertices (n)	Number of triangles	Angle Sum (m)
Triangle	3	1	1(180) = 180
Quadrilateral	4	2	2(180) = 360
Pentagon	5	3	3(180) = 540
Hexagon	6	4	4(180) = 720
Heptagon	7	5	5(180) = 900
...	...	...	...
decagon	10	8	8(180) = 1440
100-gon	100	?	?
n-gon	**n**	**n - 2**	**(n – 2)180**

Have students report out to create a chart similar to that above. Ask how doubling the number of sides related to the angle sum. Ask what patterns they notice. As they express their thoughts, ask what the pattern implies about finding the angle sum for a polygon with any number of sides.

Once they have discussed this in informal terms (but before they share a formula), ask students to write a formula they could use to find the angle sum of any polygon. There are several ways to express this, such as (n-2)*180, n*180-360, etc. If different formulas emerge, use these formulas as an opportunity to show how they are equivalent expressions.

e) Return to the homework assignment about arranging pattern blocks around a point to form a 360° angle. Did some students use only the paper cutouts? Did others use the virtual pattern blocks from the Internet? What were their discoveries? How did using a recording sheet help organize their thinking and encourage predictions? What were their findings? Were there any surprises?

5) Assignments and Resources:

a) Have students bring crayons or felt-tip markers to the next class.

b) Have students review the following websites that will review concepts taught in this lesson:

1) Triangle dissection of polygons
   Review of dissecting polygons into triangles: **http://tinyurl.com/AlgLab-Angle-Sum**
2) Angle sums
   Using patterns to discover the angle sum formula (video): **http://tinyurl.com/Angle-Sum-Patterns**
3) Developing the angle sum formula: **http://tinyurl.com/Angle-Sum-Formula1**

4) Using the angle sum formula:

http://tinyurl.com/Angle-Sum-Formula2

http://tinyurl.com/Angle-Sum-Formula3

5) The Angle sum is consistent for any polygon of “n” sides

Interactive applet**:** http://tinyurl.com/Angles-Around-Point

6) There are 360-degrees around a point

http://tinyurl.com/Angles-Around-Point1

http://tinyurl.com/Angles-Around-Point2

Unit 3 Geometry
Week 2, Session 3: Tessellations, Summary

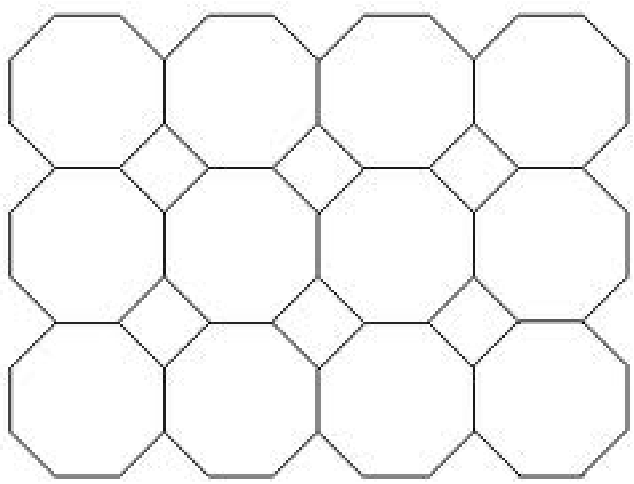
1. What are the important concepts?

a) Some polygons, combination of polygons, or other non-polygonal shapes that are based on polygons can tile a plane and cover the surface without any gaps or overlaps.

Those shapes are said to *tessellate* if they can be arranged in this manner in this manner.

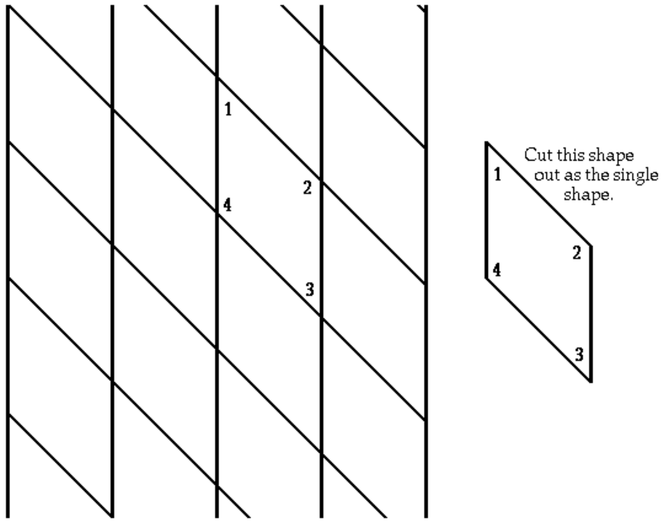
b) Certain regular (equilateral, equiangular) polygons can tessellate using only themselves. These are called regular tessellations. There are only three of these regular tessellations: those composed of equilateral triangles, those composed of squares, and those composed of regular hexagons.
Here is a visual explanation of how these regular polygons form a regular tessellation:
http://tinyurl.com/Tessel-Visual

c) *Semi*-regular tessellations can be created by combining regular polygons, as in the octagon/square design below.



d) Consider the case of a parallelogram where a "slide" of the shape is enough to create a tessellation. However, while all triangles and quadrilaterals *can* tile a plane some may need to be rotated or "flipped" in order to form a tessellation.

See a visual about this at: http://tinyurl.com/Tessel-Tranformations



e) Still other tessellations can be created by distorting triangles, quadrilaterals, and certain hexagons to create carefully designed shapes such as the polygons in the fish pattern and the non-polygonal shapes in M. C. Escher's men on horseback:

f) Tessellations depend on the previously discussed 360° angle around a point in a plane. Unless a point in a plane can be completely surrounded by shapes (without gaps or overlaps) a tessellation cannot exist. This is *not* to say that all arrangements of polygons around a point where the angles total 360° are tessellations.

This is an important mathematical point for pre-service teachers to grasp: the difference between *necessary and sufficient requirements.* While it is *necessary* that all tessellations require 360° surrounding a point, not all combinations totaling 360° around a point are *sufficien*t to create a tessellation.

g) Tessellations are not simply an abstract mathematical construct. They are found in everyday life, such as in these floor tiles from ancient Pompeii (tessellating parallelograms) and this design from a modern tile catalog (the octagon and square pattern mentioned above).

Tessellations are especially prominent in Islamic art. Look at this slide show: http://tinyurl.com/Tessel-Islamic-Art

h) Tessellations beyond the regular and semi-regular are based on distorting certain basic shapes. For example, a square can "squashed" to become a rhombus. Then the opposite sides of that rhombus can be extended to different lengths in order to become a generalized parallelogram. For more ideas about how regular triangles, squares, and hexagons can be changed into new tessellating shapes, see the interactive applet at: http://tinyurl.com/Tessel-Applet

2. How do children think about these concepts?

a) When asked to use polygons to tile a plane, covering it with polygons so that there are no gaps between shapes, children instinctively think about squares, such as those on a chessboard.

Because of their lack of experience, however, children do not realize that *any* triangle or quadrilateral can be repeated to tile a plane. If given a cutout of a quadrilateral (such as the parallelogram above), children can manipulate it and trace it repeatedly to determine that the shape can tile a plane.

b) If students are given a familiar shape (such as an regular octagon), can they predict its ability to tessellate? We often ask children to make similar predictions (in many areas of mathematics) without hands-on experience. In fact, most adults if shown a regular octagon (a "stop sign" shape) assume it will tessellate. It is only by manipulating and tracing a cut out of a regular octagon that they discover that it does *not* form a regular tessellation.

c) Just as children need to be alerted to the polygons and angles they see on an everyday basis, they also need guidance in seeing tessellations in their everyday world.

Regular tessellations of squares are as common as floor tiles, or perhaps a chessboard. Although the design is on a sphere (rather than a plane), children may be familiar with the arrangement of hexagons and pentagons on a soccer ball. Teachers should continue to find opportunities to point out tessellations both in art and in their students' environment.

d) Children can design their own tessellations even before they understand the 360-degree rule. Even children as young as 7-years of age can be given instructions that will allow them to create a tessellation from one of the basic tessellating shapes. Read through the activity Creating Tessellations (which will be used during the class session) that shows how young children, using only scissors, can create unique tessellations.

3. What is essential to know or do in class?

a) Some shapes can tessellate to tile a plane, whereas some cannot.
b) Three regular polygons can tile a plane, resulting in regular tessellations. Several other regular polygons can be combined to create semi-regular tessellations.
c) Tessellations are based on shapes creating a precise 360-degree angle around a point, with no gaps or overlaps
d) Tessellations are common both in real life (such as floor tiles) and in art.
e) Creating a new tessellation is based on distorting a basic geometric shape: triangle, quadrilateral, or hexagon.

4. Class Activities:

a) Begin class by dividing students into groups of four so they can use the cutout pattern blocks they prepared.

b) To introduce the concept of tessellations, refer to the "around-a-point" activity that students discussed in depth during the prior session. Have students recall which one-colour pattern blocks could be used to surround a point. Ask which of those were regular (equilateral, equiangular) shapes.

Do students realize the distinction between the three *regular* polygons (green) equilateral triangle, (orange) square, and (yellow) regular hexagon versus the (blue) rhombus and (red) trapezoid?

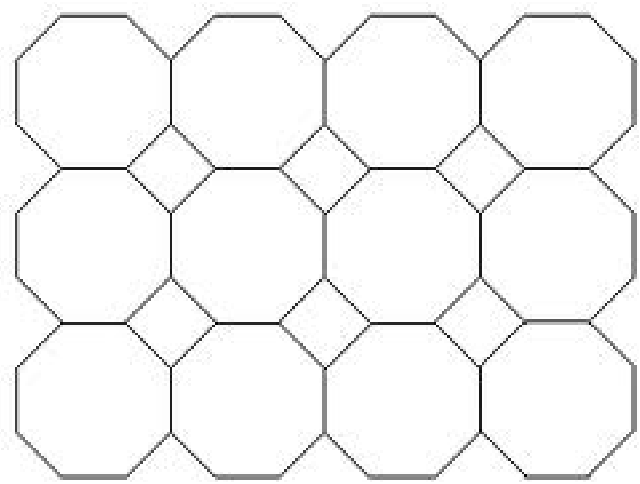
(Note that even if the blue rhombus and red trapezoid are not regular polygons, they are still quadrilaterals and as such, with reorientation, can tile a plane.)

c) To extend their around-a-point experience, introduce the terms *tessellation* and *tiling the plane*, defining them as necessary to make sure students understand that tessellating shapes cannot overlap or have gaps between them.

d) Distribute the pattern block cutouts and have students work in their groups to find the *regular* shapes that tessellate. They should find that the triangles, squares, and hexagons can tessellate. Describe these as *regular* tessellations.

e) Next, ask them if the triangle and square be used together to create a tessellation. If a group discovers only one way that the shapes can form a tessellation, challenge them to find another. (These will be shown in the handouts for the for the next activity.)

Explain that a tessellation that occurs when more than one regular polygon can create a tessellation is called a *semi-regular* tessellation. Even though there is no pattern block for a regular octagon, have students sketch a lattice of regular octagons that are connected to each other. What do they notice about the spaces between the octagons? What type of tessellation is this?



f) Give students the opportunity to work with the remaining pattern blocks to see if they can create tessellations with them. Since the two rhombuses and the trapezoid are quadrilaterals, they can tessellate. However, because they are not regular polygons (equilateral and equiangular) the results are neither regular nor semi-regular. These types of tessellations are termed *non-regular*.

g) To follow up on activity e), where students used equilateral triangles and squares to create semi-regular tessellations, distribute the following two worksheets where two different designs result from triangles and squares: http://tinyurl.com/Tessel-Coloring-Sheets

Have students circle one 360-degree angle on each sheet, then note the polygons that create the angle. How many sides does each have? Have students write that number in the shape. What do they notice about the numbers? (There are four "3"s and two "4"s.)

What is *different* about the numbers on the two sheets. (They are in a different order.) Have students note the header on each page, explaining that they just discovered the notation by which a particular arrangement of equilateral triangles and squares can be specified.

To finish this activity, distribute the colour copies made from http://budurl.com/ColorTessell to each group. Have each student in the group choose a different colour pattern and quickly colour in one of their handouts. As they begin to complete this colouring activity, ask how the tessellation takes on a "look" that was different from when it was simply black lines on white paper.

Remind students that when working with children, using colour to highlight is an important instructional practice.

h) Using the Creating Tesselations handout have each student create his or her own unique tessellating tile. This involves a student's drawing a square, drawing a jagged or undulating line between the two vertices at the top of the square, cutting out the shape defined by the line between it and the "top" of the square, and finally taping the cutout to the "bottom" side of the square. This tile then can be replicated, by tracing, to form a tessellation.

i) Dedicate time at the end of this session to engage in a student-led summary of what they learned about polygons and angles during the past two weeks. Compare their current responses to those that were recorded during the pre-assessment on the first day of Week 1.

What new things have they learned? Which of their personal misconceptions were addressed? How was the manner in which they learned about polygons and angles different from the way they were taught about these topics in high school?

j) Distribute the Regular and Semi-regular homework assignment from: http://tinyurl.com/Tessell-Assign and http://tinyurl.com/Tessell-Cutouts

5. Assignments and Resources

a) Note that the questions on the homework sheet begin with angle sums for n-gons, with the additional stipulation that these are *regular* (equilateral, equiangular) polygons.

How can students use what they know about the triangle dissection of polygons to find the number of degrees for each of the angles in a *regular* polygon?

After they complete the chart, have them use this interactive applet (Tessellation Creator) http://tinyurl.com/Angle-Sum-Applet to answer the remaining questions.

b) Have students read through the following websites to review concepts discussed in class. Make sure they know to click on the images at the bottom of these pages to see real life examples.

1) Regular tessellations: http://tinyurl.com/Tessel-Regular
2) Regular and semi-regular tessellations: http://tinyurl.com/Tessel-SemiRegular
3) Tessellations of quadrilaterals: http://tinyurl.com/Tessel-Quadrilat

c) Have students use this interactive applet to create their own personal tessellations from triangles, squares, and hexagons: http://tinyurl.com/Tessel-Applet

d) Have students review additional coloured-in tessellations at http://budurl.com/ColorTessell as well as the slideshow on patterns (including tessellations) in Islamic art: http://tinyurl.com/Tessel-Islamic-Art

Faculty Notes
Unit 3 Geometry
Week 3: Geometric Measurement--Area, Perimeter, Relationships between Area and Perimeter

Weeklong Overview:

Session 1: Introduction to Area
Session 2: Introduction to Perimeter
Session 3: Exploring the relationship of Area to Perimeter and vice versa

Faculty Preparation for Upcoming Class (1-2 hours)

Review the following article and website on perimeter, area, and the relation between the two:

- Using Representations to Explore Perimeter and Area: http://www.math.ccsu.edu/mitchell/math_409_technology_article_pres.htm (or available at http://tinyurl.com/Perim-vs-Area-409)
- Area and Perimeter of Irregular Shapes: http://tinyurl.com/Area-Perimeter-Irregular

Bring to class:

- Graph paper
- Rulers
- Scissors
- String
- Assorted boxes to measure (Students should be asked to bring these in)

Read through the plans for this week's three sessions

Weeklong Overview:

Session 1 this week begins with the concept of area. Many textbooks start a unit on geometric measurement with perimeter, because it is "one-dimensional" linear measurement.

But children do not instinctively think about measurement "around" something. Rather, living in a world where they see surfaces, they think "two-dimensionally," intuitively understanding area.

This is similar to the way angles and polygons were approached, with t*he way children think rather* than the traditional textbook format, which often begins with the more abstract concept of angle and then moves to polygons, which children see in real life situations every day. Therefore, we will begin with polygons so that children will begin to see how angles relate to the shapes they see around them.

This is why this third week of the Geometry Unit begins with area, then moves to perimeter, and finally explores the relationship between the two.

Two-dimensional shapes are not all polygons. They might be irregularly shaped, such as a lake, one's hand, or a circle. When trying to measure the area of these shapes, it is important that pre-service teachers understand the major mathematical concept that *a measurement is always an estimate*.

Measurement also depends on determining a unit of measurement (which could be standard such as a centimetre, or non-standard such as a paper clip) and then repeating that unit. For adults, this seems obvious, but confusion about this occurs not just with small children but also with older youngsters.

Session 2 is devoted to perimeter. The classroom approach begins with a hands-on experience: exploring perimeter by "wrapping" objects with string and then using a ruler to measure the string's length in order to determine perimeter.

Session 3 addresses the relationship between area and perimeter. The idea of "same area, different perimeter" will be explored in a simple manner by using graph paper. If I trace my hand with fingers closed, I can estimate my hand's area and perimeter. But what happens when I extend my fingers? What does this new tracing look like? What is the area of this new drawing? What is its perimeter? How are the two tracings the same? How are they different?

To explore the opposite concept, "same perimeter, different area," something as simple as a large loop of string or yarn can be illustrative. The string is a given perimeter. But when held by a group of four students, it can be shaped into quadrilaterals with various areas.

Unit 3 Geometry
Week 3, Session 1: Geometric Measurement: Area of Irregular Shapes and Polygons

1. What are the important concepts?

a) Area is a measurement of a two-dimensional surface and is expressed in square units.
b) All two-dimensional surfaces (irregular figures, polygons, circles, and ellipses) can be measured to estimate their area.
c) Various methods and tools can be used to estimate area.
d) All measurements (not just of area) are considered estimates and depend on the level of accuracy of the tools used.
e) Measurement depends on determining a unit of measure and then repeating that unit until an estimated measurement has been obtained.
f) Formulas for finding the area of a rectangle can be used to determine the area of other polygons such as triangles, parallelograms, and trapezoids.

2. How do children think about these concepts?

a) Children initially tend to confuse the concepts of area and perimeter. Once they acquire these misconceptions it is *extremely* difficult for youngsters to "unlearn" them. Therefore they need extensive, hands-on activities in measuring area and perimeter so that they have a clear understanding of how area and perimeter are different and what each of these measurements entails.

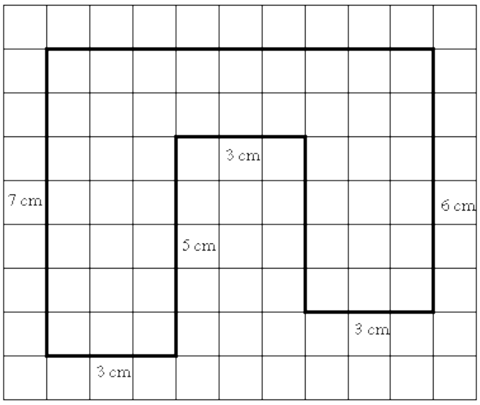
b) If area is introduced in textbooks beginning with squares and rectangles with their dimensions labeled, children tend to assume that area only applies to squares and rectangles.

This is why it is important to begin with the generalized concept of area as the measurement within *any* shape, and why this topic begins by measuring the area inside irregular shapes.

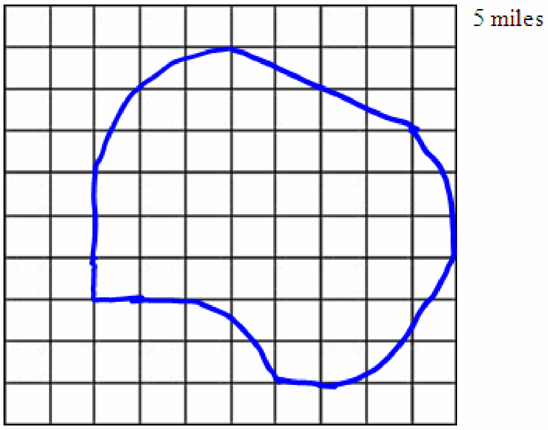
c) Once children hear that area is expressed in "square units" they assume that only "square" (or at least rectangular) surfaces have area. This is another reason to begin with the area of irregular shapes. Beginning with the area of irregular shapes also leads to the idea that circles (which are certainly not square) have an area measured in square units.

d) Textbooks often label rectangles and squares with their dimensions, allowing children to simply multiply the given side lengths in order to find area. However, even if youngsters are able to do this accurately, it does not guarantee that they understand the fundamental concept of area.

e) Elementary school textbooks usually present formulas for area: Length x Width for rectangles, Side x Side for squares, and 1/2 (Base x Height) for triangles. This is often extended to polygons that are divided into simpler shapes where each "sub-shape's" area can be calculated, with the resulting measurements added to find the polygon's total area.



However, this formulaic model does not allow for calculating/estimating the area of non-polygonal two-dimensional shapes such as this "lake":



3. What is essential to know or do in class?

a) Introduce area as a generalized concept, not just a way of measuring the surface of polygons.

b) Have students use various tools, strategies, and formulas to determine the area of irregular shapes, various quadrilaterals, compound polygons, and triangles.

c) Emphasize the three fundamental aspects of measurement:

- Choosing a unit
- Using the unit repeatedly when measuring
- That all measurements are estimates, depending on the how "fine-grained" the measuring instrument is.

4. Class Activities

These include:

a) Using a grid to estimate area of irregular shapes
b) Using a grid to measure areas of square and rectangles

c) Devising a formula to calculate the area of squares and rectangles
d) Cutting and rearranging shapes in order to see the relationship of area for rectangles, triangles, and parallelograms
d) Using the model and formula for calculating the area of rectangles to determine the area of parallelograms and triangles with an understanding of the term "height"
e) Using the model and formula for determining the area of simpler shapes to calculate the area of compound shapes

5. Assignments

a) Have each student bring an empty box to the next class session. (These boxes will be used to explore the concept of perimeter as well as next week's topics: surface area and volume.)

b) Have students visit this website:

- Finding the area and perimeter of irregular shapes: http://tinyurl.com/Area-Perimeter-Irregular

Unit 3 Geometry
Week 3, Session 2: Geometric Measurement, Perimeter of polygons and irregular shapes

1. What are the important concepts?

a) Perimeter is a one-dimensional (linear) measurement that surrounds a two-dimensional figure or three-dimensional object.

b) Perimeter can be measured around irregular figures, polygons, circles, and ellipses (and three-dimensional objects).

c) Various methods and tools can be used to find perimeter.

d) Formulas for finding perimeter may be expressed in various ways.

e) While it may be relatively easy to estimate the area of certain basic shapes (such as triangles, trapezoids, and parallelograms other than squares and rectangles), it is significantly more difficult to estimate their perimeter.

This is because these shapes have sides that do not meet at right angles. (Or as children would say, sides that are "slanted.")

f) If students inquire about the perimeter (circumference) of circles, note that this concept will be explored in detail next week.

2. How do children think about these concepts?

a) If perimeter is introduced by measuring and adding the dimensions of squares and rectangles or by using a formula, children tend to assume that perimeter is a characteristic of polygons.

This is why it is important to begin with the <u>*generalized concept*</u> of perimeter as the measurement around *any* shape. Thus, this topic begins with measuring the perimeter of irregular shapes and then moves to discovering the area of various polygons. (Later in this unit students will investigate the perimeter of circles, a measurement we call circumference.)

b) Textbooks often label shapes with their dimensions, making it relatively simple for children to add the side lengths to find the perimeter.

However, this is not how measuring perimeter occurs in the real world. In everyday situations involving perimeter, the dimensions are *not* known. They need to be measured by using various tools: rulers or tape measures, string for irregular shapes (which then needs to be measured by a ruler or tape measure), and for large shapes (such as the classroom floor) by a tool such as a trundle wheel.

Practicing these types of hands-on measurement activities in the classroom (and outside on the street or play area), not only prepares children to work with perimeter in real life situations, but also prepares them for generalizing their real life experiences into mathematical formulas.

c) Textbooks may give only limited formulas for perimeter. Children need to understand that the textbook's formula is only one of several valid methods of expressing a way to calculate perimeter from given dimensions.

For example, textbooks may express the formula for the area of a square as 4S, implying multiplication. However, the additive formula S + S + S + S is equivalent to 4S, and is equally valid.

Similarly, the perimeter of a rectangle can be expressed as L + L + W + W as well as 2L + 2W or 2 (L + W).

As youngsters begin to develop their own ways of expressing formulas for perimeter, teachers need to emphasize the notion of equivalent expressions, an important mathematical concept addressed in the Algebra unit.

d) As mentioned above, youngsters often have a real difficulty perceiving that the "slanted" side of a figure cannot be measured by counted squares. (This is not so much of a mis*conception* as it is a mis*perception*. Later, this results in the older student's common misconception of the "equilateral right triangle." Try to envision how a youngster might come to this mistaken conclusion.

3. What is essential to know or do in class?

a) Introduce perimeter as a generalized concept, not just a way of measuring around polygons.

b) Have students use various tools to determine the perimeter of irregular shapes, regular polygons, and irregular polygons.

c) Have students develop multiple formulas to express equally valid ways of calculating perimeter.

4. Class Activities

These will include:

a) Using string to measure the perimeter of an irregular shape

b) Using string and a ruler (or a tape measure) to measure the perimeter around the boxes that students brought to class

c) Determining various formulas for calculating the perimeter of squares and rectangles

d) Considering how to determine the perimeter of basic shapes (such as triangles, parallelograms, and trapezoids) whose sides do not meet at right angles

5. Assignments: To be determined by instructor

Unit 3 Geometry
Week 3, Session 3: Relationship between Area and Perimeter

1. What are the important concepts?

a) Area and perimeter are two different types of measurement
b) Shapes with a constant perimeter can vary in their area
c) Shapes with a constant area can vary in their perimeter
d) The mathematical concepts of maximum and minimum can be visualized when working with area and perimeter and charting the resulting measurements

2. How do children think about these concepts?

a) Even when children understand the difference between area and perimeter, they may assume that any rectangle with an area of, for example, 24 square units will have a fixed perimeter.

This is why it is important to have them experiment with making all possible rectangles with a fixed area and whole number sides. This will allow them to see that different shaped rectangles (with different perimeters) can be made from a given number of square units.

b) Research has shown that it is more difficult for youngsters to hold perimeter constant and area variable. This is why it is important to have pre-service teachers use a variety of techniques to explore their own understanding of rectangles with the same perimeter but different areas

3. What is essential to know or do in class?

a) Students will explore the relationship of area and perimeter from two different perspectives:

- Area as a constant with varying perimeters
- Perimeter as constant with varying areas

b) Have students use various tools, strategies, and formulas to determine the area of irregular shapes, various quadrilaterals, compound polygons, and triangles.

4. Class Activities

a) Trace one's hand on grid paper, first with the fingers closed, then with the fingers spread apart. (Constant area, different perimeter.)

b) Use a given number of square tiles (constant area) and arrange them in different configurations to design polygons with different perimeters.

c) Use a 2-metre loop of string or yarn (constant perimeter) and have four students hold it to create rectangles of different areas.

d) Have students chart the results for their findings to discover patterns that imply minimum and maximum.

5. Assignments

a) Have students bring cylindrical objects to the next class session.

Faculty Notes
Unit 3 Geometry
Week 4, Geometric Measurement: Circles (circumference and area), Surface Area of Cuboids and Cylinders

Weeklong Overview:

Session 1: Developing an understanding of pi (π) and its use in determining the perimeter (circumference) of circles
Session 2: Using the concept of "radius squares" to approximate the area of circles
Session 3: Introduction to the surface area of cubes, cuboids, and cylinders

Faculty Preparation for Upcoming Class (1-2 hours)

Read the following article and look through the following websites that address circumference, area of a circle, and surface area of cuboids and cylinders:

- How Many Times does a Radius Square Fit into a Circle? http://mathforum.org/%7Eregis/Flores_Regis.pdf (Also available at: http://tinyurl.com/Radius-Squares-Concept)
- Finding Circumference: http://tinyurl.com/FInd-Circumference
- What is a "radius square"? http://tinyurl.com/What-Is-Radius-Sq
- The Parallelogram method for approximating the area of a circle:
  - http://tinyurl.com/Circle-Parallelogram-1
  - http://tinyurl.com/Circle-Parallelogram-2
- Interactive website that addresses surface area and volume http://tinyurl.com/Surface-Area-vs-Volume

Download and print out for student use:
- Centimetre grid paper http://tinyurl.com/Cm-Grid-Paper
- Radius Squares Handout: http://tinyurl.com/RadiusSq

Bring to class:
- Tape measures
- String
- Rulers
- Scissors
- Glue sticks

- Compasses
- Assorted cylinders and boxes (Have students bring these to class.)

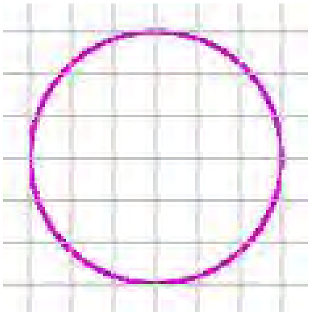
Read through the plans for this week's three sessions

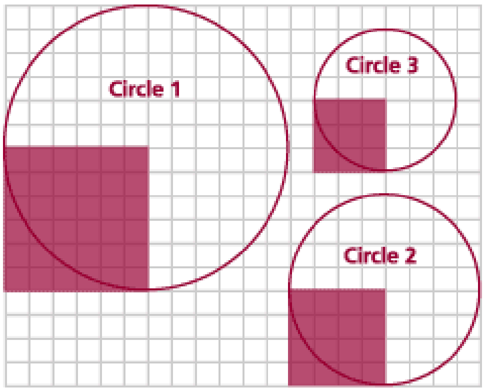
Weeklong Overview:

Session 1 begins by using string to surround various cylinders and by comparing the length of the string to the cylinder's diameter. By charting the results, students should come to the conclusion that there is a relationship between the diameter and the perimeter of the cylinder (which is called *circumference*). They may express this relationship as "three and a little bit more," which will help them understand pi.

During Session 2, students will extend what they learned about pi to find the approximate area of circles.

Just as they explored circumference prior to using a formula, students will explore the area of circles by using a counting method and a novel visualization (radius squares) to make sense of the formula $A = pi * r^2$

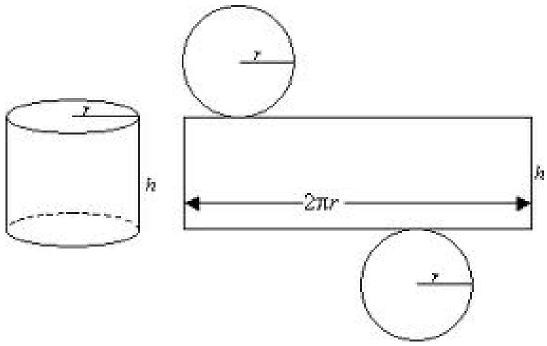




Session 3, the last for this week, builds on what students know about area of both rectangles and circles to introduce the concept of surface area.

Up to now the emphasis has been on the measurement of 2-dimensional plane figures. Surface area, however, relates to the multiple areas that cover a 3-dimensional object.

This means that for cuboids students need to consider not only length and width, but also height. For cylinders they will be dealing with a rectangular piece (like the paper label on a can) as well as a circular top and bottom.



Once again students will derive formulas for the surface area of cuboids and cylinders after measuring and calculating the surface area of various boxes and cans. Students should come up with several different formulas to express the surface area of cuboids. This is an opportunity to discuss how the various formulas are equivalent.

It is also important that students realize that *every* 3-dimensional object in our world has surface area. That means not only other geometric figures such as pyramids, spheres, and dodecahedrons but also tables, cooking pots, and mangoes. To illustrate this point, ask students to consider how many separate areas they would cover if they were painting an entire table (the top, 4 sides, the table's underneath, plus its 4 legs).

Unit 3 Geometry
Week 4, Session 1: Geometric Measurement: Circles, Circumference, Developing Pi

1. What are the important concepts?

a) Circles have a *diameter*, the widest width from edge to edge measured through the circle's center.

A *radius* is the measurement from the center to the edge. It is half the length of the diameter. (Conversely, the diameter is twice the length of the radius.)

b) Circles have a perimeter, which is referred to as *circumference*.

b) Circumference can be measured directly by using flexible tools such as a tape measure.

c) There is a relationship between a circle's diameter and its circumference, which is known as pi.

d) In cases where circumference cannot be measured directly, it can be calculated by using pi.

2. How do children think about these concepts?

a) Since the perimeter of polygons is typically taught first before that of circles, children may think circumference is a totally new topic. They need to understand that circumference is just a specialized term for a circle's perimeter.

b) Textbooks usually treat circumference by giving an approximation of pi (3.14), then giving the formula pi(d) = C or pi(2r) =C, and then finally asking students to calculate the circumference of several circles with different diameters.

However, in order to understand pi and how it relates to *any* circle, regardless of size, youngsters need to measure circular and cylindrical objects, chart their findings, and analyse their data in order to come to the conclusion that pi is a constant that has a meaningful relationship to the diameter of the circle.

c) Children need to envision that when they measure around a 3-dimensional cylinder they are actually measuring a 1-dimensional length.

d) Children need to have a clear understanding of the difference between diameter and radius. This is because the circumference is usually expressed as a function of its diameter, while the same circle's area usually is calculated by using its radius.

3. What is essential to know or do in class?

a) Introduce the vocabulary of circles: diameter, radius, circumference. But do ***not*** introduce pi.

b) Have students measure around cylindrical objects, chart their findings, analyse their results in order to come to an approximation of pi ("three and a little bit more").

c) Allow students to develop the formula for circumference pi(d) = C.

d) Help students understand the difference between direct measurement and calculation by formula.

<u>4. Class Activities</u>

a) Begin by referring to last week's focus on the area and perimeter of polygons and irregular surfaces. Note that those same concepts also apply to another type of figure: circles.

b) Students were asked to bring in cylindrical objects. After having mentioned the vocabulary relating to circles (diameter, radius, circumference) have students work in groups of four to measure the diameter and circumference of the items they brought to class.

Provide some groups with a tape measure. Give other groups string and a ruler.

Give each group chart paper with three columns labeled: *diameter* (d), *Circumference* (C), *C divided by d*. Have each group chart their findings.

c) Discuss their "C divided by d" columns. What pattern do they see? What do they remember about learning about pi in school? How does their direct measurement experiment relate to what they had been taught? How did this activity help them understand what children need to experience in order to understand pi?

<u>5. Assignments (to be determined by instructor)</u>

Unit 3 Geometry
Week 4, Session 2: Geometric Measurement: Area of Circles

1. What are the important concepts?

a) Circles, like irregular figures, have areas that are expressed in square units.
b) The area of a circle, like those of irregular figures, can be estimated by direct measurement: laying a grid over the circle and counting the squares within.
c) There is a relationship between the radius of a circle, its area, and pi.
d) There are various visual images of the formula [pi (r^2) = Area] that can help youngsters understand what this formula means.

2. How do children think about these concepts?

a) Having had the experience of using a grid to estimate the area of irregular shapes, children can transfer that understanding to the area of circles.

b) Textbooks usually give the formula for calculating the area of a circle [pi (r^2) = Area] before children understand why this is so.

Once again, the goal is for youngsters to have firsthand experience before the formula is introduced. In this way they will know what a "radius square" looks like and how it is related to finding the area of a circle.

3. What is essential to know or do in class?

a) Ensure that students understand that the area of a circle (like the area of any irregular surface), when measured by various tools or the approximation of 3.14, can only be an estimate. However, describing a circle's area in terms of pi is mathematically accurate.

b) Have students engage in the following hands-on activities that will help them make sense of the formula.

Begin with the counting the number of squares on grid paper method, and then move to the "radius square" method for approximating the area of a circle.

4. Class Activities

a) Refer to last week's discussion for finding the area of irregular 2-dimensional surfaces. Ask about how using a grid to estimate the area of an irregular surface might relate to today's task: finding the area of a circle.

b) Distribute cm grid paper and ask students to place the point of their compass on an intersection and then draw a circle. (If compasses are unavailable, students can trace around the cylinders that were used in the prior session to find circumference.) How can they estimate the area of their circle?

c) As you move about the room, notice how students deal with the grid's *partial squares*.

d) After students share their direct measurement strategies for estimating the area of a circle, distribute the "squaring a circle" handout. Note their reaction to how "r-squared" can be visualized as an overlay on a circle.

e) Have students develop a formula to express how pi and "radius squares" (or radius "squared") relate to a circle's area.

f) If there is time, introduce the "parallelogram" method” of estimating a circle's area.

g) End the session by asking students how today and yesterday’s activities informed their understanding of how pi relates to a circle’s circumference and area.

5. Assignment

Have students bring a box and a cylinder to the next class session for use in calculating surface area.

Unit 3 Geometry
Week 4, Session 3: Geometric Measurement: Surface Area of Cuboids and Cylinders

1. What are the important concepts?

a) All 3-dimensional objects (not just geometric shapes such as cubes, cuboids, and cylinders) have surface area.
b) Surface area is the sum of the all the areas covering any 3-dimensional object.
c) A "net" is a 2-dimensional representation of a 3-dimensional object's surface area.
d) Formulas for finding the surface area of cuboids may be expressed in various ways.

2. How do children think about these concepts?

a) Just as children tend to confuse the concepts of area and perimeter they also confuse the concepts of surface area and volume.

This may be attributed to a textbook's introducing these two models of measuring 3-dimensional objects in quick succession.

b) Although surface area is a measurement of a 3-dimensional object, its measurement is expressed in *square*, not cubic units.

c) It is helpful for children to envision surface area not as a geometric abstraction but as the simple act of wrapping a box with paper.

Omitting the overlap required for taping, how much paper is needed to cover the box?

For cylinders, what is the area of a can's paper label added to the area of its circular top and bottom?

d) For youngsters, the adjective "surface" in the term "surface area" can imply "only one surface." Thus, they may assume that they need find only one area of the surface rather than *the sum* of all the object's surface areas.

e) In most textbooks, the section on surface area includes diagrams of cuboids and cylinders labeled with their dimensions. This makes it relatively easy for youngsters to simply insert numbers into a surface area formula and perform the calculations. However, this is not how the surface area of boxes and cylinders is found in the real world.

3. What is essential to know or do in class?

a) Introduce surface area as a generalized concept that applies to all 3-dimensional objects, not something unique to cuboids and cylinders.

b) Introduce the idea of a "net" by cutting apart a box to demonstrate how a 3-dimensional object can be transformed into a 2-dimensional plane figure.

c) Have students work with boxes and cans to explore the surface area of cuboids and cylinders.

d) Have students develop multiple formulas to express equally valid ways of calculating surface area.

4. Class Activities

a) Begin by reminding students of how they explored the area of polygons and circles, developing methods for direct measurement and mathematical formulas. Mention how they learned that even irregular figures have a perimeter and area.

Point out several examples in the classroom that have multiple surfaces and thus several areas that would need to be added to calculate surface area, e.g., four walls that need to be painted.

b) Display a cardboard box and discuss that it has *three* dimensions, not only length and width but also height. Create a net by cutting apart the box. Ask students to discuss how an object they originally perceived as 3-dimensional now has only 2 dimensions. Ask how they might find the area of the cardboard.

c) Divide students into groups of four, half the groups will work with a box, the other half with a cylinder. Have them use various measurement tools (ruler, tape measure) to discover the surface area of their object.

Ask them to find a generalized formula for their experiment.

d) Have students share their experiences and their formulas. If students devise several valid formulas, show how these are equivalent. (If students have not come up with various formulas, introduce at least one other.) Ask if some formulas are more efficient than others.

5. Assignments (to be determined by instructor)

Faculty Notes
Unit 3 Geometry
Week 5: Geometric Measurement: Volume, Square Roots (Surds), and Right Triangles

Weeklong Overview:

Session 1: Introduction to the volume of cuboids and cylinders
Session 2: Introduction to square root
Session 3: Introduction to the Pythagorean Theorem

Faculty Preparation for Upcoming Class (1-2 hours)

Look through the following websites:

- Difference between Volume and Capacity: http://tinyurl.com/Volume-vs-Capacity
- Pythagorean Theorem: http://tinyurl.com/Alt-Pythag-Theorem

Download and print out for student use:

- Geoboard dot paper: http://tinyurl.com/Dot-Paper-Geoboard
- Plain dot paper: http://tinyurl.com/Dot-Paper-Plain
- Alternative Proof of the Pythagorean Theorem: http://tinyurl.com/PythagProof2

Bring to class:

- Graph paper
- Rulers

Read through the plans for this week's three sessions

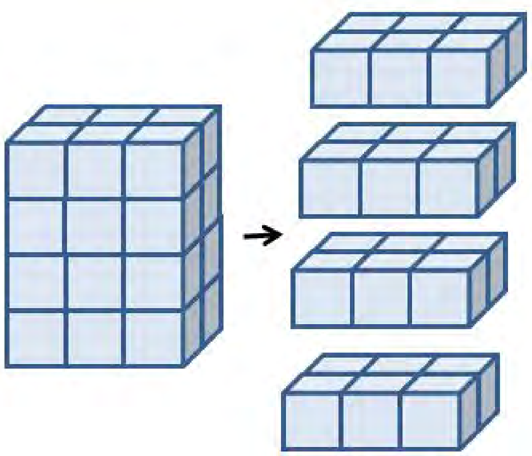
Weeklong Overview:

Session 1 addresses the concept of volume, noting that volume is a characteristic of ***all*** 3-dimensional objects, not just the cubes, cuboids, and cylinders that will be studied in this session.

The volume of a rectangular box is expressed in cubic units, meaning how many "unit cubes" would fill the box.

To visualize this, students should think of covering the bottom of the box with a layer unit cubes. This would be the area of the base. From that base, additional layers would be built until the box is full. Thus, the volume of the box would be the area of the base multiplied by the number of layers (the height in units of the box).

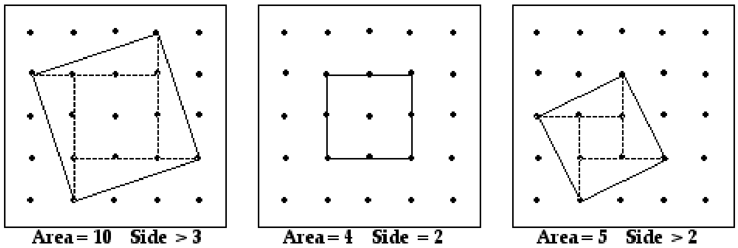
In the following illustration, the base of the cuboid is covered by 6 cubes. Then three more layers are added. The area of the base (6 cubic units) multiplied by a height of (4 layers) results in a volume of 24 cubic units.



This "layering" model helps students visualize why multiplying the length, width, and height is how volume is calculated.

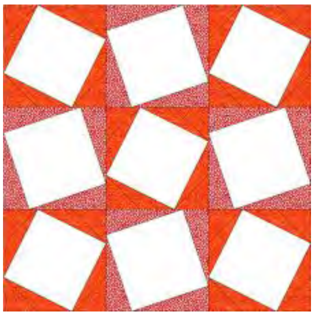
Students will extend this understanding of volume by finding the volume of cylinders. Although the base of a cylinder cannot be covered with a whole number of cubes, students can apply their knowledge of how to find the area of a circle and then determine how many layers of that area would fill the cylinder.

In Session 2, students will be introduced to the concept of square root as the side of a given square. By drawing squares on 5 x 5 dot paper, students will discover that in addition to the 1 x 1, 2 x 2, 3 x 3, and 4 x 4 "upright squares," there are also "tilted squares." Although the *area* of the tilted square can be expressed as a whole number, its side length is <u>*not*</u> a whole number.

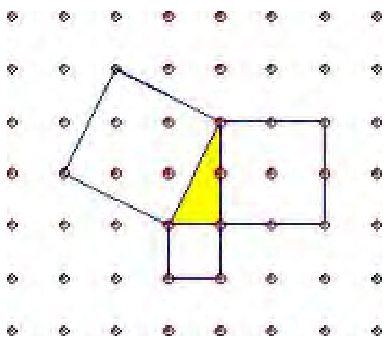


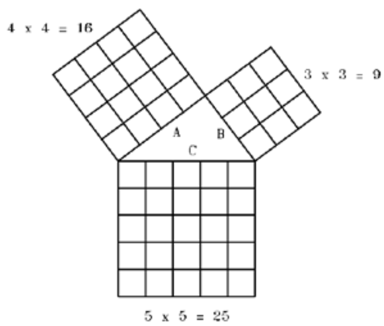
This gives rise to the concept of square root.

Notice the use of both upright and tilted squares in this quilt design. This design is actually an alternate proof for the Pythagorean Theorem.

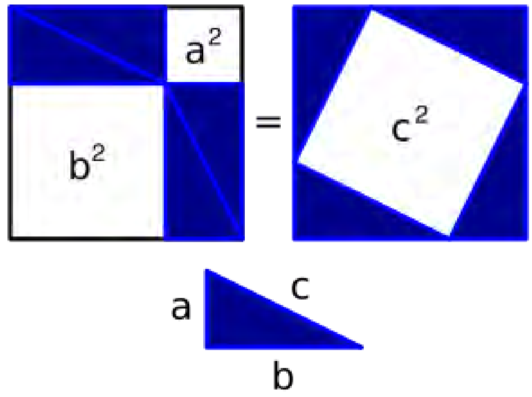


Exploring the nature of squares and square roots leads to the week's Session 3, a brief introduction to the Pythagorean Theorem. Students will not be told the formula $A^2 + B^2 = C^2$; by having students "build" squares on the sides of right triangles they will see how the hypotenuse is related to the triangle's other two sides. Loretta, as meant?





Here is the proof for the “quilt” above:



Unit 3 Geometry
Week 5 Session 1: Geometric Measurement: Volume of Cuboids and Cylinders

1. What are the important concepts?

a) Volume is the amount of space taken up by a 3-dimensional object.
b) Volume can also refer to the *capacity* of an object (such as an empty box or cup) that can be filled.
c) Volume is expressed in cubic units.
d) The volume of a prism or cylinder is calculated by multiplying the area of its base by its height.

2. How do children think about these concepts?

a) Youngsters often confuse surface area and volume, both attributes of 3-dimensional objects and usually taught in tandem.

Children need to work with "nets" to understand surface area, and then fill boxes and cylinders to understand volume. In this way they can see that 1) an empty box can be filled to illustrate volume and 2) the same box, flattened into a net, shows surface area.

b) Children tend to wonder why the volume of a cylinder can be expressed in cubic units when the cylinder is "round," and cannot be tightly packed with unit cubes to calculate volume.

This is why it was important to stress (during the sessions on the area of irregular figures and circles) that "square units" can be used as a form of measurement even if the shape under consideration is not "square."

The same can be true for “cubic units.” A 3-dimensional figure does not have to be a cuboid in order to discuss its volume in cubic units.

c) Volume is something children experience every day without realizing it. Their mothers may use measuring cups when cooking; bottles may have their volume noted on the label. These examples illustrate how an object's capacity can be expressed in cubic units even if the item isn't a cuboid.

3. What is essential to know or do in class?

a) Introduce volume as a generalized concept that applies to both 3-dimensional solids as well as any 3-dimensional object that has a capacity that can be filled.

b) Using what students know about area, have them calculate the volume of the boxes and cylinders that they brought to class.

c) Have students develop generalized expressions that express how to calculate the volume of boxes and cylinders.

4. Class Activities

a) Begin by reminding students of their work finding the <u>*area*</u> of 2-dimensional irregular shapes, polygons, and circles. Note that this will be a key element as they work with the 3-dimensional concept of *volume.*

b) Divide the students into groups of four, giving each group both a box and a cylinder. Ask them to calculate their objects' volume in cubic units.

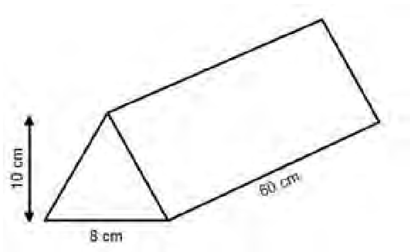
As you circulate around the room notice if they use the area of the base as a starting point and how they incorporate the object's height into their calculation.

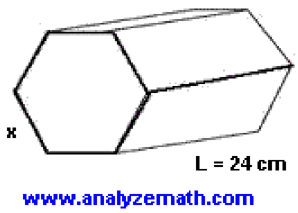
c) Have students record their findings on chart paper and present them to their classmates.

- For cuboids: length, width, height, and volume
- For cylinders: radius, area of the base, height, and volume

As students present their work, note if their method was to build layers upon the object's base area in order to find volume. Ask students to derive a generalized formula for their work.

d) Finally, fold one sheet of computer paper into thirds and another into sixths, creating triangular and hexagonal prisms.





Ask students how they might find the volume of these two new prisms, given what they discovered by working with rectangular boxes and cylinders. What generalities do they notice?

5. Assignments

Have students visit this online tutorial about Volume vs. Capacity:
http://tinyurl.com/Volume-vs-Capacity

Unit 3 Geometry
Week 5, Session 2: Squares, Tilted Squares, Square Roots (Surds)

1. What are the important concepts?

a) The side length of some squares will be rational numbers. Other squares will have side lengths that can only be expressed as an approximation or as the square root of the square's area.

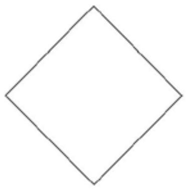
b) Numbers have two square roots, one positive and the other negative.

2. How do children think about these concepts?

a) When asked to draw squares on grid paper or dot paper, most youngsters draw only “upright squares.” Until prompted, they do not consider “tilted squares.”

This is similar to a very young child's perception of a triangle, not seeing a three-sided figure as a triangle unless it is oriented with its base "down."

Even older children may look at a tilted square and call it a "diamond," not perceiving that it has all the characteristics of a square.



b) When asked to name *the length of the side* of a square with *an area* of 2, youngsters usually say, "One." This is because they do not have the words to describe an irrational number.

c) When looking at a square cut in half along its diagonal, youngsters often incorrectly say that this results in "two *equilateral right triangles*" with sides of 1, 1, and 1.

They do not perceive that the hypotenuse is longer than the other two sides.

3. What is essential to know or do in class?

a) Have students use dot paper grids to draw both upright and tilted squares.

b) Introduce the concept and notation of square root.

c) Introduce the idea that the square root of certain numbers are irrational numbers and can only be expressed as approximations.

4. Class Activities

a) Distribute dot paper that is arranged in 5-dot squares. Have students draw as many different size squares as possible, having them label the area of each. (The upright squares will have areas of 1, 4, 9, and 16. The tilted squares will have areas of 2, 5, 8, and 10.)

Note which students stop at 1, 4, 9, and 16 and think they have discovered all possibilities. Prompt them to go beyond these, letting them know there are several more squares for them to discover. (Do not be surprised if they draw the same 1, 4, and 9 squares in different places on their dot paper.)

b) Begin by discussing the area and side lengths of the upright squares, noting that the *area* of square is found by multiplying its side lengths, and this *side length* is called the *square root* (or surd) of a given square.

Ask for the square root of each of the upright squares and introduce the format $\sqrt{16}$ as a way of expressing "the square root of 16." This can be thought of as "4 is the side length of square with an area of 16."

c) Next, turn students attention to the tilted squares they drew. Ask them to consider the square with an area of 2. What is its side length? What number multiplied by itself equals 2? If students have calculators they may try to find this number by guess-and-check, coming up with an approximation of 1.41.

However, this is not exactly the square root of 2. In fact, the most accurate way to express "the square root of 2" is to write it as $\sqrt{2}$. Following this line of thought, ask for the side lengths (square roots) of the other tilted squares.

Mention that $\sqrt{2}$ is called an irrational number, because it cannot be written as a terminating or repeating decimal. Remind students that fractions and terminating or repeating decimals are called "rational" numbers.

You may want to note that pi is also an irrational number, although for practical purposes when measuring circular objects in the real world we tend to use pi's approximation 3.14. (Mention that computers have calculated pi to over 2577 billion decimal places (and $\sqrt{2}$ to over a million decimal places) without finding a repeating pattern of digits.)

d) Remind students of their work with the multiplication of integers, and ask what factors could produce the whole number 4. Since both 2 x 2 and -2 x -2 equal 4, 4 can be said to have two square roots, one positive (2) and the other negative (-2). In fact, every positive number has two square roots. If students ask about the square roots of negative numbers, briefly mention that these are called imaginary numbers.

5. Assignments (to be determined by instructor)

Unit 3 Geometry
Week 5, Session 3: Introduction to the Pythagorean Theorem

1. What are the important concepts?

a) Right triangles have a base, a height, and a hypotenuse.

b) A right triangle's hypotenuse can be calculated by using the squares of its base and height. This is known as the Pythagorean Theorem.

c) There are certain right triangles whose base, height, and hypotenuse are whole numbers (or multiples of whole numbers).

2. How do children think about these concepts?

a) Most adults remember "A squared + B squared = C squared" from high school geometry. However few adults know why this equation makes sense. Thus, when youngsters are given this formula without having opportunities to explore proofs of the Theorem, they have little understanding of it.

b) Even if youngsters have seen a visual proof of the Pythagorean Theorem (such as squares being drawn on each side of a triangle), they are unaware that there are literally hundreds of proofs for the Theorem. It is mathematically important that students realize a theorem can be proved in multiple ways.

c) Youngsters may overgeneralize and assume that if the Pythagorean Theorem applies to all right triangles, then it must apply to *all* triangles. Having youngsters explore this will help them understand the power of a counterexample.

3. What is essential to know or do in class?

a) Have students use what they learned about building upright and tilted squares and the distance between points on a grid to build squares on the sides of right triangles.

b) Have students notice the additive relationship among the squares they drew on the sides of right triangles.

c) Have students devise the formula for the Pythagorean Theorem.

4. Class Activities

a) Remind students of how they were able to draw a "tilted square" with an area of 2, 5, 8, and 10 units in the prior session.

b) Distribute dot paper and ask students to draw a right triangle with a base of 1 and a height of 1. Then direct them to draw a square on each side of the triangle. What do they notice about the three squares that they drew? What is each square's area? Is there a relationship between the three numbers? How does this relate to one of the tilted squares they drew in the prior session?

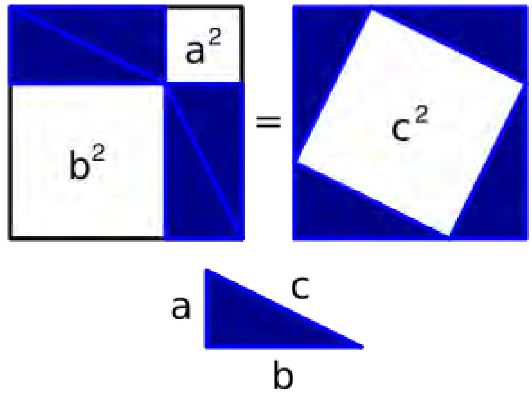
c) Have students draw a right triangle with a base of 3 and a height of 4, then draw squares on each side. Ask if they notice a pattern between their drawings. What is it?

How can it be expressed using the terminology about squares and square roots presented in the prior session?

d) Suggest using symbolic notation for their discovery: that the square on the base + the square on the height equals the square drawn on the hypotenuse. Note that while we could write this as b (base) squared x h (height) squared = H (hypotenuse) squared, it is customary to label the base and the height A and B, and the hypotenuse C, leading to the formula A squared + B squared = C squared or $A^2 + B^2 = C^2$.

e) Note that students can use this formula along with what was learned in the Number and Operations unit about "number families": you can find any one of the three terms by using the two other terms (e.g., C squared - B squared = A squared).

f) Tell students that this method of using squares drawn on a triangle's sides is only one method proving the Pythagorean Theorem. Distribute the handout showing this diagram and ask them to think through this particular proof.



g) Ask students if they think the Pythagorean Theorem works for all triangles, not just right triangles. How might they test this conjecture?

<u>5. Assignments (to be determined by instructor)</u>

Faculty Notes
Unit 4 Information Handling
Week # 1, Faculty Notes: Name?

Weeklong Overview

Session 1: The Data Process, Reading Displays of Data (Information Design), Numerical vs. Categorical Data
Session 2: Displaying Data in the Primary Grades: Tally Marks, Pictographs, Line Plots, Bar Graphs; the "Shape" of the Data

Session 3: Displaying Data in the Elementary Grades: Scatter Plots, Line Graphs; Interpreting Data?

Faculty Preparation for Upcoming Class (1-2 hour)

Look through the following websites that address graphical displays of data:

- From Line Plots to Bar Graphs: http://tinyurl.com/Data-Org-Rep
- Bar Graph Investigations http://tinyurl.com/Bar-Graph-Invest
- Overall graphs: http://tinyurl.com/Menu-of-Graphs
- Misleading graphs: http://tinyurl.com/Misleading-Graphs
- Create-A-Graph applet: http://tinyurl.com/Create-A-Graph-Applet
- Interactive Graph Applet: http://tinyurl.com/Graph-Applet

Download and print out for student use:

- Bar Graph Paper: http://tinyurl.com/Bar-Gr-Paper
- A *colour* copy of Graph Analysis 1 (1 copy per every two students): http://tinyurl.com/GraphAnalysis1
- A *colour* copy of Graph Analysis 2 (1 copy per every two students): http://tinyurl.com/GraphAnalysis2

Bring to class:

- Graph chart paper
- Chart paper markers
- Graph paper
- Centimetre graph paper
- Post-it Notes
- Crayons

Read through the plans for this week's three sessions

Weeklong Overview:

Session 1 will introduce the data process, which is essentially a research process:

1. Posing a question
2. Determining the data needed to answer the question
3. Creating a data collection plan
4. Collecting the data
5. Organizing the data
6. Displaying the data
7. Interpreting the data

In order to prepare for the remainder of the activities in this unit, students will engage in an activity that will allow them to move through all seven steps during this first class session.

Students also will:

- Look at misleading graphs that serve as warnings of what *not* to do when representing data

- Differentiate between categorical and numerical data

- Consider other forms of data displays such as weather maps and dynamic electronic information displays (e.g., a heart monitor) that use mathematical data to provide information quickly and in real time.

Session 2 is devoted to basic displays of data (tally marks, pictographs, line plots, and bar graphs) that primary grade students can be expected to develop from their own questions and own experience. When interpreting these graphs, children will be introduced to the idea that there may be a "shape" or pattern to the data being displayed.

Session 3 will address two types of graphs (scatter plot and line graph) that upper elementary and middle grade children can create when given a data set. Both scatter plots and line graphs need students to plot points on a coordinate plane as they did in the Algebra unit. An informal introduction to "line of best" fit will ask pre-service teachers to notice and interpret a trend in a scatterplot.

Unit 4 Information Handling
Week 1, Session 1: The Data Process, Categorical vs. Numerical Data

1. What are the important concepts?

a) Data can be represented in a variety of ways.

b) When collecting data, there needs to be an agreed-upon protocol.

c) Organizing raw data is a crucial step in determining how it will be represented.

d) Creating displays of data is a means to an end. The display is simply a visual convenience so that the data can be more easily interpreted.

2. How do children think about these concepts?

a) Even very young children can collect data on a routine basis. This can be something as simple as posting a weather icon and the morning temperature on a calendar.

b) As children get older and begin to work more formally with data, they need to "read" data from existing charts and graphs. This sets the stage for their understanding that the fundamental role of a data display is to help them interpret information and then make decisions on the basis of the data.

c) Textbooks usually spend a disproportionate amount of time having youngsters create graphs, and not enough time helping them interpret data displays. To rectify this, teachers need to be alert to graphs and charts in the media that can be brought into class for youngster's discussion and analysis.

3. What is essential to know or do in class?

a) Introduce the data process, emphasizing that this is really a *research* process.

b) Have students work though the process in order to create a bar graph showing their favourite subjects in high school.

c) Introduce the concept of categorical vs. numerical data.

d) Have students consider the implications of the data they collected.

4. Class Activities

a) In order to prepare for the rest of the activities in this unit, students will engage in an activity that will allow them to move through all seven steps of the data process during this first class session:

1. Posing a question
2. Determining the data needed to answer the question
3. Creating a data collection plan

4. Collecting the data
5. Organizing the data
6. Displaying the data
7. Interpreting the data

Students also will 1) look at misleading graphs that serve as warnings of what *not* to do when representing data, 2) learn to differentiate between categorical and numerical data, and 3) consider other forms of data displays such as weather maps and dynamic electronic information displays (e.g., a heart monitor) that use mathematical data to provide information quickly and in real time.

Have students sit in 4-6 person groups for this activity.

1) Pose a question: What do the students in this class think about the subjects they are preparing to teach?

2) Determine the data needed to answer the following survey question such as: When you were in high school, what was your favourite subject: literature, mathematics, social studies, science, or something else?

3) Create a data collection plan: Have students raise their hands as you ask about their favourite subject in the above five categories.

How did students feel about this informal method of data collection? Did it happen too quickly? Did they note the number of students who favoured each topic? Were there some students who liked more than one subject? How could data be collected—on paper—to reflect what just happened?

4) Collect the data: Ask students in their table groups to note their favourite subjects and put that information on paper.

Do not tell students how to do this, but observe how they complete the task.

Did they use numbers? Tally marks? Another graphic model? Did they consider it important to note each others' names? If there were students who enjoyed more than one subject, how did the group members resolve that issue?

5) Organize the data: Bring the class back together and ask how the data collected from multiple table groups can be organized to create a class profile?

Have each group share the way they collected their data. Frequently table groups will use several different methods, which can emphasize that if data is to be combined from several sources there needs to be a uniform data collection system.

Once they discover how other groups handled the information, what do they suggest for aggregating their data? Once they have come to agreement, combine the data.

6) Create a data display: Ask students to confer in their small groups regarding how the whole class data could be displayed. Some may want to create a tally chart. Others may suggest a bar graph. Or a circle graph, based on percentages. (If someone suggests a line graph, which is what youngsters might do, just note the idea at this

time.) Mention that these types of displays will be discussed in the next two class sessions.

At this point, ask students to think about the audience for their information. Is it only for themselves, as an item of interest? Or might their data be combined with those from other university classes to inform educators at the national level? How does considering their audience influence their vision of how their data might look?

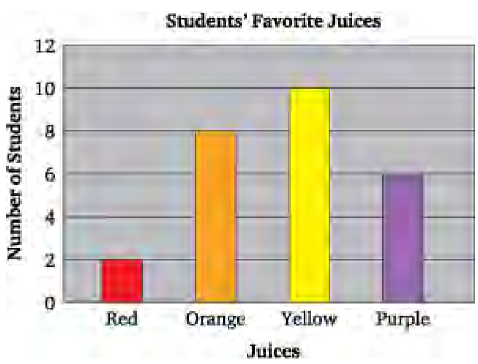
After noting their ideas, say that you would like them to use tally marks to organize the whole class data and then create a bar graph.

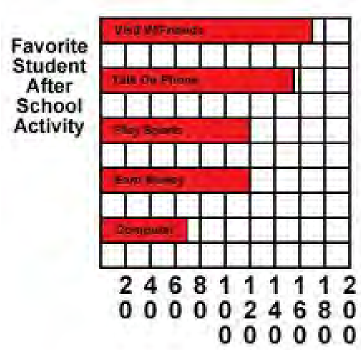
This will give you the opportunity to discuss the difference between categorical and numerical data. Mention that when creating a bar graph they will need to note the categories (subject areas) on one axis, and a number (those who favoured a particular category) on the other.

The difference between categorical and numerical data may be new to students, so take time to help them understand that categorical data can be thought of a "words," such as months of the year for students' birthdays, kinds of foods people prefer, etc. For this activity, the category is "Favourite High School Subjects."

Numerical data involves "numbers" such as how many siblings students have, their height, weight, time spent commuting to class, etc.

It is likely that students are familiar with bar graphs, but ask half of the class to create their graph with the categories on the horizontal axis and the other half with the categories on the vertical axis. These are both formats seen in graphs in news articles and reports. However, many teachers only have children work with the "bars upright" model.





After students have created their graphs on centimetre grid paper, ask them to consider the implications of their data. What trends do they notice? How might the information be useful to you as their instructor? What does the data imply for their becoming teachers of mathematics?

b) End the class by asking them to think of other displays of data that they see in everyday life, in newspapers, on the Internet, in the doctor's office, etc.

5. Assignments

Have students look at this weather site. How many different aspects of weather are represented on the page? Is the communication clear?

http://tinyurl.com/Weather-Images

Unit 4 Information Handling
Week 1 Session 2: Graphing in the Primary Grades

1. What are the important concepts?

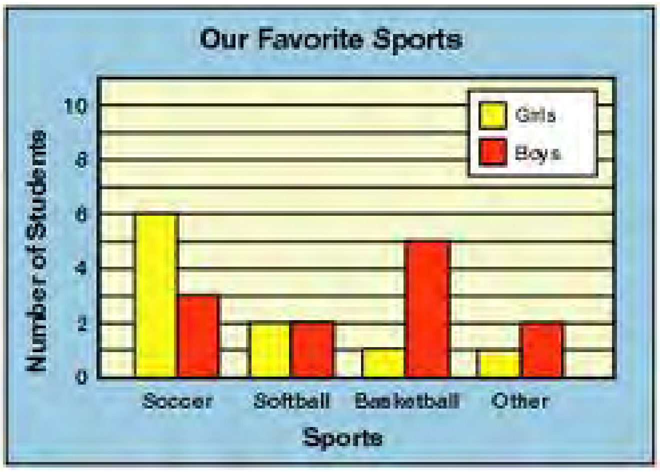
a) Tally marks, pictographs, bar graphs, and line plots can only be used to display discrete, not continuous, data.

b) Tally marks are a simple way to record information visually and then assign a numerical value to the tallies. Each item counted receives one mark; the cross mark needs to be included in the count so that each set of tallies equals five.

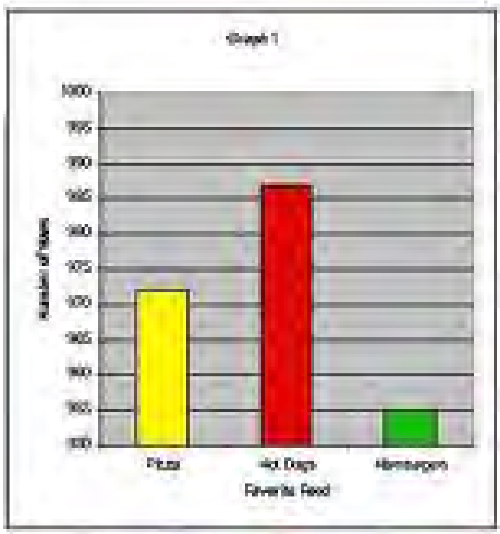
c) Each icon used in a pictograph frequently does not represent a single count. Rather the icons usually refer to multiples of numbers, such as 5, 10, 100, etc. Because of this, there needs to be a *key* accompanying the graph that indicates how many each icon stands for.

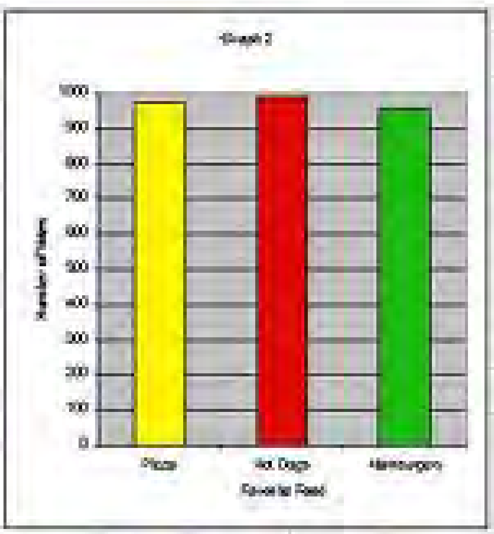
If, for example the key is 1 icon per 100 count, a half icon can be used to denote 50. Also if multiple icons are used, they should be the same size to make the graph easy to interpret.

d) Bar graphs are a visually strong communication device when kept simple. They also can be used to show two data sets side by side as in this illustration.



A problem with bar graphs, however, is that they are often truncated. When the bars do not begin at 0, the height difference between the bars can be misinterpreted as in this illustration.





e) Line plots (as opposed to line graphs) are based on a segment of the number line with an "x" written above a point on the number line for each data entry. Although they are best used for a limited range of data, they quickly show the range, minimum, maximum, clusters, outliers, etc. (These are all vocabulary words that students need to know.) Line plots also begin to help students notice the "shape" of the data.

f) After data have been collected, the information needs to be organized before a display can be designed.

g) Some types of graphs are more suitable than others for conveying information.

h) These simple types of graphs involving discrete data can provide answers to "counting" questions such as, "What was the most popular...?" "How many people chose...?" "Were some things chosen the same number of times?"

<u>2. How do children think about these concepts?</u>

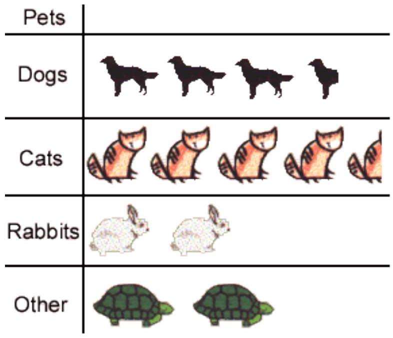
a) Children can begin creating displays of data by using tally marks to record information. Although tally marks can be jotted down informally, children need to see how tally marks can be organized into a more formal display of data such as this chart about cats, dogs, and hamsters (which if plotted on a bar graph would display categorical data).

Favorite Pets

Pet	Tally Marks	Number
	卌 卌	10
	||||	4
	卌 |	6

b) Pictographs are helpful as an introduction to data display for young children. However, consider the following image, which again deals with pets.

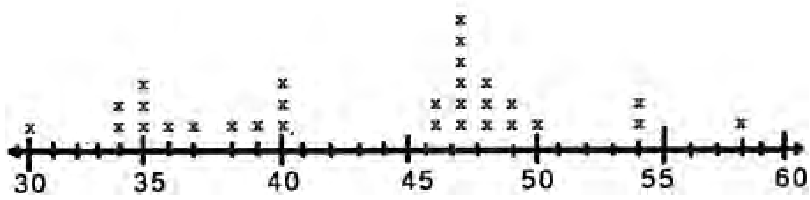
How many pets does each icon stand for? What does the half-image imply? How many respondents might there have been? Might there have been respondents who had more than one pet? Does the size of one pet's image as compared to the others help or confuse the data? These are all essential information handling issues that youngsters need to have help decoding.



c) As mentioned in the past session, bar graphs can be developed with the bars arranged either vertically or horizontally. Youngsters need to see both formats.

d) Line plots are a simple way of displaying data, but not regularly used in the media. Hence there are limited models for teachers to bring to class for students to interpret.

However, because of its simplicity of display and it ability to show trends, line plots are important in the early grades, since they can be used to create bar graphs and they are a way to visualize measures of central tendency, the focus of next week's sessions.



3. What is essential to know or do in class?

a) Introduce tally charts, pictographs, bar graphs, and line plots as a way to display discrete data.

b) Have students read, discuss, and analyse samples of these types of graphs.

c) Have students construct a line plot.

d) Discuss the shape of the data as displayed in various graphs.

e) Discuss both the questions that graphs can answer but also the questions they might raise.

4. Class Activities:

a) Begin by reminding students of what they learned in the algebra unit about discrete vs. continuous graphs. Note that the four types of graphs being studied today are all designed to display discrete data.

b) Have students recall how they used tally marks to record data and create a bar graph in the last class session. Note that in the prior session the emphasis was on constructing a graph, but that it is equally important to be able to read and analyse a graph and raise questions about what the data imply.

Without giving any further directions or comments, have students work in pairs to discuss the graph analysis sheet. What comments and questions do you notice as you circulate about the room? Call the class together for a whole group discussion of their thoughts.

On the pictograph, did they notice the key and the distortion caused by the varying size of the icons? How many responses were represented on this graph? Could this be a chart about pets? Or about favourite animals? How might 125 horses be represented?

For the bar charts, ask if they thought the double bar graphs were clear. How did colour contributed to communication?

What was their first impression when looking at the favourite food graphs? What did they notice on further analysis when comparing them? What is an approximate numerical difference among the three bars? How did the displays either represent or misrepresent the data? Why did this occur?

Line plots may be a new type of display to some students. Ask them what the data might be representing. How does the lack of a title or labels affect their understanding of what the graph means? What do they notice about the shape of the data? If students do not use terms such as range, minimum, maximum, outlier, cluster, gaps, etc., be sure to use them in your own description of the graph. Did they notice that the base of the graph was a segment of a number line? Ask if the data being shown was categorical or numerical.

c) While still working with the whole group, begin the following activity that will result in the creation of a line plot. Ask the students to count the number of letters in their first and last names. To illustrate how raw data needs to be organized, poll students individually (rather than have them raise their hands if they had a certain number of letters in their names). This should result in a disorganized list of data.

How do students suggest the data be organized in order to translate it to a line plot? Once this has been determined, ask what they would suggest for the range of numbers displayed on the number line segment. Should they begin at 0 or some other number? How far do they *need* to go? How far would make a good display? Draw the number line segment they suggested on chart paper. Then have each student come up and put his or her "x" atop their number. Do students realize that in order for the line plot to display data accurately all their "x's" need to be the same size? Have students copy the line plot into their notebooks. Once the graph is complete, begin asking questions about its range, shape, etc.

d) End the session by asking students how they might use their line plot to create a bar graph. If time allows, have them translate the line plot into a bar graph. Otherwise have them do this as a homework assignment. Ask which graph (the line plot or the bar graph) they think more clearly shows the data.

5. Assignments

a) Create a bar graph from the name length line plot.

b) Look through these two websites on bar graphs:

Bar Graph Investigations http://tinyurl.com/Bar-Graph-Investigate

Comparing Columns on a Bar Graph: http://tinyurl.com/Bar-Graph-Compare

Unit 4 Information Handling
Week 1 Session 3: Graphing in the Elementary Grades

1. What are the important concepts?

a) Scatter plots are created on a coordinate plane, with data points plotted on the grid. Like a line plot they show minimum and maximum points, clusters, and outliers. Scatter plots can be used for larger data sets than line plots, and a trend line can be used to show relationships between data.

b) Line graphs, as noted in the Algebra unit, are also composed of points plotted on a coordinate grid. However, because the data are continuous over an interval, the points are connected. An example of this would be distance traveled at a constant rate over time, growth of a plant (again over time), and the time of sunrise over several days.

c) Because most line graphs involve the aspect of time, they have a predictive quality: what will happen after (or between) the graph's data? This can be an opportunity to introduce the idea of extrapolation and interpolation.

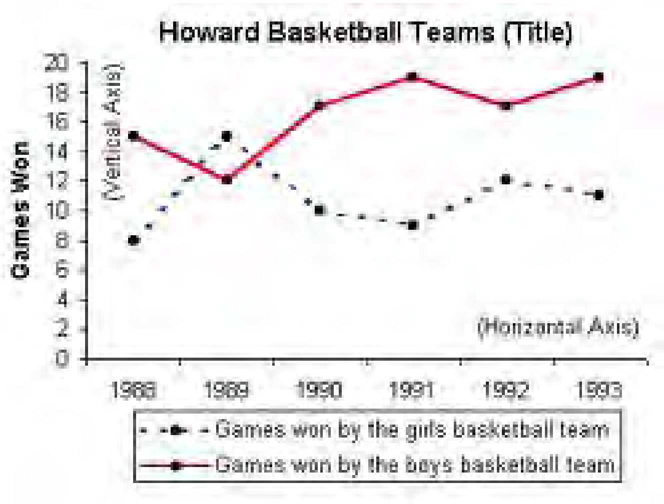
2. How do children think about these concepts?

a) Scatter plots require that youngsters are familiar with plotting points on a coordinate plane. Even if they do not have a formal (x,y) mindset, they can be taught to plot the points from the data by using the axes labels and a 1-to-1 scale on the x and y axes.

b) When looking at the points on a scatter plot, youngsters soon become aware that joining points by a line on this type of graph is not warranted. This is an important step in their understanding the difference between discrete data and continuous data (which can be portrayed on a line graph, described below).

c) Youngsters may think that a line graph needs to be a single straight line, as would occur for a linear function. They may not differentiate between a line graph that is designed for a general audience, which are usually composed of a series of lines even if each line does not really represent consistent change over a given interval. This is seen in the graph below where the data points are given for the end of the season. But no games take place between seasons. In this case the line segments on the line graph attempt to indicate trends informally vs. the way a formal regression line on a scatter plot would.

d) Just as youngsters learned to read and interpret double bar graphs, they need to do the same for multiple line graphs on the same grid such as the image below. It is important that they note the key (in this cases two colours and a straight or broken line) in order to understand the two different data sets displayed.



3. What is essential to know or do in class?

a) Introduce scatter plots and line graphs by relating them to students' prior work plotting points on a coordinate plane in the Algebra unit.

b) Have students read, analyse, and interpret line graphs and scatter plots

c) Introduce the idea of a trend line and the concept of correlation.

d) Help students distinguish between graphs that are developed to show algebraic functions and from those that are designed to communicate information to general audiences.

4. Class Activities

a) This session begins by having students work in pairs to read and interpret the line graph and scatter plot on the handout entitled Graph Analysis 2. Because these are both coordinate graphs, students should be familiar with how to plot points on a coordinate grid and to read a coordinate graph, even when the grid is not visible.

After students have worked in pairs to discuss the line graph and scatter plot, have a whole class discussion of what they discovered, both their questions and the questions at the end of the handout. In particular, note new questions that prior students had not suggested before.

b) Take the opportunity, if no one has raised it, to point out that although the weather graph is called a "line graph" it is not a graph of a single linear function. This is a graph displaying observational data from real world sources in a user-friendly manner for a general audience.

Note, too, that line graphs suggest predictions. This predictive quality allows for extrapolation (what might happen next?) and interpolation (what may have happened between data points?).

Emphasize that line graphs designed for a general audience usually connect data points that *do not* indicate mathematically consistent change over a given interval. Therefore the line graphs being studied today fall into the area of practical Information Handling rather than Algebra.

c) When students consider the scatter plot of plant growth, continue the class discussion in the same inquisitive manner as above, asking: what questions did they think the graph answered? Did the graph answer the student's questions that were included on the handout? Which of these students' questions did they find intriguing? Did the answers pose any new questions ? How did they decide where to draw a trend line? Does this scatter plot show discrete or continuous data?

5. Assignments (to be determined by instructor)

Faculty Notes
Unit 4 Information Handling
Week 2: Measures of Central Tendency

Weeklong Overview:

Session 1: Measures of Central Tendency 1: Frequency Tables, Range, Mode, Median
Session 2: Measures of Central Tendency 2: Mean, Selecting a Model
Session 3: Measures of Central Tendency 3: End of Course Reflection

Faculty Preparation for Upcoming Class (1-2 hours)

Read the following article and look through the following websites that address measures of central tendency:

- Teaching the Mean Meaningfully: http://www.learner.org/courses/learningmath/data/pdfs/session5/mean_1.pdf (or available at http://tinyurl.com/TeachingTheMean)
- Finding the Mean: Distribution Method: http://tinyurl.com/Mean-Distrib-Method

Download and print out for student use:

- Bar Graph Paper: http://tinyurl.com/Bar-Gr-Paper
- Frequency Table & Mean handout: http://tinyurl.com/FreqTable-and-Mean
- Frequency Table & Mode handout: http://tinyurl.com/FreqTableandMode
- End of Course Reflection (2 pages): http://tinyurl.com/EndCourseRefl

Bring to class:

- Chart-sized graph paper
- Scissors
- Enough large beans so that each group of 4 has 24 of them
- Crayons, coloured pencils, or markers

Read through the plans for this week's three sessions

Weeklong Overview:

Session 1 begins by revisiting the work students did with line plots (recording the letters in their names) in order to discuss frequency tables and the range of the data. Note that when they created their original bar graph from *categorical* data in the first class session, the order of the bars was not important.

However, when dealing with *numerical* data on a line plot, the resulting bar graph should show the "shape" of the data. They also will consider data sets where the median is *not* a number in the data set, which will lead them to ask if the median of a data set of whole numbers needs to be a whole number.

Session 2 will address the measure of central tendency that most students will probably recall: the arithmetic mean or what tends to be called the "average." Although calculating the mean usually means doing division, there is a manipulative, more visual model that students can use to explore this topic.

This will be an opportunity to explain that mean, median, and mode are *all* types of "averages" because they denote the "middle" of a data set. Depending on the data's context, one of these measures may be more useful than the others when communicating data. This is why statistics (both in research and in the media) need to be considered carefully, with the reader noting which measure of central tendency the writer is using.

Session 3 is the last session of the entire course. This will be a time to reflect on both the mathematical content students learned and the way that content was presented. There will be a handout for students to write out their thoughts. Then, because group work and full class discussions have been emphasized in this course, the reflection process will be 1) individual written reflection, 2) small group discussion, and 3) whole class discussion.

This session also includes a post-assessment for students to complete.

Unit 4 Information Handling
Week 2 Session 1: Measures of Central Tendency—Frequency Tables, Range, Mode, Median

1. What are the important concepts?

a) Frequency charts show the same data as a horizontal line plot only in vertical tabular form. The numbers on a line plot's number line segment are arranged in order in the left-hand column, while the number of "x's" on the line plot is listed in the right hand column as tallies The line plot and the frequency table show the same distribution of data.

Note that a frequency table may include not just single numbers in order, but ordered *intervals* of numbers as in the second table below.

Score	Tally	Frequency
1	/	1
2	/	1
3	///	3
4	/	1
5	////	4
6	𝍸	5
7	𝍸 /	6
8	𝍸	5
9	///	3
10	/	1

Class interval	Tally	Frequency
0 - 39	I	1
40 - 79	𝍸	5
80 - 119	𝍸 𝍸 II	12
120 - 159	𝍸 III	8
160 - 199	IIII	4
200 - 239	I	1
	Sum =	31

Note the following frequency table, which makes the mistake of having the same number, (20 and other multiples of 10) in two different intervals.

Classes	Frequency	Cumulative Frequency
0 - 10	1	1
10 - 20	4	5
20 - 30	3	8
30 - 40	7	15
40 - 50	7	22
50 - 60	7	29
60 - 70	1	30
Total	30	

b) There are several commonly used measures of central tendency, each of which can be considered an "average" because it denotes the "middle" of a data set.

c) The *range* of the data is found by subtracting the lowest value from the highest value.

d) The *mode* is the category or number that occurs most frequently in a distribution.

Usually, when looking at a line plot or bar graph, the mode would be seen as the tallest "stack" in a line plot or the tallest bar in a bar graph. This does not mean that the mode has the highest *value* in a data set, only that it occurs *most frequently*.

For example, if children among several families have the ages 5, 9, 6, 1, 4, 6, 7, 6, 3, the mode of their ages is 6 years old because that is the number that occurs three times, even though the oldest child is 9. If the data were put into numerical order on a line plot it would show 1, 3, 4, 5, 6, 6, 6, 7, 9 with 6 having the highest frequency (or "stack" of "x's").

e) There can be more than one mode. This type of distribution would be termed bi-modal, as in the following case where the greatest number (330) appears twice (in 1999 and 2000), while the other values appear only once.

f) The median is the number halfway between the minimum and maximum numbers listed in an *ordered* data set. Thus, in the data set above (about children's ages) there are nine numbers. When ages are organized from youngest to oldest (1, 3, 4, 5, ***6,*** 6, 6, 7, 9), the median would be 6 because half the data (four values) are to the left of the highlighted ***6***, whereas the other four values are to the right of that ***6***.

g) Note that in the data set about children's ages there was an *odd* number of values, which allowed for finding the median by counting off an equal number of values to the right and to the left. However, the median in a data set with an *even* number of values might not be a number in the data set. It might not even be an integer.

2. How do children think about these concepts?

a) When youngsters create frequency tables for *categorical* data used to create a tally charts, bar graphs, or pictographs, they do not need to design their table in ascending or descending order.

Favorite Pets		
Pet	Tally Marks	Number
	𝍸 𝍸	10
	||||	4
	𝍸 |	6

However, when working with *numerical* data, it is crucial for youngsters to order their numbers in frequency tables. This will allow them to begin seeing the "shape of the data" as it would look in a line plot or on a bar graph.

b) Youngsters may think that a frequency table can only be created with single numbers such as the “scores” in the table on the left. However, sometimes (especially with large data sets) it is necessary to organize the data in consistent *intervals*, as in the table on the right.

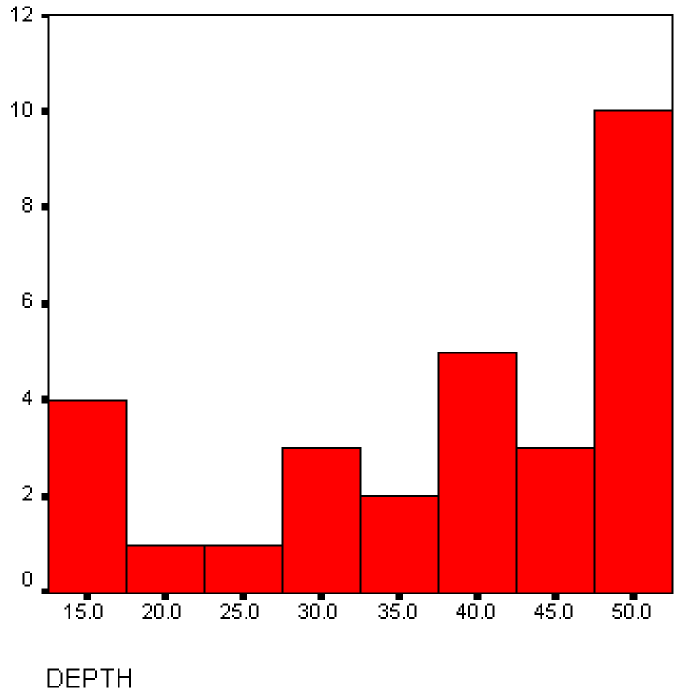
Score	Tally	Frequency
1	/	1
2	/	1
3	///	3
4	/	1
5	////	4
6	𝍸	5
7	𝍸 /	6
8	𝍸	5
9	///	3
10	/	1

Class interval	Tally	Frequency
0 - 39	|	1
40 - 79	𝍸	5
80 - 119	𝍸 𝍸 ||	12
120 - 159	𝍸 |||	8
160 - 199	||||	4
200 - 239	|	1
	Sum =	31

c) When considering the *range* of data, youngsters tend to say the data "range from *a* (the lowest value) to *b* (the greatest value)." While this is an acceptable way to describe the range, youngsters need to know that there is another way to describe it: (*b* - *a*).

d) When considering the mode, a clear data display allows youngsters to have a visual model of the highest value. Confusion occurs, however, when a display of the data set has two "peaks" of the same height. Youngsters need to be assured that there can be more than one mode in a data set.

e) Once youngsters know that there can be more than one mode, they may see a graph such as the one below and become confused when they see that there are values with the same height (as in the intervals labeled 20.0 and 25.0, and 30.0 and 45.0). They may assume these are multiple modes, not remembering that the mode needs to be the "most frequent" in the data set (50.0), not just the "height of the bars that appear most frequently."



Loretta, what does "DEPTH" mean on the above bar graph?

f) If a data set contains an ***odd*** number of entries, it is relatively easy to find the median. Consider this data set: 1, 1, 2, 2, 3, 3, 4, 4, 4, 5, 26.

Youngsters can use the "counting from both ends" strategy to arrive at the correct median: 3.

Other youngsters, however, may add the numbers in the data set and divide that sum by 2. (When the sum for the above 11 entries (55) is divided by 2, the result (27.5) is incorrect.)

g) If a data set contains an ***even*** number of entries, things become more complicated.

There is no "middle value" in the data set when "counting from the ends." Instead, the median is somewhere *between* the middle two values.

In the following data set (3, 7, 16, 25, 32, 39), the median is "somewhere" between 16 and 25. The median is found by adding 16 and 25, then dividing the sum by 2, resulting in 20.5, which not a number in the data set, nor an integer.

Youngsters need help knowing how to interpret their accurate calculations, especially if an integer would be the only *sensible* solution to a real-world data situation. For example: one family has 3 children; another family has 2. The median number of children per family is 2.5—surely not a realistic result, but a valid mathematical one.

3. What is essential to know or do in class?

a) Building on what students have learned informally about data tables when creating tally charts for categorical data, introduce the necessity of putting numerical data in numerical sequence to show patterns, both numerical and visual (to show the shape of the data).

b) Introduce the concept of range.

c) In order to introduce the mode, have students read and analyse several charts and graphs to see how the mode appears in various data displays.

d) Provide opportunities for students to ask questions about multi-modal graphs.

e) Introduce the concept of the median and have students use three methods to find the median of a data set.

f) Provide students with opportunities to analyse situations in which the median is part of both odd and even numbered data sets and where the median is not an integer.

g) Note that both the mode and the median (not just the arithmetic mean) are "averages" that show what is "typical" about the data set.

4. Class Activities

a) Frequency Tables: To build on what students have learned informally about data tables when creating tally charts for categorical data (and function tables in the Algebra unit), use the Frequency Table & Mode handout to introduce the necessity of putting numerical data in a numerical sequence (either ascending or descending).

This will allow students to discern both numerical patterns and the shape of the data.

b) Ask students how they would describe the range (or spread) of their name-length data that they recorded in their notebooks. Note if they use the conversational ("from".... "to"....) format or if they suggest a formula they learned: subtract the lowest value from the highest value.

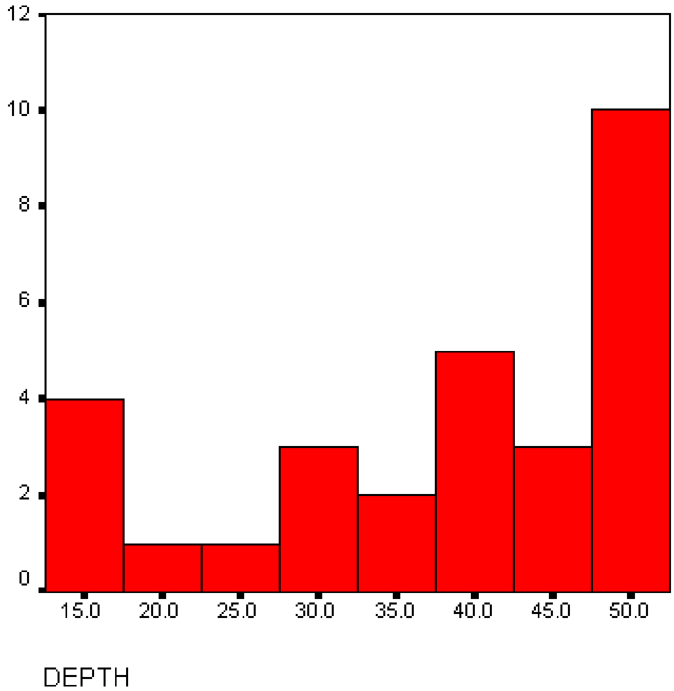
Discuss how both strategies are valid, but that one is more conversational and the other more formally mathematical.

c) Introduce the mode by having students refer to the "name length" line plot and bar graph they created last week. Which number of letters has the highest "stack of x's" in the line plot, and the highest bar on the bar graph?

If there is only one answer to this question, this number is the mode. If there are more than one stack/bar *with the same highest frequency*, the data set is multi-modal (bi-modal if there were two). Emphasize that the mode is the data that *occur most frequently* in a given data set.

Perhaps in students' name length data there are two other stacks/bars of the same height—but not as high as the single tallest one, such as in the graph below.

This does not mean that the two shorter stacks (with a frequency of 1 and 3) are the modes. Although 1 and 3 each occur twice on the histogram, they have a lower frequency than the highest frequency (represented by the tallest bar) at 10. In this case the mode is 10.



Loretta, Can you delete "DEPTH"?
For someone with a very long name, it is likely that the line plot and bar graph will show a very short stack or bar. Students may wonder why, if a name is so long, its stack or bar is so short.

This is an opportunity to clarify that the height of stack or bar does *not* represent name length. Rather, it represents *the number of people* with a particular name length. This is called *frequen*cy because it shows how *frequen*tly a particular number occurs in the data set.

Thus, a long 20-letter name may have a frequency (bar height) of 1, while a 12-letter name may be more typical of the data set and occur 8 times (and have a bar height of 8).

d) To introduce the concept of the median, define it as the value halfway from each end of the data set. Provide a short data set with an *odd* number of entries such as: 2, 2, 3, 5, 7, 8, 9.

Because there are an odd number of entries, the median can be found by counting off a pair of numbers, one from each end, until the single middle value is reached. Ask students how they might describe the mean to youngsters (e.g., half the numbers are to the left of the median, the other half are to its right).

Next, have the students write the numbers (which are already in ascending order) in a horizontal line on centimetre grid paper, one number in each cell. Have them cut out the strip of paper and fold it in half. What do they notice? (That the crease falls on the middle number, in this case, 5.)

The point of using these two hands-on methods is to give students a direct experience in understanding the median. However, with larger data sets counting from each end could cause errors and a strip of paper 43 cells long would be unwieldy.

Now that they have an understanding of the median as the middle number of a data set, it is appropriate to introduce the formula where "n" is the number of entries: (n + 1)/2. In the case of the above data set with 7 entries, this would be (7 + 1)/2 or 8/2. This gives the number 4, indicating that the median is the fourth number in the data set, which in this case is 5.

Note that all of three of the above examples referred to an odd-numbered data set.

Ask students what would happen if you added another entry to the set, perhaps the outlier, 20. The new data set would be 2, 2, 3, 5, 7, 8, 9, 20. What is the median now?

Have them explore this question by using the above three methods: 1) counting from the end, 2) folding the numbered paper strip, and 3) using the formula. What do they notice?

Given that the median is *between* two numbers, ask what they think they should do now? (This is relatively easy to do, since the number 6 is right between the 5 and the 7.) Ask if they think that the median of a data set can be a number *not* in the data set.

Challenge them to find the median in this short data set: 8, 12, 20, 38. In this case the median is still halfway between the 12 and the 20, but it is *not* part of a natural sequence as was 5, 6, 7.

Have students find the median of 2 and 5 as in the case of the median number of children per family in two families, one of which has 2 children and the other has 5 children. In this case the median in this even-numbered data set will be 3.5. Ask students to interpret this answer as it applies 1) to a real life situation and 2) to mathematics without a context.

e) End the session by noting that when students found the median for a data set with an even number of entries they were using a strategy that will be discussed more fully in the next session: the arithmetic mean.

5. Assignments (to be determined by instructor)

Unit 4 Information Handling
Week 2 Session, 2: Measures of Central Tendency—The Arithmetic Mean, Deciding on Measures of Central Tendency

1. What are the important concepts?

a) The arithmetic mean is calculated by adding all the items in the data set, then dividing by the number of items. This measure of central tendency implies that each item in the data set has the same value. Sometimes the mean is not an integer.

b) When calculating the arithmetic mean, teachers need to think about whether they consider this a practice activity in calculation (with sums and division being done by hand), or if this is an activity in Information Handling, which would be an appropriate place for students to use what they know about addition and division and to use a basic hand-held calculator.

c) The arithmetic mean is often called the "average." However, all measures of central tendency that consider what is typical of a data set can be considered to be an average. Hence the mode and the median should be considered averages, too.

d) When reading articles containing statistics, students need to be alert to which measure of central tendency the writer has chosen to use, and for what purpose. Consider the following data set:

0, 0, 0, 0, 0, 50, 50, 100, 100, 100, 4000.

1) *The mean* is a useful measure of central tendency to communicate an even distribution.

However, the mean is influenced by *all* the data, including extremes and outliers. In the above data set, the many 0s and the large outlier have a dramatic effect on the mean, 440.

If these data show salaries of 11 people in a work group (5 volunteers earning no money, 5 interns receiving a small amount (50 or 100), and a project director earning 4000), then the mean average salary for the organization would be 440. Using the mean would make it appear that the five persons represented by 0 had a 440 income, which is not true.

On the other hand, if the data represents 11 people contributing money to share with each other on the basis of need, each person would receive a 440 share. Which would be true.

2) In situations where the data is "skewed," as in the data set above, *the median* may be a more realistic measure, because it is *not* influenced by outliers. Thus, the median of this data set is 50, quite different from the mean of 440.

This is why the median is often used in situations such as real estate values since in a neighbourhood there may be many similarly priced homes and an outlier such as a single house that is quite expensive.

3) *The mode* is 0, because that is the number that occurs most frequently (5 times). If the mode were used as the measure of central tendency and this data set related to earnings, it would show that the unemployed are the most typical segment of the data set.

Students need to ask which measure of central tendency is most appropriate in a given context. Which "average" is most "typical" of the above data set? 440? 50? 0? There is no easy answer to this. It depends on the point of view that the writer wishes to communicate and illustrates why a reader must note the context of the situation and the statistical choice the writer has made.

2. How do children think about these concepts?

a) Simple arithmetic means can be calculated by hand as an introduction to the concept. When students are presented with real life data, however, the paper-and-pencil method becomes cumbersome.

This is where the use of a simple, basic, handheld calculator can make the task not only easy and quick, but move the students to think about the implications of the mean.

b) Just as youngsters find it difficult to believe that the *median* can be a number not in the data set (or not an integer), they are can be equally confused when the arithmetic *mean* turns out to be a fractional or decimal number. ("How can someone have 2.3 siblings?")

They need to understand that they probably did the calculation correctly, but that the mean represents what would happen if all the data were "equalized" for each item in the data set. This becomes more understandable when students have another context for their data such as discovering their test scores resulted in a 93.2 "average" for the term.

c) The definition of the arithmetic mean as "equalizing" each item in a data can be modeled with manipulatives.

d) Youngsters can learn to calculate the mode, median, and arithmetic mean as measures of central tendency. However, their procedural ability does not ensure that they understand why one measure of central tendency might be preferable to another in a particular context.

Because data sets without context become purely procedural for youngsters, they need to work with data sets linked to a real world situations in order to understand the implications of the data. (This is similar to youngsters needing a context when interpreting relationships displayed in graphs.)

Thus, to help youngsters interpret data sets, teachers need to find (or create) data sets that have a real world connection to their students' lives.

e) A data set that includes negative integers will have an impact on the mean.

3. What is essential to know or do in class?

a) Discuss the arithmetic mean, noting that this is the third type of "average" students have been studying.

b) Have students use simple manipulatives to explore the concept of the arithmetic mean.

c) Have students calculate the arithmetic mean of a data set, preferably by using a basic handheld calculator.

d) Discuss how negative numbers, extremes, and outliers can influence the mean.

e) Discuss why a writer might select a particular measure of central tendency (mode, median, or mean) to communicate data to a given audience.

f) Discuss the use of technology in Information Handling.

4. Class Activities

a) Have students work in pairs on Activity 1 on the "Frequency Tables, Arithmetic Mean" handout to create a frequency table organizing those entries in ascending order and to find various measures of central tendency.

23, 25, 32, 32, 33, 40, 40, 41, 41, 41, 49, 49, 50, 51, 52, 58, 67, 73, 77

After students have found the median and mode, ask how they could find the mean of the data set? (They will probably say that they would add the numbers in the data set and then divide that sum by 19.) This is a formulaic strategy: add all the numbers, and divide by the number of entries.

This is an opportunity to discuss using available technology when dealing with Information Handling. Have students who have a handheld calculator with them compute the mean. Ask the remaining students to calculate the mean of the data set by pencil and paper. Have students raise their hands when they find the mean. Overall, which group finished first?

Ask students to consider the goal of Information Handling activities. Is it for Number and Operations practice? Or is it for dealing with data quickly and efficiently so that youngsters have enough time in class to analyse the data and discuss its implications?

b) Students used the arithmetic mean earlier in this unit when they found the median of an even-numbered data set by adding the two numbers in the middle of the set and dividing that sum by two.

However, being able to do this (adding and dividing either by hand or by using a calculator) does not ensure that students understand the concept of "distributing extras" from greater values in the data set to those entries with lesser values, thereby "leveling" all the entries to a common number.

Activity 2 on the "Frequency Tables, Arithmetic Mean" handout will help students visualize how the "distributing and leveling" process works to find the arithmetic mean.

Tell students that they will be using the data set (2, 3, 3, 4, 6, 6), which represents the number of people living in six different families. Ask for the total number of people in these six families. How did students arrive at that number? What is the average number of people in the six families? (Most likely students will refer to the traditional algorithm of adding the six numbers and then dividing by six). Ask why this algorithm works.

Then give students, working in groups of four, 24 large beans. Have them use crayons, coloured pencils, or markers to "colour code" their beans:

2 orange, 3 yellow, 3 green, 4 red, 6 purple, 6 blue

Have students arrange their beans according to color, each color in a vertical line. Then ask them to "distribute" beans from the larger "families" (red, purple, and blue) to the smaller "families" (orange, yellow, and green), trying to make all six lines of beans "level."

What do students discover? Why did this "distribution" result in an "average" of 4?

This "leveled," "evened-out," or "balanced" number is the arithmetic mean. (It is also a number that is part of the data set. However, the mean could be either a number not in the data set or not an integer.)

At this point, ask students how their "leveling" the lines of beans connects to the algorithm for finding the arithmetic mean (adding all the entries in the data set (2 + 3 + 3 + 4 + 6 + 6), and then dividing that sum by 6, the number of entries) .

c) It is likely that students have not considered why, in a particular situation, the mode, median, or arithmetic mean might be the preferred measure of central tendency. If so, be explicit in explaining how these measures are dissimilar by using this set of data:

0, 0, 0, 0, 0, 50, 50, 100, 100, 100, 4000.

Using ideas from the "What are the important concepts?" section above, explain how the *mean* is a useful measure of central tendency when you want an even distribution. But that it is influenced by *all* the data, including extremes and outliers. In the above data set, the many 0s and the large outlier have a dramatic effect on the mean (440).

If the above numbers related to earnings, then it would appear that the five persons with 0 income had an income, which is *not* true. However, if the data represented 11 people sharing their money with each other on the basis of need, each person would receive a 440 share. Which *would* be true.

In situations where the data is skewed, the *median* may be a more realistic measure because it is *not* influenced by outliers. Thus, in the above data set the median is 50—quite a difference from 440.

This is why the median is often used in situations such as real estate values since in a neighbourhood there may be many moderately priced homes with an outlier, a single house that is quite expensive.

The *mode* the number that occurs most frequently (5 times) is 0. If the mode were used as the measure of central tendency and this data set related to earnings, it would show that the unemployed or volunteers were the most typical of the data set.

Have students discuss which measure of central tendency, which "average" do they think is typical of this data set? 440? 50? Or 0? Ask why writers might use a particular measure of central tendency to communicate to their audience.

5. Assignments (to be determined by instructor)

Unit 4 Information Handling
Week 2 Session 3: End of Course Reflection

1. Class Activities

a) In their end of course reflection, students will both write about and discuss the course's mathematical content and the way it was presented.

Distribute the "End of Course Reflection Sheet."

Modelling the way the course was designed, students will:

1) Write their individual reflection anonymously

2) Discuss their thoughts in small group discussion

3) Engage in a whole class discussion.

It is especially important to let students know that their written reflections and the whole class discussion will allow you to collect data about both their mathematical learning and their reaction to pedagogy used during the course.

b) There will be a post-course assessment that students will take after completing their reflection and discussions.

End of Course Reflection

During this course you worked through the four units below.

1) Think back… What stood out for you in each of the four units? What did you find new or interesting? Did you re-think the way you had thought about a particular concept?

- Number and Operations:
- Algebraic Thinking and Algebra:
- Geometry and Geometric Measurement:
- Information Handling:

2) Did you *learn maths better* as a result of taking this course? If so, why do you think this happened? What mathematical topics do you feel you understand more deeply now?

3) *Are you more comfortable with mathematics* (and the idea of teaching mathematics to children) after having taken this course? Why or why not?

4) Reflect on the way the class was structured (whole class introduction, group work, manipulative materials, multiple representations, small group + whole class discussions, etc.)

5) Was this course different from other maths courses you have taken? If so, in what ways?

6) Did the course's interactive lesson design help you have a deeper mathematical understanding of certain topics? Which ones?

7) When you become a teacher, what ideas from this course will you bring into your classroom?